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A note on a class of double well potential problems

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"Dedicated to Julian Lopez-Gomez on the occasion of his 60th birthday"

Abstract. *It is well known that under appropriate conditions on a double well potential, the associated Hamiltonian system possesses a pair of heteroclinic solutions joining the minima of the potential in addition to infinitely many other homoclinics and heteroclinics that oscillate between these minima. This paper studies the effect on such solutions of replacing the temporal domain, $\mathbb{R}$, by a finite but long time interval.*

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1. Introduction

Consider the Hamiltonian system:

$$-\ddot{q} + V_q(t, q) = 0, \ t \in \mathbb{R} \tag{HS}$$

where V is a double well potential. Several papers, [8, 9, 11, 12, 15, 19, 34, 35] have used variational methods to treat the existence and multiplicity of solutions of (HS) that are heteroclinic or homoclinic to the the points a^- and a^+ corresponding to the bottoms of the potential wells. See also [1–7, 10, 13, 14, 16–18, 20–33] for the use of such methods for related problems. The main goal of this note is to study (i) the extent to which these solutions persist qualitatively if (HS) is replaced by a large time boundary value problem with a^- and a^+ as boundary states and (ii) the behavior of these finite time solutions as the time interval tends to $\mathbb{R}$. To be more precise, suppose that V satisfies

(V_1) $V \in C^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ and is 1-periodic in $t \in \mathbb{R}$.

(V_2) There are points $a^-, a^+ \in \mathbb{R}^m$ such that $V(t, q) > V(t, a^\pm) = 0$ for any $t \in \mathbb{R}$ and $q \in \mathbb{R}^m \setminus \{a^-, a^+\}$.

(V_3) There is a constant, $V_0 > 0$, such that $\liminf_{|q|\to+\infty} V(t, q) \geq V_0$.

Associated with (HS) is the Lagrangian, $L(q) = \frac{1}{2}|\dot{q}|^2 + V(t,q)$, and the functional

$$I(q) = \int_{\mathbb{R}} L(q)\, dt.$$

For $i \in \mathbb{Z}$, let $T_i = [i, i+1]$. Set

$$E \equiv \left\{ q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m) \mid \int_{\mathbb{R}} |\dot{q}|^2\, dt + \int_{T_0} |q|^2\, dt < \infty \right\}.$$

E is a Hilbert space under the inner product associated with the norm

$$\|q\|^2 = \int_{\mathbb{R}} |\dot{q}|^2\, dt + \int_{T_0} |q|^2\, dt.$$

Consider I on E and set

$$\Gamma(a^-, a^+) = \left\{ q \in E \mid q(\pm\infty) = a^\pm \right\}$$

where by $q(\pm\infty) = a^\pm$ is meant $\lim_{t\to\pm\infty} q(t) = a^\pm$. In the present setting, this condition is equivalent to requiring that, as in [15], $\lim_{i\to\pm\infty} \|q - a^\pm\|_{L^2(T_i,\mathbb{R}^m)} = 0$. Define

$$c(a^-, a^+) = \inf_{q \in \Gamma(a^-,a^+)} I(q). \tag{1}$$

Let

$$\mathcal{M}(a^-, a^+) = \left\{ q \in \Gamma(a^-, a^+) \mid I(q) = c(a^-, a^+) \right\}.$$

It was shown in [15] that $\mathcal{M}(a^-, a^+) \neq \emptyset$ and any $Q \in \mathcal{M}(a^-, a^+)$ is a C^2 solution of (HS) heteroclinic from a^- to a^+. Likewise reversing the roles of a^- and a^+ in $\Gamma(a^-, a^+), c(a^-, a^+)$ and $\mathcal{M}(a^-, a^+)$ yields solutions of (HS) heteroclinic from a^+ to a^-.

It was further shown in [15] that there are many other heteroclinics joining a^- and a^+ as well as homoclinic solutions to a^- and to a^+ provided that the sets, $\mathcal{M}(a^-, a^+)$ and $\mathcal{M}(a^+, a^-)$ are not too degenerate. Indeed, when the corresponding nondegeneracy condition is satisfied, for any $k \in \mathbb{N}$, there are infinitely many solutions that oscillate k times between small neighborhoods of a^- and a^+, the solutions being distinguished by the amount of time they spend near the intermediate equilibria. Similar statements apply to the other possibilities for such connecting orbits. The nondegeneracy requirements will be described more fully in Section 2. These requirements lead to new multi-transition solutions that are obtained as local minima of I.

Turning now to our main goal, let $\sigma = (\sigma^-, \sigma^+)$. The analogue of (HS) that will be studied here is

$$-\ddot{q} + V_q(t,q) = 0,\ t \in \sigma,\ \ q(\sigma^-) = a^-, q(\sigma^+) = a^+. \tag{2}$$

The corresponding functional is

$$I_\sigma(q) = \int_\sigma L(q)\, dt$$

where

$$q \in \Gamma_\sigma(a^-, a^+) \equiv \left\{ q \in E \mid q(t) = a^- \text{ for } t \le \sigma^-;\ q(t) = a^+ \text{ for } t \ge \sigma^+ \right\}.$$

Due to the periodicity of V in t, the problem (2) is equivalent to the analogous one on the translated interval $\sigma + k$ for any $k \in \mathbb{Z}$. Thus, without loss of generality, we can normalize the choice of the interval, σ, by assuming that its center belongs to $[0, 1)$. With this choice, when $|\sigma| > 1$, we have $\sigma^- < 0 < \sigma^+$.

Let

$$c_\sigma(a^-, a^+) = \inf_{q \in \Gamma_\sigma(a^-,a^+)} I_\sigma(q) = \inf_{q \in \Gamma_\sigma(a^-,a^+)} I(q). \tag{3}$$

Thus $\Gamma_\sigma(a^-, a^+) \subset \Gamma(a^-, a^+)$ and $c_\sigma(a^-, a^+) \ge c(a^-, a^+)$.

In Section 2, it will be shown that for any $\sigma \subset \mathbb{R}$, there is a global minimizer, $Q_\sigma \in \Gamma_\sigma(a^-, a^+)$, of I_σ. In addition, under the same nondegeneracy condition on $\mathcal{M}(a^-, a^+)$ and $\mathcal{M}(a^+, a^-)$ that leads to the infinitude of local minima of I, it will be proved that there are also local minimizers of I_σ whenever $|\sigma|$ is sufficiently large. These local minimizers are near elements of $\mathcal{M}(a^-, a^+)$ since, as will be proved, the local minimizers converge along subsequences to members of $\mathcal{M}(a^-, a^+)$ as $|\sigma| \to +\infty$. Then in Section 3, the same nondegeneracy assumption leads to analogous results in the setting of multitransition local minima solutions. In particular as $\sigma^+ - \sigma^-$ increase, there appear more and more local minima of I_σ and associated multitransition solutions of (2). Moreover as $\sigma^+, -\sigma^- \to \infty$, again any corresponding sequence of such solutions has a subsequence converging to a solution of the same type of (HS).

2. One transition local minima of I_σ

In this section the existence of minima of I_σ and their behavior for large σ will be studied.

LEMMA 2.1. *For all $\sigma = (\sigma^-, \sigma^+)$ with $\sigma^+ > \sigma^-$, there is a $Q_\sigma \in \Gamma_\sigma$ such that $I_\sigma(Q_\sigma) = c_\sigma(a^-, a^+)$. Any such minimizer is a (classical) solution of* (HS).

Proof. The existence is immediate since I_σ is weakly lower semicontinuous and Γ_σ is weakly closed. That the minimizer is C^2 and satisfies (HS) follows from standard arguments. □

To study the behavior of the solutions, Q_σ of (2) as $\sigma^+, -\sigma^- \to \infty$, some a priori bounds for these functions will be obtained. For convenience, suppose that $\sigma^+ \geq 1$ and $\sigma^- \leq 0$. For $t \in [0,1]$, set $\varphi(t) = ta^+ + (1-t)a^-$. Extend the domain of φ to $\mathbb{R}$ via $\varphi(t) = a^-$ for $t \leq 0$ and $\varphi(t) = a^+$ for $t \geq 1$. Thus $\varphi \in \Gamma_\sigma$ for all such σ and

$$c_\sigma(a^-, a^+) \leq I_\sigma(\varphi) = \int_0^1 L(\varphi)\, dt \equiv M_0 \tag{4}$$

independently of σ.

Proposition 2.2. *There is a constant $M > 0$ such that*

$$\|Q_\sigma\|_{W^{1,2}(T_i,\mathbb{R}^m)} \leq M \tag{5}$$

independently of i and σ.

Proof. From (4), we find

$$\|\dot{Q}_\sigma\|^2_{L^2(\sigma,\mathbb{R}^m)} \leq 2M_0. \tag{6}$$

With this initial estimate, the argument of the proof of Proposition 2.24 of [15] can be followed yielding (5). □

Taking advantage of Proposition 2.2, a natural approach to study the behavior of the minima, c_σ, and minimizers, Q_σ, of I_σ for large σ is to begin by taking any sequence, $\sigma_k = (\sigma_k^-, \sigma_k^+)$ with $-\sigma_k^-, \sigma_k^+ \to \infty$ as $k \to \infty$ together with corresponding sequences, c_{σ_k} and Q_{σ_k}. The minima, c_{σ_k} are easy to deal with:

Proposition 2.3. *Suppose that $\sigma_k \subset \sigma_{k+1}$ and $\sigma_k^+, -\sigma_k^- \to \infty$ as $k \to \infty$. Then $c_{\sigma_k} \geq c_{\sigma_{k+1}} \to c(a^-, a^+)$ as $k \to \infty$.*

Proof. Let $Q_{\sigma_k} \in \Gamma_{\sigma_k}$ such that $I_{\sigma_k}(Q_{\sigma_k}) = c_{\sigma_k}$. Then, since $\Gamma_{\sigma_k} \subset \Gamma_{\sigma_{k+1}}$,

$$c_{\sigma_k} = I_{\sigma_{k+1}}(Q_{\sigma_k}) \geq c_{\sigma_{k+1}}.$$

To show that $c_{\sigma_k} \to c(a^-, a^+)$ as $k \to \infty$, let $q^* \in \mathcal{M}(a^-, a^+)$ and $\varepsilon > 0$. Choosing $s \in \mathbb{N}$, define q_s^* where

$$q_s^*(t) = \begin{cases} a^-, & t \leq -s-1, \\ (-s-t)a^- + (t+s+1)q^*(t), & -s-1 \leq t \leq -s, \\ q^*(t), & -s \leq t \leq s, \\ (t-s)a^+ + (s+1-t)q^*(t), & s \leq t \leq s+1, \\ a^+, & s+1 \leq t. \end{cases}$$

Then $q_s^* \in \Gamma_\sigma$ for any σ for which $-\sigma^-, \sigma^+ \geq s+1$. For $s = s(\varepsilon)$ sufficiently large, it can be assumed that

$$\int_{s \leq |t| \leq s+1} L(q_s^*)\, dt \leq \varepsilon. \tag{7}$$

Choose k so that $-\sigma_k^-, \sigma_k^+ \geq s(\varepsilon)+1$. Then $q_{\sigma_k}^* \in \Gamma_{\sigma_k}$ and by (7)

$$c_{\sigma_k} \leq I_{\sigma_k}(q_{s_k}^*) = \int_{|t|<s_k} L(q^*)\, dt + \int_{s_k \leq |t| \leq s_k+1} L(q_{s_k}^*)\, dt$$
$$\leq I(q^*) + \varepsilon = c(a^-, a^+) + \varepsilon \tag{8}$$

and the Proposition follows from (8) and the fact that $c_{\sigma_k} \geq c(a^-, a^+)$ □

Next we would like to show that a subsequence of the functions, Q_{σ_k}, converges to a member of $\mathcal{M}(a^-, a^+)$. The bounds of (5) imply there is a function, $Q \in E$ such that along a subsequence, Q_{σ_k} converges to Q weakly in E. Unfortunately it may be the case that $Q = a^-$ or $Q = a^+$. This possibility was excluded in the proof in [15] showing that $I(q)$ has a minimizer in $\Gamma(a^-, a^+)$ by exploiting the fact that $\Gamma(a^-, a^+)$ is invariant under the family of integer phase shifts $q(t) \to q(t+j)$ for $j \in \mathbb{Z}$. This invariance property no longer holds for $\Gamma_\sigma(a^-, a^+)$. Nevertheless as the next result shows, more can be said about the convergence of the sequence, Q_{σ_k}. For $z \in \mathbb{R}$, let $[z]$ denote the integer part of z.

Proposition 2.4. *Suppose that $\sigma_k \subset \sigma_{k+1}$ and $\sigma_k^+, -\sigma_k^- \to \infty$ as $k \to \infty$. Let $Q_{\sigma_k} \in \Gamma_{\sigma_k}$ be such that $I_{\sigma_k}(Q_{\sigma_k}) = c_{\sigma_k}$. Then there is a $\tau_k \in \sigma_k$ for each $k \in \mathbb{N}$, and there is a $Q \in \mathcal{M}(a^-, a^+)$ such that along a subsequence, $Q_{\sigma_k}(\cdot + [\tau_k]) - Q \to 0$ in E as $k \to \infty$.*

Remark 2.5: Note that $Q_{\sigma_k} \in \Gamma_{\sigma_k} \subset \Gamma(a^-, a^+)$ and by Proposition 2.3, $I(Q_{\sigma_k}) = I_{\sigma_k}(Q_{\sigma_k}) = c_{\sigma_k} \to c(a^-, a^+)$. Hence the sequence (Q_{σ_k}) is a minimizing sequence for I on $\Gamma(a^-, a^+)$. Consequently the conclusion of Proposition 2.4 can be interpreted as a variant of the Palais-Smale condition for minimizing sequences in the current setting. Similar conclusions have been obtained in related settings. See e.g. Proposition 2.50 of [31] or Theorem 2.7 of [24].

Proof of Proposition 2.4. As has just been noted, the sequence (Q_{σ_k}) is a minimizing sequence for I on $\Gamma(a^-, a^+)$. By Proposition 2.2, for any $i \in \mathbb{Z}$, $\|Q_{\sigma_k}\|_{W^{1,2}((i,i+1),\mathbb{R}^m)} \leq M$.

Choose $\tau_k \in \sigma_k$ so that $|Q_{\sigma_k}(\tau_k) - a^-| = 1/2|a^+ - a^-|$. Then via (V_1),

(i) $q_k \equiv Q_{\sigma_k}(\cdot + [\tau_k]) \in \Gamma(a^-, a^+)$,

(ii) $|q_k(\tau_k - [\tau_k]) - a^-| = |Q_{\sigma_k}(\tau_k) - a^-| = 1/2|a^+ - a^-|$,

(iii) $I(q_k) = I(Q_{\sigma_k}) = c_{\sigma_k} \to c(a^-, a^+)$,

(iv) $\|q_k\|_{W^{1,2}((i,i+1),\mathbb{R}^m)} = \|Q_{\sigma_k}\|_{W^{1,2}((i+[\tau_k],i+1+[\tau_k]),\mathbb{R}^m)} \le M$ for any $i \in \mathbb{Z}$.

Therefore as in the paragraph before this Proposition, there exists a $Q \in E$ such that along a subsequence still denoted by (q_k), as $k \to \infty, q_k \to Q$ weakly in $W^{1,2}(T, \mathbb{R}^m)$ for any bounded interval $T \subset \mathbb{R}$. Item (iv) and the fact that $\int_{\mathbb{R}} |\dot{q}_k|^2\, dt \le 2I(q_k)$, show (q_k) is bounded in E. Hence $q_k \to Q$ weakly in E. We claim

$$Q \in \Gamma(a^-, a^+). \tag{9}$$

Assuming (9) for the moment, the rest of Proposition 2.4 follows. Indeed (9) implies $I(Q) \ge c(a^-, a^+)$. By the weak lower semicontinuity of I,

$$I(Q) \le \liminf_{k\to+\infty} I(q_k) = c(a^-, a^+).$$

Thus $I(Q) = c(a^-, a^+)$, and $Q \in \mathcal{M}(a^-, a^+)$.

Next to show that $q_k - Q \to 0$ in E, it suffices to verify that

$$\int_{\mathbb{R}} |\dot{q}_k|^2\, dt \to \int_{\mathbb{R}} |\dot{Q}|^2\, dt \tag{10}$$

as $k \to \infty$. Towards this end, observe that by weak lower semicontinuity again,

$$\int_{\mathbb{R}} V(t, Q)\, dt \le \liminf_{k\to+\infty} \int_{\mathbb{R}} V(t, q_k)\, dt$$

and

$$\int_{\mathbb{R}} |\dot{Q}|^2\, dt \le \liminf_{k\to+\infty} \int_{\mathbb{R}} |\dot{q}_k|^2\, dt.$$

Thus combining these estimates gives

$$\begin{aligned}\limsup_{k\to+\infty} \int_{\mathbb{R}} |\dot{q}_k|^2\, dt &= \lim_{k\to+\infty} 2I(q_k) - 2 \liminf_{k\to+\infty} \int_{\mathbb{R}} V(t, q_k)\, dt \\ &\le 2I(Q) - 2\int_{\mathbb{R}} V(t, Q)\, dt = \int_{\mathbb{R}} |\dot{Q}|^2\, dt \le \liminf_{k\to+\infty} \int_{\mathbb{R}} |\dot{q}_k|^2\, dt\end{aligned}$$

from which (10) follows.

It remains to prove (9). To do so, let $B_r(a^\pm)$ denote the open ball of radius r in $\mathbb{R}^m$ centered at $a^\pm$. Let $r_0 \in (0, \frac{1}{2}|a^+ - a^-|)$ be such that

$$\max\{V(t, \xi) \mid t \in \mathbb{R}, \xi \in \bar{B}_{r_0}(a^+) \cup \bar{B}_{r_0}(a^-)\} < V_0.$$

For $r \in (0, r_0)$ set

$$\underline{\omega}_r = \min\{V(t, \xi) \mid t \in \mathbb{R}, \xi \in \mathbb{R}^m \setminus (B_r(a^+) \cup B_r(a^-))\} \text{ and}$$

$$\overline{\omega}_r = \max\{V(t,\xi) \mid t \in \mathbb{R}, \xi \in \bar{B}_r(a^+) \cup \bar{B}_r(a^-)\}.$$

By $(V_1) - (V_3)$, $\underline{\omega}_r > 0$ and $\overline{\omega}_r \to 0$ as $r \to 0$. Moreover if $(\alpha, \beta) \subset \mathbb{R}$ is such that $q_k(t) \in \mathbb{R}^m \setminus (B_r(a^+) \cup B_r(a^-))$ for any $t \in (\alpha, \beta)$, then

$$\begin{aligned} I_{(\alpha,\beta)}(q_k) &= \tfrac{1}{2}\|\dot{q}_k\|^2 + \int_{(\alpha,\beta)} V(t, q_k)\, dt \\ &\geq \tfrac{1}{2(\beta-\alpha)}|q_k(\beta) - q_k(\alpha)|^2 + \underline{\omega}_r(\beta - \alpha) \geq \sqrt{2\underline{\omega}_r}|q_k(\beta) - q_k(\alpha)|. \end{aligned} \tag{11}$$

Set $\omega = \underline{\omega}_{|a^+ - a^-|/4}$ and define a constant, Δ, by

$$\Delta = \sqrt{2\omega}\, |a^+ - a^-|/8.$$

Since $\overline{\omega}_\varepsilon \to 0$ as $\varepsilon \to 0$, ε can be chosen in $(0, |a^+ - a^-|/8)$ so that

$$\varepsilon^2 + 2\overline{\omega}_\varepsilon < 2\Delta. \tag{12}$$

Set $C = \max_{k\in\mathbb{N}} I(q_k)$ and $T_\varepsilon = 2C/\underline{\omega}_\varepsilon$. Due to (ii), $q_k(\tau_k - [\tau_k]) \notin B_\varepsilon(a^+) \cup B_\varepsilon(a^-)$. Suppose $q_k(t) \notin B_\varepsilon(a^+) \cup B_\varepsilon(a^-)$ for any $t \in (\tau_k - [\tau_k], \tau_k - [\tau_k] + T_\varepsilon)$. Then by (11),

$$I_{(\tau_k - [\tau_k], \tau_k - [\tau_k] + T_\varepsilon)}(q_k) \geq \underline{w}_\varepsilon T_\varepsilon \geq 2C. \tag{13}$$

which is not possible by the definition of C. Hence for any $k \in \mathbb{N}$,

$$\text{there is an } \ell_k^+ \in (\tau_k - [\tau_k], \tau_k - [\tau_k] + T_\epsilon) \text{ with } q_k(\ell_k^+) \in B_\epsilon(a^+) \cup B_\epsilon(a^-). \tag{14}$$

Similarly for any $k \in \mathbb{N}$

$$\text{there is an } \ell_k^- \in (\tau_k - [\tau_k] - T_\epsilon, \tau_k - [\tau_k]) \text{ with } q_k(\ell_k^-) \in B_\epsilon(a^+) \cup B_\epsilon(a^-). \tag{15}$$

The next step in proving (9) is to verify that for any $k \in \mathbb{N}$,

$$q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^-) \text{ for } t \geq \ell_k^+ \text{ and} \tag{16}$$

$$q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^+) \text{ for } t \leq \ell_k^-. \tag{17}$$

Their proofs being the same, only (16) will be proved. Recall that by definition, $q_k(t) = a^+$ for any $t \geq \sigma_k^+ - [\tau_k]$ and by (ii), $|q_k(\tau_k - [\tau_k]) - a^-| = \frac{1}{2}|a^+ - a^-|$, for any $k \in \mathbb{N}$. Arguing indirectly, assume that for some $k \in \mathbb{N}$, there exists an $\ell_0 \in [\ell_k^+, \sigma_k^+ - [\tau_k])$ for which $q_k(\ell_0) \in B_\epsilon(a^-)$. Since $|q_k(\tau_k - [\tau_k]) - a^-| = \frac{1}{2}|a^+ - a^-|$ and $q_k(\ell_0) \in B_\epsilon(a^-)$, there is an interval $(\alpha, \beta) \subset (\tau_k - [\tau_k], \ell_0)$ such that $q_k(t) \notin B_{|a^+-a^-|/4}(a^-) \cup B_{|a^+-a^-|/4}(a^+)$ for any $t \in (\alpha, \beta)$ and $|q(\beta) - q(\alpha)| \geq |a^+ - a^-|/4$. Then, by the definition of ω, $V(t, q_k(t)) \geq \omega$ for any $t \in (\alpha, \beta)$ so by (11),

$$I_{(\alpha,\beta)}(q_k) \geq \sqrt{2\omega}\, |q_k(\beta) - q_k(\alpha)| \geq \sqrt{2\omega}\, |a^+ - a^-|/4 = 2\Delta. \tag{18}$$

Thus if $q_k(\ell_0) \in B_\epsilon(a^-)$, (18) provides a positive lower bound for $I_{(\alpha,\beta)}(q_k)$. Next it will be shown that the same is true for $I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k)$. Indeed, consider the function

$$\bar{q}_k(t) = \begin{cases} a^- & t \le \ell_0 - 1, \\ (\ell_0 - t)a^- + (t+1-\ell_0)q_k(\ell_0) & \ell_0 - 1 \le t \le \ell_0, \\ q_k(t) & \ell_0 \le t. \end{cases}$$

Then $\bar{q}_k \in \Gamma_{\sigma_k+[\tau_k]}(a^-,a^+)$ and $\bar{Q}_k(t) \equiv \bar{q}_k(t-[\tau_k]) \in \Gamma_{\sigma_k}(a^-,a^+)$. Hence $I(\bar{q}_k) = I_{\sigma_k}(\bar{Q}_k) \ge c_{\sigma_k}$. Since $I(\bar{q}_k) \ge I_{(\ell_0-1,\ell_0)}(\bar{q}_k) + I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k)$, it follows that

$$I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k) \ge c_{\sigma_k} - I_{(\ell_0-1,\ell_0)}(\bar{q}_k). \tag{19}$$

Using the definition of $\bar{q}_k$ on (ℓ_0-1,ℓ_0) and that $q_k(\ell_0) \in B_\epsilon(a^-)$ gives

$$I_{(\ell_0-1,\ell_0)}(\bar{q}_k) \le \int_{\ell_0-1}^{\ell_0} \tfrac{1}{2}|q_k(\ell_0)-a^-|^2 + \max_{(t,\xi)\in[\ell_0-1,\ell_0]\times B_\epsilon(a^-)} V(t,\xi)\,dt \le \tfrac{1}{2}\epsilon^2 + \overline{\omega}_\epsilon.$$

This estimate together with (12) and (19) implies

$$I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k) \ge c_{\sigma_k} - \Delta. \tag{20}$$

Combining (iii), (18) and (20) then yields

$$c_{\sigma_k} = I(q_k) \ge I_{(\alpha,\beta)}(q_k) + I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k) \ge 2\Delta + c_{\sigma_k} - \Delta,$$

a contradiction. Thus (16) is proved.

Since $-T_\epsilon - 1 < \ell_k^- < \tau_k - [\tau_k] < \ell_k^+ < T_\epsilon + 1$, by (16) and (17),

$$q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^-) \text{ for } t \ge T_\epsilon + 1 \text{ and } q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^+) \text{ for } t \le -T_\epsilon - 1.$$

The convergence of $q_k(t)$ to $Q(t)$ for any $t \in \mathbb{R}$ then shows

$$Q(t) \in \mathbb{R}^m \setminus B_\epsilon(a^-) \text{ for } t \ge T_\epsilon + 1 \quad \text{and } Q(t) \in \mathbb{R}^m \setminus B_\epsilon(a^+) \text{ for } t \le -T_\epsilon - 1. \tag{21}$$

But $I(Q) < +\infty$ so by Proposition 2.3 of [15], there are points, $\varphi^\pm \in \{a^-,a^+\}$ such that $Q(\pm\infty) = \varphi^\pm$. Consequently (21) shows $\varphi^\pm = a^\pm$. Then (9) follows and the proposition is proved. □

In general the sequence (τ_k) given by Proposition 2.4 may not be bounded. Thus the sequence Q_{σ_k} may not converges in E to a $Q \in \mathcal{M}(a^-,a^+)$. In other words we cannot guarantee that the problem (2) has solutions which approximate fixed elements of $\mathcal{M}(a^-,a^+)$ as $|\sigma| \to +\infty$. We do not know if it is essential, but to get around this difficulty, we require a further condition. As was shown in [15], in order to obtain multitransition solutions of (HS), some nondegeneracy conditions are required for $\mathcal{M}(a^-,a^+)\cup\mathcal{M}(a^+,a^-)$ and they also suffice to overcome the present difficulty. To introduce them, some results from [15] will be recalled. Set

$$\mathcal{S}(a^-,a^+)=\{q|_{T_0}\mid q\in\mathcal{M}(a^-,a^+)\}.$$

The subset $\mathcal{S}(a^-,a^+)$ of $W^{1,2}(T_0,\mathbb{R}^m)$ possesses the following properties:

- $\bar{\mathcal{S}}(a^-,a^+)=\mathcal{S}(a^-,a^+)\cup\{a^-\}\cup\{a^+\}$,
- $\bar{\mathcal{S}}(a^-,a^+)$ is compact in $W^{1,2}(T_0,\mathbb{R}^m)$.

For the details, see [15].

Let $\mathcal{C}_{a^-}(a^-,a^+)$ be the component of $\bar{\mathcal{S}}(a^-,a^+)$ containing a^- and let $\mathcal{C}_{a^+}(a^-,a^+)$ be the component of $\bar{\mathcal{S}}(a^-,a^+)$ containing a^+. Then from e.g. [15], we have

Proposition 2.6. *Either*

(i) $\mathcal{C}_{a^-}(a^-,a^+)=\mathcal{C}_{a^+}(a^-,a^+)$, *or*

(ii) $\mathcal{C}_{a^-}(a^-,a^+)=\{a^-\}$ *and* $\mathcal{C}_{a^+}(a^-,a^+)=\{a^+\}$.

If (ii) holds, there exist nonempty disjoint compact sets,

$$K_{a^-}(a^-,a^+),K_{a^+}(a^-,a^+)\subset\bar{\mathcal{S}}(a^-,a^+)$$

such that

a) $a^-\in K_{a^-}(a^-,a^+),\ \ a^+\in K_{a^+}(a^-,a^+)$,

b) $\bar{\mathcal{S}}(a^-,a^+)=K_{a^-}(a^-,a^+)\cup K_{a^+}(a^-,a^+)$,

c) $\mathrm{dist}(K_{a^-}(a^-,a^+),K_{a^+}(a^-,a^+))\equiv 5r(a^-,a^+)>0$.

Remark 2.7: The splitting, $K_{a^-}(a^-,a^+),K_{a^+}(a^-,a^+)$, of $\bar{\mathcal{S}}(a^-,a^+)$ is not unique. Indeed subjecting each of the functions, q that make up these sets to the same integer phase shift produces a new such splitting. For what follows, we fix the choice of this splitting.

Remark 2.8: Reversing the roles of a^- and a^+ yields $\mathcal{C}_{a^+}(a^+,a^-)$, $\mathcal{C}_{a^-}(a^+,a^-)$, $K_{a^+}(a^+,a^-)$, $K_{a^-}(a^+,a^-)$, namely the analogous sets for heteroclinics from a^+ to a^- of what we have obtained for heteroclinics from a^- to a^+.

The nondegeneracy conditions that we impose are those of alternative (ii) of Proposition 2.6. They will be used to construct a subset of $\Gamma_\sigma(a^-,a^+)$ in which a new family of local minima of I_σ will be found that have the convergence properties that we were unable to verify for the functions, Q_σ.

To carry out the new construction, select a $\delta \in (0, r(a^-,a^+))$ and let $q^* \in \mathcal{M}(a^-,a^+)$. Then with $K_{a^-}(a^-,a^+), K_{a^+}(a^-,a^+)$ as in Proposition 2.6, there is an $s_0 \in \mathbb{N}$ depending on δ and q^* such that for all $i \in \mathbb{Z}$ with $|i| \geq s_0$,

$$\|q^* - K_{a^-}(a^-,a^+)\|_{L^2(T_{-i})}, \|q^* - K_{a^+}(a^-,a^+)\|_{L^2(T_i)} \leq \delta. \tag{22}$$

Fix such an i and choose σ so that $[-i, i+1] \subset \sigma$. Define

$$\Gamma_{\sigma,i}(a^-,a^+) = \{q \in \Gamma_\sigma(a^-,a^+) \mid q \text{ satisfies } (22)\}.$$

Then $\Gamma_{\sigma,i}(a^-,a^+) \neq \emptyset$. Set

$$c_{\sigma,i}(a^-,a^+) = \inf_{q \in \Gamma_{\sigma,i}(a^-,a^+)} I_\sigma(q) \tag{23}$$

The existence of a minimizer, $Q_{\sigma,i}$, in (23) follows as in Lemma 2.1 and standard regularity arguments imply it is a solution of (2) except possibly in the constraint intervals, $T_{-i} \cup T_i$. Letting σ_k be as earlier, we will show that for large k, there is strict inequality for $Q_{\sigma,i}$ in (22). Towards that end, an analogue of results from [24] or [15] is needed. Set

$$\Lambda(a^-,a^+) = \{q \in \Gamma(a^-,a^+) \mid \ \|q - K_{a^-}(a^-,a^+)\|_{L^2(T_{-i})} = \delta \\ \text{or } \|q - K_{a^+}(a^-,a^+)\|_{L^2(T_i)} = \delta\}.$$

Note that $\Lambda(a^-,a^+)$ also depends on δ. Define

$$d(a^-,a^+) = \inf_{q \in \Lambda(a^-,a^+)} I(q). \tag{24}$$

Then the arguments of Proposition 2.47 of [24] show

$$d(a^-,a^+) > c(a^-,a^+). \tag{25}$$

Now we have:

THEOREM 2.9. *Suppose that* $\sigma_k \subset \sigma_{k+1}$ *and* $\sigma_k^+, -\sigma_k^- \to \infty$ *as* $k \to \infty$. *Then*

$$c_{\sigma_k,i}(a^-,a^+) \geq c_{\sigma_{k+1},i}(a^-,a^+) \to c(a^-,a^+) \text{ as } k \to \infty. \tag{26}$$

Moreover for any k *for which* $c_{\sigma_k,i}(a^-,a^+) < d(a^-,a^+)$ *and in particular for large* k, *any minimizer of* I_{σ_k} *in* $\Gamma_{\sigma_k,i}(a^-,a^+)$ *is a solution of* (2). *In addition, there is a* $Q \in \mathcal{M}(a^-,a^+)$ *such that along a subsequence,* $Q_{\sigma_k,i} \to Q$ *in* $W^{1,2}_{loc}(\mathbb{R},\mathbb{R}^m)$.

Proof. The argument of Proposition 2.3 shows that (26) holds. To show that the constraints are satisfied with strict inequality provided that k is sufficiently large, arguing indirectly, suppose that

$$\|Q_{\sigma_k,i} - K_{a^-}(a^-,a^+)\|_{L^2(T_{-i})} = \delta \text{ or } \|Q_{\sigma_k,i} - K_{a^+}(a^-,a^+)\|_{L^2(T_i)} = \delta.$$

Then $Q_{\sigma_k,i} \in \Lambda(a^-,a^+)$ and by (23) and (25),

$$c_{\sigma_k,i}(a^-,a^+) = I_\sigma(Q_{\sigma_k,i}) = I(Q_{\sigma_k,i}) \geq d(a^-,a^+). \tag{27}$$

But (25) and (26) show (27) is not possible.

Thus for any k for which $c_{\sigma_k,i}(a^-,a^+) < d(a^-,a^+)$ and in particular for large k, any minimizer of I_{σ_k} in $\Gamma_{\sigma_k,i}(a^-,a^+)$ is a solution of (2).

It remains to establish the convergence of the solutions, $Q_{\sigma_k,i}$, along a subsequence. By the argument of Proposition 2.24 of [15] again, $\|Q_{\sigma_k,i}\|_{W^{1,2}(T_j,\mathbb{R}^m)}$ is bounded independently of k and j. As in Corollary 2.42 of [15], this leads to a k-independent bound for $\|Q_{\sigma_k,i}\|_{L^\infty(\sigma_k,\mathbb{R}^m)}$ and then via (2), a k-independent bound for $\|Q_{\sigma_k,i}\|_{C^2(\sigma_k,\mathbb{R}^m)}$. Thus by the Arzela-Ascoli Theorem and (2), as $k \to \infty$, for any subsequence of $Q_{\sigma_k,i}$, there is a solution, Q of (HS) such that along a further subsequence, $Q_{\sigma_k,i}$ converges to Q in $C^2_{loc}(\mathbb{R},\mathbb{R}^m)$. Moreover restricting ourselves to this subsequence, for any $p \in \mathbb{N}$, by (26),

$$\begin{aligned}\int_{-p}^{p} L(Q)\,dt &\leq \liminf_{k\to\infty} \int_{-p}^{p} L(Q_{\sigma_k,i})\,dt \leq \liminf_{k\to\infty} I_{\sigma_k}(Q_{\sigma_k,i}) \\ &= \liminf_{k\to\infty} c_{\sigma_k,i}(a^-,a^+) = c(a^-,a^+).\end{aligned}$$

Letting $p \to \infty$ further shows

$$I(Q) \leq c(a^-,a^+) < \infty. \tag{28}$$

Thus by (28) and Proposition 2.3 of [15], there are points, $\varphi^\pm \in \{a^-,a^+\}$ such that $Q(\pm\infty) = \varphi^\pm$.

We claim that $\varphi^\pm = a^\pm$. It then follows that $Q \in \Gamma(a^-,a^+)$ with $I(Q) = c(a^-,a^+)$ so $Q \in \mathcal{M}(a^-,a^+)$. Towards proving that $\varphi^\pm = a^\pm$, let

$$\mathcal{P} = \{Q_{\sigma,i} \mid c_{\sigma,i}(a^-,a^+) < d(a^-,a^+)\}.$$

Then we have

Proposition 2.10. *There is a $\beta = \beta(\delta) > 0$ such that*

$$\beta = \inf_{q\in\mathcal{P}} \int_{-i}^{i+1} L(q)\,dt. \tag{29}$$

Proof. As was noted earlier, the set of functions, $\mathcal{P}$ is a bounded subset of $C^2([-i, i+1], \mathbb{R}^m)$. Choose a minimizing sequence (q_l) for (29). Therefore as $l \to \infty$,

$$\int_{-i}^{i+1} L(q_l)\, dt \to \beta.$$

By the Arzela-Ascoli Theorem, there is a function , $\hat{q} \in C^1([-i, i+1], \mathbb{R}^m)$, such that along a subsequence, as $l \to \infty$,

$$\int_{-i}^{i+1} L(q_l)\, dt \to \int_{-i}^{i+1} L(\hat{q})\, dt = \beta.$$

Moreover $\hat{q}$ is a solution of (HS) on $[-i, i+1]$ and satisfies the constraints in (22). If $\beta = 0$, $\hat{q} \equiv a^-$ or $\hat{q} \equiv a^+$. But by the choice of δ, the function $\hat{q}$ cannot satisfy both constraints. Therefore $\beta > 0$. □

Conclusion of the proof of Theorem 2.9. Returning to our claim that $\varphi^{\pm} = a^{\pm}$, and arguing indirectly, suppose that the pair of equalities is not satisfied. Then there are three possibilities: (i) $\varphi^- = a^+$ and $\varphi^+ = a^+$, (ii) $\varphi^- = a^+$ and $\varphi^+ = a^-$, or (iii) $\varphi^- = a^-$ and $\varphi^+ = a^-$. Now a comparison argument will be employed. Suppose e.g. that (i) occurs. For $l \in \mathbb{Z}$, set $X_l = \cup_{j=l-1}^{l+1} T_j$. Pick an $\varepsilon > 0$. Then there is a $p = p(\varepsilon) \in \mathbb{N}$ with $p \le \min(-\sigma_k^-, \sigma_k^+)$ such that for all large k in our subsequence, $\|Q_{\sigma_k,i} - a^+\|_{C^2(X_{-p},\mathbb{R}^m)} \le \varepsilon$. Define $v_k \in \Gamma(a^-, a^+)$ via modifying $Q_{\sigma_k,i}$ in X_{-p}:

$$v_k(t) = \begin{cases} Q_{\sigma_k,i}(t), & t \le -p-1, \\ (-p-t)Q_{\sigma_k,i}(-p-1) + (t+p+1)a^+, & -p-1 \le t \le -p, \\ a^+, & -p \le t. \end{cases}$$

Then

$$\int_{X_p} L(v_k) dt \le \kappa(\varepsilon) \tag{30}$$

where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now using (30) and Proposition 2.10 yields

$$\begin{aligned} c_{\sigma_k,i} = I(Q_{\sigma_k,i}) = \int_{-\infty}^{-p+1} L(v_k)\, dt - \int_{-p-1}^{-p+1} L(v_k)\, dt + \int_{-p-1}^{\infty} L(Q_{\sigma_k,i})\, dt \\ \ge c(a^-, a^+) - \kappa(\varepsilon) + \beta(\delta). \quad (31) \end{aligned}$$

Choose ε so that $\kappa(\varepsilon) < \frac{1}{2}\beta(\delta)$. Since the first assertion of this theorem shows $c_{\sigma_k,i} \to c(a^-, a^+)$ as $k \to \infty$, (31) and case (i) are not possible. A similar argument excludes case (ii) and likewise case (iii) is excluded by doing the cutting and pasting near $t = \infty$. Thus Theorem 2.9 is proved. □

Remark 2.11: There is an analogous result on interchanging the roles of a^- and a^+.

Remark 2.12: There is another approach we could have taken to the material in this section. Replacing $\mathcal{S}(a^-, a^+)$ by

$$\mathcal{T}(a^-, a^+) = \{q(0) \mid q \in \mathcal{M}(a^-, a^+)\},$$

then $\bar{\mathcal{T}}(a^-, a^+) = \mathcal{T}(a^-, a^+) \cup \{a^-, a^+\}$ and $\bar{\mathcal{T}}$ is compact in $\mathbb{R}^m$. Moreover the analogue of Proposition 2.6 with $\mathcal{T}$ replacing $\mathcal{S}$ holds for this new setting. Then in (22) and the definition of Λ, replace the L^2 norm by the $\mathbb{R}^m$ norm leading to a variant of Theorem 2.9 (although one can no longer invoke [15] for part of the proof). A similar approach using pointwise constraints can likewise be made the in the next section where local minima of I that are multitransition solutions of (HS) are obtained. However unfortunately this replacement can no longer be made when dealing with mountain pass solutions of (HS) and its finite time relative, (2). The reason it fails is that for mountain pass solutions, again an appropriate version of Proposition 2.6 is needed. To obtain it, one has to work with the map from the set of solutions of (HS) in say $\Gamma(a^-, a^+)$ with $I(q) \leq d$ to

$$\mathcal{T}^d(a^-, a^+) = \{q(0) \mid q \in \Gamma(a^-, a^+),\ \text{satisfies (HS), and } I(q) \leq d\}$$

where d is greater than the mountain pass minimax value (see [25, 26]). Unfortunately, unlike the case where $d = c(a^-, a^+)$, this map is not one to one causing the earlier proof of Proposition 2.6 to break down. This failure does not occur when working with

$$\mathcal{S}^d(a^-, a^+) = \{q|_{[0,1]} \mid q \in \Gamma(a^-, a^+),\ \text{satisfies (HS), and } I(q) \leq d\}.$$

The analogues of the results of this paper for mountain pass solutions of (2) will be explored in a future publication.

3. Multitransition local minima

In this section, the existence and multiplicity of solutions of (2) that undergo multiple transitions will be studied. As in Section 2, the results here will follow with the aid of comparison arguments involving multitransition local minima for (HS). Such results were obtained in [15] where it was shown that there is an infinitude of $k-$transition solutions of (HS) for each $k \in \mathbb{N}$. In [15], the same ideas were employed to treat $k = 2$ as for general $k > 2$. This is also the case in the current setting. In [15], the main concern was the solution of a PDE problem in a cylindrical domain for which (HS) occurs as a degenerate special case. We begin here by stating a slightly stronger version of the result of [15] specialized to (HS) when $k = 2$, show how to use it to obtain a related result for (2) and then discuss the case of $k > 2$.

To formulate the result for (HS), choose $\mathbf{m} = (m_1, \cdots, m_4) \in \mathbb{Z}^4$ and $l \in \mathbb{N}$ so that

$$m_1 + 2l < m_2 - 2l < m_2 + 2l < m_3 - 2l < m_3 + 2l < m_4 - 2l. \tag{32}$$

The integers m_i and l will depend on a parameter, ε, that will be introduced in the next theorem. For $r > 0$ and $A \subset W^{1,2}(T_0, \mathbb{R}^m)$, let

$$N_r(A) \equiv \{q \in W^{1,2}(T_0, \mathbb{R}^m) \mid \mathrm{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(q, A) \le r\}.$$

Set $\delta = \min(\rho, r(a^-, a^+), r(a^+, a^-))$ where $\rho \in (0, \frac{1}{4}|a^- - a^+|)$. As the class of admissible functions in which local minima of I will be sought, take

$$\mathcal{A}_2 = \mathcal{A}_2(\mathbf{m}, l) = \{q \in E \mid q \ \text{ satisfies (33)}\}$$

where

$$q(\cdot + j)|_{T_0} \in \begin{cases} N_\delta(K_{a^-}(a^-, a^+)), & j < m_1 + l, \\ N_\delta(K_{a^+}(a^-, a^+)), & m_2 - l \le j < m_2 + l, \\ N_\delta(K_{a^+}(a^+, a^-)), & m_3 - l \le j < m_3 + l, \\ N_\delta(K_{a^-}(a^+, a^-)), & m_4 - l \le j. \end{cases} \tag{33}$$

Define

$$b_2 = b_2(\varepsilon) = \inf_{q \in \mathcal{A}_2} I(q). \tag{34}$$

This setting was studied in Section 5 of [15]. As a somewhat more quantitative version of the result there, we have:

Theorem 3.1. *Let* $(V_1) - (V_3)$ *and the four conditions* $\mathcal{C}_{a^\pm}(a^-, a^+) = \{a^\pm\}$, $\mathcal{C}_{a^\pm}(a^+, a^-) = \{a^\pm\}$ *be satisfied. For any* $\varepsilon \in (0, \delta/16)$, *there exists an* $m_0 = m_0(\varepsilon) \in \mathbb{N}$, *an* $l = l(\varepsilon) \in \mathbb{N}$, $l \ge m_0$ *and a* $\zeta_0 = \zeta_0(\varepsilon) > 0$ *with* $\zeta_0 \to 0$ *as* $\varepsilon \to 0$ *such that*

1^o *for each* $\mathbf{m} = (m_1, m_2, m_3, m_4)$ *satisfying*

$$m_{j+1} - m_j - 6l \ge m_0 \text{ for } j = 1, 2, 3, \tag{35}$$

the set

$$\mathcal{M}(b_2) \equiv \{q \in \mathcal{A}_2 \mid I(q) = b_2\} \ne \emptyset.$$

2^o *Any* $Q \in \mathcal{M}(b_2)$ *satisfies the constraints,* (33), *with strict inequality and is a classical solution of* (HS) *that is homoclinic to* a^-.

3^o *If* $Q \in \mathcal{M}(b_2)$,

$$\|Q - a^-\|_{L^\infty((-\infty,m_1-l],\mathbb{R}^m)} < \zeta_0\,, \|Q - a^+\|_{L^\infty([m_2+l,m_3-l],\mathbb{R}^m)} < \zeta_0$$
$$\textit{and}\ \|Q - a^-\|_{L^\infty([m_4-l,+\infty),\mathbb{R}^m)} < \zeta_0.$$

4^o *As* $\varepsilon \to 0$, $b_2(\varepsilon) \to c(a^-, a^+) + c(a^+, a^-)$.

Remark 3.2: The statement of Theorem 3.1 is rather technical due to the presence of so many parameters, but the geometrical content of the result, both for $k = 2$ and more generally, is simple. Heuristically the result says that if the constraint regions are far enough apart, there exists an interior minimizer of the associated variational problem and this minimizer is a classical solution of (2). Moreover in the region between each pair of constraint regions involving the same point, the point being a^+ for Theorem 3.1, the solution remains close to that point.

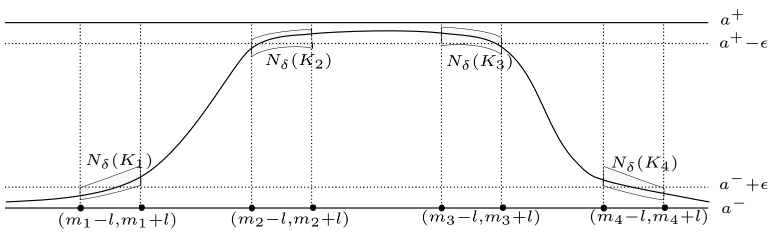

Figure 1: *In the diagram* $K_1 = K_{a^-}(a^-, a^+), K_2 = K_{a^+}(a^-, a^+), K_3 = K_{a^+}(a^+, a^-), K_4 = K_{a^-}(a^+, a^-)$

Proof of Theorem 3.1. Fix $\mathbf{m}$ satisfying (35). Both 1^o and 2^o are either part of the statement or the proof of Theorem 5.16 of [15]. To verify 3^o, note first that for m_0 is sufficiently large, one can choose members of $\mathcal{M}(a^-, a^+)$ and $\mathcal{M}(a^+, a^-)$, modify them slightly to obtain a member of $\mathcal{A}_2$, and use it to find an upper bound for b_2 as in (5.19) of [15]:

$$b_2 < c(a^-, a^+) + c(a^+, a^-) + 2 \tag{36}$$

independently of $\mathbf{m}$. Next we will show that in any interval, $\mathcal{I}$, of length at least m_0, there is a subinterval, $X_i = \cup_{k=i-2}^{i+2} T_k \subset \mathcal{I}$ such that either

$$\|Q - a^-\|_{L^\infty(T_j,\mathbb{R}^m)} < \varepsilon \text{ or } \|Q - a^+\|_{L^\infty(T_j,\mathbb{R}^m)} < \varepsilon \tag{37}$$

for $T_j \in X_i$. Indeed if both inequalities in (37) fail for some T_j, by (2.12) $I_{T_j}(Q) \geq \min_{(t,\xi)\in T_j\times(\mathbb{R}^m\setminus B_\varepsilon(a^-)\cup B_\varepsilon(a^+))} V(t, \xi) \equiv \gamma(\varepsilon) > 0$. Thus if there were no such $X_i \subset \mathcal{I}$,

$$b_2 = I(Q) \geq \frac{1}{5} m_0(\varepsilon)\gamma(\varepsilon), \tag{38}$$

which is contrary to the upper bound for b_2 for m_0 sufficiently large. It is here that the dependence of m_0 (and hence l) on ε first enters.

Since $l \geq m_0$, applying this observation to the constraint regions associated with m_1, m_2, m_3 and m_4 yields intervals, $X_1 \subset [m_1 - l, m_1]$, $X_2 \subset [m_2, m_2 + l]$, $X_3 \subset [m_3 - l, m_3]$ and $X_4 \subset [m_4, m_4 + l]$ in which (37) holds. In fact due to the definition of δ in the constraint, (37) can be strengthened to

$$\|Q(t) - a^-\|_{L^\infty(X_i,\mathbb{R}^m)} < \varepsilon,\ i = 1, 4,\ \|Q(t) - a^+\|_{L^\infty(X_i,\mathbb{R}^m)} < \varepsilon,\ i = 2, 3. \quad (39)$$

Now we verify 3^o, i.e. there is a $\zeta_0 = \zeta_0(\varepsilon)$ such that $\zeta_0 \to 0$ as $\varepsilon \to 0$ for which

(i) $\|Q - a^-\|_{L^\infty((-\infty,m_1-l],\mathbb{R}^m)} < \zeta_0$,

(ii) $\|Q - a^+\|_{L^\infty([m_2+l,m_3-l],\mathbb{R}^m)} < \zeta_0$, and

(iii) $\|Q - a^-\|_{L^\infty([m_4-l,+\infty),\mathbb{R}^m)} < \zeta_0$.

We will only prove (ii). The proofs of properties (i) and (iii) are similar and simpler and will be omitted. Set

$$\bar{\alpha} = \sup X_2,\ \bar{\beta} = \inf X_3,$$

and the function

$$\bar{Q}(t) = \begin{cases} Q(t) & t \leq \bar{\alpha} - 1, \\ Q(\bar{\alpha} - 1) + (t + 1 - \bar{\alpha})(a^+ - Q(\bar{\alpha} - 1)) & \bar{\alpha} - 1 \leq t \leq \bar{\alpha}, \\ a^+ & \bar{\alpha} \leq t \leq \bar{\beta} \\ a^+ + (t - \bar{\beta})(Q(\bar{\beta} + 1) - a^+) & \bar{\beta} \leq t \leq \bar{\beta} + 1, \\ Q(t) & t \geq \bar{\beta} + 1. \end{cases}$$

We claim the function $\bar{Q}$ satisfies the constraint (33). Indeed on the intervals T_j for $j \leq \bar{\alpha} - 2$ or $j \geq \bar{\beta} + 1$, $\bar{Q}(t) = Q(t)$ so the constraint is satisfied. The same is true when $\bar{\alpha} \leq j \leq \bar{\beta} - 1$ since on the corresponding intervals T_j, $\bar{Q}(t) = a^+$. If $j = \bar{\alpha} - 1$, then $\bar{Q}(t) = Q(\bar{\alpha} - 1) + (t + 1 - \bar{\alpha})(a^+ - Q(\bar{\alpha} - 1))$ for $t \in T_j$. Thus by (39)

$$\begin{aligned} \mathrm{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(\bar{Q}(\cdot + j), K_{a^+}(a^-, a^+))^2 &\leq \|\bar{Q}(t) - a^+\|^2_{W^{1,2}(T_j,\mathbb{R}^m)} \\ &= \int_{T_j} |a^+ - Q(\bar{\alpha} - 1)|^2 + |t - \bar{\alpha}|^2 |Q(\bar{\alpha} - 1) - a^+|^2 \leq 2\varepsilon^2 < \delta^2. \end{aligned}$$

Likewise another application of (39) shows $\|\bar{Q}(t) - a^+\|^2_{W^{1,2}(T_j,\mathbb{R}^m)} \leq 2\varepsilon^2 < \delta^2$ for $j = \beta$. Hence all the inequalities in (33) hold for the function $\bar{Q}$ so it belongs to the class $\mathcal{A}_2$. Consequently, since Q minimizes I on $\mathcal{A}_2$,

$$0 \leq I(\bar{Q}) - I(Q) \leq \int_{\bar{\alpha}-1}^{\bar{\alpha}} L(\bar{Q})\,dt + \int_{\bar{\beta}}^{\bar{\beta}+1} L(\bar{Q})\,dt - I_{(\bar{\alpha},\bar{\beta})}(Q) \quad (40)$$

and so

$$I_{(\bar{\alpha},\bar{\beta})}(Q) \leq \int_{\bar{\alpha}-1}^{\bar{\alpha}} L(\bar{Q})\,dt + \int_{\bar{\beta}}^{\bar{\beta}+1} L(\bar{Q})\,dt. \tag{41}$$

To conclude (ii) from (41), the definition of $\bar{Q}$ on the intervals $[\bar{\alpha}-1, \bar{\alpha}]$ and $[\bar{\beta}, \bar{\beta}+1]$ as well as (39) will be used. Recalling the function, $\overline{\omega}_\varepsilon$, in the proof of Proposition 2.4:

$$\overline{\omega}_\varepsilon = \max\{V(t,\xi) \mid t \in \mathbb{R}, \xi \in \bar{B}_\varepsilon(a^+) \cup \bar{B}_\varepsilon(a^-)\},$$

by (39) we have

$$\int_{\bar{\alpha}-1}^{\bar{\alpha}} L(\bar{Q})\,dt \leq \int_{\bar{\alpha}-1}^{\bar{\alpha}} \tfrac{1}{2}|a^+ - Q(\bar{\alpha}-1)|^2 + \max_{(t,\xi)\in[\bar{\alpha}-1,\bar{\alpha}]\times B_\varepsilon(a^+)} V(t,\xi)\,dt$$
$$\leq \tfrac{1}{2}\varepsilon^2 + \overline{\omega}_\varepsilon.$$

Similarly

$$\int_{\bar{\beta}}^{\bar{\beta}+1} L(\bar{Q})\,dt \leq \tfrac{1}{2}\varepsilon^2 + \overline{\omega}_\varepsilon$$

so by (41)

$$I_{(\bar{\alpha},\bar{\beta})}(Q) \leq \varepsilon^2 + 2\overline{\omega}_\varepsilon. \tag{42}$$

To conclude the proof of property (ii), recall the function, $\underline{\omega}_\zeta$, defined for $\zeta > 0$:

$$\underline{\omega}_\zeta \equiv \min\{V(t,\xi) \mid t \in \mathbb{R}, \mathrm{dist}(\xi, \{a^-, a^+\}) \geq \zeta\}.$$

The function $\underline{\omega}_\zeta$ is increasing and continuous on $[0, \frac{1}{4}|a^+ - a^-|)$. Define ζ_0 by

$$\zeta_0 = \zeta_0(\varepsilon) = \min\{\zeta \in [2\varepsilon, \tfrac{1}{4}|a^+ - a^-|) \mid \sqrt{2\underline{\omega}_{\zeta/2}}\,\zeta \geq 2(\varepsilon^2 + 2\overline{\omega}_\varepsilon)\}.$$

Since $\overline{\omega}_\varepsilon \to 0$ as $\varepsilon \to 0$, ζ_0 is well defined for small ε. Note that $\zeta_0(\epsilon) \to 0$ as $\epsilon \to 0$. Finally to verify Property (ii), i.e.

$$\|Q - a^+\|_{L^\infty([m_2+l, m_3-l],\mathbb{R}^m)} < \zeta_0, \tag{43}$$

we argue indirectly. Assume that there exists a $\mu \in [m_2+l, m_3-l]$ for which $|Q(\mu) - a^+| \geq \zeta_0$. See Figure 3. Since by (39), $|Q(\bar{\alpha}) - a^+| < \varepsilon$, there exists an interval $(\alpha,\beta) \subset (\bar{\alpha},\mu)$ such that $Q(t) \notin B_{\zeta_0/2}(a^-) \cup B_{\zeta_0/2}(a^+)$ for any $t \in (\alpha,\beta)$ and $|Q(\beta) - Q(\alpha)| \geq \zeta_0$. Then $V(t, Q(t)) \geq \underline{\omega}_{\zeta_0/2}$ for any $t \in (\alpha,\beta)$ and by (11),

$$I_{(\alpha,\beta)}(Q) \geq \sqrt{2\underline{\omega}_{\zeta_0/2}}\,|Q(\beta) - Q(\alpha)| \geq \sqrt{2\underline{\omega}_{\zeta_0/2}}\,\zeta_0. \tag{44}$$

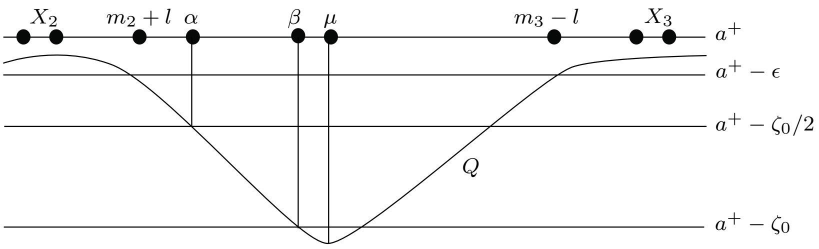

Figure 2: *The indirect argument*

Hence, by (42), (44), $\sqrt{2\underline{\omega}_{\zeta_0/2}}\,\zeta_0 \le I_{(\alpha,\beta)}(Q) \le I_{(\bar\alpha,\bar\beta)}(Q) \le \varepsilon^2 + 2\overline{\omega}_\varepsilon$ which is not possible by the definition of ζ_0. This completes the proof of (ii) and of 3^o.

It remains to prove 4^o. For m_0 possibly still larger, there is a $q^- \in \mathcal{M}(c(a^-,a^+))$ such that q^- satisfies the m_1 and m_2 constraints in (33) and there is a $q^+ \in \mathcal{M}(c(a^+,a^-))$ such that q^+ satisfies the m_3 and m_4 constraints in (33). Moreover it can be assumed that

$$\|q^- - a^+\|_{L^\infty([m_2+l,\infty),\mathbb{R}^m)} < \varepsilon \text{ and } \|q^+ - a^+\|_{L^\infty((-\infty,m_3-l],\mathbb{R}^m)} < \varepsilon.$$

Therefore appropriately modifying q^- for $t > m_2 + l$ and q^+ for $t < m_3 - l$ yields a function $q_2 \in \mathcal{A}_2$ satisfying the improved version of (36):

$$I(q_2) \le c(a^-,a^+) + c(a^+,a^-) + \kappa_1(\varepsilon)$$

where $\kappa_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence

$$b_2(\varepsilon) \le c(a^-,a^+) + c(a^+,a^-) + \kappa_1(\varepsilon). \tag{45}$$

To obtain a lower bound for $b_2(\varepsilon)$, let $Q \in \mathcal{M}(b_2(\varepsilon))$. Define a function, $\bar Q$ as in the proof of 3^o where now $\bar\alpha$ and $\bar\beta$ are replaced respectively by α_1 and α_1+1 where these points are integers interior to (m_2+l, m_3-l). By its definition, $I(\bar Q) \ge c(a^-,a^+) + c(a^+,a^-)$. Thus in the spirit of (30) and (40), there is a function, $\kappa_2(\varepsilon) \to 0$ as $\varepsilon \to 0$ with

$$c(a^-,a^+) + c(a^+,a^-) - b_2(\varepsilon) \le I(\bar Q) - b_2(\varepsilon) = I(\bar Q) - I(Q) \le \kappa_2(\varepsilon)$$

or

$$c(a^-,a^+) + c(a^+,a^-) - \kappa_2(\varepsilon) \le b_2(\varepsilon). \tag{46}$$

Combining (45) and (46) then gives 4^o and completes the proof of Theorem 3.1. □

Now the results just mentioned for (HS) can be used as a tool to obtain 2-transition solutions of (2). Continuing with $\mathbf{m}$ as in Theorem 3.1, let $\sigma^- < m_1$ and $\sigma^+ > m_4$, and define

$$\mathcal{A}_{2,\sigma} = \mathcal{A}_{2,\sigma}(\mathbf{m}, l) = \{q \in \mathcal{A}_2 \mid q(t) = a^- \text{ for } t \notin \sigma\}.$$

Then for $\sigma^+, -\sigma^-$ sufficiently large, $\mathcal{A}_{2,\sigma} \neq \emptyset$. Define

$$b_{2,\sigma} = b_{2,\sigma}(\mathbf{m}, l) = \inf_{q \in \mathcal{A}_{2,\sigma}} I(q). \tag{47}$$

Then parallelling Theorem 3.1, we have:

THEOREM 3.3. *Suppose the hypotheses of Theorem 3.1 are satisfied. Then*

1^o *For $\sigma^+, -\sigma^-$ sufficiently large,*

$$\mathcal{M}(b_{2,\sigma}) \equiv \{q \in \mathcal{A}_{2,\sigma} \mid I(q) = b_{2,\sigma}\} \neq \emptyset.$$

2^o *Any $Q_\sigma \in \mathcal{M}(b_{2,\sigma})$ is a solution of* (2).

Proof. The existence of $Q_\sigma \in \mathcal{M}(b_{2,\sigma})$ follows as in the proof of Lemma 2.1. As in earlier arguments, it is a solution of (2) in any of the intervals where there is no constraint and also in any constraint interval in which the constraint is satisfied with strict inequality. Thus to complete the proof of item 2^o, it must be shown that strict inequality holds in the 4 constraint regions. The argument involving (36)-(39) also holds in the current setting so for $\varepsilon < \delta$, again there are intervals, $X_j = \cup_{k=i_j-2}^{i_j+2} T_k$ for $j = 2, 3$ and $i_j \subset [m_j - l, m_j + l]$, in which

$$\|Q_\sigma - a^+\|_{L^\infty(T_i, \mathbb{R}^m)} < \varepsilon \text{ for } T_i \in X_j. \tag{48}$$

Now suppose that Q_σ satisfies one of the m_1 or m_2 constraints with equality. Cutting and pasteing in X_3 yields a pair of functions,

$$f(t) = \begin{cases} Q_\sigma(t), & t \leq i_3 - 1, \\ (i_3 - t)Q_\sigma(i_3 - 1) + (t - i_3 + 1)a^+, & i_3 - 1 \leq t \leq i_3, \\ a^+, & t \geq i_3, \end{cases}$$

$$g(t) = \begin{cases} a^+, & t \leq i_3 + 1, \\ (i_3 + 2 - t)a^+ + (t - i_3 - 1))Q_\sigma(i_3 + 2), & i_3 + 1 \leq t \leq i_3 + 2, \\ Q_\sigma(t), & t \geq i_3 + 2. \end{cases}$$

Note that $f \in \Gamma(a^-, a^+)$, $g \in \Gamma(a^+, a^-)$ and since Q_σ satisfies one of the m_1 or m_2 constraints with equality, $f \in \Lambda(a^-, a^+)$. Hence $I(g) = I_{(i_3+1,+\infty)}(g) \geq c(a^+, a^-)$ and by (25), $I(f) = I_{(-\infty,i_3)}(f) \geq d(a^-, a^+)$. Moreover by (48), arguing as in the proof of Theorem 3.1 for the function $\bar{Q}$ shows

$$I_{(i_3-1,\leq i_3)}(f) \leq e_1(\varepsilon) \text{ and } I_{(i_3+1,i_3+2)}(g) \leq e_1(\varepsilon)$$

with $e_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By the above observations

$I_{(-\infty,i_3-1)}(Q_\sigma) = I_{(-\infty,i_3-1)}(f) \geq I(f) - e_1(\varepsilon) \geq d(a^-,a^+) - e_1(\varepsilon)$ and
$I_{(i_3+2,+\infty)}(Q_\sigma) = I_{(i_3+2,+\infty)}(g) \geq I(g) - e_1(\varepsilon) \geq c(a^+,a^-) - e_1(\varepsilon)$

so that

$$\begin{aligned} I(Q_\sigma) &\geq I_{(-\infty,i_3-1)}(Q_\sigma) + I_{(i_3+2,+\infty)}(Q_\sigma) \\ &\qquad \geq d(a^-,a^+) + c(a^+,a^-) - 2e_1(\varepsilon). \end{aligned} \quad (49)$$

On the other hand, by 4^o of Theorem 3.1, as $\varepsilon \to 0$, $b_2 \to c(a^-,a^+)+c(a^+,a^-)$. Thus for small ε and $-\sigma^-,\sigma^+$ sufficiently large, there is a function, $e_2(\varepsilon)$ with $e_2(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$I(Q_\sigma) \leq c(a^-,a^+) + c(a^+,a^-) + e_2(\varepsilon). \quad (50)$$

But (49) and (50) are incompatible for small ε since $d(a^-,a^+) > c(a^-,a^+)$. A similar argument establishes the result if Q_σ satisfies one of the m_3 or m_4 constraints with equality and item 2^o is proved. □

The existence of the 2−transition solutions having been established, now their behavior as $-\sigma^-,\sigma^+ \to \infty$ will be studied. We will show

THEOREM 3.4. *Let the hypotheses of Theorem 3.3 be satisfied for a fixed admissible ε. If $-\sigma_i^-,\sigma_i^+ \to \infty$ as $i \to \infty$, then $b_{2,\sigma_i} \to b_2$. Moreover if $Q_{\sigma_i} \in \mathcal{M}(b_{2,\sigma_i})$, then there is a $Q \in \mathcal{M}(b_2)$, such that along a subsequence, $Q_{\sigma_i} \to Q$ in $C^2_{loc}(\mathbb{R},\mathbb{R}^m)$ as $i \to \infty$.*

Proof. Since $\mathcal{M}(b_{2,\sigma_i}) \subset \mathcal{M}(b_2)$, $b_2 \leq b_{2,\sigma_i}$. Let $\bar\varepsilon > 0$ be small. Choose any $Q \in \mathcal{M}(b_2)$. Then Q can be modified near $t = \pm\infty$ to produce $Q_{\bar\varepsilon} \in \mathcal{M}(b_{2,\sigma_i})$ for all large $|\sigma_i^\pm|$ and $b_{2,\sigma_i} \leq I(Q_{\bar\varepsilon}) \leq b_2 + \bar\varepsilon$. Thus the first assertion of the theorem follows. To prove the second assertion, since $Q_{\sigma_i} \in \mathcal{A}_2$, by earlier arguments, there is an $M > 0$ such that $\|Q_{\sigma_i}\|_{C^2(\sigma_i,\mathbb{R}^m)} \leq M$ for all $i \in \mathbb{N}$. Thus by the Arzela-Ascoli Theorem, a subsequence of Q_{σ_i} converges in $C^2_{loc}(\mathbb{R},\mathbb{R}^m)$ to a function $Q \in C^2_{loc}(\mathbb{R},\mathbb{R}^m) \cap \mathcal{A}_2$ so $I(Q) \geq b_2$, But along our subsequence, for any $p \in \mathbb{N}$,

$$\int_{-p}^{p} L(Q)\,dt \leq \liminf_{i\to\infty} \int_{-p}^{p} L(Q_{\sigma_i})\,dt \leq \liminf_{i\to\infty} I(Q_{\sigma_i}) = \liminf_{i\to\infty} b_{2,\sigma_i} = b_2.$$

Thus $I(Q) = b_2$ and $Q \in \mathcal{M}(b_2)$. □

Next the case of $k > 2$ transitions will be discussed briefly. See [24] for a detailed argument in a related case. Again one takes $l \in \mathbb{N}$ and now chooses

$\mathbf{m} = (m_1, \cdots, m_{2k}) \in \mathbb{Z}^{2k}$ with $m_j - m_{j-1} > 4l$ for $j = 2, \cdots, 2k$. (We note at this point a typo in the first line of page 1763 of [15] where $> 2l$ is written rather than $> 4l$). To describe the analogue of the condition, (33), choose $\{a_1, \dots, a_{2k}\} \in \{a^-, a^+\}^{2k}$ so that

$$a_1 \neq a_2 = a_3 \neq \dots \neq a_{2k-2} = a_{2k-1} \neq a_{2k}$$

and define the family of sets $\{K_1, \dots, K_{2k}\}$ by

$$K_{2j-1} = K_{a_{2j-1}}(a_{2j-1}, a_{2j}) \text{ and } K_{2j} = K_{a_{2j}}(a_{2j-1}, a_{2j}), \quad j = 1, \dots, k.$$

Then the class of admissible functions for the $k-$transition problem is

$$\mathcal{A}_k = \mathcal{A}_k(\mathbf{m}, l) = \{q \in E \mid q \text{ satisfies (51)}\}$$

where

$$q(\cdot + p)|_{T_0} \in \begin{cases} N_\delta(K_1), & p \in (-\infty, m_1 + l) \cap \mathbb{Z}, \\ N_\delta(K_j), & p \in [m_j - l, m_j + l) \cap \mathbb{Z},\ 2 \leq j \leq 2k-1, \\ N_\delta(K_{2k}), & p \in [m_{2k} - l, +\infty) \cap \mathbb{Z}. \end{cases} \tag{51}$$

Now set

$$b_k = b(k, \mathbf{m}, l) = \inf_{q \in \mathcal{A}(k, \mathbf{m}, l)} I(q). \tag{52}$$

Then we have

Theorem 3.5. *Under the hypotheses of Theorem 3.1,*

$$\mathcal{M}(b_k) \equiv \{Q \in \mathcal{A}(k, \mathbf{m}, l) \mid I(Q) = b(k, \mathbf{m}, l)\} \neq \emptyset$$

and any $Q \in \mathcal{M}(b_k)$ is a classical solution of (PDE) satisfying (BC).

Remark 3.6: There are also analogues of $3^o - 4^o$ of Theorem 3.1.

To state the result corresponding to Theorem 3.5 for (2), let $\sigma^- < m_1$ and $\sigma^+ > m_{2k}$, and set

$$\mathcal{A}_{k,\sigma} = \mathcal{A}_{k,\sigma}(\mathbf{m}, l) = \{q \in \mathcal{A}_k \mid q(t) = a^- \text{ for } t \notin \sigma\}.$$

As earlier for $\sigma^+, -\sigma^-$ sufficiently large, $\mathcal{A}_{k,\sigma} \neq \emptyset$. Define

$$b_{k,\sigma} = b_{k,\sigma}(\mathbf{m}, l) = \inf_{q \in \mathcal{A}_{k,\sigma}} I(q). \tag{53}$$

Then we have:

Theorem 3.7. *Suppose the hypotheses of Theorem 3.5 are satisfied. Then*

1^o *For* $\sigma^+, -\sigma^-$ *sufficiently large,*

$$\mathcal{M}(b_{k,\sigma}) \equiv \{q \in \mathcal{A}_{k,\sigma} \mid I(q) = b_{k,\sigma}\} \neq \emptyset.$$

2^o *Any* $Q_\sigma \in \mathcal{M}(b_{k,\sigma})$ *is a solution of* (2).

The proof is quite similar to that of Theorem 3.3, relying on (25) and a cutting and pasteing argument.

REMARK 3.8: In conclusion, we note that the natural version of Theorem 3.4 holds in the $k > 2$ setting.

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