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On graded classical 2-absorbing submodules of graded modules over graded commutative rings

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Abstract. *Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we will introduce the concept of graded classical 2-absorbing submodules of graded modules over a graded commutative ring as a generalization of graded classical prime submodules and investigate some basic properties of these classes of graded modules.*

Keywords: graded 2-absorbing submodule, graded classical prime submodule, graded classical 2-absorbing submodule.
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1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary. Badawi in [8] introduced the concept of 2-absorbing ideals of commutative rings. We recall from [8] that a proper ideal I of R is called *a 2-absorbing ideal of* R if whenever $r, s, t \in R$ and $rst \in I$ implies $rs \in I$ or $rt \in I$ or $st \in I$. Later on, Anderson and Badawi in [7] generalized the concept of 2-absorbing ideals of commutative rings to the concept of n-absorbing ideals of commutative rings for every positive integer $n \geq 2$. We recall from [7] that a proper ideal I of R is called an n-absorbing ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. In light of [8] and [7], many authors studied the concept of 2-absorbing submodules and n-absorbing submodules. Recently, H. Mostafanasab, U. Tekir and K.H. Oral in [12] studied classical 2-absorbing submodules of modules over commutative rings. Let M be an R-module. A proper submodule N of M is called classical 2-absorbing submodule, if whenever a, b, $c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ $bcm \in N$.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, in particular, we are dealing with graded classical 2-absorbing submodules of graded modules over graded commutative rings. The notion of graded 2-absorbing ideals as a generalization of graded prime ideals was introduced and studied in [3, 13]. The notion of graded 2-absorbing ideals was extended to graded 2-absorbing submodules in [2, 11]. The notion of graded classical prime submodules as a generalization of graded prime submodules was introduced in [9] and studied in [1, 4, 5]. The purpose of this paper is to introduced the concept of graded classical 2-absorbing submodules as a generalization of graded classical prime submodules and give a number of its properties (see sec. 2).

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [10, 14, 15, 16] for these basic properties and more information on graded rings and modules.

Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a *G-graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded.

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus\limits_{g\in G} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup\limits_{g\in G} M_g$ and the elements of $h(M)$ are called to be *homogeneous*. Let $M = \bigoplus\limits_{g\in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \bigoplus\limits_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$.

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \underset{g\in G}{\oplus}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \;\; and \; g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called module of fractions, if $S^{-1}M = \underset{g\in G}{\oplus}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S \;\; and \; g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \underset{g\in G}{\cup}(S^{-1}R)_g$ and $h(S^{-1}M) = \underset{g\in G}{\cup}(S^{-1}M)_g$. Consider the graded homomorphism $\eta : M \to$

$S^{-1}M$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Similar to non graded case, one can prove that $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$ and that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N :_R M) = \phi$. If K is a graded submodule of $S^{-1}R$-module $S^{-1}M$, then $K \cap M$ will denote the graded submodule $\eta^{-1}(K)$ of M. Moreover, similar to the non graded case one can prove that $S^{-1}(K \cap M) = K$.

Let R be a G-graded ring and M a graded R-module.

A proper graded ideal P of R is said to be *a graded prime ideal* if whenever $rs \in P$, we have $r \in P$ or $s \in P$, where $r, s \in h(R)$ (see [18].) It is shown in [6, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R : rN \subseteq M\}$ is a graded ideal of R.

A proper graded submodule P of M is said to be *a graded prime submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M)$ or $m \in P$ (see [6, 17].)

A proper graded ideal I of R is said to be *a graded 2-absorbing ideal of* R if whenever $r, s, t \in h(R)$ with $rst \in I$, then $rs \in I$ or $rt \in I$ or $st \in I$ (see [3, 13].)

A proper graded submodule N of M is called *a graded 2-absorbing submodule of* M if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$ (see [2].)

A proper graded submodule N of M is called *a graded classical prime* submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ (see [4, 9].)

2. Results

Definition 2.1. *Let R be a G-graded ring, M a graded R-module, C a graded submodule of M and let $g \in G$.*

(i) *We say that C_g is a classical g-2-absorbing submodule of R_e-module M_g if $C_g \neq M_g$; and whenever $r, s, t \in R_e$ and $m \in M_g$ with $rstm \in C_g$, then either $rsm \in C_g$ or $rtm \in C_g$ or $stm \in C_g$.*

(ii) *We say that C is a graded classical 2-absorbing submodule of M if $C \neq M$; and whenever $r, s, t \in h(R)$ and $m \in h(M)$ with $rsm \in C$, then either $rsm \in C$ or $rtm \in C$ or $stm \in C$.*

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module and C a graded submodule of M. If C is a graded classical 2-absorbing submodule of M, then C_g is a classical g-2-absorbing R_e-submodule of M_g for every $g \in G$.*

Proof. Suppose that C is a graded classical 2-absorbing submodule of M. For $g \in G$ assume that $rstm \in C_g \subseteq C$ where $r, s, t \in R_e$ and $m \in M_g$. Since C

is a graded classical 2-absorbing submodule of M, we have either $rsm \in C$ or $rtm \in C$ or $stm \in C$. Since $M_g \subseteq M$ and $C_g = C \cap M_g$, we conclude that either $rsm \in C_g$ or $rtm \in C_g$ or $stm \in C_g$. So C_g is classical g-2-absorbing R_e-submodule of M_g. □

THEOREM 2.3. *Let R be a G-graded ring, M a graded R-module and C a proper graded submodule of M. Then the following statements hold:*

(i) *If C is a graded 2-absorbing submodule of M, then C is a graded classical 2-absorbing submodule of M.*

(ii) *C is a graded classical prime submodule of M if and only if C is a graded 2-absorbing submodule of M and $(C :_R M)$ is a graded prime ideal of R.*

Proof. (i) Assume that C is a graded 2-absorbing submodule of M. Let r, s, $t \in h(R)$ and $m \in h(M)$ such that $rstm \in C, rtm \notin C$ and $stm \notin C$. Since C is a graded 2-absorbing submodule of M, we conclude that $rs \in (C :_R M)$ and hence $rsm \in C$. Thus C is a graded classical 2-absorbing submodule of M.

(ii) Assume that C is a graded classical prime submodule of M. It is clear that C is a graded 2-absorbing submodule of M. Also by [4, Lemma 3.1.], $(C :_R M)$ is a graded prime ideal of R. Conversely, assume that C is a graded 2-absorbing submodule of M and $(C :_R M)$ is a graded prime ideal of R. Let r, $s \in h(R)$ and $m \in h(M)$ such that $rsm \in C$, $rm \notin C$ and $sm \notin C$. Since C is a graded 2-absorbing submodule of M, $rs \in (C :_R M)$. It follows that either $r \in (C :_R M)$ or $s \in (C :_R M)$ and hence $rm \in C$ or $sm \in C$, which is a contradiction. Thus C is a graded classical prime submodule of M. □

The following example shows that the converse of theorem 2.3(i) is not true.

EXAMPLE 2.4. Let $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, .)$. Define

$R_g = \left\{ \begin{array}{cc} \mathbb{Z} & \text{if } g = 0 \\ 0 & \text{otherwise} \end{array} \right\}$. Then R is a G-graded ring. Let $M = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Q}$. Then M is a G-graded R-module with

$$M_g = \left\{ \begin{array}{cc} \{0\} \times \mathbb{Z}_3 \times \mathbb{Q} & \text{if } g = 0 \\ \mathbb{Z}_2 \times \{0\} \times \mathbb{Q} & \text{if } g = 1 \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \{0\} & \text{if } g = 2 \\ \{0\} \times \{0\} \times \{0\} & \text{otherwise} \end{array} \right\}.$$

Now consider a graded submodule $C = \{(0, 0, 0)\}$. One can easily see that C is a graded classical 2-absorbing submodule of M. Since $2.3.(1, 1, 0) = (0, 0, 0)$, but $3.(1, 1, 0) \notin C$, $2.(1, 1, 0) \notin C$ and $2.3.(1, 1, 1) \notin C$, we get C is not a graded 2-absorbing submodule. Also, part (ii) of theorem 2.3(ii) shows that C is note a graded classical prime submodule. Hence the two concepts of graded classical prime submodules and of graded classical 2-absorbing submodules are different in general.

Recall that a graded zero-divisor on a graded R-module M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$ (see [2].)

The following result studies the behavior of graded 2-absorbing submodules under localization.

THEOREM 2.5. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ a multiplication closed subset of R. Then the following hold:*

(i) *If C is a graded classical 2-absorbing submodule of M such that $(C :_R M) \cap S = \phi$, then $S^{-1}C$ is a graded classical 2-absorbing submodule of $S^{-1}M$.*

(ii) *If $S^{-1}C$ is a graded classical 2-absorbing submodule of $S^{-1}M$ and $S\cap$ $G\text{-}Zdv_R(M/C) = \phi$, then C is a graded classical 2-absorbing submodule of M.*

Proof. (i) Let C be a graded classical 2-absorbing submodule of M and $(C :_R M)\cap S = \phi$. Suppose that $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{r_3}{s_3}\frac{m}{s_4} \in S^{-1}C$ for some $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in h(S^{-1}R)$ and for some $\frac{m}{s_4} \in h(\ S^{-1}M)$. Hence there exists $k \in S$ such that $r_1r_2r_3(km) \in C$. Since C is a graded classical 2-absorbing submodule of M, we conclude that either $r_1r_2(km) \in C$ or $r_1r_3(km) \in C$ or $r_2r_3(km) \in C$. Thus $\frac{r_1r_2(km)}{s_1s_2s_4k} = \frac{r_1}{s_1}\frac{r_2}{s_2}\frac{m}{s_4} \in S^{-1}C$ or $\frac{r_1r_3(km)}{s_1s_3s_4k} = \frac{r_1}{s_1}\frac{r_3}{s_3}\frac{m}{s_4} \in S^{-1}C$ or $\frac{r_2r_3(km)}{s_2s_3s_4k} = \frac{r_2}{s_2}\frac{r3}{s_3}\frac{m}{s_4} \in S^{-1}C$. Therefore $S^{-1}C$ is a graded classical 2-absorbing submodule of $S^{-1}M$.

(ii) Assume that $S^{-1}C$ is a graded classical 2-absorbing submodule of $S^{-1}M$ and $S\cap$ $G\text{-}Zdv_R(M/C) = \phi$. Let $r_1r_2r_3m \in C$ for some $r_1, r_2, r_3 \in h(R)$ and for some $m \in h(M)$. Then $\frac{r_1r_2r_3m}{1} = \frac{r_1}{1}\frac{r_2}{1}\frac{r_3}{1}\frac{m}{1} \in S^{-1}C$. Since $S^{-1}C$ is a graded classical 2-absorbing submodule of $S^{-1}M$, we conclude that either $\frac{r_1}{1}\frac{r_2}{1}\frac{m}{1} = \frac{r_1r_2m}{1} \in S^{-1}C$ or $\frac{r_1}{1}\frac{r_3}{1}\frac{m}{1} = \frac{r_1r_3m}{1} \in S^{-1}C$ or $\frac{r_2}{1}\frac{r_3}{1}\frac{m}{1} = \frac{r_2r_3m}{1} \in S^{-1}C$. If $\frac{r_1r_2m}{1} \in S^{-1}C$, then there exists $s \in S$ such that $sr_1r_2m \in C$ and since $S\cap$ $G\text{-}Zdv_R(M/C) = \phi$, we have $r_1r_2m \in C$. With a same argument, we can show that if $\frac{r_1r_3m}{1} \in S^{-1}C$, then $r_1r_3m \in C$ and also we can show if $\frac{r_2r_3m}{1} \in S^{-1}C$, then $r_2r_3m \in C$. Therefore C is a graded classical 2-absorbing submodule of M. □

LEMMA 2.6. *Let R be a G-graded ring, M a graded R-module and C a graded classical 2-absorbing submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. Then for every $r, s \in h(R)$, $m \in h(M)$ and $g \in G$ with $rsI_gm \subseteq C$, either $rsm \in C$ or $rI_gm \subseteq C$ or $sI_gm \subseteq C$.*

Proof. Let $r, s \in h(R)$, $m \in h(M)$ and $g \in G$ such that $rsI_gm \subseteq C$, $rsm \notin C$, $rI_gm \nsubseteq C$ and $sI_gm \nsubseteq C$. Then there exist i_{1g}, $i_{2g} \in I_g$ such that $ri_{1g}m \notin C$ and $si_{2g}m \notin C$. Since C is a graded classical 2-absorbing submodule, $rsi_{1g}m \in$

C, $rsm \notin C$ and $ri_{1g}m \notin C$, we have $si_{1g}m \in C$. Also $rsi_{2g}m \in C$ implies that $ri_{2g}m \in C$, since C is a graded classical 2-absorbing submodule. Since $rs(i_{1g} + i_{2g})m \in C$, we conclude that $r(i_{1g} + i_{2g})m \in C$ or $s(i_{1g} + i_{2g})m \in C$ or $rsm \in C$ and hence either $rsm \in C$ or $ri_{1g}m \in C$ or $si_{2g}m \in C$, which is a contradiction. □

THEOREM 2.7. *Let R be a G-graded ring, M a graded R-module and C a graded classical 2-absorbing submodule of M. Let $I = \bigoplus_{g\in G} I_g$ and $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then for every $r \in h(R)$, $m \in h(M)$ and $g, h \in G$ with $rI_gJ_hm \subseteq C$, either $rI_gm \subseteq C$ or $rJ_hm \subseteq C$ or $I_gJ_hm \subseteq C$.*

Proof. Let $r \in h(R)$, $m \in h(M)$ and $g, h \in G$ such that $rI_gJ_hm \subseteq C$, $rI_gm \nsubseteq C$ and $rJ_hm \nsubseteq C$. We have to show that $I_gJ_hm \subseteq C$. Assume that $i_g \in I_g$ and $j_h \in J_h$. By assumption there exist $i'_g \in I_g$ and $j'_h \in J_h$ such that $ri'_gm \notin C$ and $rj'_hm \notin C$. Since $ri'_gJ_hm \subseteq C$, $ri'_gm \notin C$ and $rJ_hm \nsubseteq C$, by Lemma 2.6, we have $i'_gJ_hm \subseteq C$. Also since $rj'_hI_g\ m \subseteq C$, $rj'_hm \notin C$ and $rI_gm \nsubseteq C$, by Lemma 2.6, we have $j'_hI_g\ m \subseteq C$. By $(i_g + i'_g) \in I_g$ and $(j_h + j'_h) \in J_h$ it follows that $r(i_g + i'_g)(j_h + j'_h)m \in C$. Since C is a graded classical 2-absorbing submodule, either $r(i_g+i'_g)m \in C$ or $r(j_h+j'_h)m \in C$ or $(i_g+i'_g)(j_h+j'_h)m \in C$. If $r(i_g+i'_g)m = ri_gm+ri'_gm \in C$, then $ri_gm \notin C$ which implies that $i_gj_hm \in C$ by Lemma 2.6. Similarly, by $r(j_h + j'_h)m \in C$, we conclude that $i_gj_hm \in C$. If $(i_g + i'_g)(j_h + j'_h)m \in C$, then $i_gj_hm + i_gj'_hm + i'_gj_hm + i'_gj'_hm \in C$ and so $i_gj_hm \in C$. Thus $I_gJ_hm \subseteq C$. □

THEOREM 2.8. *Let R be a G-graded ring, M a graded R-module and C a proper graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{g\in G} J_g$ and $K = \bigoplus_{g\in G} K_g$ be a graded ideals of R. Then the following statement are equivalent:*

(i) *C is a graded classical 2-absorbing submodule of M;*

(ii) *For every $g, h, \lambda \in G$ and $m \in h(M)$ with $I_gJ_hK_\lambda m \subseteq C$, either $I_gJ_hm \subseteq C$ or $I_gK_\lambda m \subseteq C$ or $J_hK_\lambda m \subseteq C$*

Proof. $(i) \Rightarrow (ii)$ Assume that C is a graded classical 2-absorbing submodule of M. Let $g, h, \lambda \in G$ and $m \in h(M)$ such that $I_gJ_hK_\lambda m \subseteq C$ and $I_gJ_hm \nsubseteq C$. Then by Theorem 2.7, for all $r_\lambda \in K_\lambda$ either $I_gr_\lambda m \subseteq C$ or $J_hr_\lambda m \subseteq C$. If $I_gr_\lambda m \subseteq C$, for all $r_\lambda \in K_\lambda$ we are done. Similarly if $J_hr_\lambda m \subseteq C$, for all $r_\lambda \in K_\lambda$ we are done. Suppose that $r_\lambda, r'_\lambda \in K_\lambda$ are such that $I_gr_\lambda m \nsubseteq C$ and $J_hr'_\lambda m \nsubseteq C$. It follows that $I_gr'_\lambda m \subseteq C$ and $J_hr_\lambda m \subseteq C$. Since $I_gJ_h(r_\lambda + r'_\lambda)m \subseteq C$, by Theorem 2.7, we have either $I_g(r_\lambda+r'_\lambda)m \subseteq C$ or $J_h(r_\lambda+r'_\lambda)m \subseteq C$. By $I_g(r_\lambda + r'_\lambda)m \subseteq C$ it follows that $I_gr_\lambda m \subseteq C$ which is a contradiction. Similarly by $J_h(r_\lambda + r'_\lambda)m \subseteq C$ we get a contradiction. Therefore $I_gK_\lambda m \subseteq C$ or $J_hK_\lambda m \subseteq C$.

$(ii) \Rightarrow (i)$Assume that (ii) holds. Let $r_g, s_h, t_\lambda \in h(R)$ and $m \in h(M)$ such that $r_gs_ht_\lambda m \in C$. Let $I = r_gR$, $J = s_hR$ and $K = t_\lambda R$ be a graded ideals of R generated by r_g, s_h and t_λ, respectively. Then $I_g J_h K_\lambda m \subseteq C$. By our assumption we obtain $I_g J_h m \subseteq C$ or $I_g K_\lambda m \subseteq C$ or $J_h K_\lambda m \subseteq C$. Hence $r_g s_h m \in C$ or $r_g t_\lambda m \in C$ or $s_h t_\lambda m \in C$. Therefore C is a graded classical 2-absorbing submodule of M. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $\varphi : M \to M'$ is a homomorphism of R-modules verifying $\varphi(M_g) \subseteq M'_g$ for every $g \in G$.

Theorem 2.9. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an epimorphism of graded modules.*

(i) *If C is a graded classical 2-absorbing submodule of M containing $Ker\varphi$, then $\varphi(C)$ is a graded classical 2-absorbing submodule submodule of M'.*

(ii) *If C' is a graded classical 2-absorbing submodule of M', then $\varphi^{-1}(C')$ is a graded classical 2-absorbing submodule of M.*

Proof. (i) Suppose that C is a graded classical 2-absorbing submodule of M and let $r, s, t \in h(R)$ and $m' \in h(M')$ such that $rstm' \in \varphi(C)$, $rsm' \notin \varphi(C)$ and $rtm' \notin \varphi(C)$. Since $rstm' \in \varphi(C)$, there exists $c \in C \cap h(M)$ such that $\varphi(c) = rstm'$. Since $m' \in h(M')$ and φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$. Then $\varphi(c) = rst\varphi(m)$ and so $\varphi(c-rstm) = 0$. Hence $c - rstm \in Ker\varphi \subseteq C$ and so $rstm \in C$. Since C is a graded classical 2-absorbing submodule of M, $rsm \notin C$ and $rtm \notin C$, we have $stm \in C$. Hence $stm' \in \varphi(C)$. Thus $\varphi(C)$ is a graded classical 2-absorbing submodule of M'.

(ii) Suppose that C' is a graded classical 2-absorbing submodule of M' and let $r, s, t \in h(R)$ and $m \in h(M)$ such that $rstm \in \varphi^{-1}(C')$, $rsm \notin \varphi^{-1}(C')$ and $rtm \notin \varphi^{-1}(C')$. Since φ is an epimorphism, $\varphi(rstm) = rst\varphi(m) \in C'$. Since C' is a graded classical 2-absorbing submodule of M', $rs\varphi(m) = \varphi(rsm) \notin C'$and $rt\varphi(m) = \varphi(rtm) \notin C'$, we have $st\varphi(m) = \varphi(stm) \in C'$ and hence $stm \in \varphi^{-1}(C')$. Thus $\varphi^{-1}(C')$ is a graded classical 2-absorbing submodule of M. □

As an immediate consequence of Theorem 2.9 we have the following corollary.

Corollary 2.10. *Let R be a G-graded ring, M a graded R-module and $K \subseteq C$ a graded submodules of M. Then C is a graded classical 2-absorbing submodule of M if and only if C/K is a graded classical 2-absorbing submodule of M/K.*

Lemma 2.11. *Let R be a G-graded ring, M a graded R-module and C a graded submodule of M. If C is an intersection of two graded classical prime submodules of M, then C is a graded classical 2-absorbing submodule of M.*

Proof. Suppose that $C = C_1 \cap C_2$, where C_1 and C_2 are graded classical prime submodules of M. Let r, s, $t \in h(R)$ and $m \in h(M)$ with $rstm \in C$. Since

C_1 is a graded classical prime submodules of M, we have either $rm \in C_1$ or $sm \in C_1$ or $tm \in C_1$. Since C_2 is a graded classical prime submodules of M, we have either $rm \in C_2$ or $sm \in C_2$ or $tm \in C_2$. It follows that $rsm \in C_1 \cap C_2$ or $rtm \in C_1 \cap C_2$ or $stm \in C_1 \cap C_2$. Thus C is a a graded classical 2-absorbing submodule of M. □

Let R_i be a graded commutative ring with identity and M_i be a graded R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded R-module and each graded submodule of M is of the form $C = C_1 \times C_2$ for some graded submodules C_1 of M_1 and C_2 of M_2.

THEOREM 2.12. *Let $R = R_1 \times R_2$ be a graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a graded R_1-module and M_2 is a graded R_2-module. Let C_1 and C_2 be a proper graded submodules of M_1 and M_2, respectively.*

(i) *C_1 is a graded classical 2-absorbing submodule of M_1 if and only if $C = C_1 \times C_2$ is a graded classical 2-absorbing submodule of M.*

(ii) *C_2 is a graded classical 2-absorbing submodule of M_2 if and only if $C = M_1 \times C_2$ is a graded classical 2-absorbing submodule of M.*

(iii) *$C = C_1 \times C_2$ is a graded classical 2-absorbing submodule of M if and only if C_1 and C_2 are graded classical prime submodules of M_1 and M_2, respectively.*

Proof. (i) Suppose that $C = C_1 \times M_2$ is a graded classical 2-absorbing submodule of M. From our hypothesis, C_1 is proper, So $C_1 \neq M_1$. Set $M' = \frac{M}{\{0\} \times M_2}$. Hence $C' = \frac{C}{\{0\} \times M_2}$ is a graded classical 2-absorbing submodule of M by Corollary 2.10. Also observe that $M' \cong M_1$ and $C' \cong C_1$. Thus C_1 is a graded classical 2-absorbing submodule of M_1. Conversely, if C_1 is a graded classical 2-absorbing submodule of $M_{1,}$ then it is clear that $C = C_1 \times M_2$ is a graded classical 2-absorbing submodule of M.

(ii) It can be easily verified similar to (i).

(iii) Assume that $C = C_1 \times C_2$ is a graded classical 2-absorbing submodule of M. We show that C_1 is a graded classical prime submodules of M_1. Since $C_2 \neq M_2$, there exists $m_2 \in M_2 \backslash C_2$. Let $rsm_1 \in C_1$ for r, $s \in h(R_1)$ and $m_1 \in h(M_1)$. Then $(r, 1)(s, 1)(1, 0)(m_1, m_2) = (rsm_1, 0) \in C = C_1 \times C_2$. Since $C = C_1 \times C_2$ is a graded classical 2-absorbing submodule of M and $m_2 \notin C_2$, either $(r, 1)(1, 0)(m_1, m_2) = (rm_1, 0) \in C = C_1 \times C_2$ or $(s, 1)(1, 0)(m_1, m_2) = (sm_1, 0) \in C = C_1 \times C_2$. Hence either $rm_1 \in C_1$ or $sm_1 \in C_1$ which shows that C_1 is a graded classical prime submodule of M_1. Similarly, one can show that C_2 is a graded classical prime submodule of M_2. Conversely, assume that $C_1$and C_2 are graded classical prime submodules of M_1 and M_2, respectively. One can easily see that $(C_1 \times M_2)$ and $(M_1 \times C_2)$ are graded classical prime submodules of M. Hence $(C_1 \times M_2) \cap (M_1 \times C_2) = C_1 \times C_2 = C$ is a graded classical 2-absorbing submodule of M by Lemma 2.11. □

References

[1] R. Abu-Dawwas and K. Al-Zoubi, *On graded weakly classical prime submodules*, Iran. J. Math. Sci. Inform. **12** (2017), no. 1, 153–161.

[2] K. Al-Zoubi and R. Abu-Dawwas, *On graded 2-absorbing and weakly graded 2-absorbing submodules*, J. Math. Sci. Adv. Appl. **28** (2014), 45–60.

[3] K. Al-Zoubi, R. Abu-Dawwas and S. Ceken, *On graded 2-absorbing and graded weakly 2-absorbing ideals*, submitted.

[4] K. Al-Zoubi, M. Jaradat and R. Abu-Dawwa, *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc. **41** (2015), no. 1, 217–225.

[5] K. Al-Zoubi and M. Al-Dolat, *On graded classical primary submodules*, Adv. Pure Appl. Math. **7** (2016), no. 2, 93–96.

[6] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci. **33** (2006), no. 1, 3–7.

[7] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra **39** (2011), 1646–1672.

[8] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Aust. Math. Soc. **75** (2007), 417–429.

[9] A. Y. Darani and S. Motmaen, *Zariski topology on the spectrum of graded classical prime submodules*, Appl. Gen. Topol. **14** (2013), no. 2, 159–169.

[10] R. Hazrat, *Graded rings and graded Grothendieck groups*, Cambridge University Press, Cambridge, 2016.

[11] H. Mohammad and A. Ashour, *On graded n-absorbing submodules*, Le Matematiche **70** (2015), no. 2, 243–254.

[12] H. Mostafanasab, U. Tekir and K.H. Oral, *Classical 2-absorbing submodules of modules over commutative rings*, Eur. J. Pure Appl. Math. **8** (2015), no. 3, 417–430.

[13] S. R. Naghani and H. F. Moghimi, *On graded 2-absorbing and graded weakly 2-absorbing ideals of a commutative ring*, Cankaya Univ. J. Sci. Eng. **13** (2016), no. 2, 011–017.

[14] C. Nastasescu and V.F. Oystaeyen, *Graded and filtered rings and modules*, Lecture Notes in Math. 758, Berlin-New York, Springer, 1982.

[15] C. Nastasescu and F. Van Oystaeyen, *Graded ring theory*, Mathematical Library 28, North-Holland, Amsterdam, 1982.

[16] C. Nastasescu and V.F. Oystaeyen, *Methods of graded rings*, Lecture Notes in Math. 1836, Springer, Berlin-Heidelberg, 2004.

[17] K. H. Oral, U. Tekir and A. G. Agargun, *On graded prime and primary submodules*, Turkish J. Math. **35** (2011), 159–167.

[18] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math. **28** (2004), no. 3, 217–229.

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