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A note on the Nielsen realization problem for connected sums of $S^2 \times S^1$

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Abstract. *We consider finite group-actions on 3-manifolds $\mathcal{H}_g$ obtained as the connected sum of g copies of $S^2 \times S^1$, with free fundamental group F_g of rank g. We prove that, for $g > 1$, a finite group of diffeomorphisms of $\mathcal{H}_g$ inducing a trivial action on homology is cyclic and embeds into an S^1-action on $\mathcal{H}_g$. As a consequence, no nontrivial element of the twist subgroup of the mapping class group of $\mathcal{H}_g$ (generated by Dehn twists along embedded 2-spheres) can be realized by a periodic diffeomorphism of $\mathcal{H}_g$ (in the sense of the Nielsen realization problem). We also discuss when a finite subgroup of the outer automorphism group* $\mathrm{Out}(F_g)$ *of the fundamental group of $\mathcal{H}_g$ can be realized by a group of diffeomorphisms of $\mathcal{H}_g$.*

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1. Introduction

All finite group-actions in the present paper will be faithful, smooth and orientation-preserving, all manifolds orientable. We are interested in finite group-actions on connected sums $\mathcal{H}_g = \sharp_g(S^2 \times S^1)$ of g copies of $S^2 \times S^1$; we will call $\mathcal{H}_g$ a *closed handle of genus* g in the following. The fundamental group of $\mathcal{H}_g$ is the free group F_g of rank g. Considering induced actions on the fundamental group and on the first homology $H_1(\mathcal{H}_g) \cong \mathbb{Z}^g$, there are canonical maps

$$\mathrm{Diff}(\mathcal{H}_g) \to \mathrm{Out}(F_g) \to \mathrm{GL}(g, \mathbb{Z})$$

where $\mathrm{Diff}(\mathcal{H}_g)$ denotes the orientation-preserving diffeomorphism group of $\mathcal{H}_g$ and $\mathrm{Out}(F_g) = \mathrm{Aut}(F_g)/\mathrm{Inn}(F_g)$ the outer automorphism group of its fundamental group.

Theorem 1.1. *Let G be a finite group acting on a closed handle $\mathcal{H}_g$ of genus $g > 1$ such that the induced action on the first homology of $\mathcal{H}_g$ is trivial. Then*

G is cyclic and a subgroup of an S^1-action on $\mathcal{H}_g$; in particular, all elements of G are isotopic to the identity.

For a description and classification of circle-actions on 3-manifolds and closed handles, see [14].

Denoting by $\mathrm{Mod}(\mathcal{H}_g)$ the *mapping class group* of isotopy classes of orientation-preserving diffeomorphisms of $\mathcal{H}_g$, there are induced maps

$$\mathrm{Mod}(\mathcal{H}_g) \to \mathrm{Out}(F_g) \to \mathrm{GL}(g, \mathbb{Z}).$$

Let $\mathrm{Twist}(\mathcal{H}_g)$ denote the subgroup of $\mathrm{Mod}(\mathcal{H}_g)$ generated by all Dehn twists along embedded 2-spheres in $\mathcal{H}_g$ (i.e., by cutting along a 2-sphere and regluing after twisting by one full turn around an axis; since such a twist represents a generator of $\pi_1(SO(3)) \cong \mathbb{Z}_2$, its square is isotopic to the identity). By classical results of Laudenbach [6, 7] there is a short exact sequence

$$1 \to \mathrm{Twist}(\mathcal{H}_g) \hookrightarrow \mathrm{Mod}(\mathcal{H}_g) \to \mathrm{Out}(F_g) \to 1;$$

moreover $\mathrm{Twist}(\mathcal{H}_g) \cong (\mathbb{Z}_2)^g$ is generated by the sphere twists around the core spheres $S^2 \times *$ of the g different $S^2 \times S^1$ summands of $\mathcal{H}_g$ (twists around separating 2-spheres instead are isotopic to the identity). It is proved in [1] that $\mathrm{Mod}(\mathcal{H}_g)$ is isomorphic to a semidirect product $\mathrm{Twist}(\mathcal{H}_g) \rtimes \mathrm{Out}(F_g)$. Theorem 1.1 has the following consequence (in the sense of the *Nielsen realization problem*).

COROLLARY 1.2. *No nontrivial element of the twist group* $\mathrm{Twist}(\mathcal{H}_g)$ *can be realized (represented) by a periodic diffeomorphism of* $\mathcal{H}_g$.

For $g > 1$ this follows from Theorem 1.1 but the methods apply also to the case $g = 1$ of $\mathcal{H}_1 = S^2 \times S^1$, using the fact that $S^2 \times S^1$ is a geometric 3-manifold belonging to the $(S^2 \times \mathbb{R})$-geometry (one of Thurston's eight 3-dimensional geometries, see [15]), and that finite group-actions on $S^2 \times S^1$ are geometric ([10, Theorem 8.4]).

For a solution of the Nielsen realization problem for aspherical and Haken 3-manifolds, see [19] (here finite groups of mapping classes can always be realized, except for a purely algebraic obstruction in the case of Seifert fiber spaces where, however, a finite inflation of the group can always be realized).

By [6], homotopic diffeomorphisms of $\mathcal{H}_g$ are isotopic but this does not remain true for arbitrary connected sums of 3-manifolds. By [4], twists around separating 2-spheres in a 3-manifold may or may not be homotopic to the identity, moreover by [3] there are sphere-twists which are homotopic but not isotopic to the identity (see also the discussion in the introduction of [1]). As an example, considering a connected sum $M = M_1 \sharp M_2$ of two closed hyperbolic 3-manifolds M_1 and M_2, the sphere-twist around the connecting 2-sphere is not homotopic to the identity; also, it cannot be realized by a periodic map (e.g., if M_1 or M_2 does not admit a nontrivial periodic map then also the connected sum $M = M_1 \sharp M_2$ has no periodic maps).

There arises naturally the question of which finite subgroups of $\mathrm{Out}(F_g)$ can be realized by a finite group of diffeomorphisms of $\mathcal{H}_g$. Finite groups G of diffeomorphisms of $\mathcal{H}_g$ which act faithfully on the fundamental group (i.e., inject into $\mathrm{Out}(F_g)$) are considered in [17] where, for $g \geq 15$, the quadratic upper bound $|G| \leq 24g(g-1)$ for their orders is obtained. Since $\mathrm{Out}(F_g)$ has finite subgroups of larger orders, these subgroups cannot be realized by finite groups of diffeomorphisms (by [16] the maximal order of a finite subgroup of $\mathrm{Out}(F_g)$ is $2^g g!$, for $g > 2$). A precise result is as follows (we refer to [17, Section 2] for definitions and the proof).

Theorem 1.3. *Let G be a finite subgroup of $\mathrm{Out}(F_g)$ and $1 \to F_g \to E \to G \to 1$ the corresponding group extension associated to G. Then G can be realized by an isomorphic group of diffeomorphisms of $\mathcal{H}_g$ if and only if E is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of a finite graph of finite groups $(\Gamma, \mathcal{G})$ in normal form associated to a closed handle-orbifold (in particular, the vertex groups of $(\Gamma, \mathcal{G})$ have to be isomorphic to finite subgroups of SO(4) and the edge groups to finite subgroups of SO(3)).*

We note that, for a finite group G acting on a closed handle $\mathcal{H}_g$, the quotient $\mathcal{H}_g/G$ has the structure of a closed handle-orbifold (see [17]). Analogous results on finite group-actions on 3-dimensional handlebodies are obtained in [8, 12] (and in [9] for finite group-actions on handlebodies in arbitrary dimensions).

The case $g = 2$ is special. By well-known results,

$$\mathrm{Out}(F_2) \cong \mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}(2, \mathbb{Z}) \cong \mathbb{D}_6 *_{\mathbb{D}_2} \mathbb{D}_4,$$

so up to conjugation the maximal finite subgroups of $\mathrm{Out}(F_2)$ are the dihedral groups $\mathbb{D}_6$ and $\mathbb{D}_4$ of orders 12 and 8, and both can be realized by diffeomorphisms of the torus with one boundary component (hence, if the realizations of the amalgamated subgroups $\mathbb{D}_2$ coincide, one obtains a realization of the whole group $\mathrm{Out}(F_2) \cong \mathbb{D}_6 *_{\mathbb{D}_2} \mathbb{D}_4$). Considering the product with a closed interval, one obtains realizations on the handlebody V_2 of genus 2 and also on its double $\mathcal{H}_2$ along the boundary.

Concerning the case $g = 3$, by [20] there are exactly five maximal finite subgroups of $\mathrm{Out}(F_3)$ up to conjugation; by an easy application of Theorem 1.3, all of these maximal finite subgroups can be realized by diffeomorphisms of the closed handle $\mathcal{H}_3$ of genus 3 (but not of a handlebody V_3 of genus 3).

2. Proof of Theorem 1.1

Let G be a finite group acting faithfully and orientation-preservingly on a closed handle $\mathcal{H}_g = \sharp_g(S^2 \times S^1)$ of genus g. By the equivariant sphere theorem (see [10] for an approach by minimal surface techniques, [2, 5] for topological-combinatorial proofs), there exists an embedded, homotopically nontrivial 2-sphere S^2 in $\mathcal{H}_g$ such that $x(S^2) = S^2$ or $x(S^2) \cap S^2 = \emptyset$ for all $x \in G$. We cut $\mathcal{H}_g$ along the system of disjoint 2-spheres $G(S^2)$, by removing the interiors of G-equivariant regular neighbourhoods $S^2 \times [-1, 1]$ of these 2-spheres, and call each of these regular neighbourhoods $S^2 \times [-1, 1]$ a 1-handle. The result is a collection of 3-manifolds with 2-sphere boundaries, with an induced action of G. We close each of the 2-sphere boundaries by a 3-ball and extend the action of G by taking the cone over the center of each of these 3-balls, so G permutes these 3-balls and their centers. The result is a finite collection of closed handles of lower genus on which G acts (cf. [17]). Applying inductively the procedure of cutting along 2-spheres, we finally end up with a finite collection of 3-spheres or 0-handles (closed handles of genus 0). Note that the construction gives a finite graph Γ on which G acts whose vertices correspond to the 0-handles and whose edges to the 1-handles. Note that Γ has no *free edges*, i.e. edges with one vertex of valence 1.

On each 3-sphere (0-handle) there are finitely many points which are the centers of the attached 3-balls (their boundaries are the 2-spheres along which the 1-handles are attached). For each of these 3-spheres, let G_v denote its stabilizer in G (by the geometrization of finite group-actions on 3-manifolds, one may assume that the action of a stabilizer G_v on the corresponding 3-sphere is orthogonal but this is not needed for the following). Denoting by G_e the stabilizer in G of a 1-handles $S^2 \times [-1, 1]$, we can assume that each stabilizer G_e preserves the product structure of $S^2 \times [-1, 1]$ of the corresponding 1-handle (by choosing small equivariant regular neighbourhoods of the 2-spheres). If some element of a stabilizer G_e acts as a reflection on [-1,1], we split the 1-handle into two 1-handles by introducing a new 0-handle obtained from a small regular neighbourhood $S^2 \times [-\epsilon, \epsilon]$ of $S^2 \times \{0\}$ by closing up with two 3-balls. Hence we can assume that each stabilizer G_e of a 1-handle $S^2 \times [-1, 1]$ does not interchange its two boundary 2-spheres; that is, G acts *without inversions* on the graph Γ.

Suppose now that $g > 1$ and that the induced action of G on the first homology of $\mathcal{H}_g$ and hence also of Γ is trivial. As before, G acts without inversions on Γ and Γ has no free edges. We will prove in next Proposition 2.1 that under these hypotheses the action of G on Γ is trivial, that is each element of G acts as the identity on Γ. Hence G fixes each vertex and each edge of Γ.

Since G fixes each 1-handle $S^2 \times [-1,1]$, it maps each 2-sphere $S^2 \times \{0\}$ to itself. By construction, G does not interchange the two sides of such a 2-sphere and acts faithfully on it (otherwise some element of G would act trivially on an invariant regular neighbourhood of such a 2-sphere and then act trivially also on all of $\mathcal{H}_g$ (well-known in particular for smooth actions)). It follows that G is isomorphic to a finite subgroup of the orthogonal group SO(3), i.e. cyclic $\mathbb{Z}_n$, dihedral $\mathbb{D}_{2n}$, tetrahedral $\mathbb{A}_4$, octahedral $\mathbb{S}_4$ or dodecahedral $\mathbb{A}_5$. It is easy to see that an orientation-preserving action of $\mathbb{D}_{2n}$, $\mathbb{A}_4$, $\mathbb{S}_4$ or $\mathbb{A}_5$ on S^3 has at most two global fixed points around which a 1-handle can be attached; but then the graph Γ would be a segment or a circle, that is $g \leq 1$. Since $g > 1$, G is a cyclic group which acts by rotations around an axis S^1 in each 0-handle S^3. By the positive solution of the Smith-conjecture [13], each of these axes is a trivial knot in S^3, and hence the action of the cyclic group G embeds into an S^1-action on each 0-handle. Since these S^1-actions on the 0-handles extend to the connecting 1-handles $S^2 \times [-1,1]$, the cyclic G-action on $\mathcal{H}_g$ embeds into an S^1-action.

To complete the proof of Theorem 1.1, it remains to prove the following proposition (which may be considered as an analogue of Theorem 1.1 for finite graphs).

Proposition 2.1. *Let G be a finite group acting faithfully on a finite connected graph Γ without free edges and of genus $g > 1$ (or cycle rank, or rank of its free fundamental group). Then also the induced action of G on the first homology $H_1(\Gamma) \cong \mathbb{Z}^g$ of Γ is faithful.*

Proof. By subdividing edges, we can assume that G acts without inversion of edges on Γ. Suppose that an element $x \in G$ acts trivially on the first homology of Γ. Then its Lefschetz number is $1-g$ which, by the Hopf trace formula, is equal to the Euler characteristic of the fixed point set of x which is a subgraph Γ' of Γ (since G acts without inversions of edges). The graph Γ of genus g has Euler characteristic $1-g$; passing from Γ' to Γ by adding successively the missing edges, the Euler characteristic remains unchanged (when adding a free edge) or decreases. Since Γ has no free edges, this implies $\Gamma' = \Gamma$, and hence x acts trivially on Γ. This completes the proof of the proposition. □

By [18, Proof of Satz 3.1], each finite subgroup of $\mathrm{Out}(F_g)$ can be realized by an action of the group on a finite graph Γ without free edges (this is a version of the *Nielsen realization problem for finite graphs* which several years later was "rediscovered" by various authors); Proposition 2.1 implies then the following well-known result.

Corollary 2.2. *The canonical projection* $\mathrm{Out}(F_g) \to \mathrm{GL}(g,\mathbb{Z})$ *is injective on finite subgroups of* $\mathrm{Out}(F_g)$.

We note that not all finite subgroups of $\mathrm{GL}(g,\mathbb{Z})$ are induced in this way by finite subgroups of $\mathrm{Out}(F_g)$; in fact, for $g = 2, 4, 6, 7, 8, 9$ and 10 there are finite subgroups of $\mathrm{GL}(g,\mathbb{Z})$ of orders larger than $2^g g!$ (which, by [16], is the maximal order of a finite subgroup of $\mathrm{Out}(F_g)$)). On the other hand, there are also small cyclic subgroups of $\mathrm{GL}(g,\mathbb{Z})$ which cannot be realized in this way, see the discussion in [21, Section 5].

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