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Monotonicity theorems and inequalities for certain sine sums

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Abstract. *Inspired by the work of Askey-Steinig, Szegö, and Schweitzer, we provide several monotonicity theorems and inequalities for certain sine sums. Among others, we prove that for $n \geq 1$ and $x \in (0, \pi/2)$, we have*

$$\frac{d}{dx}\frac{C_n(x)}{1-\cos(x)} < 0 \quad \textit{and} \quad \frac{d}{dx}\left(1-\cos(x)\right)C_n(x) > 0,$$

where

$$C_n(x) = \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}$$

denotes Carslaw's sine polynomial. Another result states that the inequality

$$\sum_{k=1}^{n}(n-k+a)(n-k+b)k\sin(kx) > 0 \quad (a, b \in \mathbb{R})$$

holds for all $n \geq 1$ and $x \in (0, \pi)$ if and only if $a = b = 1$.
Many corollaries and applications of these results are given. Among them, we present a two-parameter class of absolutely monotonic rational functions.

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1. Introduction and statement of the results

I. A classical result in the theory of trigonometric polynomials states that

$$F_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} > 0 \quad (n \geq 1;\, 0 < x < \pi). \tag{1}$$

Fejér conjectured the validity of (1) in 1910. The first proof was published by Jackson [21] one year later. Since then, more than 20 proofs of the Fejér-Jackson inequality were discovered. A remarkable stronger result than (1) was given by Askey and Steinig [13] in 1976. They proved the monotonicity property

$$\frac{d}{dx}\frac{F_n(x)}{\sin(x/2)} < 0 \quad (n \geq 1;\, 0 < x < \pi). \tag{2}$$

Some related theorems were published by Gasper [18] and Alzer and Koumandos [2].

The inequality

$$C_n(x) = \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1} > 0 \quad (n \geq 1;\, 0 < x < \pi) \tag{3}$$

is an elegant counterpart of (1). It is due to Carslaw [15]. We note that (3) is equivalent to the functional inequality

$$F_n(2x) < 2F_{2n}(x) \quad (n \geq 1;\, 0 < x < \pi).$$

Extensions and refinements of (3) as well as various similar results can be found in Alzer and Koumandos [1], Alzer and Kwong [4, 5, 7], Koschmieder [23], Meynieux and Tudor [25], Ruscheweyh and Salinas [29]; see also Milovanović et al. [26, p. 317].

In view of (2) it is natural to ask: do there exist monotonicity properties of functions which are defined in terms of $C_n(x)$? Our first theorem gives an affirmative answer to this question.

THEOREM 1.1. *Let $n \geq 1$ be an integer. Then, for $x \in (0, \pi/2)$,*

$$\frac{d}{dx}\frac{C_n(x)}{1-\cos(x)} < 0 \quad \textit{and} \quad \frac{d}{dx}(1-\cos(x))\,C_n(x) > 0. \tag{4}$$

For $x \in (\pi/2, \pi)$, we have

$$\frac{d}{dx}\frac{C_n(x)}{1+\cos(x)} > 0 \quad \textit{and} \quad \frac{d}{dx}(1+\cos(x))\,C_n(x) < 0.$$

REMARK 1.2. (i) It follows from the formula $C_n(\pi - x) = C_n(x)$ that each of the two different sets of inequalities in Theorem 1.1 can be derived from the other.
(ii) As an immediate consequence of the monotonicity results we obtain the estimates

$$(1-\cos(x))\,L_n < C_n(x) < \frac{L_n}{1-\cos(x)} \quad (n \geq 1;\, 0 < x < \pi/2),$$

where

$$L_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1}$$

denotes the n-th partial sum of the classical Leibniz series for $\pi/4$.

A new lower bound for $C_n(x)$, given in the next theorem, plays a crucial role in the proof of (4).

THEOREM 1.3. *Let $n \geq 1$ be an integer. For $x \in (0, \pi)$, we have*

$$|\sin(2nx)|\frac{1 - |\cos(x)|}{1 - \cos(2x)} < C_n(x). \tag{5}$$

II. The inequality

$$\sum_{k=1}^{n}(n-k+1)\sin(kx) > 0 \quad (n \geq 1;\, 0 < x < \pi) \tag{6}$$

was first published by Fejér [17] in 1928. It is due to Lukács. Fejér offered a proof of (6) by using properties of power series. An elegant short proof and an extension involving a binomial coefficient were given by Turán [32]; see also Alzer and Kwong [3]. Askey and Gasper [12] pointed out that (6) is a special case of an inequality for the sum of Jacobi polynomials. We define

$$S_n(x) = \sum_{k=1}^{n}(n-k+1)^2 k \sin(kx).$$

Here, we present a companion to (6).

THEOREM 1.4. *Let $n \geq 1$ be an integer. For $x \in (0, \pi)$, we have $S_n(x) > 0$.*

The following representation for $S_n(x)$ plays a key role in our proof of Theorem 1.4.

THEOREM 1.5. *Let $n \geq 1$ be an integer. For $x \in \mathbb{R}$, we have*

$$\begin{aligned} 16\sin^4(x/2)S_n(x) = {} & 4(n+1)\sin(x) - (n+2)\sin(nx) \\ & - 4\sin((n+1)x) + n\sin((n+2)x). \end{aligned} \tag{7}$$

REMARK 1.6. Integrating $S_n(t)$ from $t = x$ to $t = y$ yields, from Theorem 1.4, an inequality involving the cosine function,

$$\sum_{k=1}^{n}(n-k+1)^2\left(\cos(kx) - \cos(ky)\right) > 0 \quad (n \geq 1;\, 0 \leq x < y \leq \pi). \tag{8}$$

REMARK 1.7. From Theorem 1.4 we conclude that the function

$$M_n(x) = \sum_{k=1}^{n}(n-k+1)^2\frac{\sin(kx)}{k}$$

is strictly concave on $[0, \pi]$. Applying the Petrović inequality (see Mitrinović [27, section 1.4.7]) gives that M_n satisfies the subadditive property

$$M_n(x+y) < M_n(x) + M_n(y) \quad (n \geq 1;\, x, y > 0, x + y \leq \pi).$$

Robertson [28] proved the inequality: For $n \ge 2$ and $0 < x < \pi$,

$$(n+1)\frac{\sin((n-1)x)}{\sin(x)} - (n-1)\frac{\sin((n+1)x)}{\sin(x)} \le 4\left(n - \frac{\sin(nx)}{\sin(x)}\right)$$

and used it to deduce properties of certain analytic functions. Askey and Gasper [11] refined this inequality by showing that the factor 4 can be replaced by $3+\cos(x)$. The inequality

$$\frac{\sin(nx)}{n\sin(x)} \le \frac{\sqrt{6}}{9} \quad (n \ge 2;\ \pi/n \le x \le \pi - \pi/n) \tag{9}$$

is due to Askey; see Jagers [22]. It plays a role in the proof of Theorem 1.4. An application of Theorem 1.4 leads to the following related result.

Corollary 1.8. *Let* $\lambda \in \mathbb{R}$ *with* $\lambda \ge 1$. *The inequality*

$$\frac{\sin(nx)}{n\sin(x)} < \frac{\lambda + \cos(nx)}{\lambda + \cos(x)} \tag{10}$$

holds for all integers $n \ge 2$ *and* $x \in (0,\pi)$ *if and only if* $\lambda \ge 2$.

A function $f : I \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is called absolutely monotonic if f has derivatives of all orders and satisfies

$$f^{(n)}(x) \ge 0 \quad (n = 0, 1, 2, ...;\ x \in I).$$

These functions have applications in probability theory and the theory of analytic functions. We refer to Widder [33, chapter IV] and Boas [14] for more information on this subject. An additional application of Theorem 1.4 provides a two-parameter class of absolutely monotonic rational functions.

Corollary 1.9. *Let* $a, b \in \mathbb{R}$ *with* $-1 < a, b < 1$. *The function*

$$R_{a,b}(x) = \left(\frac{1+x}{1-x}\right)^2 \frac{x}{(x^2+2ax+1)(x^2+2bx+1)} \tag{11}$$

is absolutely monotonic on $[0,1)$.

We discovered Theorem 1.4 when studying a remarkable paper published by Szegö [31] in 1941. His work on univalent functions led Szegö to the inequality

$$\sum_{k=1}^{n}(n-k+1)(n-k+2)k\sin(kx) > 0 \quad (n \ge 1;\ 0 < x \le \tau), \tag{12}$$

where $\tau = 1.98...$ is defined by the equation $\sin^2(\tau/2) = 7/10$. Schweitzer [30] improved this result. He showed that (12) is valid for all $n \ge 1$, $x \in (0, 2\pi/3)$ and that $2\pi/3$ cannot be replaced by a larger constant. Applications and counterparts of (12) can be found in Askey and Fitch [10] and Alzer and Kwong [6]. The following companion to (12) is valid.

THEOREM 1.10. *Let $a, b \in \mathbb{R}$. The inequality*

$$\sum_{k=1}^{n} (n-k+a)(n-k+b)k\sin(kx) > 0 \tag{13}$$

holds for all integers $n \geq 1$ and $x \in (0, \pi)$ if and only if $a = b = 1$.

III. In the literature, we can find numerous papers on inequalities for trigonometric sums. The main reason for the great interest is that these results have applications in various fields, like, for instance, geometric function theory, numerical analysis, and number theory. Detailed information on this subject with interesting historical comments and many references are given in Askey [8], Askey and Gasper [12], Milovanović et al. [26, chapter 4]; see also Askey [9], Dimitrov and Merlo [16], Gluchoff and Hartmann [19], and Koumandos [24].

IV. Our proofs of the stated theorems and corollaries are given in Sections 2-8. The algebraic and numerical computations have been carried out by using the computer program MAPLE 13.

2. Proof of Theorem 1.3

Let $n \geq 1$, $x \in (0, \pi)$ and

$$B_n(x) = C_n(x) - |\sin(2nx)| \frac{1 - |\cos(x)|}{1 - \cos(2x)}.$$

Since $B_n(\pi - x) = B_n(x)$, it suffices to prove that B_n is positive on $(0, \pi/2]$.

Let $x \in (0, \pi/2]$. Then,

$$B_n(x) = C_n(x) - \frac{|\sin(2nx)|}{2(1 + \cos(x))}. \tag{14}$$

We obtain

$$B_1(x) = \frac{\sin(x)}{1 + \cos(x)} > 0.$$

Let $t = \cos(x)$. If $x \in (0, \pi/4]$, then

$$B_2(x) = \frac{2\sin(x)(1 + 2\cos(x))}{3(1 + \cos(x))} p(t),$$

and if $x \in (\pi/4, \pi/2]$, then

$$B_2(x) = \frac{2\sin(x)}{3(1 + \cos(x))} q(t)$$

with

$$p(t) = -2t^2 + 2t + 1 \quad \text{and} \quad q(t) = 8t^3 + 2t^2 - 2t + 1.$$

Since p is positive on $[\sqrt{2}/2, 1]$ and q is positive on $[0, \sqrt{2}/2]$, we conclude that $B_2(x) > 0$ for $x \in (0, \pi/2]$.

Next, let $n \geq 3$. We consider two cases.

Case 1. $x \in (0, \pi/(2n))$.

We have

$$B_n(x) = C_n(x) - \frac{\sin(2nx)}{2(1+\cos(x))}.$$

Using

$$C_n'(x) = \sum_{k=1}^{n} \cos((2k-1)x) = \frac{\sin(2nx)}{2\sin(x)} \tag{15}$$

gives

$$2\sin(x)B_n'(x) = \sin(2nx)\eta(x) - 2n\tan(x/2)\cos(2nx)$$

with

$$\eta(x) = 1 - \left(\frac{\sin(x)}{1+\cos(x)}\right)^2.$$

Since η is decreasing on $(0, \pi)$, we conclude from $0 < x < \pi/(2n) \leq \pi/6$ that

$$\eta(x) \geq \eta(\pi/6) > 0.92.$$

It follows that

$$2\sin(x)B_n'(x) > 0.92\sin(2nx) - 2n\tan(x/2)\cos(2nx). \tag{16}$$

Case 1.1. $x \in (0, \pi/(4n))$.

From (16) we obtain

$$2\sin(x)B_n'(x) > \cos(2nx)\sigma_n(x)$$

with

$$\sigma_n(x) = 0.92\tan(2nx) - 2n\tan(x/2).$$

Since

$$\frac{1}{n}\sigma_n'(x) = \frac{1.84}{\cos^2(2nx)} - \frac{1}{\cos^2(x/2)} > \frac{1}{\cos^2(2nx)} - \frac{1}{\cos^2(x/2)} > 0,$$

we get $\sigma_n(x) > \sigma_n(0) = 0$. Thus, $B_n'(x) > 0$.

Case 1.2. $x \in [\pi/(4n), \pi/(2n))$.

Since $\sin(2nx) > 0 \geq \cos(2nx)$, we get from (16) that $B_n'(x) > 0$.

From Case 1.1 and Case 1.2 we conclude that B_n' is positive on $(0, \pi/(2n)]$. This leads to $B_n(x) > B_n(0) = 0$.

Case 2. $x \in [\pi/(2n), \pi/2]$.
From (15) we obtain the integral representation

$$C_n(x) = \int_0^x \frac{\sin(2ns)}{2\sin(s)} ds.$$

Carslaw [15] proved that in $[\pi/(2n), \pi/2]$, C_n attains its global minimum at $x = \pi/n$. Thus,

$$C_n(x) \ge C_n(\pi/n) = Y_n + Z_n, \tag{17}$$

where

$$Y_n = \int_0^{\pi/(2n)} \frac{\sin(2ns)}{2\sin(s)} ds = \frac{1}{4n} \int_0^{\pi} \frac{\sin(t)}{\sin(t/(2n))} dt$$

and

$$Z_n = \int_{\pi/(2n)}^{\pi/n} \frac{\sin(2ns)}{2\sin(s)} ds = \frac{1}{4n} \int_{\pi}^{2\pi} \frac{\sin(t)}{\sin(t/(2n))} dt.$$

Using the estimate

$$\frac{2n}{t} \le \frac{1}{\sin(t/(2n))} \quad (0 < t < \pi)$$

gives

$$Y_n \ge \frac{1}{4n} \int_0^{\pi} \frac{2n}{t} \sin(t) dt > 0.92.$$

Since $t \mapsto \sin(t)/t$ is decreasing on $(0, \pi)$, we obtain for $t \in (\pi, 2\pi)$,

$$\frac{2n}{t} \sin\left(\frac{t}{2n}\right) \ge \frac{3}{\pi} \sin\left(\frac{\pi}{3}\right).$$

Thus,

$$\frac{1}{\sin(t/(2n))} \le \frac{4\pi n}{3\sqrt{3}t} \quad (\pi < t < 2\pi).$$

This leads to

$$Z_n \ge \frac{\pi}{3\sqrt{3}} \int_{\pi}^{2\pi} \frac{\sin(t)}{t} dt > -0.27.$$

It follows that

$$Y_n + Z_n > \frac{1}{2}. \tag{18}$$

Moreover, we have

$$\frac{|\sin(2nx)|}{2(1+\cos(x))} \le \frac{1}{2(1+\cos(x))} \le \frac{1}{2}. \tag{19}$$

From (14), (17), (18) and (19) we conclude that $B_n(x) > 0$. The proof of Theorem 1.3 is complete.

3. Proof of Theorem 1.1

Let $n \geq 1$. We define

$$G_n(x) = \frac{C_n(x)}{1-\cos(x)}, \quad H_n(x) = (1-\cos(x))C_n(x).$$

Using (15) and (5) gives, for $x \in (0, \pi/2)$,

$$\begin{aligned}\frac{(1-\cos(x))^2}{\sin(x)} \frac{d}{dx} G_n(x) &= \frac{1}{\sin(x)} \big[(1-\cos(x))C_n'(x) - \sin(x)C_n(x)\big] \\ &= \sin(2nx)\frac{1-|\cos(x)|}{1-\cos(2x)} - C_n(x) < 0\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sin(x)} \frac{d}{dx} H_n(x) &= \frac{1}{\sin(x)} \big[(1-\cos(x))C_n'(x) + \sin(x)C_n(x)\big] \\ &= C_n(x) + \sin(2nx)\frac{1-|\cos(x)|}{1-\cos(2x)} > 0.\end{aligned}$$

We define

$$G_n^*(x) = \frac{C_n(x)}{1+\cos(x)}, \quad H_n^*(x) = (1+\cos(x))C_n(x).$$

Since $G_n^*(x) = G_n(\pi - x)$ and $H_n^*(x) = H_n(\pi - x)$, we obtain, for $x \in (\pi/2, \pi)$,

$$\frac{d}{dx} G_n^*(x) = -G_n'(\pi - x) > 0 \quad \text{and} \quad \frac{d}{dx} H_n^*(x) = -H_n'(\pi - x) < 0.$$

4. Proof of Theorem 1.5

We have

$$\sum_{k=1}^{n} k\sin(kx) = \frac{\sin((n+1)x)}{4\sin^2(x/2)} - (n+1)\frac{\cos((n+1/2)x)}{2\sin(x/2)} \tag{20}$$

and

$$\sum_{k=1}^{n} k\cos(kx) = (n+1)\frac{\sin((n+1/2)x)}{2\sin(x/2)} - \frac{1-\cos((n+1)x)}{4\sin^2(x/2)}; \tag{21}$$

see Gradshteyn and Ryzhik [20, p. 38]. Next, we set

$$s(k) = \sin(kx) \quad \text{and} \quad T(k) = (2\sin(x/2))^k.$$

By differentiation, we obtain from (20) and (21),

$$\sum_{k=1}^{n} k^2 s(k) = \frac{A_n^*}{T(4)}\,, \tag{22}$$

where

$$\begin{aligned} A_n^* &= -2s(1) - (n+1)^2 s(n-1) + n(3n+4)s(n) \\ &\quad - (n+1)(3n-1)s(n+1) + n^2 s(n+2) \end{aligned}$$

and

$$\sum_{k=1}^{n} k^3 s(k) = \frac{B_n^*}{T(4)}\,, \tag{23}$$

where

$$\begin{aligned} B_n^* &= -(n+1)^3 s(n-1) + (3n^3 + 6n^2 - 4)s(n) \\ &\quad - (3n^3 + 3n^2 - 3n + 1)s(n+1) + n^3 s(n+2). \end{aligned}$$

Moreover, (20) can be written as

$$\sum_{k=1}^{n} k s(k) = \frac{C_n^*}{T(4)}\,, \tag{24}$$

where

$$C_n^* = -(n+1)s(n-1) + (3n+2)s(n) - (3n+1)s(n+1) + ns(n+2).$$

Applying (22), (23), (24) and the representation

$$S_n(x) = (n+1)^2 \sum_{k=1}^{n} k s(k) - 2(n+1) \sum_{k=1}^{n} k^2 s(k) + \sum_{k=1}^{n} k^3 s(k)$$

we conclude that (7) holds.

5. Proof of Theorem 1.4

Using (7) we obtain

$$2\frac{(1-\cos(x))^2}{\sin(x)} S_n(x) = A_n(x),$$

where

$$A_n(x) = 2(n+1) - \frac{\sin(nx)}{\sin(x)} - 2\frac{\sin((n+1)x)}{\sin(x)} + n\cos((n+1)x).$$

We show that $A_n(x) > 0$ for $n \ge 1$ and $x \in (0, \pi)$. First, we consider the cases $n = 1, 2, 3, 4, 5, 6$. We set $t = \cos(x) \in (-1, 1)$. Then,

$$A_1(x) = 2(1-t)^2 > 0 \quad \text{and} \quad A_2(x) = 8(1+t)(1-t)^2 > 0.$$

Moreover,

$$A_3(x) = 4(1-t)^2 p_3(t), \qquad A_4(x) = 8(1+t)(1-t)^2 p_4(t),$$
$$A_5(x) = 2(1-t)^2 p_5(t), \qquad A_6(x) = 16(1+t)(1-t)^2 p_6(t)$$

with

$$p_3(t) = 6t^2 + 8t + 3, \qquad p_4(t) = 8t^2 + 4t + 1,$$
$$p_5(t) = 80t^4 + 128t^3 + 48t^2 + 3, \qquad p_6(t) = 24t^4 + 16t^3 - 4t^2 - 2t + 1.$$

A short calculation yields that the polynomials p_3, p_4, p_5 and p_6 are positive on $(-1, 1)$. It follows that A_3, A_4, A_5 and A_6 are positive on $(0, \pi)$.

Let $n \ge 7$. We consider five cases.

Case 1. $x \in (0, \pi/n]$.

We set $x = s/(n+1)$ with $s \in (0, (n+1)\pi/n]$ and define

$$\begin{aligned} J_n(s) &= \sin\left(\frac{s}{n+1}\right) A_n\left(\frac{s}{n+1}\right) \\ &= (2n+2+n\cos(s))\sin\left(\frac{s}{n+1}\right) - \sin\left(\frac{ns}{n+1}\right) - 2\sin(s). \end{aligned} \tag{25}$$

Case 1.1. $s \in (0, 3\pi/4]$.

Using

$$1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 - \frac{1}{720}\theta^6 \le \cos(\theta) \quad (\theta \ge 0)$$

and

$$\theta - \frac{1}{6}\theta^3 \le \sin(\theta) \le \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 \quad (\theta \ge 0)$$

we obtain the estimates

$$\sin\left(\frac{s}{n+1}\right) \ge \frac{s}{n+1} - \frac{s^3}{6(n+1)^3},$$
$$-\sin\left(\frac{ns}{n+1}\right) \ge -\frac{ns}{n+1} + \frac{n^3 s^3}{6(n+1)^3} - \frac{n^5 s^5}{120(n+1)^5}.$$

It follows that

$$\begin{aligned} J_n(s) \geq & \left(2n+2+n\left(1-\frac{1}{2}s^2+\frac{1}{24}s^4-\frac{1}{720}s^6\right)\right)\left(\frac{s}{n+1}-\frac{s^3}{6(n+1)^3}\right) \\ & -\frac{ns}{n+1}+\frac{n^3s^3}{6(n+1)^3}-\frac{n^5s^5}{120(n+1)^5}-2s+\frac{1}{3}s^3-\frac{1}{60}s^5 \\ = & \frac{s^5 P_n(s)}{4320(n+1)^5} \end{aligned}$$

with

$$\begin{aligned} P_n(s) = & n(n+1)^2 s^4 - 6n(n+1)^2(n^2+2n+6)s^2 + 72n^5 \\ & + 360n^4 + 720n^3 + 720n^2 + 180n - 72. \end{aligned}$$

It remains to show that $P_n(s) > 0$, or, equivalently, after replacing s^2 by $t \in (0, (3\pi/4)^2) \subset (0, 6)$,

$$Q_n(t) = t^2 - 6(n^2+2n+6)t + \frac{72n^5+360n^4+720n^3+720n^2+180n-72}{n(n+1)^2} > 0.$$

Since

$$Q_n'(t) = 2t - 6(n^2+2n+6) < 0 \quad (0 < t < 6),$$

we obtain

$$Q_n(t) > Q_n(6) = \frac{36(n^5+6n^4+10n^3+8n^2-2)}{n(n+1)^2} > 0.$$

Case 1.2. $s \in [3\pi/4, (n+1)\pi/n]$.
Applying

$$\sin\left(\frac{ns}{n+1}\right) \leq \sin\left(\frac{3n\pi}{4(n+1)}\right) \leq \sin\left(\frac{21\pi}{32}\right) < 0.882$$

and

$$2\sin(s) \leq 2\sin\left(\frac{3\pi}{4}\right) < 1.415$$

leads to

$$-\sin\left(\frac{ns}{n+1}\right) - 2\sin(s) > -2.297. \tag{26}$$

Using the monotonicity of $x \mapsto \sin(x)/x$ we obtain

$$\begin{aligned} (n+2)\sin\left(\frac{s}{n+1}\right) & \geq (n+2)\sin\left(\frac{3\pi}{4(n+1)}\right) \\ & \geq \frac{8(n+2)}{n+1}\sin\left(\frac{3\pi}{32}\right) > 2.321. \end{aligned} \tag{27}$$

From (25), (26) and (27) we get

$$J_n(s) \geq (n+2)\sin\left(\frac{s}{n+1}\right) - \sin\left(\frac{ns}{n+1}\right) - 2\sin(s) > 0.$$

Case 2. $x \in [\pi/n, \pi - \pi/n]$.
Using (9) we obtain

$$\begin{aligned} A_n(x) &\geq 2(n+1) - \frac{\sqrt{6}}{9}n - \frac{2\sqrt{6}}{9}(n+1) - n \\ &= \left(1 - \frac{\sqrt{6}}{3}\right)n + 2\left(1 - \frac{\sqrt{6}}{9}\right) > 0\,. \end{aligned}$$

Case 3. n is odd and $x \in [\pi - \pi/n, \pi - \pi/(n+1)]$.
Since $x \mapsto -\sin(nx)$ is decreasing on $I = [\pi - \pi/n, \pi - \pi/(n+1)]$, we obtain

$$-\sin(nx) \geq -\sin\left(n\pi - \frac{n\pi}{n+1}\right) = -\sin\left(\frac{\pi}{n+1}\right).$$

Moreover, we have

$$\sin(x) \geq \sin\left(\pi - \frac{\pi}{n+1}\right) = \sin\left(\frac{\pi}{n+1}\right). \tag{28}$$

This gives

$$-\frac{\sin(nx)}{\sin(x)} \geq -1. \tag{29}$$

The function $x \mapsto -\sin((n+1)x)$ is increasing on I. Thus,

$$-2\sin((n+1)x) \geq -2\sin\left((n+1)\pi - \frac{n+1}{n}\pi\right) = -2\sin\left(\frac{\pi}{n}\right). \tag{30}$$

Using (28) and (30) gives

$$\begin{aligned} -2\frac{\sin((n+1)x)}{\sin(x)} \geq -2\frac{\sin(\pi/n)}{\sin(x)} &\geq -2\frac{\sin(\pi/n)}{\sin(\pi/(n+1))} \\ &\geq -2\frac{n+1}{n} \geq -\frac{16}{7}. \end{aligned} \tag{31}$$

From (29) and (31) we conclude that

$$A_n(x) \geq 2(n+1) - 1 - \frac{16}{7} - n = n - \frac{9}{7} > 0.$$

Case 4. n is odd and $x \in (\pi - \pi/(n+1), \pi)$.

We have $\sin((n+1)x) < 0$, and since $0 < \pi - x < \pi/(n+1) < \pi/n$, we conclude from Case 1 that $A_n(\pi - x) > 0$. It follows that

$$A_n(x) = A_n(\pi - x) - 4\frac{\sin((n+1)x)}{\sin(x)} > 0.$$

Case 5. n is even and $x \in (\pi - \pi/n, \pi)$.
We have

$$A_n(x) = A_n(\pi - x) + \frac{n(n+2)}{\sin(x)}\omega_n(x), \tag{32}$$

where

$$\omega_n(x) = \frac{\sin((n+2)x)}{n+2} - \frac{\sin(nx)}{n}.$$

Then,

$$\omega_n'(x) = -2\sin(x)\sin((n+1)x).$$

It follows that ω_n' is positive on $(\pi - \pi/n, n\pi/(n+1))$ and negative on $(n\pi/(n+1), \pi)$. This leads to

$$\omega_n(x) \geq \min\left(\omega_n(\pi - \pi/n), \omega_n(\pi)\right) = 0.$$

Moreover, from Case 1 we obtain that $A_n(\pi - x) > 0$. Applying (32) gives $A_n(x) > 0$. This completes the proof of Theorem 1.4.

6. Proof of Corollary 1.8

Let $n \geq 2$ and $x \in (0, \pi)$. We define for $\lambda \geq 2$,

$$D(\lambda) = D_n(\lambda, x) = \lambda + \cos(nx) - (\lambda + \cos(x))\frac{\sin(nx)}{n\sin(x)}.$$

Applying Theorem 1.4 and Theorem 1.5 gives

$$\begin{aligned} D(2) &= 2 + \cos(nx) - (2 + \cos(x))\frac{\sin(nx)}{n\sin(x)} \\ &= \frac{2\sin(x)(1-\cos(x))}{n(1+\cos(x))}S_{n-1}(x) > 0. \end{aligned}$$

Since

$$D'(\lambda) = 1 - \frac{\sin(nx)}{n\sin(x)} > 0,$$

we obtain $D(\lambda) \geq D(2) > 0$. This leads to (10). Next, we assume that (10) is valid for all $n \geq 2$ and $x \in (0, \pi)$. We define

$$E(x) = E_n(\lambda, x) = n(\lambda + \cos(nx))\sin(x) - (\lambda + \cos(x))\sin(nx).$$

Then, $E(x) > 0$. Since $E(0) = E'(0) = E''(0) = 0$, we get

$$E'''(0) = n(n^2 - 1)(\lambda - 2) \geq 0.$$

This yields $\lambda \geq 2$.

7. Proof of Corollary 1.9

Let $x \in (-1, 1)$ and $t \in (0, \pi)$. We define

$$U(x) = \sum_{k=0}^{\infty} (k+1)^2 x^k = \frac{1+x}{(1-x)^3}$$
$$\text{and} \quad V_t(x) = \sum_{k=1}^{\infty} \cos(kt) x^k = \frac{x(\cos(t) - x)}{x^2 - 2x\cos(t) + 1}.$$

Let $0 < \alpha < \beta < \pi$ and $W_{\alpha,\beta}(x) = U(x)\left(V_\alpha(x) - V_\beta(x)\right)$. Then,

$$\begin{aligned} W_{\alpha,\beta}(x) &= \sum_{k=0}^{\infty} (k+1)^2 x^k \sum_{k=1}^{\infty} \left(\cos(k\alpha) - \cos(k\beta)\right) x^k \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} (n-k+1)^2 \left(\cos(k\alpha) - \cos(k\beta)\right) x^n \\ &= \left(\cos(\alpha) - \cos(\beta)\right) \left(\frac{1+x}{1-x}\right)^2 \frac{x}{\phi(x)}, \end{aligned} \tag{33}$$

where

$$\phi(x) = (x^2 - 2x\cos(\alpha) + 1)(x^2 - 2x\cos(\beta) + 1).$$

Moreover, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=1}^{n} (n-k+1)^2 k \frac{\sin(k\beta)}{\sin(\beta)} x^n &= \lim_{\alpha \to \beta} \frac{W_{\alpha,\beta}(x)}{\cos(\alpha) - \cos(\beta)} \\ &= \left(\frac{1+x}{1-x}\right)^2 \frac{x}{(x^2 - 2x\cos(\beta) + 1)^2}. \end{aligned} \tag{34}$$

Applying (8) and Theorem 1.4 we conclude that the power series in (33) and (34) have positive coefficients. We set $a = -\cos(\alpha) \in (-1, 1)$ and $b = -\cos(\beta) \in (-1, 1)$. It follows that the function $R_{a,b}$, defined in (11), is absolutely monotonic on $[0, 1)$.

8. Proof of Theorem 1.10

We denote the sum in (13) by $K_n(a,b;x)$. From Theorem 1.4 we conclude that $K_n(1,1;x) > 0$ for $n \geq 1$ and $x \in (0,\pi)$. Next, we assume that (13) is valid for all $n \geq 1$ and $x \in (0,\pi)$. From $K_1(a,b;x) = ab\sin(x) > 0$ we conclude that $ab > 0$. For $n = 2$ we obtain

$$K_2(a,b;x) = (1 + a + b + ab(1 + 4\cos(x)))\sin(x) > 0.$$

This gives $1 + a + b - 3ab \geq 0$. Thus,

$$0 < 3ab \leq 1 + a + b. \tag{35}$$

Since $K_n(a,b;\pi) = 0$, we obtain for $n \geq 1$,

$$\frac{d}{dx}K_n(a,b;x)\Big|_{x=\pi} = \sum_{k=1}^{n}(-1)^k(n-k+a)(n-k+b)k^2 \leq 0.$$

We consider two cases.

Case 1. $n = 2N$.
We obtain

$$\frac{d}{dx}K_{2N}(a,b;x)\Big|_{x=\pi} = N^2(2ab - a - b) + N(ab - 1) \leq 0.$$

This gives

$$2ab - a - b \leq 0. \tag{36}$$

Case 2. $n = 2N + 1$.
Then,

$$\frac{d}{dx}K_{2N+1}(a,b;x)\Big|_{x=\pi} = N^2(a + b - 2ab) + N(a + b - 3ab) - ab \leq 0.$$

It follows that

$$a + b - 2ab \leq 0. \tag{37}$$

From (36) and (37) we get

$$a + b = 2ab. \tag{38}$$

Using (35) and (38) we conclude that $a, b > 0$. This gives

$$ab = \frac{a+b}{2} \geq \sqrt{ab}.$$

Thus, $ab \geq 1$. Using (35) and (38) yields

$$1 + 2ab = 1 + a + b \geq 3ab.$$

Hence, $ab \leq 1$. It follows that $ab = 1$. Applying this result and (38) leads to

$$0 = a + b - 2ab = \frac{1}{a}(a-1)^2.$$

Thus, $a = 1$. It follows that $b = 1$.

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