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Existence of attractors when diffusion and reaction have polynomial growth

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Dedicated to our 60 years young friend, Julian Lopez Gomez, whose accomplishments have well exceeded his age.

Abstract. *We study an interesting model, with reaction terms of Lotka-Volterra type, where diffusion and reaction have polynomial growth of any order. We establish existence of global attractors as well as exponential attractors. In the sequel we study the long time dynamics of an appropriate semigroup and show that it possesses a global attractor (and exponential attractors) in a certain Banach space.*

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1. Introduction

Consider the following model introduced in [16]:

$$\begin{cases} u_t = \mathrm{div}[\nabla(a_1u + \alpha_{11}u^2 + \alpha_{12}uv) + b_1u\nabla\Phi(x)] + f_1(u,v), \\ v_t = \mathrm{div}[\nabla(a_2v + \alpha_{21}uv + \alpha_{22}v^2) + b_2v\nabla\Phi(x)] + f_2(u,v), \end{cases} \tag{1}$$

where $f_i(u,v)$ are reaction terms of Lotka-Volterra type and quadratic in u,v. The unknowns $u(x,t), v(x,t)$ denote the densities of two species at time t and location $x \in \Omega$, a bounded domain in $\mathbb{R}^2$. Dirichlet or Neumann boundary conditions were usually assumed for (1). This model was used to describe the population dynamics of the species u,v which move under the influence of population pressures and of the environmental potential $\Phi(x)$.

Under suitable assumptions on the coefficients in (1), Yagi [18] proved the global existence of strong solutions (their first derivatives are bounded and their second spatial derivatives exist) to the above system for a planar domain Ω (i.e. $n = 2$).

Let us consider the following system of m equations ($m \geq 2$)

$$u_t = \Delta(\mathcal{P}(u)) + \hat{f}(u, Du)\,, \qquad (x,t) \in Q = \Omega \times (0,T), \tag{2}$$

where $u:\Omega\to{\rm I\!R}^m$, $\mathcal{P}:{\rm I\!R}^m\to{\rm I\!R}^m$ and $f:{\rm I\!R}^m\times{\rm I\!R}^{nm}\to{\rm I\!R}^m$ are vector valued functions. It is clear that (1) is a special case of the above when $m=2$, the components of $\mathcal{P}(u),\hat{f}(u,Du)$ are *quadratic* in u and the terms with the potential $\Phi(x)$ are incorporated in $\hat{f}(u,Du)$, which has *linear* growth in Du.

For simplicity, we assume the homogeneous Dirichlet boundary condition for u (see Section 6 for other boundary conditions) and rewrite (2) as

$$\begin{cases} u_t=\mathrm{div}(A(u)Du)+\hat{f}(u,Du) & (x,t)\in Q=\Omega\times(0,T),\\ u(x,0)=U_0(x) & x\in\Omega\,,\\ u=0 & \text{on } \partial\Omega\times(0,T)\,. \end{cases} \tag{3}$$

Here, Ω is a bounded domain with smooth boundary $\partial\Omega$ in ${\rm I\!R}^2$; $A(u)=\partial_u\mathcal{P}(u)$ is a full matrix $m\times m$ and $\hat{f}:{\rm I\!R}^m\times{\rm I\!R}^{2m}\to{\rm I\!R}^m$. The initial data U_0 is given in $W^{1,p_0}(\Omega)$ for some $p_0>2$, the dimension of Ω. In this paper we will allow $\mathcal{P},\hat{f}$ to have *polynomial growth of any order* in u. We then refer to the above system as the *generalized (1) system.*

We will show that (3) defines a global semiroup $\{\mathcal{S}(t)\}_{t\ge0}$ in the Banach space $X=W^{1,p_0}(\Omega)$, namely

$$\mathcal{S}(0)U_0=U_0,\quad \mathcal{S}(t)U_0(x)=u(x,t)$$

is defined for all $t>0$ with u being the solution of (3). Moreover, this semigroup possesses a global attractor and exponential attractors – a result that, to the best of our knowledge, is established for the first time for (SKT) systems under such general setting.

Let us recall the definition of a global attractor. The notion and conditions for the existence of exponential attractors in Hilbert spaces were introduced in [4]: a set $\mathcal{A}\subset X$ is an exponential attractor if 1) $\mathcal{A}$ is a positively invariant set ($S(t)\mathcal{A}\subset\mathcal{A}$, $\forall t\ge0$), 2) For any $U_0\in X$, $S(t)U_0$ converges expontiallly to $\mathcal{A}$ as $t\to\infty$. We refer the readers to [12] for the notion and existence of exponential attractors of semigroups in Banach spaces.

First of all, we need to establish the global existence result for (3) in order to verify that the associated semigroup is global. In the last few decades, papers concerning strongly coupled parabolic systems like (3) usually relied on a result of Amann in [1, 2] which showed that a solution to (3) exists globally if its $W^{1,p_0}(\Omega)$ norm does not blow up in finite time. This requires the existence of a continuous function $\mathcal{C}$ on $(0,\infty)$ such that for $p_0>n$, the dimension of Ω,

$$\|u(\cdot,t)\|_{W^{1,p_0}(\Omega)}\le\mathcal{C}(t)\,,\qquad \forall t\in(0,T_0). \tag{4}$$

The checking of (4) is very difficult and equivalently requires Hölder continuity of the solution u. The latter is a hard problem in the theory of PDEs as known techniques for the regularity of solutions to scalar equations could not be extended to systems and counterexamples were available. Maximum or comparison principles for systems are also unavailable so that the boundedness of solutions to (3) are generally unknown. The conditions for comparison principles in [14, 15] do not apply to the structure of (2) or even (1). In a recent work by the second author [11], he considered (3) on a domain in $\mathbb{R}^n$ $(n \geq 2)$ and was able to relax the condition (4) by

$$\|u(\cdot,t)\|_{W^{1,n}(\Omega)} \leq \mathcal{C}(t)\,, \qquad \forall t \in (0,T_0). \tag{5}$$

By checking this condition when $n = 2$ in [13], he established the global existence of classical solutions for the generalized (SKT) systems (3) on bounded planar domains. Obviously, (5) does not imply that $|u|$ is bounded so that (3) is not necessarily regularly elliptic, i.e. eigenvalues of $A(u)$ can be unbounded. In fact, in [11] we allowed $A(u)$ to have a polynomial growth of *any* order and assumed that the eigenvalues of $A(u)$ grow like $(\lambda_0 + |u|)^k$ for *any* positive reals λ_0, k.

In this paper, we study long time dynamics of the semigroup defined in (2), a special case of (3), and show that it possesses a global attractor (and exponential attractors) in the Banach space $X = W^{1,p_0}(\Omega)$, for some $p_0 > 2$, if λ_0 is sufficiently large.

To that aim we will make use of the following well known result (e.g., see [4]).

Theorem 1.1. *Let X be a Banach space. The semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ on X possesses a global attractor $\mathcal{A}$ if*

i) there exists an absorbing ball $\mathcal{B}_0$ contained in X for $\{\mathcal{S}(t)\}_{t\geq 0}$. That is, $\mathcal{B}_0$ is bounded and for any bounded subset $\mathcal{B}$ of X there exists $T(\mathcal{B})$ such that $\mathcal{S}(t)(\mathcal{B}) \subset \mathcal{B}_0$ for $t \geq T(\mathcal{B})$,

ii) for some $t_1 > 0$, $\mathcal{S}(t_1) : X \to X$ is compact.

Our paper is organized as follows. In Section 2 we introduce some notations, assumptions, and the statement of our main result on the existence of global attractors. Section 3 establishes uniform estimates for various weighted norms of the solutions in $W^{1,2}(\Omega)$. Since we have to work in the space $X = W^{1,p_0}(\Omega)$ with $p_0 > 2$, we establish similar uniform estimates for the norm in X; thus giving the existence of an absorbing ball required by i) of Theorem 1.1. The proof is fairly technical and will be presented in Section 4. Finally, we establish the required compactness in ii) and prove our main result in Section 5. We also show that the global existence result for (3) can be obtained by modifying some of our argument.

2. Preliminaries and Main Results

Throughout this paper, Ω is a bounded domain with smooth boundary in $\mathbb{R}^2$. For any smooth (vector valued) function u defined on $\Omega \times (0,T)$, $T > 0$, its temporal and spatial derivatives are denoted by u_t, Du respectively. If A is a C^1 function in u then we also abbreviate $\frac{\partial A}{\partial u}$ by A_u. In the sequel, we will write $a \sim b$ if there are two generic positive constants C_1, C_2 such that $C_1 b \le a \le C_2 b$.

As usual, $W^{1,p}(\Omega, \mathbb{R}^m)$, $p \ge 1$, will denote the standard Sobolev spaces whose elements are vector valued functions $u : \Omega \to \mathbb{R}^m$ with finite norm

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}.$$

We assume the following structural conditions.

(A) $A(u)$ is C^1 in u. Moreover, there are constants $\lambda_0, C, k > 0$ and a scalar C^1 function $\lambda(u)$ such that $\lambda(u) \sim (\lambda_0 + |u|)^k$ for all $u \in \mathbb{R}^m$. Furthermore, for any $\zeta \in \mathbb{R}^{nm}$

$$\lambda(u)|\zeta|^2 \le \langle A(u)\zeta, \zeta\rangle \text{ and } |A(u)| \le C\lambda(u). \tag{6}$$

We also assume $|A_u| \le C|\lambda_u|$ and

$$|\lambda_u(u)| \sim C(\lambda_0 + |u|)^{k-1}. \tag{7}$$

For the sake of simplicity, we will consider first the case where the reaction term $\hat{f}$ in (3) does not depend on Du. That is $\hat{f}(u, Du) = f(u)$, satisfying the following growth condition.

(F) There are positive constants ε_0, C and nonnegative C^1 functions $P, F : \mathbb{R}^m \to \mathbb{R}^+$ satisfying $F(0) = P(0) = 0$ and for all $u \in \mathbb{R}^m$

$$|F_u(u)| \le C\lambda^{\frac{1}{2}}(u), \tag{8}$$

$$|P_u(u)| \le C\lambda(u) \tag{9}$$

such that

$$|f(u)||u| \le \varepsilon_0 F^2(u) + C, \tag{10}$$

$$\frac{\lambda^{\frac{1}{2}}(u)|f(u)|}{P(u)+1} \le C(F(u)+1). \tag{11}$$

More generally, we can replace $f(u)$ by a function $\hat{f}$ depending on u, Du and satisfying a linear growth in Du. Namely, we will assume the following.

(F') There exist a constant C and a function $f(u)$ satisfying (F) such that

$$|\hat{f}(u, Du)| \le C\lambda^{\frac{1}{2}}(u)|Du| + f(u), \tag{12}$$

$$|f_u(u)| \le C\lambda(u). \tag{13}$$

Our main result is the following.

THEOREM 2.1. *Assume that (A) and (F') hold. Then there exists $p_0 > 2$ such that (3) defines a semigroup $\{\mathcal{S}(t)\}_{t\ge 0}$ on $X = W^{1,p_0}(\Omega)$, namely*

$$\mathcal{S}(0)U_0 = U_0, \quad \mathcal{S}(t)U_0(x) = u(x,t)$$

and (u(x, t), the solution of (3), is defined for all $t > 0$. Furthermore, if we assume that either ε_0 (see (10)) or the diameter $d(\Omega)$ is small, then for λ_0 is sufficiently large in term of the geometry of Ω and other parameters in (A) and (F), this semigroup possesses a global attractor in X.

Let us discuss some situations where the conditions (A) and (F) can be verified for the generalized (SKT) model, with $A(u), f(u)$ having polynomial growths in u. In many applications, in particular if the components of u are all nonnegative densities of species or chemicals, we have nonnegative numbers k and $\lambda_0 > 0$ such that $\lambda(u) \sim (\lambda_0 + |u|)^k$ and $|f(u)| \sim (\lambda_0 + |u|)^{k+1}$. It is reasonable to assume the growth of $|\lambda_u|$ to be like (7) in (A). Concerning (F), we can take $F(u) = |u|^{\frac{k+2}{2}}$ and $P(u) = |u|^{k+1}$. It is clear that (8) and (9) hold for such choices of F, P. We also have $|f(u)||u| \le (\lambda_0 + |u|)^{k+2} \sim (F(u) + 1)^2$. Thus, (10) is satisfied with ε_0 being the coefficient of the highest power of u in $f(u)$. The (SKT) system

$$\begin{cases} u_t = \mathrm{div}[\nabla(a_1 u + \alpha_{11}u^2 + \alpha_{12}uv) + b_1 u\nabla\Phi(x)] + f_1(u,v), \\ v_t = \mathrm{div}[\nabla(a_2 v + \alpha_{21}uv + \alpha_{22}v^2) + b_2 v\nabla\Phi(x)] + f_2(u,v) \end{cases}$$

introduced in the Introduction section is a special case when $k = 1$.

We easily see that (11) is satisfied because

$$\frac{\lambda^{\frac{1}{2}}(u)|f(u)|}{P(u)+1} \le C\frac{(1+|u|)^{\frac{k}{2}+k+1}}{(1+|u|)^{k+1}} \sim (1+|u|)^{k+1-\frac{k}{2}-1} \le C(1+|u|)^{\frac{k}{2}+1} = C(F(u)+1).$$

Hence, it is clear that the main assumptions in (F) and (13) in (F') are verified.

3. Uniform Estimates in $W^{1,2}(\Omega)$

In this section, we will consider a classical solution to (3) that exists in its maximal time interval $(0, T_0)$. Refering to the results in [13], or Section 5 of this work, we see that $T_0 = \infty$ under the assumptions (A) and (F').

In the proof, when there is no ambiguity C, C_i will denote universal constants that can change from line to line in our argument. Furthermore, $C(\cdots)$ is used to denote quantities which are bounded in terms of theirs parameters.

For any $T_1, T_2, T_3 > 0$ such that $0 < T_1 < T_2 < T_3$ we will say that η is a cutoff function for $[T_1, T_2]$ and $[T_1, T_3]$ if η is a nonnegative C^1 function satisfying $\eta(s) \in [0,1]$ for all s and

$$\eta(s) = \begin{cases} 0 & s \le T_1, \\ 1 & s \ge T_2, \end{cases} \quad \text{and } |\eta'(s)| \le \frac{1}{T_2 - T_1}. \tag{14}$$

In particular, for $T_2 = (T_1 + T_3)/2$ we simply say that such η is a cutoff function for $[T_1, T_3]$.

LEMMA 3.1. *Assume that F satisfies (8) and (10)); and let $T, \tau_0 > 0$. If either ε_0 or $d(\Omega)$ is sufficiently small, then then there is a constant $C(|\Omega|, r)$, which depends also on the parameters in (A) and (F) but not on λ_0, such that*

$$\iint_{\Omega\times[T+\tau_0, T+2\tau_0]} \lambda(u)|Du|^2 \, dz \le C(|\Omega|, \tau_0), \tag{15}$$

$$\iint_{\Omega\times[T+\tau_0, T+2\tau_0]} \lambda^2(u)|Du|^2 \, dz \le C(|\Omega|, \tau_0)(\lambda_0^k + 1). \tag{16}$$

Proof. For any $l \in [0,k]$ we multiply the i^th equation of the system for u with $|u|^l u_i \eta^p(t)$, where η is a cutoff function for $[T, T+2\tau_0]$ and $p > 1$ to be determined later, and integrate over $Q = \Omega \times [T, T+2\tau_0]$. Integrating by parts in x, adding the results and using the fact that $|\eta_t| \le 1/\tau_0$, we easily obtain

$$\begin{aligned} \frac{2}{l+2} \sup_{t\in[T+\tau_0, T+2\tau_0]} \int_\Omega |u|^{l+2} \, dx + \iint_Q \langle A(u)Du, D(|u|^l u)\rangle \eta^p \, dz \\ \le C \iint_Q \left[\langle f(u), |u|^l u\rangle \eta^p + \frac{1}{\tau_0}|u|^{l+2}\eta^{p-1} \right] dz. \end{aligned} \tag{17}$$

Since $n = 2$, we have $\langle A(u)Du, D(|u|^l u)\rangle \ge C(l)\lambda(u)|u|^l|Du|^2$ for some positive constant $C(l)$ (see also (37) below).

By (8) and (10) of (F) we can take $F(u) \sim |u|^{\frac{k+2}{2}}$ and then find a constant C such that

$$\int_\Omega \langle f(u), |u|^l u\rangle \eta^p \, dx \le \varepsilon_0 \int_\Omega |u|^{k+l+2}\eta^p \, dx + C|\Omega|.$$

Here, $|\Omega|$ is the Lebesgue measure of Ω. Using Poincaré's inequality for $|u|^{\frac{k+l}{2}+1}$ and the fact that $\lambda(u) \sim (\lambda_0 + |u|)^k$, we have

$$\begin{aligned} \int_\Omega |u|^{k+l+2}\eta^p \, dx \le Cd^2(\Omega) \int_\Omega |u|^{k+l}|Du|^2\eta^p \, dx \\ \le Cd^2(\Omega) \int_\Omega |u|^l \lambda(u)|Du|^2\eta^p \, dx. \end{aligned} \tag{18}$$

On the other hand, since $k > 0$, if we now fix a p such that $k+2 < (p-1)k$ then $l+2 < (p-1)k$ for all $l \in [0,k]$. For such p, it is clear that $p-1 > p\frac{l+2}{k+l+2}$ so that we can write $p-1 = p\frac{l+2}{k+l+2} + \varepsilon(p)\frac{k}{k+l+2}$ for some $\varepsilon(p) > 0$. Hence, we can use Young's inequality to find some positive constant $C(\varepsilon_0, k, \tau_0)$ such that

$$\frac{1}{\tau_0}|u|^{l+2}\eta^{p-1} \le \varepsilon_0|u|^{k+l+2}\eta^p + C(\varepsilon_0,k,\tau_0)\eta^{\varepsilon(p)} \le \varepsilon_0|u|^{k+l+2}\eta^p + C(\varepsilon_0,k,\tau_0).$$

The integral of the first term on the right can be treated by Poincaré's inequality as before.

Therefore, if either ε_0 or $d(\Omega)$ is sufficiently small then we can deduce from (17) the following estimate.

$$\sup_{t\in[t+\tau_0,t+2\tau_0]}\int_\Omega |u|^{l+2}\,dx + \iint_Q |u|^l\lambda(u)|Du|^2\eta^p\,dz \le C(|\Omega|,\tau_0). \tag{19}$$

For $l=0$ the above implies (15) of the lemma, using the property of η. Multipyling (19) when $l=0$ with λ_0^k and add the result to (19) with $l=k$, we get

$$\iint_{\Omega\times[T+\tau_0,T+2\tau_0]} (\lambda_0^k + |u|^k)\lambda(u)|Du|^2\,dz \le C(|\Omega|,\tau_0)(\lambda_0^k+1).$$

Because $\lambda(u) \sim (\lambda_0 + |u|)^k$, we can use Young's inequality to see that $\lambda(u) \le C(\lambda_0^k + |u|^k)$ for some constant C depending on k. The above then yields

$$\iint_{\Omega\times[T+\tau_0,T+2\tau_0]} \lambda^2(u)|Du|^2\,dz \le C(|\Omega|,\tau_0)(\lambda_0^k+1).$$

This gives (16) and the proof is then complete. □

To proceed, we need the following elementary fact. If $U=0$ on the boundary $\partial\Omega$ then the Sobolev's imbedding theorem for planar domains gives

$$\left(\int_\Omega |U|^4\,dx\right)^{\frac12} \le \int_\Omega |DU|^2\,dx. \tag{20}$$

Hence, if U,V vanish on the boundary $\partial\Omega$ then

$$\int_\Omega |U|^2|V|^2\,dx \le \left(\int_\Omega |U|^4\,dx\right)^{\frac12}\left(\int_\Omega |V|^4\,dx\right)^{\frac12} \le C\int_\Omega |DU|^2\,dx\int_\Omega |DV|^2\,dx. \tag{21}$$

We also note that $\lambda(u)$ is the smallest eigenvalue of $(A+A^T)/2$ and $\Lambda(u)$ is the smallest eigenvalue of A^TA. Thus, if $\mu(u)$ is the eigenvalue of A with smallest real part then $\lambda(u) = \Re(\mu(u))$ and $\Lambda(u) = |\mu(u)|^2$. Therefore,

$$|A(u)\zeta|^2 = \left\langle A^T(u)A(u)\zeta, \zeta\right\rangle \geq \Lambda(u)|\zeta|^2 \geq \lambda^2(u)|\zeta|^2. \tag{22}$$

LEMMA 3.2. *Under the assumptions of Lemma 3.1, there exists a constant $C(|\Omega|, \tau_0)$, which depends also on the parameters in (A) and (F) but not on λ_0, such that*

$$\int_{\Omega\times\{T\}} \lambda^2(u)|Du|^2\, dx \leq C(|\Omega|, \tau_0)(\lambda_0^k + 1)\,, \qquad \forall T > 2\tau_0. \tag{23}$$

Proof. For any $t > 0$ we test the system for u by $A(u)u_t$ (i.e. multiplying the i^{th} equation of (3) by $\sum_j a_{ij}(u)(u_j)_t$, $A(u) = (a_{ij}(u))$, integrating over Ω and summing the results) and integrate by parts to get

$$\int_\Omega \left(\langle A(u)u_t, u_t\rangle + \langle A(u)Du, D(A(u)u_t)\rangle\right) dx = \int_\Omega \langle f(u), A(u)u_t\rangle\, dx. \tag{24}$$

Because $A(u) = \mathcal{P}_u$, we have $\langle A(u)Du, D(A(u)u_t)\rangle = \frac{1}{2}\frac{\partial}{\partial t}|D\mathcal{P}(u)|^2$. Thus, we can rewrite (24) as

$$\int_\Omega \left(\langle A(u)u_t, u_t\rangle + \frac{1}{2}\frac{\partial}{\partial t}|ADu|^2\right) dx = \int_\Omega \langle f(u), A(u)u_t\rangle\, dx.$$

The ellipticity of $A(u)$ then gives

$$\int_\Omega \lambda(u)|u_t|^2\, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |ADu|^2\, dx \leq C\int_\Omega \lambda(u)|f(u)||u_t|\, dx.$$

Using Young's inequality, we find a constant $C(\varepsilon)$ such that for any $\varepsilon > 0$

$$|f(u)|\lambda(u)|u_t| \leq \varepsilon\lambda(u)|u_t|^2 + C(\varepsilon)\lambda(u)|f(u)|^2.$$

For sufficiently small and fixed ε we then have

$$\frac{d}{dt}\int_\Omega |ADu|^2\, dx \leq C\int_\Omega \lambda(u)|f(u)|^2\, dx. \tag{25}$$

Now, let $U = P(u)$ be the function described in (F) and $V = \frac{\lambda^{\frac{1}{2}}(u)|f(u)|}{P(u)+1}$.

Then (11) gives $|V| \leq CF(u)$ ($F(u)$ was also defined in (A)). We observe that

$$\begin{aligned}\int_\Omega \lambda(u)|f(u)|^2\,dx &\leq C\int_\Omega (U^2+1)(F(u)^2+1)\,dx\\ &= C\int_\Omega P(u)^2F(u)^2\,dx + C\int_\Omega P(u)^2\,dx + C\int_\Omega F(u)^2\,dx + C(|\Omega|)\\ &\leq C\int_\Omega |DP(u)|^2\,dx\int_\Omega |DF(u)|^2\,dx\\ &\qquad + C\int_\Omega (|DP(u)|^2+|DF(u)|^2)\,dx + C(|\Omega|),\end{aligned}$$

where we used (21) and then Poincaré's inequality for $P(u), F(u)$ in the last estimate, noting that $P(u), F(u)$ vanish on the boundary of Ω. By (9) and (22), we have

$$\begin{aligned}|DP(u)|^2 \leq |P_u(u)|^2|Du|^2 &\leq C\lambda^2(u)|Du|^2\\ &\leq C\left\langle A^T(u)A(u)Du, Du\right\rangle = C|A(u)Du|^2.\end{aligned}$$

Since $|DF(u)|^2 \leq C\lambda(u)|Du|^2$ by (8), we can use the above estimates in (25) to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega |ADu|^2\,dx \leq C\left[\int_\Omega \lambda(u)|Du|^2\,dx + 1\right]\int_\Omega |A(u)Du|^2\,dx\\ + \int_\Omega \lambda(u)|Du|^2\,dx + C(|\Omega|). \quad (26)\end{aligned}$$

We now set

$$y(t) = \int_\Omega |A(u)Du(x,t)|^2\,dx,\ \alpha(t) = \int_\Omega \lambda(u)|Du(x,t)|^2\,dx + C(|\Omega|),$$

and

$$\beta(t) = \int_\Omega \lambda(u)|Du(x,t)|^2\,dx + 1.$$

From (26) we obtain

$$y'(t) \leq \alpha(t) + C\beta(t)y(t) \quad \forall t \in (0,\infty).$$

By (15) and (16), we have, for any $\tau_0 > 0$ and $t > \tau_0$, the followings

$$\int_t^{t+\tau_0} \beta(s)ds \leq a_1,\ \int_t^{t+\tau_0} \alpha(s)ds \leq a_2,\ \text{and}\ \int_t^{t+\tau_0} y(s)ds \leq a_3,$$

where $a_1 = a_2 = C(|\Omega|, \tau_0)$ and $a_3 = C(|\Omega|, \tau_0)(\lambda_0^k + 1)$.

The uniform Gronwall inequality (see [17, Lemma 1.1] which was used in [5] in the context of Navier Stokes equation) then gives

$$y(t+\tau_0) \le \left[\frac{a_3}{\tau_0} + a_2\right] \exp(a_1) \le C(|\Omega|, \tau_0)(\lambda_0^k + 1)\,, \qquad \forall t > \tau_0.$$

Using the definition of $y(t+r)$ and (22) we complete the proof. □

REMARK 3.3: If we asume (F') and replace $f(u)$ by $\hat{f}(u, Du)$ satisfying

$$|\hat{f}(u, Du)| \le C\lambda^{\frac{1}{2}}(u)|Du| + f(u)$$

then the result continues to hold. Firstly, by Young's inequality and (F')

$$\left\langle \hat{f}(u, Du), |u|^l u \right\rangle \le \varepsilon\lambda(u)|u|^l|Du|^2 + C(\varepsilon)|u|^{l+2} + |f(u)||u|^{l+1}.$$

For suficiently small ε, there will be an extra term $|u|^{l+2}$ in the last integral of (17) in the argument in the proof of Lemma 3.1. Since $k > 0$ we can use Young's inequality again to have $|u|^{l+2} \le \varepsilon|u|^{k+l+2} + C(\varepsilon)$ and to obtain (19) again. The proof continues as before.

Next,

$$\begin{aligned} |\hat{f}(u, Du)|\lambda(u)|u_t| &\le C\lambda^{\frac{1}{2}}(u)\lambda(u)|Du||u_t| + C|f(u)|\lambda(u)|u_t| \\ &\le \varepsilon\lambda(u)|u_t|^2 + C(\varepsilon)\lambda^2(u)|Du|^2 + C|f(u)|\lambda(u)|u_t|. \end{aligned}$$

As $f(u)$ satisfies (F), for small ε in the above, the proof of Lemma 3.2 can continue.

4. Absorbing Balls in $W^{1,p_0}(\Omega)$

In this section we will establish a uniform bound for the $W^{1,p_0}(\Omega)$ norms of solutions when t is sufficiently large. To begin, let us fix a number $R > 0$ such that

(C) Ω can be covered by finitely many balls $B_{\frac{R}{4}}(x_i)$, $i = 1, \dots, n(R)$, with the property that either $x_i \in \partial\Omega$ or $x_i \in \Omega$ and $B_{2R}(x_i) \subset \Omega$.

The main result of this section is the following.

PROPOSITION 4.1. *For any $T, r_0 > 0$ and $p > 1$, if p is close to 1 and λ_0 is sufficiently large (in terms of $|\Omega|, r_0$ and the parameters in (A) and (F)) then there is a constant $C(\Omega, r_0, p)$ such that*

$$\sup_{t\in[T,T+r_0]} \int_\Omega |Du|^{2p}\, dx + \iint_{\Omega\times[T,T+r_0]} \lambda(u)|Du|^{2p-2}|D^2u|^2\, dz \le C(\Omega, r_0, p). \quad (27)$$

The constant $C(\Omega, r_0, p)$ depends on the geometry of Ω as well, namely, the numbers R and $n(R)$ in (C).

We will establish local estimates for the gradients of our solutions in these balls and then add up the results to obtain their global estimates. In the proof, we will only consider the case when $B_{2R}(x_i) \subset \Omega$. The boundary case ($x_i \in \partial\Omega$) is similar, invoking a reflection argument and using the fact that $\partial\Omega$ is smooth.

In the sequel, we will denote $\Phi(u) = \frac{|\lambda_u(u)|^2}{\lambda(u)}$. Before going to the proof of the proposition, we need some estimates for the integral of $\Phi(u)|Du|^4$.

LEMMA 4.2. *Assume (7) in (A). For any $R, r > 0$ and any nonnegative function $\psi \in C_0^1(B_R)$ there is a constant $C(|\Omega|, r)$, as in (23) of Lemma 3.2, such that for $t > r$*

$$\int_{B_R} \Phi(u)|Du|^4\psi^4 \, dx \le \frac{C(|\Omega|, r)}{\lambda_0^{k+2}} \int_{B_R} (\lambda(u)|D^2u|^2\psi^2 + \Phi(u)|Du|^4\psi^2) \, dx + \frac{C(|\Omega|, r)}{\lambda_0^{k+2}} \int_{B_R} \lambda(u)|D\psi|^2|Du|^2 \, dx. \quad (28)$$

Furthermore,

$$\int_{B_R} \lambda^2(u)|Du|^4\psi^4 \, dx \le C(|\Omega|, r) \int_\Omega \left[\lambda(u)|D^2u|^2\psi^2 + \Phi(u)|Du|^4\psi^2 + \lambda(u)|D\psi|^2|Du|^2\right] dx. \quad (29)$$

Proof. We establish (29). First we recall Ladyzhenskaya's inequality

$$\left(\int_\Omega |U|^4 \, dx\right)^{\frac{1}{2}} \le C \left(\int_\Omega |U|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |DU|^2 \, dx\right)^{\frac{1}{2}}, \quad (30)$$

if $U = 0$ on $\partial\Omega$. Using the above with $\Omega = B_R$ and $U = \lambda^{\frac{1}{2}}(u)\psi Du$ we have

$$\int_{B_R} \lambda^2(u)|Du|^4\psi^4 \, dx = \int_{B_R} |U|^4 \, dx \le C \int_{B_R} |U|^2 \, dx \int_{B_R} |DU|^2 \, dx \le C \int_{B_R} \lambda(u)|Du|^2\psi^2 \, dx \int_{B_R} |D(\lambda^{\frac{1}{2}}(u)\psi Du)|^2 \, dx.$$

It is clear that there is a constant C_2 such that

$$|D(\lambda^{\frac{1}{2}}(u)\psi Du)|^2 \le C_2 \left[\lambda(u)|D^2u|^2\psi^2 + \frac{|\lambda_u(u)|^2}{\lambda(u)}|Du|^4\psi^2 + \lambda(u)|D\psi|^2|Du|^2\right].$$

Since $\lambda(u) \sim (\lambda_0 + |u|)^k$ and $|\lambda_u(u)| \sim (\lambda_0 + |u|)^{k-1}$, we have $\lambda(u) \le C\lambda_0^{-k}\lambda^2(u)$, and (23) implies

$$\int_{B_R} \lambda(u)|Du|^2\psi^2 \, dx \le C\lambda_0^{-k} \int_{B_R} \lambda^2(u)|Du|^2\psi^2 \, dx \le C(|\Omega|, r) \quad (31)$$

for all $t > r$. We then obtain

$$\int_{B_R} \lambda^2(u)|Du|^4\psi^4 \, dx \\ \le C(|\Omega|, r)\int_\Omega \left[\lambda(u)|D^2u|^2\psi^2 + \Phi(u)|Du|^4\psi^2 + \lambda(u)|D\psi|^2|Du|^2\right] \, dx.$$

The above is (29). We also note that

$$\Phi(u) = \frac{|\lambda_u(u)|^2}{\lambda(u)} = \frac{|\lambda_u(u)|^2}{\lambda^3(u)}\lambda^2(u) \le \frac{C}{\lambda_0^{k+2}}\lambda^2(u).$$

Using (29) and the above estimate, we obtain (28) and prove the lemma. □

In the sequel, let $B_{2R}(x_0)$ be a fixed ball in the condition (C). For any s, t such that $0 \le s < t \le 2R$, let ψ be a cutoff function for two balls B_s, B_t centered at x_0. That is, ψ is nonnegative, $\psi \equiv 1$ in B_s and $\psi \equiv 0$ outside B_t with $|D\psi| \le 1/(t-s)$. We also let $r_0 > 0$ be a positive constant. For any $T > r_0$ let η be a cutoff function for $[T - r_0, T + r_0]$, see (14).

For any $p \ge 1$ and $t > 0$ we denote $Q_t = B_t \times [T - r_0, T + r_0]$ and

$$\mathcal{A}_p(t) = \sup_{\tau\in[T-r_0,T+r_0]} \int_{B_t} |Du|^{2p}\eta \, dx, \ \mathcal{H}_p(t) \\ = \iint_{Q_t} \lambda(u)|Du|^{2p-2}|D^2u|^2\eta \, dz, \tag{32}$$

$$\mathcal{B}_p(t) = \iint_{Q_t} \Phi(u)||Du|^{2p+2}\eta \, dz, \ \mathcal{G}_p(t) \\ = \iint_{Q_t} \lambda(u)|Du|^{2p} \, dz, \ \mathcal{J}_p = \frac{1}{r_0}\iint_{Q_t} |Du|^{2p} \, dz. \tag{33}$$

We then have the following local energy estimate result.

Lemma 4.3. *For any $p \ge 1$ and sufficiently close to 1 such that (36) holds, there is a constant C_1 depending on p such that*

$$\mathcal{A}_p(s) + \mathcal{H}_p(s) \le C_1\mathcal{B}_p(t) + \frac{C_1}{(t-s)^2}\mathcal{G}_p(t) + C_1\mathcal{J}_p(t), \quad 0 < s < t \le 2R. \tag{34}$$

Proof. Since u is a strong solution, we can differentiate (3) with respect to x to get

$$(Du)_t = \mathrm{div}((A(u)D^2u + A_u(u)DuDu) + Df(u, Du). \tag{35}$$

By the uniform ellipticity of $A(u)$, we can find a constant C such that $|A(u)\zeta| \le C\lambda(u)|\zeta|$. Thus, for some $p > 1$ and sufficiently close to 1 there is

$\delta \in (0,1)$ such that $\alpha = 2p-2$ satisfies

$$\frac{\alpha}{2+\alpha} = \frac{2p-2}{2p} = \delta C^{-1}. \tag{36}$$

We then recall the following simple algebraic fact in [3, Lemma 2.1]. If A is a matrix satisfying $\lambda_0|\zeta|^2 \le \langle A\zeta, \zeta\rangle$ and $|A\zeta| \le \Lambda_0|\zeta|$ then for any $\alpha > 0$ if the number $\delta_\alpha := \frac{\alpha}{2+\alpha}\frac{\Lambda_0}{\lambda_0} \in (0,1)$ then

$$\langle AD\zeta, D(|\zeta|^\alpha\zeta)\rangle \ge \widehat{\lambda}|\zeta|^\alpha|D\zeta|^2, \quad \hat{\lambda} = (1-\delta_\alpha^2)\lambda_0. \tag{37}$$

We test (35) with $|Du|^{2p-2}Du\psi^2\eta$ and integration by parts in x. By (36) and (37), with $\zeta = Du$, it is standard to see that there is a positive constant $C(p)$ such that for $Q = \Omega \times [T-r_0, T+r_0]$

$$\begin{aligned}
&\sup_{\tau\in(T-r_0,T+r_0)}\int_\Omega |Du|^{2p}\psi^2\eta\,dx + C(p)\iint_Q \lambda(u)|Du|^{2p-2}|D^2u|^2\psi^2\eta\,dz\\
&\le \iint_Q |A(u)||D^2u||Du|^{2p-1}\psi|D\psi|\eta\,dz\\
&\quad - \iint_Q A_u(u)DuDuD(|Du|^{2p-2}Du\psi^2)\eta\,dz\\
&\qquad + \iint_Q D\hat{f}(u,Du)|Du|^{2p-2}Du\psi^2\eta\,dz + \frac{1}{r_0}\iint_Q |Du|^{2p}\psi^2\,dz.
\end{aligned}$$

For simplicity, we will assume in the sequel that $\hat{f} \equiv 0$. The presence of $\hat{f}$ will be discussed later in Remark 4.4. For any given positive ε we use Young's inequality to find a constant $C(\varepsilon)$ such that

$$\begin{aligned}
&|A(u)||D^2u||Du|^{2p-1}\psi|D\psi|\\
&\qquad \le \varepsilon\lambda(u)|Du|^{2p-2}|D^2u|^2\psi^2 + C(\varepsilon)\lambda(u)|Du|^{2p}|D\psi|^2,\\
&|A_u(u)DuDuD(|Du|^{2p-2}Du\psi^2)|\\
&\qquad \le |A_u(u)||Du|^{2p}|D^2u|\psi^2 + |A_u(u)||Du|^{2p+1}\psi|D\psi|\\
&\qquad \le \varepsilon\lambda(u)|Du|^{2p-2}|D^2u|^2 + C(\varepsilon)\frac{|A_u|^2}{\lambda(u)}|Du|^{2p+2}\psi^2\\
&\qquad\quad + C(\varepsilon)\lambda(u)|Du|^{2p}|D\psi|^2.
\end{aligned}$$

Therefore, taking ε small and using the above two inequalities in the previous one, we easily deduce

$$\sup_{\tau\in[T-r_0,T+r_0]}\int_{B_s}|Du|^{2p}\eta\,dx+\iint_{Q_s}\lambda(u)|Du|^{2p-2}|D^2u|^2\eta\,dz$$
$$\le C_1\iint_{Q_t}\Phi(u)|Du|^{2p+2}\psi^2\eta\,dz$$
$$+C_1\iint_{Q_t}\left(\frac{1}{(t-s)^2}\lambda(u)+\frac{1}{r_0}\right)|Du|^{2p}\,dz. \quad (38)$$

Here, we used the definition of ψ and $\Phi(u)$ and the fact that $|A_u|\sim\lambda_u$. From the notations (32) and (33), the above gives the lemma. □

Remark 4.4: If we replace $f(u)$ by a function $\hat{f}$ depending on u, Du and satisfying a linear growth in Du then the proof can go on with minor modification. Namely, there exist a constant C and a function $f(u)$ satisfying (F) such that

$$|\hat{f}(u,Du)|\le C\lambda^{\frac{1}{2}}(u)|Du|+f(u).$$

Formally, we can assume that $|D\hat{f}(u,Du)|\le C|D(\lambda^{\frac{1}{2}}(u)|Du|+|f_u(u)||Du|$ so that by Young's inequality ($\Phi(u)=\frac{|\lambda_u(u)|^2}{\lambda(u)}$)

$$|D\hat{f}(u,Du)|\le C\lambda^{\frac{1}{2}}|D^2u|+C\Phi^{\frac{1}{2}}(u)|Du|^2+|f_u(u)||Du|.$$

Therefore, the extra term $|D\hat{f}(u,Du)||Du|^{2p-1}\psi^2$ in the proof can be handled by using the following estimates, which are the results of a simple use of Young's inequality.

$$|D\hat{f}(u,Du)||Du|^{2p-1}\le C[\lambda^{\frac{1}{2}}|D^2u|+C\Phi^{\frac{1}{2}}(u)|Du|^2+|f_u(u)||Du|]|Du|^{2p-1}$$
$$\le\varepsilon\lambda|Du|^{2p-2}|D^2u|^2+C(\varepsilon)\lambda|Du|^{2p}$$
$$+C\Phi(u)|Du|^{2p+2}+C|Du|^{2p}+C|f_u||Du|^{2p}.$$

We can then assume that $|f_u|\le C\lambda(u)$ for some constant C and see that the proof can continue to obtain the energy estimate (38).

Next, we also need the following elementary iteration result (e.g., see [8, Lemma 6.1, p.192]).

Lemma 4.5. *Let f,g,h be bounded nonnegative functions in the interval $[\rho,R]$ with g,h being increasing. Assume that for $\rho\le s<t\le R$ we have*

$$f(s)\le[(t-s)^{-\alpha}g(t)+h(t)]+\varepsilon f(t)$$

with $\alpha>0$ and $0\le\varepsilon<1$. Then

$$f(\rho)\le c(\alpha,\varepsilon)[(R-\rho)^{-\alpha}g(R)+h(R)].$$

The constant $c(\alpha,\varepsilon)$ *can be taken to be* $(1-\nu)^{-\alpha}(1-\nu^{-\alpha}\nu_0)^{-1}$ *for any* ν *satisfying* $\nu\in(0,1)$ *and* $\nu^{-\alpha}\nu_0<1$.

We then have

LEMMA 4.6. *If* λ_0 *is sufficiently large such that for some* $\mu_0\in(0,1)$

$$C_1\frac{C(|\Omega|,r_0)}{\lambda_0^{k+2}}\le\frac{\mu_0}{2}, \tag{39}$$

where $C_1, C(|\Omega|,r_0)$ *are the constants in (34) and (28) (with* $r=r_0$*), then there is a constant* C *such that*

$$\iint_{Q_{\frac{R}{2}}}\Phi^2(u)|Du|^4\,dz\le\frac{C}{\lambda_0^4}\iint_{Q'_{2R}}\left(\frac{1}{R^2}\lambda(u)+\frac{1}{r_0}\right)|Du|^2\,dz\,,\quad\forall T>2r_0. \tag{40}$$

Here, $Q_{R/2}=B_{R/2}\times[T-r_0,T+r_0]$ *and* $Q'_{2R}=B_{2R}\times[T-2r_0,T+r_0]$.

Proof. Let ψ be the cutoff function for $B_s(x_0), B_t(x_0)$ in Lemma 4.2. Fix a number $\mu_0\in(0,1)$. Multiplying (28) by η and integrating the result over $[T-r_0,T+r_0]$ we see that if λ_0 satisfies (39) then, with the notations (32) and (33), we have

$$C_1\mathcal{B}_1(s)\le\frac{\mu_0}{2}\left(\mathcal{H}_1(t)+\mathcal{B}_1(t)+\frac{1}{(t-s)^2}\mathcal{G}_1(t)\right) \tag{41}$$

for all s,t such that $0<s<t<R$.

For $p=1$, (34) gives

$$\mathcal{H}_1(s)\le C_1\mathcal{B}_1(t)+\frac{C_1}{(t-s)^2}\mathcal{G}_1(t)+C_1\mathcal{J}_1(t),\quad 0<s<t<R.$$

Let $t_1=(s+t)/2$ and use (41) with s being t_1 and the above with t being t_1 to obtain

$$\mathcal{H}_1(s)\le\frac{\mu_0}{2}[\mathcal{H}_1(t)+\mathcal{B}_1(t)]+\frac{C_2}{(t-s)^2}\mathcal{G}_1(t)+C_2\mathcal{J}_1(t). \tag{42}$$

Obviously, we can assume that $C_1\ge1$. Thus, we can add (41) and (42) to have

$$\mathcal{H}_1(s)+\mathcal{B}_1(s)\le\mu_0[\mathcal{H}_1(t)+\mathcal{B}_1(t)]+\frac{C_3}{(t-s)^2}\mathcal{G}_1(t)+C_3\mathcal{J}_1(t),\quad 0<s<t<R.$$

Since $\mu_0\in(0,1)$, we can use Lemma 4.5 with $f(t)=\mathcal{H}_1(t)+\mathcal{B}_1(t)$, $h(t)=\mathcal{J}_1(t)$, $g(t)=\mathcal{G}_1(t)$ and $\alpha=2$ to obtain a constant C_4 depending on μ_0, C_3 such that

$$\mathcal{H}_1(s)+\mathcal{B}_1(s)\le\frac{C_4}{(t-s)^2}\mathcal{G}_1(t)+C_4\mathcal{J}_1(t),\quad 0<s<t<R.$$

For $s = R$ and $t = 2R$ the above gives

$$\mathcal{H}_1(R) + \mathcal{B}_1(R) \le C_4 \iint_{Q_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz. \tag{43}$$

Now, if $T > 2r_0$ and η is a cutoff function for $[T-2r_0, T]$ ($\eta \equiv 1$ in $[T-r_0, T]$) then the above and the definitions (32) and (33) give the estimate

$$\iint_{B_R \times [T-r_0, T+r_0]} [\lambda(u)|D^2u|^2 + \Phi(u)|Du|^4] \, dz \le 2C_4 \iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz,$$

where $Q'_{2R} = B_{2R} \times [T - 2r_0, T + r_0]$.

Integrating (29) over $[T - r_0, T + r_0]$ and using the above estimate in the result with $s = R/2$ and $t = R$, we have

$$\iint_{Q_{\frac{R}{2}}} \lambda^2(u)|Du|^4 \, dz \le 2C_5 \iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz \,, \quad \forall T > 2r_0. \tag{44}$$

Because $\Phi^2(u) = \frac{|\lambda_u(u)|^4}{\lambda^2(u)} = \frac{|\lambda_u(u)|^4}{\lambda^4(u)}\lambda^2(u) \le \frac{C}{\lambda_0^4}\lambda^2(u)$, we have

$$\iint_{Q_{\frac{R}{2}}} \Phi^2(u)|Du|^4 \, dz \le \frac{C}{\lambda_0^4} \iint_{Q_{\frac{R}{2}}} \lambda^2(u)|Du|^4 \, dz.$$

Combining the above with (44), we obtain the lemma. □

REMARK 4.7: We can choose $r_0 = R^2$ to obtain a uniform bound for the integral on the right of (40).

We are now giving the uniform estimate for $W^{1,p}(\Omega)$ of our solutions.

Proof of Proposition 4.1. For any $p > 1$ we have by Hölder's inequality

$$\iint_{Q_t} \Phi(u)|Du|^{2p+2}\psi^2\eta \, dz \le \left(\iint_{Q_t} \Phi^2(u)|Du|^4 \, dz\right)^{\frac{1}{2}} \left(\iint_{Q_t} |Du|^{4p}\psi^4\eta^2 \, dz\right)^{\frac{1}{2}}. \tag{45}$$

Using Ladyzhenskaya's inequality (30) with $U = |Du|^{p-1}Du\psi$, multiplying the result with η^2 and integrating over $[T - r_0, T + r_0]$, we have

$$\iint_{Q_t} |Du|^{4p}\psi^4\eta^2 \, dz \le C \sup_{\tau \in [T-r_0, T+r_0]} \int_{B_t} |Du|^{2p}\psi^2\eta \, dx \iint_{Q_t} |DU|^2\eta \, dz.$$

Since $\lambda(u) \sim (\lambda_0 + |u|)^k$ there is a constant C such that

$$|DU|^2 \le \frac{C}{\lambda_0^k}[\lambda(u)|Du|^{2p-2}|D^2u|^2\psi^2 + \lambda(u)|Du|^{2p}|D\psi|^2].$$

Therefore,

$$\begin{aligned}\iint_{Q_t} |Du|^{4p}\psi^4\eta^2 \, dz &\le \frac{C}{\lambda_0^k}\mathcal{A}_p(t)\left[\mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right] \\ &\le \frac{C}{\lambda_0^k}\left[\mathcal{A}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right]^2.\end{aligned}$$

Here, Cauchy's inequality was used in the last inequality.

We now assume $T > 2r_0$, $p > 1$ sufficiently close to 1, and λ_0 is sufficiently large as in Lemma 4.6. Applying the above inequality in (45) for $t \in (0, R/2)$ we derive, using (40) of Lemma 4.6 to estimate the first factor on the right of (45),

$$\begin{aligned}&\iint_{Q_t} \Phi(u)|Du|^{2p+2}\psi^2 \, dz \\ &\le C\left(\frac{C_*(u,R,r_0)}{\lambda_0^4}\right)^{\frac{1}{2}} \lambda_0^{\frac{-k}{2}}\left[\mathcal{A}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right], \quad (46)\end{aligned}$$

where we denote

$$C_*(u,R,r_0) = C\iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right)|Du|^2 \, dz.$$

By (15) of Lemma 3.1 we can find a constant $C(|\Omega|, r_0)$ such that

$$C_*(u,R,r_0) \le \max\left\{\frac{1}{R^2}, \frac{1}{r_0}\right\} C(|\Omega|, r_0).$$

Recall that $C(|\Omega|, r_0)$ does not depend on λ_0. Hence, for any given $\mu_1 \in (0,1)$ and $k \ge 0$ if λ_0 is sufficiently large such that, with C_1 being the constant in (38),

$$C_1 C\left(\max\left\{\frac{1}{R^2}, \frac{1}{r_0}\right\} C(|\Omega|, r_0)\right)^{\frac{1}{2}} \lambda_0^{\frac{-k}{2}-2} \le \mu_1, \quad (47)$$

then (46) gives

$$\begin{aligned}&C_1\iint_{Q_t} \Phi(u)|Du|^{2p+2}\psi^2 \, dz \\ &\le \mu_1\left[\mathcal{A}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right] \quad 0 < s < t < \frac{R}{2}. \quad (48)\end{aligned}$$

If $p > 1$ and satisfies (36), we then have from (38) and the above inequality the following.

$$\mathcal{A}_p(s)+\mathcal{H}_p(s) \le \mu_1(\mathcal{A}_p(t)+\mathcal{H}_p(t))+\frac{C_2}{(t-s)^2}\mathcal{G}_p(t)+C_2\mathcal{J}_p(t), \quad 0 < s < t < \frac{R}{2}.$$

For $f(t) = \mathcal{A}_p(t) + \mathcal{H}_p(t)$, $h(t) = \mathcal{J}_p(t)$, $g(t) = \mathcal{G}_p(t)$ and $\alpha = 2$ we can use Lemma 4.5, as $\mu_1 \in (0, 1)$, to obtain

$$f(s) \le \frac{C_3(\mu_1)}{(t-s)^2}\mathcal{G}_p(t) + C_3(\mu_1)\mathcal{J}_p(t), \quad 0 < s < t < \frac{R}{2}.$$

For $s = R/4$ and $t = R/2$ the above yields, recalling $\eta \equiv 1$ in $[T, T + r_0]$,

$$\sup_{t\in[T,T+r_0]} \int_{B_{\frac{R}{4}}} |Du|^{2p}\,dx + \mathcal{H}_p(\tfrac{R}{4}) \le \frac{C_3(\mu_1)}{R^2}\mathcal{G}_p(\tfrac{R}{2}) + C_3(\mu_1)\mathcal{J}_p(\tfrac{R}{2}). \tag{49}$$

If $2p < 4$ then a simple use of Hölder's inequality and the uniform bound in Lemma 4.6 for $\|Du\|_{L^4(B_R\times[T-r_0,T+r_0])}$ show that the right hand side can be bounded by a constant depending only on R, r_0. Thus, for some p such that $p \in (1, 2)$ a finite covering of Ω with balls $B_{R/4}$ as in (C) yields

$$\sup_{t\in[T,T+r_0]} \int_{\Omega} |Du|^{2p}\,dx + \iint_{\Omega\times[T,T+r_0]} \lambda(u)|Du|^{2p-2}|D^2u|^2\,dz \le C(\Omega, R, r_0).$$

This is (27) of the proposition and completes the proof. □

5. Proof of the Main Result

We are now ready to present the proof of our main result, Theorem 2.1, by verifying the conditions of Theorem 1.1 giving the existence of global attractors.

Proof of Theorem 2.1. By Amann's results in [1, 2], (3) defines a local semigroup on $W^{1,p}(\Omega)$ for any $p > 2$. By the proof of Proposition 4.1, the norm $\|u\|_{W^{1,p}(\Omega)}$ never blows up in any finite interval $(0, T)$ so that u exists globally . Thus, under the assumption (A) and (F'), (3) defines a semigroup $\{\mathcal{S}(t)\}_{t\ge0}$ on $X = W^{1,p_0}(\Omega)$, with $p_0 = 2p$ for some $p > 1$.

The existence of an absorbing ball in i) of Theorem 1.1 is also established by Proposition 4.1. Indeed, as the solutions of (3) exist globally, we can choose a uniform $r_0 = R^2$, where R is described in (C). Hence, the key assumption on the existence of μ_0, μ_1 in (39) and (47) can be guaranteed if λ_0 is sufficiently large in terms the parameters in (A), (F) and R, or the geometry of Ω. Therefore, for any solution u of (3) Proposition 4.1 shows that there is a uniform constant

C depending only on the geometry of Ω such that $\|u(\cdot,t)\|_{W^{1,p_0}(\Omega)} \le C$ if t is sufficiently large. We have proved the existence of an absorbing ball in X.

Concerning the compactness of the $\mathcal{S}(t_1)$ in ii) of Theorem 1.1, we fix a $t_1 > 0$. By (27), with $p = 1$ and $p > 1$, we have the bound for the following quantities

$$\sup_{t\in[t_1,2t_1]} \int_\Omega |Du|^2\, dx,\ \sup_{t\in[t_1,2t_1]} \int_\Omega |Du|^{2p}\, dx,\ \iint_{\Omega\times[t_1,2t_1]} \lambda(u)|D^2u|^2\, dz \le C(\Omega, t_1, p).$$

From the system for u we can estimate $|u_t|^p$ in terms of $|Du|^{2p}, |D^2u|^p$ and powers of $|u|$. Since $2p > 2$, the above and Sobolev's imbedding theorem show that $u(\cdot,t)$ is Hölder continuous in x and thus bounded. Therefore, by the above estimates, u_t is in $L^p(Q)$. It is now standard to show that u is Hölder in (x,t) and then Du is Hölder continuous (see [6]). The compactness of $\mathcal{S}(t_1)$ then follows. □

6. Mixed and Neumann Boundary Conditions

We notice that the only place in the proof we need the boundary condition $u = 0$ is the validity of the Poincaré inequality

$$\int_\Omega |U|^2\, dx \le Cd^2(\Omega)\int_\Omega |DU|^2\, dx, \quad U \in W_0^{1,2}(\Omega), \tag{50}$$

in the proof of Lemma 3.1. The above inequality also yields Sobolev's inequality of the form (20) and Lemma 3.2 to obtain the key uniform Gronwall inequality (26).

It is well known that (50) continues to hold if $U = 0$ on a nonempty relatively open set $\partial\Omega_1$ of $\partial\Omega$ and Ω is starshaped with respect to $\partial\Omega_1$. This comes from a simple modification of the proof of [7, Lemma 7.14] and then the use of Riesz's potential estimates in [7, Lemma 7.12]. Hence, we can assume mixed boundary conditions for (3). Namely, u satifies the homogenous Dirichlet condition on a nonempty relatively open set $\partial\Omega_1$ of $\partial\Omega$, Ω is starshaped with respect to $\partial\Omega_1$ and the homogeneous Neumann condition on $\partial\Omega \setminus \partial\Omega_1$. We then see that our results still hold in this case.

On the other hand, if u satifies the homogenous Neumann condition on the boundary $\partial\Omega$ then we have to assume that the semigroup defined by (3) *has an absorbing ball in* $L^1(\Omega)$. That is, for any bounded set $B \subset W^{1,p_0}(\Omega)$ there are $T(B) > 0$ and a universal constant C such that

$$\forall U_0 \in B,\ \|S(t)U_0\|_{L^1(\Omega)} \le C \quad t \ge T(B). \tag{51}$$

Indeed, by the compactness of the imbedding $W^{1,2}(\Omega) \to L^p(\Omega)$ $(n = 2, p > 1)$ and a simple argument by contradiction shows that for any positive reals ε, q there is a constant $C(\varepsilon, q)$ such that

$$\left(\int_\Omega |U|^p \, dx\right)^{\frac{1}{p}} \le \varepsilon \left(\int_\Omega |DU|^2 \, dx\right)^{\frac{1}{2}} + C(\varepsilon, p, q) \left(\int_\Omega |U|^q \, dx\right)^{\frac{1}{q}}. \tag{52}$$

Therefore, if U_0 belongs to a bounded set in $W^{1,p_0}(\Omega)$ and U is a polynomial in u, with u being the solution to (3), then by choosing q small in the above and using (51) we can easily see that

$$\left(\int_\Omega |U|^p \, dx\right)^{\frac{1}{p}} \le \varepsilon \left(\int_\Omega |DU|^2 \, dx\right)^{\frac{1}{2}} + C(\varepsilon, p, q, B). \tag{53}$$

Using the above for $p = k + l + 2$ and $U = u$ in the proof of Lemma 3.1 we see that the proof can go on. Similarly, we use the above for $p = 4$ and U being $P(u)$ or $F(u)$ in the proof of Lemma 3.2, the uniform Gronwall inequality (26).

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