

Two Moore’s theorems for graphs

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*Devoted to the 70-th anniversary of our friend and colleague
Bruno Zimmermann*

ABSTRACT. *Let X be a finite connected graph, possibly with loops and multiple edges. An automorphism group of X acts purely harmonically if it acts freely on the set of directed edges of X and has no invertible edges. Define a genus g of the graph X to be the rank of the first homology group. A finite group acting purely harmonically on a graph of genus g is a natural discrete analogue of a finite group of automorphisms acting on a Riemann surface of genus g . In the present paper, we investigate cyclic group \mathbb{Z}_n acting purely harmonically on a graph X of genus g with fixed points. Given subgroup $\mathbb{Z}_d < \mathbb{Z}_n$, we find the signature of orbifold X/\mathbb{Z}_d through the signature of orbifold X/\mathbb{Z}_n . As a result, we obtain formulas for the number of fixed points for generators of group \mathbb{Z}_d and for genus of orbifold X/\mathbb{Z}_d . For Riemann surfaces, similar results were obtained earlier by M. J. Moore.*

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1. Introduction

Let X be a finite connected graph, possibly with loops and multiple edges. We provide each edge of X (including loops) by two possible orientations. Define the genus g of the graph X to be the rank of its first homology group. An automorphism group of a graph is said to act *harmonically* if it acts freely on the set of its directed edges and *purely harmonically* if it also has no invertible edges. By [2] and [4], a finite group acting harmonically on a graph of genus g is a natural discrete analogue of a finite group of homeomorphisms acting on a closed orientable topological surface of genus g . In particular, in papers [4, 5], a discrete version of the classical $84(g - 1)$ Hurwitz theorem is established. Also discrete versions of the Oikawa [15] and the Arakawa [1] theorems which sharpen the Hurwitz upper bound for various classes of groups were found in [9, 13].

An automorphism of a graph X is said to be *harmonic* if it generates a

cyclic group acting harmonically on X . In paper [12], the authors have found a discrete analogue of the Wiman theorem by which the order of a harmonic automorphism of a graph X of genus $g \geq 2$ does not exceed $2g + 2$ and this bound is achieved for any even g . However, in contrary to the Riemann surface case considered by Wiman, an automorphism of maximal order acts without fixed points. The size of cyclic group acting harmonically on X with given number of fixed points was estimated from the above in [6].

In the present paper, we investigate cyclic group \mathbb{Z}_n acting purely harmonically on a graph X of genus g with fixed points. Given subgroup $\mathbb{Z}_d < \mathbb{Z}_n$, we find the signature of orbifold X/\mathbb{Z}_d through the signature of X/\mathbb{Z}_n . To do this, we use an approach developed by the first named author [10] for the case of Riemann surfaces. As a result, we obtain formulas for the number of fixed points for generators of group \mathbb{Z}_d and for genus of orbifold X/\mathbb{Z}_d . For Riemann surfaces, similar results were obtained earlier by M. J. Moore [14].

2. Basic definitions and preliminary facts

In this paper, a graph X is a finite connected multigraph, possibly with loops. We provide each edge of X including loops, by the two possible orientations. Denote by $V(X)$ the set of vertices and by $E(X)$ the set of directed edges of X . Given $e \in E(X)$, by \bar{e} we denote edge e taking with the opposite orientation. Let $G \leq \text{Aut}(X)$ be a group of automorphisms of a graph X . An edge $e \in E(X)$ is called *invertible* if there is $\varphi \in G$ such that $\varphi(e) = \bar{e}$. Let G act without invertible edges. Define the quotient graph X/G so that its vertices and edges are G -orbits of the vertices and edges of X . Note that if the endpoints of an edge $e \in E(X)$ lie in the same G -orbit then the G -orbit of e is a loop in the quotient graph X/G . We say that the group G acts *harmonically* on a graph X if it acts freely on the set of directed edges $E(X)$ which simply means that, each element of G that fixes an edge $e \in E(X)$ is the identity. If G acts harmonically and without invertible edges, we say that G acts *purely harmonically* on X .

Let G be a finite group acting purely harmonically on a graph X . For every $\tilde{v} \in V(X)$ denote by $G_{\tilde{v}}$ the stabilizer of \tilde{v} in the group G and by $|G_{\tilde{v}}|$ its order. Next to each vertex $v \in V(X/G)$ we prescribe the number $m_v = |G_{\tilde{v}}|$, where $\tilde{v} \in \varphi^{-1}(v)$. Since G acts transitively on each fibre of φ , these numbers are well-defined. The point v , for which $m_v \geq 2$, will be called *branch point of order m_v* . Defining the *genus* of a graph as its cyclomatic number or Betti number (equivalently, rank of the first homology group) we have the following version of the Riemann-Hurwitz formula that can be found in [2, 4, 11].

PROPOSITION 2.1. *Let G be a finite group acting purely harmonically on a graph*

X of genus g . Denote by γ genus of the factor graph X/G . Then

$$g - 1 = |G| \left(\gamma - 1 + \sum_{v \in V(X/G)} \left(1 - \frac{1}{m_v} \right) \right). \quad (1)$$

Observe that actually v in the above sum run over the branch points of X/G .

We prefer to look on the quotient graph X/G as on a one-dimensional orbifold. In this case, the notion of signature is very important. If the group G acts purely harmonically on X , the signature is defined as the sequence $(\gamma; m_1, \dots, m_r)$, where γ is the genus of X/G and m_1, m_2, \dots, m_r are branch orders used in Proposition 2.1. In the case of repetition, we will use power mark to indicate the number of equal entries. For example, we write $(3; 1^3, 2^2, 3^1)$ instead of $(3; 1, 1, 1, 2, 2, 3)$.

3. Finite group acting purely harmonically on a graph

The main technique of the present paper is the uniformization theory of graphs and their coverings [7, 12, 9].

Let G be a finite group acting purely harmonically on a graph X of genus g with the factor space X/G of signature $(\gamma; m_1, \dots, m_r)$. In what follows, we suppose that all vertex stabilizers of G on graph X are cyclic groups.

Let \mathcal{X} be a graph of groups with a trivial group assigned to each vertex and each edge of X . Consider a graph of groups \mathcal{Y} obtained by prescribing the respective group Z_{m_i} , $i = 1, \dots, r$ to each of r points of the branch set and trivial groups to all other vertices and edges of Y . Then the map $\varphi : X \rightarrow Y$ can be naturally extended to a covering $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ of graph of groups. Denote by $\Delta = \pi_1(\mathcal{X})$ and $\Gamma = \pi_1(\mathcal{Y})$ the fundamental groups and by $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ the universal covering trees of graphs of groups \mathcal{X} and \mathcal{Y} respectively. By the Bass uniformization theorem [3, Proposition 2.4], there exists a lift of Φ to an isomorphism $\tilde{\Phi} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$ between covering trees equivariant under the action of Δ and Γ on $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ respectively. We note that $\mathcal{X} \cong \tilde{\mathcal{X}}/\Delta$ and $\mathcal{Y} \cong \tilde{\mathcal{Y}}/\Gamma$. Identifying $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ via isomorphism $\tilde{\Phi}$ we replace the covering $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ by the covering $\tilde{\mathcal{X}}/\Delta \rightarrow \tilde{\mathcal{X}}/\Gamma$ induced by a group inclusion $H \triangleleft \Gamma$ with $\Gamma/\Delta \cong G$.

By [3, p. 7], Δ is a free group of the rank g and $\Gamma = \mathbb{Z}^{*\gamma} * \mathbb{Z}_{m_1} * \dots * \mathbb{Z}_{m_r}$, where $\mathbb{Z}^{*\gamma}$ is a free product of γ copies of \mathbb{Z} . Let \tilde{X} be the universal covering tree of the graph X . Note that \tilde{X} is the underlying graph of the graph of group $\tilde{\mathcal{X}}$. Following traditions in the Riemann surface theory (see, for example, [8]), one can refer to Γ as a *universal covering group* of the orbifold X/G .

4. Cyclic group action on Riemann surfaces and graphs

First of all, we are going to find a discrete version of the following result established earlier in [10]. Then, we will use the obtained result to establish graph-theoretical versions of two Moore's theorems proved in [14].

THEOREM 4.1 (A. D. Mednykh, 1980). *Let S be a compact Riemann surface and \mathbb{Z}_n be a cyclic group of conformal automorphisms of S . Denote by $(\gamma; m_1, \dots, m_r)$ the signature of orbifold S/\mathbb{Z}_n . Let \mathbb{Z}_d be a subgroup of \mathbb{Z}_n of order d . Then the orbifold S/\mathbb{Z}_d has the signature*

$$\left(\gamma_d; (m_1, d)^{n/[m_1, d]}, \dots, (m_r, d)^{n/[m_r, d]} \right),$$

where $(m_i, d) = \gcd(m_i, d)$, $[m_i, d] = \text{lcm}(m_i, d)$, $i = 1, 2, \dots, r$, while γ and γ_d are genera of the respective orbifolds.

We prove the following theorem.

THEOREM 4.2. *Let X be a finite connected graph and \mathbb{Z}_n be a cyclic group acting purely harmonically on X . Denote by $(\gamma; m_1, \dots, m_r)$ the signature of orbifold X/\mathbb{Z}_n . Let \mathbb{Z}_d be a subgroup of \mathbb{Z}_n of order d . Then the orbifold X/\mathbb{Z}_d has the signature*

$$\left(\gamma_d; (m_1, d)^{n/[m_1, d]}, \dots, (m_r, d)^{n/[m_r, d]} \right),$$

where $(m_i, d) = \gcd(m_i, d)$, $[m_i, d] = \text{lcm}(m_i, d)$, $i = 1, 2, \dots, r$, while γ and γ_d are genera of the respective orbifolds.

Proof. Let \tilde{X} be the universal covering graph of orbifold $\mathbb{O} = X/\mathbb{Z}_n$. See [12] for detailed definition. Then there is an action of the group $\Gamma = \mathbb{Z}^{*\gamma} * \mathbb{Z}_{m_1} * \dots * \mathbb{Z}_{m_r}$ on \tilde{X} such that the factor graph \tilde{X}/Γ is isomorphic to \mathbb{O} . That is, Γ is the universal covering group of \mathbb{O} . Moreover, there exists an order preserving epimorphism $\theta : \Gamma \rightarrow \mathbb{Z}_n$ whose kernel is a free group of rank g , where g is genus of graph X .

Denote by H preimage $\theta^{-1}(\mathbb{Z}_d)$. Then H is the universal covering group of orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$. Identifying \mathbb{O}_d with \tilde{X}/H and \mathbb{O} with \tilde{X}/Γ we have the sequence of orbifold coverings $\tilde{X} \xrightarrow{p} \mathbb{O}_d \xrightarrow{q} \mathbb{O}$ induced by group inclusions $I < H < \Gamma$. Since θ is order preserving we have $|\Gamma : H| = |\mathbb{Z}_n : \mathbb{Z}_d| = n/d$. This number coincides with multiplicity of covering q . That is, each edge of \mathbb{O} has exactly n/d preimages in \mathbb{O}_d . Consider a vertex $x \in \mathbb{O}_d$. By [12, Example 1], branch order of x in orbifold \mathbb{O}_d is equal to the size of stabilizer $|H_{\tilde{x}}|$ of group H in any preimage $\tilde{x} \in p^{-1}(x)$. In the same time, branch order of $y = q(x)$

in orbifold \mathbb{O} is equal to the size of stabilizer $|\Gamma_{\bar{x}}|$. Avoiding the points with trivial stabilizer, we can assume that $|\Gamma_{\bar{x}}| = m_j$ for some $j = 1, 2, \dots, r$ and y is a branch point of orbifold \mathbb{O} . Recall [12] that $\Gamma_{\bar{x}}$ is a cyclic group of order m_j . The stabilizer $H_{\bar{x}}$ is formed by those elements of $\Gamma_{\bar{x}} = \mathbb{Z}_{m_j}$ whose images under epimorphism θ belong to \mathbb{Z}_d . Since θ preserves the order of elements, we have $H_{\bar{x}} = \mathbb{Z}_{(m_j, d)}$, where (m_j, d) is the greatest common divisor of m_j and d . So, each element in the fiber $q^{-1}(y)$ of the n/d -fold covering q has branch order (m_j, d) . Notice there are exactly $(n/d) : (m_j/(m_j, d)) = n(m_j, d)/(m_j d) = n/[m_j, d]$ of them. This gives the proof of the theorem. \square

As a consequence of the above result, we obtain the following version of the Moore formula for the number of fixed points well known in the Riemann surface theory [8, 14]. An alternative proof of this result can be obtained by making use of discrete version of Macbeath's formula given in [6].

THEOREM 4.3 (Moore's formula for graphs). *Let \mathbb{Z}_n be a cyclic group acting purely harmonically on a graph X and h be an element of order d , $d > 1$ in the group \mathbb{Z}_n . Denote by $(\gamma; m_1, \dots, m_r)$ the signature of the orbifold X/\mathbb{Z}_n . Then the number of fixed points of h is given by the formula*

$$\sum_{d|m_i} \frac{n}{m_i}.$$

Proof. Consider the canonical map $\varphi : X \rightarrow X/\mathbb{Z}_d$, where \mathbb{Z}_d is a cyclic group generated by h . By Theorem 4.2, the signature of orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$ is equal to

$$\left(\gamma_d; (m_1, d)^{n/[m_1, d]}, \dots, (m_r, d)^{n/[m_r, d]} \right),$$

where γ_d is genus of \mathbb{O}_d .

Let $x \in \mathbb{O}_d$ and $F_x = \varphi^{-1}(x)$ be the fiber of x . Since the covering φ is regular, the group \mathbb{Z}_d acts transitively of the set F_x . So, the fiber F_x contains a fixed point of h if and only if it consists of one element. If x is an ordinary point of orbifold \mathbb{O}_d (that is, branch point of order 1), the fiber F_x consists of $d > 1$ elements and has no fixed points of h . According to the signature of \mathbb{O}_d , for any $i = 1, 2, \dots, r$ we have $n/[m_i, d]$ branch points of order (m_i, d) . The fiber of such a point has length one if and only if $d/(m_i, d) = 1$. The latter is equivalent to $d|m_i$. As a result, the number of fixed points of h is given by

$$\sum_{d|m_i} \frac{n}{[m_i, d]} = \sum_{d|m_i} \frac{n}{m_i}.$$

\square

As a direct consequence of Theorem 4.3 we have the following statement.

COROLLARY 4.4. *Let \mathbb{Z}_n be a cyclic group acting purely harmonically on a graph X and denote by $(\gamma; m_1, \dots, m_r)$ the signature of the orbifold X/\mathbb{Z}_n . Then the number of fixed points of a generator of \mathbb{Z}_n coincides with the number of entities m_i in the signature which are equal to n .*

Now our aim is to obtain Moore's formula for the genus of X/\mathbb{Z}_d . The original result [14, Theorem 4], rewritten in terms of orbifolds, is given by the theorem below. The proof of this theorem in the paper [14] is quite complicated. Our approach, based on Theorems 4.1 and 4.2, allows to get the result quickly.

THEOREM 4.5. *Let G be a cyclic group of automorphisms of order n acting on a Riemann surface S of genus g at least two. Suppose that the signature of orbifold S/G has r_b periods b , for each b dividing n , and, for $d|n$, let G_d be the subgroup of G of order d . Then the orbit space S/G_d has genus γ_d , given by*

$$\gamma_d = 1 + \frac{1}{d}(g-1) - \frac{1}{2d} \sum_{bb'=n} b' r_b ((b, d) - 1),$$

where (b, d) denotes the greatest common divisor of b and d .

The following theorem is a discrete version of the above Moore's theorem for cyclic orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$.

THEOREM 4.6. *Let X be a finite graph of genus g and \mathbb{Z}_n be a cyclic group acting purely harmonically on X . Denote by $(\gamma; m_1, \dots, m_r)$ be the signature of orbifold X/\mathbb{Z}_n . Let \mathbb{Z}_d be a subgroup of \mathbb{Z}_n of order d . Then genus g of graph X and genus γ_d of orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$ are related by the formula*

$$g - 1 = d(\gamma_d - 1) + \sum_{i=1}^r \frac{n}{m_i} ((m_i, d) - 1).$$

where $(m_i, d) = \gcd(m_i, d)$.

Proof. By Theorem 4.2, orbifold $\mathbb{O}_d = X/\mathbb{Z}_d$ has the following signature

$$\left(\gamma_d; (m_1, d)^{n/[m_1, d]}, \dots, (m_r, d)^{n/[m_r, d]} \right).$$

By the Riemann-Hurwitz formula (1) we obtain

$$g - 1 = d \left(\gamma_d - 1 + \sum_{i=1}^r \frac{n}{[m_i, d]} \left(1 - \frac{1}{(m_i, d)} \right) \right). \quad (2)$$

We note that $m_i, d = m_i d$ and

$$\begin{aligned} d \left(\frac{n}{[m_i, d]} \left(1 - \frac{1}{(m_i, d)} \right) \right) &= d \left(\frac{n}{[m_i, d]} - \frac{n}{m_i, d} \right) = \frac{nd}{[m_i, d]} - \frac{nd}{m_i d} \\ &= \frac{n}{m_i} \left(\frac{m_i d}{[m_i, d]} - 1 \right) = \frac{n}{m_i} ((m_i, d) - 1). \end{aligned}$$

Hence,

$$d \sum_{i=1}^r \frac{n}{[m_i, d]} \left(1 - \frac{1}{(m_i, d)} \right) = \sum_{i=1}^r \frac{n}{m_i} ((m_i, d) - 1)$$

and the result follows from (2). \square

Assuming that signature $(\gamma; m_1, \dots, m_r)$ has r_b periods $m_j = b$, for each b dividing n , we restate the main result of Theorem 4.6 in the form

$$\gamma_d = 1 + \frac{1}{d}(g-1) - \frac{1}{d} \sum_{bb'=n} b' r_b ((b, d) - 1).$$

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