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Nonlinear boundary value problems relative to the one dimensional heat equation

Laurent Véron

To Julian with high esteem and sincere friendship

Abstract. *We consider the problem of existence of a solution u to $\partial_t u - \partial_{xx} u = 0$ in $(0,T) \times \mathbb{R}_+$ subject to the boundary condition $-u_x(t,0) + g(u(t,0)) = \mu$ on $(0,T)$ where μ is a measure on $(0,T)$ and g a continuous nondecreasing function. When $p > 1$ we study the set of self-similar solutions of $\partial_t u - \partial_{xx} u = 0$ in $\mathbb{R}_+ \times \mathbb{R}_+$ such that $-u_x(t,0) + u^p = 0$ on $(0,\infty)$. At end, we present various extensions to a higher dimensional framework.*

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1. Introduction

Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous nondecreasing function. Set $Q^T_{\mathbb{R}_+} = (0,T) \times \mathbb{R}_+$ for $0 < T \leq \infty$ and $\partial_\ell Q^T_{\mathbb{R}_+} = \overline{\mathbb{R}_+} \times \{0\}$. The aim of this article is to study the following 1-dimensional heat equation with a nonlinear flux on the parabolic boundary

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^T_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) + g(u(\,\cdot\,,0)) &= \mu && \text{in } [0,T) \\ u(0,\,\cdot\,) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \tag{1}$$

where ν, μ are Radon measures in $\mathbb{R}_+$ and $[0,T)$ respectively. A related problem in $Q^\infty_{\mathbb{R}_+}$ for which there exist explicit solutions is the following,

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+} \\ -u_x(t,0) + |u|^{p-1}u(t,0) &= 0 && \text{for all } t > 0 \\ \lim_{t\to 0} u(t,x) &= 0 && \text{for all } x > 0, \end{aligned} \tag{2}$$

where $p>1$. Problem (2) is invariant under the transformation T_k defined for all $k>0$ by
$$T_k[u](t,x)=k^{\frac{1}{p-1}}u(k^2t,kx). \tag{3}$$
This leads naturaly to look for existence of self-similar solutions under the form
$$u_s(t,x)=t^{-\frac{1}{2(p-1)}}\omega\left(\frac{x}{\sqrt{t}}\right). \tag{4}$$
Putting $\eta=\frac{x}{\sqrt{t}}$, ω satisfies
$$\begin{aligned}-\omega''-\frac{1}{2}\eta\omega'-\frac{1}{2(p-1)}\omega=0\,,\quad \text{in }\ \mathbb{R}_+\,,\\ -\omega'(0)+|\omega|^{p-1}\omega(0)=0\,,\\ \lim_{\eta\to\infty}\eta^{\frac{1}{p-1}}\omega(\eta)=0\,.\end{aligned} \tag{5}$$
Self-similar solutions of non-linear diffusion equations such as porous-media or fast-diffusion equation were discovered long time ago by Kompaneets and Zeldovich and a thourougful study was made by Barenblatt, reducing the study to the one of integrable ordinary differential equations with explicit solutions. Concerning semilinear heat equation Brezis, Terman and Peletier opened the study of self-similar solutions of semilinear heat equations in proving in [5] the existence of a positive strongly singular function satisfying
$$u_t-\Delta u+|u|^{p-1}u=0\quad\text{in }\ \mathbb{R}_+\times\mathbb{R}^n, \tag{6}$$
and vanishing at $t=0$ on $\mathbb{R}^n\setminus\{0\}$. They called it the *very singular solution*. Their method of construction is based upon the study of an ordinary differential equation with a phase space analysis. A new and more flexible method based upon variational analysis has been provided by [7]. Other singular solutions of (6) in different configurations such as boundary singularities have been studied in [13]. We set $K(\eta)=e^{\eta^2/4}$ and
$$L^2_K(\mathbb{R}_+)=\left\{\phi\in L^1_{loc}(\mathbb{R}_+):\int_{\mathbb{R}_+}\phi^2Kdx:=\|\phi\|^2_{L^2_K}<\infty\right\},$$
and, for $k\geq 1$,
$$H^k_K(\mathbb{R}_+)=\left\{\phi\in L^2_K(\mathbb{R}_+):\sum_{\alpha=0}^{k}\left\|\phi^{(\alpha)}\right\|^2_{L^2_K}:=\|\phi\|^2_{H^k_K}<\infty\right\}.$$
Let us denote by $\mathcal{E}$ the subset of $H^1_K(\mathbb{R}_+)$ of weak solutions of (5) that is the set of functions satisfying
$$\int_0^\infty\left(\omega'\zeta'-\frac{1}{2(p-1)}\omega\zeta\right)K(\eta)d\eta+\left(|\omega|^{p-1}\omega\zeta\right)(0)=0,$$

and by $\mathcal{E}_+$ the subset of nonnegative solutions. The next result gives the structure of $\mathcal{E}$.

THEOREM 1.1. *1. If $p \geq 2$, then $\mathcal{E} = \{0\}$.*

2. If $1 < p \leq \frac{3}{2}$, then $\mathcal{E}_+ = \{0\}$

3. If $\frac{3}{2} < p < 2$ then $\mathcal{E} = \{\omega_s, -\omega_s, 0\}$ where ω_s is the unique positive solution of (5). Furthermore there exists $c > 1$ such that

$$c^{-1}\eta^{\frac{1}{p-1}-1} \leq e^{\frac{\eta^2}{4}}\omega_s(\eta) \leq c\eta^{\frac{1}{p-1}-1} \quad \text{for all} \quad \eta > 0. \tag{7}$$

Whenever it exists the function u_s defined in (4) is the limit, when $\ell \to \infty$ of the positive solutions $u_{\ell\delta_0}$ of

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^{\infty}_{\mathbb{R}_+} \\ -u_x(t,\,\cdot\,) + |u|^{p-1}u(t,\,\cdot\,) &= \ell\delta_0 \quad \text{in } [0,T) \\ \lim_{t\to 0} u(t,x) &= 0 \quad \text{for all } x \in \mathbb{R}_+. \end{aligned}$$

When such a function u_s does not exits the sequence $\{u_{\ell\delta_0}\}$ tends to infinity. This is a charateristic phenomenon of an underlying fractional diffusion associated to the linear equation

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^{\infty}_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) &= \mu \quad \text{in } [0,\infty) \\ u(0,\,\cdot\,) &= 0 \quad \text{in } \mathbb{R}_+. \end{aligned}$$

More generaly we consider problem (1). We define the set $\mathbb{X}(Q^T_{\mathbb{R}_+})$ of test functions by

$$\mathbb{X}(Q^T_{\mathbb{R}_+}) = \left\{\zeta \in C^{1,2}_c([0,T)\times[0,\infty)) : \zeta_x(t,0) = 0 \text{ for } t \in [0,T]\right\}.$$

DEFINITION 1.2. *Let ν, μ be Radon measures in $\mathbb{R}_+$ and $[0,T)$ respectively. A function u defined in $\overline{Q^T_{\mathbb{R}_+}}$ and belonging to $L^1_{loc}(\overline{Q^T_{\mathbb{R}_+}}) \cap L^1(\partial_\ell Q^T_{\mathbb{R}_+}; dt)$ such that $g(u) \in L^1(\partial_\ell Q^T_{\mathbb{R}_+}; dt)$ is a weak solution of (1) if for every $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$ there holds*

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})u\,dxdt + \int_0^T (g(u)\zeta)(t,0)dt = \int_0^\infty \zeta d\nu(x) + \int_0^T \zeta(t,0)d\mu(t). \tag{8}$$

We denote by $E(t,x)$ the Gaussian kernel in $\mathbb{R}_+\times\mathbb{R}$. The solution of

$$\begin{aligned} v_t - v_{xx} &= 0 \quad \text{in } Q^{\infty}_{\mathbb{R}_+} \\ -v_x &= \delta_0 \quad \text{in } \overline{\mathbb{R}_+} \\ v(0,\cdot) &= 0 \quad \text{in } \mathbb{R}_+, \end{aligned}$$

has explicit expression

$$v(t,x) = 2E(t,x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

If $x,y>0$ and $s<t$ we set $\tilde{E}(t-s,x,y) = E(t-s,x-y) + E(t-s,x+y)$. When $\nu\in\mathfrak{M}^b(\mathbb{R}_+)$ and $\mu\in\mathfrak{M}^b(\overline{\mathbb{R}_+})$ the solution of

$$\begin{aligned} v_t - v_{xx} &= 0 \quad \text{in } Q^{\infty}_{\mathbb{R}_+} \\ -v_x(\cdot,0) &= \mu \quad \text{in } \overline{\mathbb{R}_+} \\ u(0,\cdot) &= \nu \quad \text{in } \mathbb{R}_+, \end{aligned} \tag{9}$$

is given by

$$\begin{aligned} v_{\nu,\mu}(t,x) &= \int_0^\infty \tilde{E}(t,x,y)d\nu(y) + 2\int_0^t E(t-s,x)d\mu(s) \\ &= \mathcal{E}_{\mathbb{R}_+}[\nu](t,x) + \mathcal{E}_{\mathbb{R}_+\times\{0\}}[\mu](t,x) = \mathcal{E}_{Q^{\infty}_{\mathbb{R}_+}}[(\nu,\mu)](t,x). \end{aligned} \tag{10}$$

We prove the following existence and uniqueness result.

THEOREM 1.3. *Let $g:\mathbb{R}\mapsto\mathbb{R}$ be a continuous nondecreasing function such that $g(0)=0$. If g satisfies*

$$\int_1^\infty (g(s)-g(-s))s^{-3}ds < \infty, \tag{11}$$

then for any bounded Borel measures ν in $\mathbb{R}_+$ and μ in $[0,T)$, there exists a unique weak solution $u := u_{\nu,\mu}\in L^1(Q^T_{\mathbb{R}_+})$ of (1). Furthermore the mapping $(\nu,\mu)\mapsto u_{\nu,\mu}$ is nondecreasing.

When $g(s)=|s|^{p-1}s$, condition (11) is satisfied if

$$0<p<2.$$

The above result is still valid under minor modifications if $\mathbb{R}_+$ is replaced by a bounded interval $I:=(a,b)$, and problem (1) by

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^T_I \\ u_x(\cdot,b) + g(u(\cdot,b)) &= \mu_1 \quad \text{in } [0,T) \\ -u_x(\cdot,a) + g(u(\cdot,a)) &= \mu_2 \quad \text{in } [0,T) \\ u(0,\cdot) &= \nu \quad \text{in } (a,b), \end{aligned}$$

where ν, μ_j $(j = 1, 2)$ are Radon measures in I and $(0, T)$ respectively.

In the last section we present the scheme of the natural extensions of this problem to a multidimensional framework

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } Q^T_{\mathbb{R}^n_+} \\ -u_{x_n} + g(u) &= \mu \quad \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0, \cdot\,) &= \nu \quad \text{in } \mathbb{R}^n_+, \end{aligned}$$

The construction of solutions with measure data can be generalized but there are some difficulties in the obtention of self-similar solutions. The equation with a source flux

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } Q^T_{\mathbb{R}^n_+} \\ u_{x_n} + g(u) &= 0 \quad \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0, \cdot\,) &= \nu \quad \text{in } \mathbb{R}^n_+, \end{aligned} \tag{12}$$

has been studied by several authors, in particular Fila, Ishige, Kawakami and Sato [8, 10, 11]. Their main concern deals with global existence of solutions.

2. Self-similar solutions

2.1. The symmetrization

We define the operator $\mathcal{L}_K$ in $C^2_0(\mathbb{R})$ by

$$\mathcal{L}_K(\phi) = -K^{-1}(K\phi')'.$$

The operator $\mathcal{L}_K$ has been thouroughly studied in [7]. In particular

$$\inf\left\{\int_{-\infty}^{\infty} \phi'^2 K(\eta)\eta : \int_{-\infty}^{\infty} \phi^2 K(\eta)d\eta = 1\right\} = \frac{1}{2}.$$

The above infimum is achieved by $\phi_1 = (4\pi)^{-\frac{1}{2}}K^{-1}$ and $\mathcal{L}_K$ is an isomorphism from $H^1_K(\mathbb{R})$ onto its dual $(H^1_K(\mathbb{R}))' \sim H^{-1}_K(\mathbb{R})$. Finally $\mathcal{L}_K^{-1}$ is compact from $L^2_K(\mathbb{R})$ into $H^1_K(\mathbb{R})$, which implies that $\mathcal{L}_K$ is a Fredholm self-adjoint operator with

$$\sigma(\mathcal{L}_K) = \left\{\lambda_j = \tfrac{1+j-1}{2} : j = 1, 2, \dots\right\},$$

and

$$ker\left(\mathcal{L}_K - \lambda_j I_d\right) = span\left\{\phi_1^{(j)}\right\}.$$

If ϕ is defined in $\mathbb{R}_+$, $\tilde{\phi}(x) = \phi(-x)$ is the symmetric with respect to 0 while $\phi^*(x) = -\phi(-x)$ is the antisymmetric with respect to 0. The operator $\mathcal{L}_K$ restricted to $\mathbb{R}_+$ is denoted by $\mathcal{L}_K^+$. The operator $\mathcal{L}_K^{+,N}$ with Neumann condition at $x=0$ is again a Fredholm operator. This is also valid for the operator $\mathcal{L}_K^{+,D}$ with Dirichlet condition at $x=0$. Hence, if ϕ is an eigenfunction of $\mathcal{L}_K^{+,N}$, then $\tilde{\phi}$ is an eigenfunction of $\mathcal{L}_K$ in $L^2_K(\mathbb{R})$. Similarly, if ϕ is an eigenfunction of $\mathcal{L}_K^{+,D}$, then ϕ^* is an eigenfunction of $\mathcal{L}_K$ in $L^2_K(\mathbb{R})$. Conversely, any even (resp. odd) eigenfunction of $\mathcal{L}_K$ in $L^2_K(\mathbb{R})$ satisfies Neumann (resp. Dirichlet) boundary condition at $x=0$. Hence its restiction to $L^2_K(\mathbb{R}_+)$ is an eigenfunction of $\mathcal{L}_K^{+,N}$ (resp. $\mathcal{L}_K^{+,D}$). Since $\phi_1^{(j)}$ is even (resp. odd) if and only if j is even (resp. odd), we derive

$$H^{1,0}_K(\mathbb{R}_+)=\bigoplus_{\ell=1}^{\infty} span\left\{\phi_1^{(2\ell+1)}\right\},$$

and

$$H^{1}_K(\mathbb{R}_+)=\bigoplus_{\ell=0}^{\infty} span\left\{\phi_1^{(2\ell)}\right\}.$$

Note that $\phi\in H^1_K(\mathbb{R}_+)$ such that $\phi_x(0)=0$ (resp. $\phi(0)=0$) implies $\tilde{\phi}\in H^1_K(\mathbb{R})$ (resp. $\phi^*\in H^1_K(\mathbb{R})$). Furthermore, ϕ_1 is an eigenfunction of $\mathcal{L}_K^{+}$ in $H^1_K(\mathbb{R}^n_+)$ with Neumann boundary condition on $\partial\mathbb{R}^n_+$ while $\partial_{x_n}\phi_1$ is an eigenfunction of $\mathcal{L}_K^{+}$ in $H^1_K(\mathbb{R}^n_+)$ with Dirichlet boundary condition on $\partial\mathbb{R}^n_+$. We list below two important properties of $H^1_K(\mathbb{R}_+)$ valid for any $\beta>0$. Actually they are proved in [7, Prop. 1.12] with $H^1_{K^\beta}(\mathbb{R})$ but the proof is valid with $H^1_{K^\beta}(\mathbb{R}_+)$.

$$\begin{array}{ll}(i) & \phi\in H^1_{K^\beta}(\mathbb{R}_+)\Longrightarrow K^{\frac{\beta}{2}}\phi\in C^{0,\frac{1}{2}}(\mathbb{R}_+)\\ (ii) & H^1_{K^\beta}(\mathbb{R}_+)\hookrightarrow L^2_{K^\beta}(\mathbb{R}_+)\quad\text{is compact for all } n\geq 1.\end{array}$$

2.2. Proof of Theorem 1.1-(i)-(ii)

Assume $p\geq 2$, then $\frac{1}{2(p-1)}\leq\frac{1}{2}$. If ω is a weak solution, then

$$\int_0^\infty\left(\omega'^2-\frac{1}{2(p-1)}\omega^2\right)Kd\eta+|\omega|^{p+1}(0)=0.$$

If $\frac{1}{2}>\frac{1}{2(p-1)}$ we deduce that $\omega=0$. Furthermore, when $\frac{1}{2}=\frac{1}{2(p-1)}$ then

$$|\omega|^{p+1}(0)=0.$$

If ω is nonzero, it is an eigenfunction of $\mathcal{L}_K^{+,D}$. Since the first eigenvalue is 1 it would imply $1=\frac{1}{2(p-1)}\leq\frac{1}{2}$, contradiction.

Assume $1<p\leq\frac{3}{2}$ and ω is a nonnegative weak solution. We take $\zeta(\eta)=\eta e^{-\frac{\eta^2}{4}}=-2\phi_1'(\eta)$, then

$$\int_0^\infty\left(-\zeta''-\frac{1}{2(p-1)}\zeta\right)\omega K(\eta)d\eta+\zeta'(0)\omega^p(0)=0.$$

Since $-\zeta'' = \zeta\lfloor_{\mathbb{R}_+} > 0$ and $\zeta'(0) = \phi_1(0) = 1$, we derive $\omega\zeta = 0$ if $1 > \frac{1}{2(p-1)}$ and $\omega(0) = 0$ if $1 = \frac{1}{2(p-1)}$. Hence $\omega'(0) = 0$ by the equation and $\omega \equiv 0$ by the Cauchy-Lipschitz theorem.

2.3. Proof of Theorem 1.1-(iii)

We define the following functional on $H^1_K(\mathbb{R}^n_+)$

$$J(\phi) = \frac{1}{2}\int_0^\infty \left(\phi'^2 - \frac{1}{2(p-1)}\phi^2\right) K d\eta + \frac{1}{p+1}|\phi(0)|^{p+1}.$$

Lemma 2.1. *The functional J is lower semicontinuous in $H^1_K(\mathbb{R}_+)$. It tends to infinity at infinity and achieves negative values.*

Proof. We write

$$J(\psi) = J_1(\psi) - J_2(\psi) = J_1(\psi) - \frac{1}{2(p-1)} \|\psi\|^2_{L^2_K}.$$

Clearly J_1 is convex and J_2 is continuous in the weak topology of $H^1_K(\mathbb{R}_+)$ since the imbedding of $H^1_K(\mathbb{R}_+)$ into $L^2_K(\mathbb{R}_+)$ is compact. Hence J is weakly semicontinuous in $H^1_K(\mathbb{R}_+)$.

Let $\epsilon > 0$, then

$$J(\epsilon\phi_1) = \left(\frac{1}{4} - \frac{1}{4(p-1)}\right)\frac{\epsilon^2\sqrt{\pi}}{2} + \frac{\epsilon^{p+1}}{p+1}.$$

Since $1 < p < 2$, $\frac{1}{4} - \frac{1}{4(p-1)} < 0$. Hence $J(\epsilon\phi_1) < 0$ for ϵ small enough, thus J achieves negative values on $H^1_K(\mathbb{R}_+)$.

If $\psi \in H^1_K(\mathbb{R}_+)$ it can be written in a unique way under the form $\psi = a\phi_1 + \psi_1$ where $a = 2\sqrt{\pi}\psi(0)$ and $\psi_1 \in H^{1,0}_K(\mathbb{R}_+)$. Hence, for any $\epsilon > 0$,

$$\begin{aligned} J(\psi) &= \frac{1}{2}\int_0^\infty \left(\psi_1'^2 - \frac{1}{2(p-1)}\psi_1^2\right) K d\eta + \frac{a^2}{2}\int_0^\infty \left(\phi_1'^2 - \frac{1}{2(p-1)}\phi_1^2\right) K d\eta \\ &\quad + a\int_0^\infty \left(\psi_1'\phi_1' - \frac{1}{2(p-1)}\psi_1\phi_1\right) K d\eta + \frac{1}{p+1}|a|^{p+1} \\ &\geq \frac{2p-3}{4(p-1)}\int_0^\infty \psi_1'^2 K d\eta - \frac{a\epsilon}{2}\int_0^\infty \left(\psi_1'^2 + \frac{1}{2(p-1)}\psi_1^2\right) K d\eta \\ &\quad + \frac{a^2(p-2)\sqrt{\pi}}{4(p-1)} - \frac{ap\sqrt{\pi}}{4(p-1)\epsilon} + \frac{1}{p+1}|a|^{p+1}. \end{aligned}$$

Note that $\|\psi\|^2_{H^1_K} \leq 4\left(\|\psi_1'\|^2_{L^2_K} + a^2\right)$. Since $2p-3>0$, we can take $\epsilon>0$ small enough in order that

$$\lim_{\|\psi\|_{H^1_K}\to\infty} J(\psi) = \infty.$$

□

By Lemma 2.1 the functional J achieves its minimum in $H^1_K(\mathbb{R}_+)$ at some $\omega_s \neq 0$, and ω_s can be assumed to be nonnegative since J is even. By the strong maximum principle $\omega_s > 0$, and by the method used in the proof of [15, Proposition 1] is is easy to prove that positive solutions belong to $H^2_K(\mathbb{R}_+)$. Assume that $\tilde\omega$ is another positive solution, then

$$\int_0^\infty \left(\frac{(K\omega_s')'}{\omega_s} - \frac{(K\tilde\omega_s')'}{\tilde\omega_s}\right)(\omega_s^2 - \tilde\omega_s^2)d\eta = 0.$$

Integration by parts, easily justified by regularity, yields

$$\begin{aligned}
&\int_0^\infty \left(\frac{(K\omega_s')'}{\omega_s} - \frac{(K\tilde\omega_s')'}{\tilde\omega_s}\right)(\omega_s^2 - \tilde\omega_s^2)d\eta \\
&= \left[K\omega_s'\left(\omega_s - \frac{\tilde\omega_s^2}{\omega_s}\right) - K\tilde\omega_s'\left(\frac{\omega_s^2}{\tilde\omega_s} - \tilde\omega_s\right)\right]_0^\infty \\
&\quad - \int_0^\infty \left(\omega_s - \frac{\tilde\omega_s^2}{\omega_s}\right)' K\omega_s' d\eta + \int_0^\infty \left(\frac{\omega_s^2}{\tilde\omega_s} - \tilde\omega_s\right)' K\omega_s' d\eta \\
&= -\left(\omega_s^{p-1} - \tilde\omega_s^{p-1}\right)\left(\omega_s^2 - \tilde\omega_s^2\right)(0) \\
&\quad - \int_0^\infty \left(\left(\frac{\omega_s'\tilde\omega_s - \omega_s\tilde\omega_s'}{\tilde\omega_s}\right)^2 + \left(\frac{\omega_s\tilde\omega_s' - \tilde\omega_s\omega_s'}{\omega_s}\right)^2\right) d\eta.
\end{aligned}$$

This implies that $\omega_s = \tilde\omega_s$. The proof of (7) is similar as the proof of estimate (2.5) in [13, Theorem 4.1].

2.4. The explicit approach

This part is an adaptation to our problem of what has been done in [9] concerning the blow-up problem in equation (12). Let ω be a solution of

$$\omega'' + \frac{1}{2}\eta\omega' + \frac{1}{2(p-1)}\omega = 0 \qquad \text{in } \mathbb{R}_+. \tag{13}$$

We set

$$r = \frac{\eta^2}{4} \text{ and } \omega(\eta) = r^{-\frac{1}{4}} e^{-\frac{r}{2}} Z(r).$$

Then Z satisfies the Whittaker equation (with the standard notations)

$$Z_{rr} + \left(-\frac{1}{4} + \frac{k}{r} + \frac{1-4\mu^2}{4r^2} \right) Z = 0$$

where $k = \frac{1}{2(p-1)} - \frac{1}{4}$ and $\mu = \frac{1}{4}$. Notice that the only difference with the expression in [9, Lemma 3.1] is the value of the coefficient k. This equation admits two linearly independent solutions

$$Z_1(r) = e^{-\frac{r}{2}} r^{\frac{1}{2}+\mu} U\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right),$$

and

$$Z_2(r) = e^{-\frac{r}{2}} r^{\frac{1}{2}+\mu} M\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right).$$

The functions U and M are the Whittaker functions which play an important role not only in analysis but also in group theory. They have the following asymptotic expansion as $r \to \infty$ (see e.g. [1]),

$$U\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right) = r^{k-\mu-\frac{1}{2}} \left(1 + O(r^{-1})\right) = r^{\frac{1}{2(p-1)}-1} \left(1 + O(r^{-1})\right),$$

and

$$\begin{aligned} M\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right) &= \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2} + \mu - k)} e^r r^{-(\mu + \frac{1}{2} + k)} \left(1 + O(r^{-1})\right) \\ &= \frac{\Gamma(\frac{3}{2})}{\Gamma(1 - \frac{1}{2(p-1)})} e^r r^{-\frac{p}{2(p-1)}} \left(1 + O(r^{-1})\right). \end{aligned}$$

Then

$$Z_1(r) = r^{\frac{1}{2(p-1)} - \frac{1}{4}} e^{-\frac{r}{2}} \left(1 + O(r^{-1})\right),$$

and

$$Z_2(r) = \frac{\Gamma(\frac{3}{2})}{\Gamma(1 - \frac{1}{2(p-1)})} r^{\frac{1}{4} - \frac{1}{2(p-1)} -} e^{\frac{r}{2}} \left(1 + O(r^{-1})\right).$$

To this correspond the two linearly independent solutions ω_1 and ω_2 of (13) with the following behaviour as $\eta \to \infty$,

$$(i) \qquad \omega_1(\eta) = c_1 \eta^{\frac{1}{p-1} - 1} e^{-\frac{\eta^2}{4}} \left(1 + O(\eta^{-2})\right),$$

$$(ii) \qquad \omega_2(\eta) = c_2 \eta^{-\frac{1}{p-1}} \left(1 + O(\eta^{-2})\right).$$

Clearly only ω_1 satisfies the decay estimate $\omega(\eta) = o(\eta^{-\frac{1}{p-1}})$ as $\eta \to \infty$. Hence the solution ω is a multiple of ω_1 and the multiplicative constant c is adjusted in order to fit the condition $\omega'(0) = \omega^p(0)$.

3. Problem with measure data

3.1. The regular problem

Set $G(r) = \int_0^r g(s)ds$. We consider the functional J in $L^2(\mathbb{R}_+)$ with domain $D(J) = H^1(\mathbb{R}_+)$ defined by

$$J(u) = \frac{1}{2}\int_0^\infty u_x^2 dx + G(v(0)).$$

It is convex and lower semicontinuous in $L^2(\mathbb{R}_+)$ and its subdifferential ∂J sastisfies

$$\int_0^\infty \partial J(u)\zeta dx = \int_0^\infty u_x\zeta_x dx + g(u(0))\zeta(0),$$

for all $\zeta \in H^1(\mathbb{R}_+)$. Therefore

$$\int_0^\infty \partial J(u)\zeta dx = -\int_0^\infty u_{xx}\zeta dx + (g(u(0)) - u_x(0))\zeta(0).$$

Hence

$$\partial J(u) = -u_{xx} \text{ for all } u \in D(\partial J) = \{v \in H^1(\mathbb{R}_+) : v_x(0) = g(v(0))\}.$$

The operator ∂J is maximal monotone, hence it generates a semi-group of contractions. Furthermore, for any $u_0 \in L^2(\mathbb{R}_+)$ and $F \in L^2(0,T;L^2(L^2(\mathbb{R}_+))$ there exists a unique strong solution to

$$\begin{aligned} U_t + \partial J(U) &= F \quad \text{a.e. on } (0,T) \\ U(0) &= u_0. \end{aligned}$$

Proposition 3.1. *Let $\mu \in H^1(0,T)$ and $\nu \in L^2(\mathbb{R}_+)$. Then there exists a unique function $u \in C([0,T];L^2(\mathbb{R}_+)$ such that $\sqrt{t}u_{xx} \in L^2((0,T)\times\mathbb{R}_+)$ which satisfies* (14). *The mapping $(\mu,\nu) \mapsto u := u_{\mu,\nu}$ is non-decreasing and u is a weak solution in the sense that it satisfies* (8).

Proof. Let $\eta \in C_0^2([0,\infty))$ such that $\eta(0) = 0$, $\eta'(0) = 1$. If $f \in H^1(0,T)$, $\nu \in L^2(\mathbb{R}_+)$, and u is a solution of

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^T_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) + g(u(\,\cdot\,,0)) &= \mu(t) \quad \text{in } [0,T) \\ u(0,\,\cdot\,) &= \nu \quad \text{in } \mathbb{R}_+, \end{aligned} \tag{14}$$

where $\nu \in L^2(\mathbb{R}_+)$, then the function $v(t,x) = u(t,x) - \mu(t)\eta(x)$ satisfies

$$\begin{aligned} v_t - v_{xx} &= F \quad \text{in } Q^T_{\mathbb{R}_+} \\ -v_x(\,\cdot\,,0) + g(v(\,\cdot\,,0)) &= 0 \quad \text{in } [0,T) \\ v(0,\,\cdot\,) &= \nu - \mu(0)\eta \quad \text{in } \mathbb{R}_+, \end{aligned}$$

with $F(t,x) = -(\mu'(t)\eta(x) + \mu(t)\eta''(x))$. The proof of the existence follows by using [3, Theorem 3.6].
Next, let $(\tilde{\mu}, \tilde{\nu}) \in H^1(0,T) \times L^2(\mathbb{R}_+)$ such that $\tilde{\mu} \le \mu$ and $\tilde{\nu} \le \nu$ and let $\tilde{u} = u_{\tilde{\mu},\tilde{\nu}}$, then

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty (\tilde{u} - u)_+^2 dx + \int_0^\infty (\partial_x(\tilde{u} - u)_+)^2\, dx - (\tilde{\mu}(t) - \mu(t))\,(\tilde{u}(t,0) - u(t,0))_+ \\ + (g(\tilde{u}(t,0)) - g(u(t,0))))(\tilde{u}(t,0) - u(t,0)) = 0.$$

Then

$$\int_0^\infty (\tilde{u} - u)_+^2 dx\lfloor_{t=0} \Longrightarrow \int_0^\infty (\tilde{u} - u)_+^2 dx = 0 \quad \text{on } [0,T].$$

We can also use (10) to express the solution of (14):

$$u(t,x) = \int_0^\infty \tilde{E}(t,x,y)\nu(y)dy + 2\int_0^t E(t-s,x)(\mu(s) - g(u(s,0)))ds.$$

In particular, if $g(0) = 0$, then

$$|u(t,x)| \le \int_0^\infty \tilde{E}(t,x,y)|\nu(y)|dy + 2\int_0^t E(t-s,x)|\mu(s)|ds.$$

The proof of (8) follows since u is a strong solution. □

Next, we prove that the problem is well-posed if $\mu \in L^1(0,T)$.

PROPOSITION 3.2. *Assume $\{\nu_n\} \subset C_c(\mathbb{R}_+)$ and $\{\mu_n\} \subset C^1([0,T])$ are Cauchy sequences in $L^1(\mathbb{R}_+)$ and $L^1(0,T)$ respectively. Then the sequence $\{u_n\}$ of solutions of*

$$\begin{aligned} u_{n\,t} - u_{n\,xx} &= 0 \quad \text{in } Q^T_{\mathbb{R}_+} \\ -u_{n\,x}(\,\cdot\,,0) + g(u_n(\,\cdot\,,0)) &= \mu_n(t) \quad \text{in } [0,T) \\ u_n(0,\,\cdot\,) &= \nu_n \quad \text{in } \mathbb{R}_+, \end{aligned} \tag{15}$$

converges in $C([0,T]; L^1(\mathbb{R}_+)$ to a function u which satisfies (8).

Proof. For $\epsilon > 0$ let p_ϵ be an odd C^1 function defined on $\mathbb{R}$ such that $p'_\epsilon \ge 0$ and $p_\epsilon(r) = 1$ on $[\epsilon, \infty)$, and put $j_\epsilon(r) = \int_0^r p_\epsilon(s)ds$. Then

$$\frac{d}{dt}\int_0^\infty j_\epsilon(u_n - u_m)dx + \int_0^\infty (u_{n\,x} - u_{m\,x})^2 p'_\epsilon(u_n - u_m)dx \\ + (g(u_n(t,0)) - g(u_m(t,0)))\, p_\epsilon(u_n(t,0) - u_m(t,0)) \\ = (\mu_n(t) - \mu_m(t))\, p_\epsilon(u_n(t,0) - u_m(t,0)).$$

Hence

$$\int_0^\infty j_\epsilon(u_n-u_m)(t,x)dx+\left(g(u_n(t,0))-g(u_m(t,0))\right)p_\epsilon(u_n(t,0)-u_m(t,0))$$
$$\leq\int_0^\infty j_\epsilon(\nu_n-\nu_m)dx+\left(\mu_n(t)-\mu_m(t)\right)p_\epsilon(u_n(t,0)-u_m(t,0)).$$

Letting $\epsilon\to 0$ implies $p_\epsilon\to sgn_0$, hence for any $t\in[0,T]$,

$$\int_0^\infty |u_n-u_m|(t,x)dx+|g(u_n(t,0))-g(u_m(t,0)|$$
$$\leq\int_0^\infty|\nu_n-\nu_m|dx+|\mu_n(t)-\mu_m(t)|.$$

Therefore $\{u_n\}$ and $\{g(u_n(\,\cdot\,,0)\}$ are Cauchy sequences in $C([0,T];L^1(\mathbb{R}_+))$ and $C([0,T])$ respectively with limit u and $g(u)$ and $u=u_{\nu,\mu}$ satisfies (8). If we assume that $(\nu,\tilde\nu)$ and $(\mu,\tilde\mu)$ are couples of elements of $L^1(\mathbb{R}_+)$ and $L^1(0,T)$ respectively and if $u=u_{\nu,\mu}$ and $\tilde u=u_{\tilde\nu,\tilde\mu}$, there holds by the above technique,

$$\int_0^\infty |u-\tilde u|(t,x)dx+|g(u(t,0))-g(\tilde u(t,0)|$$
$$\leq\int_0^\infty|\tilde\nu-\tilde\nu|dx+|\tilde\mu(t)-\tilde\mu(t)|\quad\text{for all }\ t\in[0,T].\tag{16}$$

□

The following lemma is a parabolic version of an inequality due to Brezis.

Lemma 3.3. *Let $\nu\in L^1(\mathbb{R}_+)$ and $\mu\in L^1(0,T)$ and v be a function defined in $[0,T)\times\mathbb{R}_+$, belonging to $L^1(Q^T_{\mathbb{R}_+})\cap L^1(\partial_\ell Q^T_{\mathbb{R}_+})$ and satisfying*

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})vdxdt=\int_0^T\zeta(\,\cdot\,,0)\mu dt+\int_0^\infty\nu\zeta dx.\tag{17}$$

Then for any $\zeta\in\mathbb{X}(Q^T_{\mathbb{R}_+})$, $\zeta\geq 0$, there holds

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})|v|dxdt\leq\int_0^\infty\zeta(\,\cdot\,,0)sign(v)\mu dt+\int_0^\infty|\nu|\zeta dx.\tag{18}$$

Similarly

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})v_+dxdt\leq\int_0^\infty\zeta(\,\cdot\,,0)sign_+(v)\mu dt+\int_0^\infty\nu_+\zeta dx.\tag{19}$$

Proof. Let p_ϵ be the approximation of $sign_0$ used in Proposition 3.2 and η_ϵ be the solution of

$$\begin{aligned} -\eta_{\epsilon\, t} - \eta_{\epsilon\, xx} &= p_\epsilon(v) \quad \text{in } Q^T_{\mathbb{R}_+} \\ \eta_{\epsilon\, x}(\,\cdot\,, 0) &= 0 \quad \text{in } [0, T] \\ \eta_\epsilon(0, \,\cdot\,) &= 0 \quad \text{in } \mathbb{R}_+. \end{aligned}$$

Then $|\eta_\epsilon| \leq \eta^*$ where η^* satisfies

$$\begin{aligned} -\eta^*_t - \eta^*_{xx} &= 1 \quad \text{in } Q^T_{\mathbb{R}_+} \\ \eta^*_x(\,\cdot\,, 0) &= 0 \quad \text{in } [0, T] \\ \eta^*(0, \,\cdot\,) &= 0 \quad \text{in } \mathbb{R}_+. \end{aligned}$$

Although η_ϵ does not belong to $\mathbb{X}(Q^T_{\mathbb{R}_+})$ (it is not in $C^{1,2}([0, T) \times \mathbb{R}_+)$), it is an admissible test function and we deduce that there exists a unique solution to (17). Thus v is given by expression (10).

In order to prove (18), we can assume that μ and ν are smooth, $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$, $\zeta \geq 0$ and set $h_\epsilon = p_\epsilon(v)\zeta$ and $w_\epsilon = v p_\epsilon(v)$, then

$$\begin{aligned} \int_0^\infty h_{\epsilon\, xx} v dx &= \int_0^\infty \left(2p'_\epsilon(v) v_x \zeta_x + p_\epsilon(v)\zeta_{xx} + \zeta(p_\epsilon(v))_{xx}\right) v dx \\ &= \int_0^\infty \left(2v p'_\epsilon(v) v_x \zeta_x - w_{\epsilon\, x}\zeta_x - (v\zeta)_x (p_\epsilon(v))_x\right) dx \\ &\qquad\qquad - \zeta(t,0) v(t,0) p'_\epsilon(v(t,0)) v_x(t,0) \\ &= -\int_0^\infty \left(\zeta_x (j_\epsilon(v))_x + \zeta p'(v)_\epsilon v_x^2\right) dx - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0) \\ &= -\int_0^\infty \left(\zeta p'(v)_\epsilon v_x^2 - j_\epsilon(v)\zeta_{xx}\right) dx - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0), \end{aligned}$$

and

$$\int_0^T h_{\epsilon\, t} v dt = \int_0^T (p_\epsilon(v)\zeta_t + p'_\epsilon(v)\zeta v_t) v dt.$$

Since v is smooth

$$\begin{aligned} 0 &= \int_0^T \int_0^\infty (v_t - v_{xx}) h_\epsilon dx dt \\ &= -\int_0^T \int_0^\infty (h_{\epsilon\, t} + h_{\epsilon\, xx}) v dx dt - \int_0^\infty h_\epsilon(0, x)\nu(x) dx \\ &\qquad - \int_0^T \left[p_\epsilon(v(t,0)) - v(t,0)p'_\epsilon(v(t,0))\right] \zeta(t,0)\mu(t) dt. \end{aligned}$$

Therefore, using (18) and (19),

$$-\int_0^T\int_0^\infty (j_\epsilon v)\zeta_{xx} + vp_\epsilon(v)\zeta_t)\,dxdt$$
$$+\int_0^T\int_0^\infty \left(\zeta p'_\epsilon(v)v_x^2 - vp'_\epsilon(v)v_t\zeta\right)dxdt$$
$$= \int_0^\infty h_\epsilon(0,x)\nu(x)dx + \int_0^T h_\epsilon(t,0)\mu(t)dt. \quad (20)$$

Put $\ell_\epsilon(s) = \int_0^s rp'_\epsilon(r)dr$, then $|\ell_\epsilon(s) \leq c\epsilon^{-1}s^2\chi_{[-\epsilon,\epsilon]}(s)|$. Since

$$\int_0^T\int_0^\infty \zeta vp'_\epsilon(v)v_t dxdt = -\int_0^\infty \ell_\epsilon(v(0,x))\zeta(x)dx - \int_0^T\int_0^\infty \zeta_t\ell_\epsilon(v)dxdt,$$

and ζ has compact support, it follows that

$$\lim_{\epsilon\to 0}\int_0^T\int_0^\infty \zeta vp'_\epsilon(v)v_t dxdt = 0.$$

Letting $\epsilon \to 0$ in (20), we derive (18) for smooth v. Using Proposition 3.2 completes the proof of (18). The proof of (19) is similar. □

Remark 3.4: Inequalities (18) and (19) hold if $\zeta(t,x)$ does not vanish if $|x| \geq R$ for some R but if it satisfies

$$\lim_{x\to\infty}\sup_{t\in[0,T]}(\zeta(t,x) + |\zeta_x(t,x)|) = 0. \quad (21)$$

The proof follows by replacing $\zeta(t,x)$ by $\zeta(t,x)\eta_n(x)$ where $\eta_n \in C_c^\infty(\mathbb{R}_+)$ with $0 \leq \eta_n \leq 1$, $\eta_n(x) = 1$ on $[0,n]$, $\eta_n(x) = 0$ on $[n+1,\infty)$, $|\eta'_n| \leq 2$, $|\eta''_n| \leq 4$. Then $\eta_n\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$ by letting $n \to \infty$ and the proof follows by letting $n \to \infty$.

3.2. Proof of Theorem 1.3

We give first some *heat-ball* estimates relative to our problem. For $r > 0$, $x \in \mathbb{R}_+$ and $t \in \mathbb{R}$ we set

$$e(t,x;r) = \left\{(s,y) \in (0,T)\times\mathbb{R}_+ : s \leq t,\ \tilde{E}(t-s,x,y) \geq r\right\}.$$

Since

$$e(t,x;r) \subset [t - \tfrac{1}{4\pi er^2}, t] \times [x - \tfrac{1}{r\sqrt{\pi e}}, x + \tfrac{1}{r\sqrt{\pi e}}],$$

there holds

$$|e(t,x;r)| \leq \frac{1}{2r^3(\pi e)^{\frac{3}{2}}},$$

and if

$$e^*(t;r) = \{s \in (0,T) : s \le t,\ E(t-s,0,0) \ge r\},$$

then we have

$$e^*(t;r) \subset [t - \tfrac{1}{4\pi e r^2}, t] \Longrightarrow |e^*(t;r)| \le \frac{1}{4r^2\pi e}. \tag{22}$$

If G is a measured space, λ a positive measure on G and $q > 1$, $M^q(G,\lambda)$ is the Marcinkiewicz space of measurable functions $f : G \mapsto \mathbb{R}$ satisfying for some constant $c > 0$ and all measurable set $E \subset G$,

$$\int_E |f| d\lambda \le c\,(\lambda(E))^{\frac{1}{p'}},$$

and

$$\|f\|_{M^q(G,\lambda)} = \inf\{c > 0 \text{ s.t. } (22) \text{ holds}\}.$$

LEMMA 3.5. *Assume μ,ν are bounded measure in $\overline{\mathbb{R}_+}$ and $\mathbb{R}_+$ respectively and u is the solution of* (9) *given by* (10) *and $v_{\nu,\mu}$ is the solution of* (9). *Then*

$$\|v_{\nu,\mu}\|_{M^3(Q^T_{\mathbb{R}_+})} + \left\| v_{\nu,\mu}\lfloor_{\partial Q^T_{\mathbb{R}_+}} \right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \le c\left(\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})} + \|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})} \right).$$

Proof. First we consider $v_{0,\mu}$

$$v_{0,\mu}(t,x) = 2\int_0^t E(t-s,x)d\mu(s).$$

If $F \subset [0,T]$ is a Borel set, than for any $\tau > 0$

$$\begin{aligned}
\int_F E(t-s,0)ds &= \int_{F\cap\{E\le\tau\}} E(t-s,0)ds + \int_{F\cap\{E>\tau\}} E(t-s,0)ds \\
&\le \tau|F| + \int_{\{E>\tau\}} E(t-s,0)ds \\
&\le \tau|F| - \int_\tau^\infty \lambda d|e^*(t,\lambda)| \\
&\le \tau|F| + \int_\tau^\infty \lambda d|e^*(t,\lambda)| \\
&\le \tau|F| + \frac{1}{4\pi e\tau}.
\end{aligned}$$

If we choose $\tau^2 = \frac{1}{4\pi e|F|}$, we derive

$$\int_F E(t-s,0)ds \le \frac{|F|^{\frac{1}{2}}}{\sqrt{\pi e}}.$$

If $F \subset (0,T)$ is a Borel set then

$$\left|\int_F v_{0,\mu}(t,0)dt\right| = 2\left|\int_0^t\int_F E(t-s,0)dtd\mu(s)\right| \leq \frac{2|F|^{\frac{1}{2}}}{\sqrt{\pi e}}\,\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

This proves that

$$\left\|v_{0,\mu}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\,\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

Similarly, if $G \subset [0,T]\times[0,\infty)$ is a Borel set, then

$$\int_G \tilde{E}(t-s,x,0)ds \leq \frac{2|G|^{\frac{1}{3}}}{\sqrt{\pi e}},$$

and

$$\|v_{0,\mu}\|_{M^3(Q^T_{\mathbb{R}_+})} \leq c\,\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

In the same way we prove that

$$\|v_{\nu,0}\|_{M^3(Q^T_{\mathbb{R}_+})} + \left\|v_{\nu,0}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\,\|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}.$$

This ends the proof of the lemma. □

Proof of Theorem 1.3. Uniqueness. Assume u and $\tilde{u}$ are solutions of (1), then $w = u - \tilde{u}$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 \quad \text{in } Q^T_{\mathbb{R}_+} \\ -w_x(\,\cdot\,,0) + g(u(\,\cdot\,,0)) - g(\tilde{u}(\,\cdot\,,0)) &= 0 \quad \text{in } [0,T) \\ w(0,\,\cdot\,) &= 0 \quad \text{in } \mathbb{R}_+. \end{aligned}$$

Applying (18), we obtain

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})|w|dxdt + \int_0^\infty (g(u(\,\cdot\,,0)) - g(\tilde{u}(\,\cdot\,,0)))sign(w)\zeta(t,0)dt \leq 0,$$

for any $\zeta \in \mathbb{X}^T_{\mathbb{R}_+}$ with $\zeta \geq 0$. Let $\theta \in C^1_c(Q^T_{\mathbb{R}_+})$, $\eta \geq 0$, we take ζ to be the solution of

$$\begin{aligned} -\zeta_t - \zeta_{xx} &= \theta \quad \text{in } (0,T)\times\mathbb{R}_+ \\ \zeta_x(\,\cdot\,,0) &= 0 \quad \text{in } (0,T) \\ \zeta(T,\,\cdot\,) &= 0 \quad \text{in } (0,\infty). \end{aligned}$$

Then ζ satisfies (21), hence

$$\int_0^T\int_0^\infty \theta|w|dxdt + \int_0^\infty (g(u(\,\cdot\,,0)) - g(\tilde{u}(\,\cdot\,,0)))sign(w)\zeta(t,0)dt \leq 0.$$

This implies $w = 0$.

Existence. Without loss of generality we can assume that μ and ν are nonnegative. Let $\{\nu_n\} \subset C_c(\mathbb{R}_+)$ and $\{\mu_n\} \subset C_c([\mathbb{R}_+]0,T))$ converging to ν and μ in the sense of measures and let u_n be the solution of (15). Then from (16),

$$\int_0^T \int_0^\infty |u_n| dxdt + \int_0^T |g(u_n(t,0))| dt \leq T \int_0^\infty |\nu_n| dx + \int_0^T |\mu_n| dt.$$

Therefore u_n and $g(u_n(\,\cdot\,,0))$ remain bounded respectively in $L^1(Q^T_{\mathbb{R}_+})$ and in $L^1(0,T)$. Furthermore, by Lemma 3.5, u_n remains bounded in $M^3(Q^T_{\mathbb{R}_+})$ and in $M^2(\partial Q^T_{\mathbb{R}_+})$. We can also write u_n under the form

$$\begin{aligned} u_n(t,x) &= \int_0^\infty \tilde{E}(t,x,y)\mu_n(y)dy + 2\int_0^t E(t-s,x)(\nu_n(t) - g(u_n(t,0)))ds \\ &= A_n(t,x) + B_n(t,x). \end{aligned} \tag{23}$$

Since we can perform the even reflexion through $y = 0$, the mapping

$$(t,x) \mapsto A_n(t,x) := \int_0^\infty \tilde{E}(t,x,y)\mu_n(y)dy,$$

is relatively compact in $C^m_{loc}(\overline{Q^T_{\mathbb{R}_+}})$ for any $m \in \mathbb{N}^*$. Hence we can extract a subsequence $\{u_{n_k}\}$ which converges uniformly on every compact subset of $(0,T] \times [0,\infty)$, hence a.e. on $(0,T]$ for the 1-dimensional Lebesque measure. Concerning the boundary term

$$(t,x) \mapsto B_n(t,x) := \int_0^t E(t-s,x)(\nu_n(t) - g(u_n(t,0)))ds,$$

it is relatively compact on every compact subset of $[0,T] \times (0,\infty)$. If $x = 0$, then

$$B_n(t,0) = \int_0^t (\nu_n(t) - g(u_n(t,0))) \frac{ds}{\sqrt{\pi(t-s)}}.$$

Since $\|\nu_n(\,\cdot\,) - g(u_n(\,\cdot\,,0))\|_{L^1(0,T)}$, $t \mapsto B_n(t,0)$ is uniformly integrable on $(0,T)$, hence relatively compact by the Frechet-Kolmogorov Theorem. Therefore there exists a subsequence, still denoted by $\{n_k\}$ such that $B_{n_k}(t,0)$ converges for almost all $t \in (0,T)$. This implies that the sequence of function $\{u_{n_k}\}$ defined by (23) converges in $\overline{Q^T_{\mathbb{R}_+}}$ up to a set $\Theta \cup \Lambda$ where $\Theta \subset Q^T_{\mathbb{R}_+}$ is neglectable for the 2-dimensional Lebesgue measure and $\Lambda \subset \partial_\ell Q^T_{\mathbb{R}_+}$ neglectable for the 1-dimensional Lebesgue measure.

From Lemma 3.5, $(u_{n,k}\lfloor_{Q^T_{\mathbb{R}_+}}, u\lfloor_{\partial_\ell Q^T_{\mathbb{R}_+}})$ converges in $L^1_{loc}(Q^T_{\mathbb{R}_+}) \times L^1(\partial_\ell Q^T_{\mathbb{R}_+})$ and the convergence of each of the components holds also almost everywhere (up to a subsequence). Since $u_{n,k}$ is a weak solution, it satisfies for any $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$

$$-\int_0^T\int_0^\infty (\zeta_t+\zeta_{xx})u_{n,k}dxdt+\int_0^T (g(u_{n,k})\zeta)(t,0)dt = \int_0^\infty \zeta\nu_{n,k}(x)dx+\int_0^T\zeta(t,0)\mu_{n,k}(t)dt.$$

In order to prove the convergence of $g(u_{n,k}(t,0))$, we use Vitali's convergence theorem and the assumption (11). Let $F\subset[0,T]$ be a Borel set. Using the fact that $0\leq u_{n,k}\leq v_{\nu_{n,k},\mu_{n,k}}$ and the estimate of Lemma 3.5, we have for any $\lambda>0$,

$$\begin{aligned}\int_F|g(u_{n,k}(t,0))|dt &\leq \int_{F\cap\{u_{n,k}(t,0)\leq\lambda\}}|g(u_{n,k}(t,0))|dt\\ &\qquad+\int_{\{u_{n,k}(t,0)>\lambda\}}|g(u_{n,k}(t,0))|dt\\ &\leq g(\lambda)|F|-\int_\lambda^\infty\sigma d|\{t:|g(u_{n,k}(t,0))|>\sigma\}|\\ &\leq g(\lambda)|F|+c\int_\lambda^\infty|g(\sigma)|\sigma^{-3}ds,\end{aligned}$$

where c depends of $\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}+\|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}$. For $\epsilon>0$ given, we chose λ large enough so that the integral term above is smaller than ϵ and then $|F|$ such that $g(\lambda)|F|+\leq\epsilon$. Hence $\{g(u_{n,k}(\,\cdot\,,0))\}$ is uniformly integrable. Therefore up to a subsequence, it converges to $g(u(\,\cdot\,,0))$ in $L^1(0,T)$. Clearly u satisfies

$$-\int_0^T\int_0^\infty (\zeta_t+\zeta_{xx})udxdt+\int_0^T (g(u)\zeta)(t,0)dt = \int_0^\infty \zeta\nu(x)dx+\int_0^T\zeta(t,0)\mu(t)dt,$$

which ends the existence proof.

Monotonicity. If $\nu\geq\tilde\nu$ and $\mu\geq\tilde\mu$; we can choose the approximations such that $\nu_n\geq\tilde\nu_n$ and $\mu_n\geq\tilde\mu_n$. It follows from (19) that $u_{\nu_n,\mu_n}\geq u_{\tilde\nu_n,\tilde\mu_n}$. Choosing the same subsequence $\{n_k\}$, the limits u, $\tilde u$ are in the same order. The conclusion follows by uniqueness. □

3.3. The case $g(u)=|u|^{p-1}u$

Condition (11) is satisfied if $p<2$. If this condition holds there exists a solution $u_{\ell\delta_0}=u_{0,\ell\delta_0}$ and the mapping $\ell\mapsto u_{\ell\delta_0}$ is increasing.

THEOREM 3.6. *(i) If* $1 < p \leq \frac{3}{2}$, $u_{\ell\delta_0}$ *tends to* ∞ *when* $k \to \infty$.
(ii) If $\frac{3}{2} < p < 2$, $u_{\ell\delta_0}$ *converges to* U_{ω_s} *defined by*

$$U_{\omega_s}(t,x) = t^{-\frac{1}{2(p-1)}}\omega_s(\tfrac{x}{\sqrt{t}}),$$

when $k \to \infty$.

Proof. By uniqueness and using (3), there holds

$$T_k[u_{\ell\delta_0}] = u_{k^{\frac{2-p}{p-1}}\ell\delta_0},$$

for any $k, \ell > 0$. Since $\ell \mapsto u_{\ell\delta_0}$ is increasing, its limit u_∞, when $\ell \to \infty$, satisfies

$$T_k[u_\infty] = u_\infty.$$

Hence u_∞ is a positive self-similar solution of (2), provided it exists. Hence $u_\infty = U_{\omega_s}$ if $\frac{3}{2} < p < 2$. If $1 < p \leq \frac{3}{2}$, $u_{k\delta_0}$ admits no finite limit when $k \to \infty$ which ends the proof. □

REMARK 3.7: As a consequence of this result, no a priori estimate of Brezis-Friedman type (parabolic Keller-Osserman) exists for a nonnegative function $u \in C^{2,1}(\overline{Q^\infty_{\mathbb{R}_+}} \setminus \{(0,0)\})$ solution of

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^\infty_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) + |u|^{p-1}u(\,\cdot\,,0) &= 0 \quad \text{for all } t > 0 \\ u(0,\,\cdot\,) &= 0 \quad \text{for all } x > 0. \end{aligned}$$

when $1 < p \leq \frac{3}{2}$. When $\frac{3}{2} < p < 2$ it is expected that

$$u(t,x) \leq \frac{c}{(|x|^2 + t)^{\frac{1}{2(p-1)}}}.$$

The type of phenomenon (i) in Theorem 3.6 is characteristic of fractional diffusion. It has already been observed in [6, Theorem 1.3] with equations

$$\begin{aligned} u_t + (-\Delta)^\alpha u + t^\beta u^p &= 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ u(0,\,\cdot\,) &= k\delta_0 \quad \text{in } \mathbb{R}^N, \end{aligned}$$

when $0 < \alpha < 1$ is small and $p > 1$ is close to 1.

4. Extension and open problems

The natural extension is to replace a one dimensional domain by a mutidimenional one. The main open problem is the question of a priori estimate as stated in the last remark above.

4.1. Self-similar solutions

Let $\eta=(\eta_1,\dots,\eta_n)$ be the coordinates in $\mathbb{R}^n$ and denote

$$\mathbb{R}^n_+=\{\eta=(\eta_1,\dots,\eta_n)=(\eta',\eta_n):\eta_n>0\}.$$

We set $K(\eta)=e^{\frac{|\eta|^2}{4}}$ and $K'(\eta')=e^{\frac{|\eta'|^2}{4}}$. Similarly to Section 2 we define $\mathcal{L}_K$ in $C^2_0(\mathbb{R}^n)$ by

$$\mathcal{L}_K(\phi)=-K^{-1}div(K\nabla\phi). \tag{24}$$

If $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we set $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_n$. We denote by ϕ_1 the function K^{-1}. Then the set of eigenvalues of $\mathcal{L}_K$ is the set of numbers $\left\{\lambda_k=\frac{n+k}{2}:k\in\mathbb{N}\right\}$ with corresponding set of eigenspaces

$$N_k=span\left\{D^\alpha\phi_1:|\alpha|=k\right\}.$$

The operators $\mathcal{L}^{+,N}_K$ and $\mathcal{L}^{+,D}_K$ are defined accordingly in $H^1_K(\mathbb{R}^n_+)$ and $H^{1,0}_K(\mathbb{R}^n_+)$ respectively, and

$$\sigma(\mathcal{L}^{+,N}_K)=\left\{\frac{n+k}{2}:k\in\mathbb{N}\right\}\quad\text{and}\quad\sigma(\mathcal{L}^{+,D}_K)=\left\{\frac{n+k}{2}:k\in\mathbb{N}^*\right\}.$$

Furthermore

$$N_{k,N}=ker\left(\mathcal{L}^{+,N}_K-\tfrac{n+k}{2}I_d\right)=span\left\{D^\alpha\phi_1:|\alpha|=k,\alpha_n=2\ell,\ \ell\in\mathbb{N}\right\},$$

and

$$N_{k,D}=ker\left(\mathcal{L}^{+,D}_K-\tfrac{n+k}{2}I_d\right)=span\left\{D^\alpha\phi_1:|\alpha|=k,\alpha_n=2\ell+1,\ \ell\in\mathbb{N}\right\}.$$

Since $\mathcal{L}^{+,N}_K$ and $\mathcal{L}^{+,D}_K$ are Fredholm operators,

$$H^1_K(\mathbb{R}^n_+)=\bigoplus_{k=0}^\infty N_{k,N}\quad\text{and}\quad H^{1,0}_K(\mathbb{R}^n_+)=\bigoplus_{k=1}^\infty N_{k,D}.$$

We define the following functional on $H^1_K(\mathbb{R}^n_+)$

$$J(\phi)=\frac{1}{2}\int_{\mathbb{R}^n_+}\left(|\nabla\phi|^2-\frac{1}{2(p-1)}\phi^2\right)Kd\eta+\frac{1}{p+1}\int_{\partial\mathbb{R}^n_+}|\phi|^{p+1}K'd\eta'.$$

The critical points of J satisfies

$$\begin{aligned}-\Delta\omega-\frac{1}{2}\eta.\nabla\omega-\frac{1}{2(p-1)}\omega=0\quad&\text{in }\ \mathbb{R}^n_+\\ -\omega_{\eta_n}+|\omega|^{p-1}\omega=0\quad&\text{in }\ \partial\mathbb{R}^n_+.\end{aligned} \tag{25}$$

If ω is a solution of (25), the function

$$u_\omega(t,x) = t^{-\frac{1}{2(p-1)}}\,\omega\left(\frac{x}{\sqrt{t}}\right)$$

satisfies

$$\begin{aligned} u_{\omega\,t} - \Delta u_\omega &= 0 \qquad \text{in } Q^\infty_{\mathbb{R}^n_+} := (0,\infty)\times\mathbb{R}^n_+ \\ -u_{\omega\,x_n} + |u_\omega|^{p-1}u_\omega &= 0 \qquad \text{in } \partial_\ell Q^\infty_{\mathbb{R}^n_+} := (0,\infty)\times\partial\mathbb{R}^n_+. \end{aligned}$$

Here we have set $\mathbb{R}^n_+ = \{x = (x_1,\ldots,x_n) = (x', x_n) : x_n > 0\}$. We denote by $\mathcal{E}$ the subset $H^1_K(\mathbb{R}^n_+)\cap L^p(\partial\mathbb{R}^n_+; d\eta')$ of solutions of (25) and by $\mathcal{E}_+$ the subset of positive solutions. As for the case $n = 1$ we have the following non-existence result

Proposition 4.1. *1. If $p \geq 1 + \frac{1}{n}$, then $\mathcal{E} = \{0\}$.*

2. If $1 < p \leq 1 + \frac{1}{n+1}$, then $\mathcal{E}_+ = \{0\}$.

The proof is similar to the one of Theorem 1.1. Hence the existence is to be found in the range $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$.

Conjecture 4.2. *Assume $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$, then the functional J is bounded from below in $H^1_K(\mathbb{R}^n_+)\cap L^p_{K'}(\partial\mathbb{R}^n_+)$. Furthermore $J(\phi)$ tends to infinity when $\|\phi\|_{H^1_K(\mathbb{R}^n_+)} + \left\|\phi\lfloor_{\partial\mathbb{R}^n_+}\right\|_{L^{p+1}_{K'}(\partial\mathbb{R}^n_+)}$ tends to infinity.*

4.2. Problem with measure data

The method for proving Theorem 1.3 can be adapted to prove the following n-dimensional result

Theorem 4.3. *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a nondecreasing continuous function such that $g(0) = 0$ and*

$$\int_1^\infty (g(s) - g(-s))s^{-\frac{2n+1}{n}}\,ds < \infty,$$

then for any bounded Radon measures ν in $\mathbb{R}^n_+$ and μ in $(0,T)\times\partial\mathbb{R}^n_+$, there exists a unique Borel function $u := u_{\nu,\mu}$ defined in $\overline{Q^{\mathbb{R}^n_+}_T} := [0,T]\times\mathbb{R}^n_+$ such that $u \in L^1(Q^{\mathbb{R}^n_+}_T)$, $u\lfloor_{(0,T)\times\partial\mathbb{R}^n_+} \in L^1((0,T)\times\partial\mathbb{R}^n_+)$ and $g(u) \in L^1((0,T)\times\partial\mathbb{R}^n_+)$ solution of

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } Q^T_{\mathbb{R}^n_+} \\ -u_{x_n} + g(u) &= \mu \quad \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0,\cdot\,) &= \nu \quad \text{in } \mathbb{R}^n_+, \end{aligned}$$

in the sense that

$$\int\int_{Q^T_{\mathbb{R}^n_+}}(-\partial_t\zeta-\Delta\zeta)udxdt+\int\int_{\partial_\ell Q^T_{\mathbb{R}^n_+}}g(u)\zeta dx'dt$$
$$=\int_{\mathbb{R}^n_+}\zeta d\nu+\int\int_{\partial_\ell Q^T_{\mathbb{R}^n_+}}\zeta d\mu,$$

for all $\zeta\in C^{1,2}_c(\overline{Q^T_{\mathbb{R}^n_+}})$ *such that* $\zeta_{x_n}=0$ *on* $(0,T)\times\partial\mathbb{R}^n_+$ *and* $\zeta(T,\cdot)=0$. *Furthermore* $(\nu,\mu)\mapsto u_{\nu,\mu})$ *is nondecreasing.*

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Author's address:

Laurent Véron
Institut Denis Poisson, CNRS UMR 7013,
Université de Tours, France.
E-mail: veronl@univ-tours.fr

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