Rend. Istit. Mat. Univ. Trieste
Volume 52 (2020), 1–10 (electronic preview)
DOI: 10.13137/2464-8728/30766

# Non-isometric hyperbolic 3-orbifolds with the same topological type and volume

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*To Bruno Zimmermann on his 70th birthday*

Abstract. *We construct pairs of non-isometric closed hyperbolic 3-orbifolds with the same topological type and volume. Topologically these orbifolds are mapping tori of pseudo-Anosov maps of the surface of genus 2, with singular locus a fibred (hyperbolic) link with five components.*

Keywords: Branched covers, mapping tori, (pseudo-)Anosov diffeomorphisms.
MS Classification 2010: 57M10, 57M50, 57M60, 37E30.

## 1. Introduction

In this brief note, building upon techniques and ideas already used in [2], we construct a hyperbolic 3-manifold that is a branched cover of a link in another hyperbolic manifold (the mapping torus of a pseudo-Anosov map of the surface of genus 2) in two different ways. More precisely we show

Theorem 1.1. *Given two integers $n > m \geq 2$, there are infinitely many pairwise distinct pairs $(M, L)$ where $M$ is a closed fibred hyperbolic 3-manifold and $L$ a five-component link contained in another closed fibred hyperbolic 3-manifold $|O|$ such that $M$ is a $2mn$-sheeted regular branched cover of $L$ in two non-equivalent ways.*

By distinct pairs $(M, L)$ and $(M', L')$ we mean that either the two manifolds are not homeomorphic (equivalently not isometric as hyperbolic manifolds, by Mostow's rigidity [3]) or that the two links $L$ and $L'$ are not equivalent, i.e. the pairs $(|O|, L)$ and $(|O'|, L')$ are not homeomorphic. In our construction, the link $L$ is transverse to the fibration of the manifold in which it is contained.

As a straightforward consequence of the above result, we obtain

Corollary 1.2. *There are infinitely many pairs of non-isometric closed hyperbolic 3-orbifolds that have the same topological type and volume.*

Indeed, let $M$ be one of the closed hyperbolic manifolds whose existence is assured by Theorem 1.1. The two orbifolds of each pair mentioned in the corollary are obtained as quotients of $M$ by the action of two groups of order $2mn$, corresponding to the two deck-transformation groups of the two non equivalent regular branched coverings. By construction, the two orbifolds have the same volume and topological type: in both cases, their underlying space, i.e. the space of orbits of the action, is $|O|$, and the points with non trivial stabilisers map precisely onto $L$. The construction will show that the orders of ramification are $(2, 2, 2m, 2m, n)$ in one case and $(2, 2, 2n, 2n, m)$ in the other so that the two orbifolds cannot be isometric. In any case, because of Mostow's rigidity [3], one can also conclude by observing that the two orbifolds would be isometric if and only if the corresponding regular branched coverings were equivalent.

In order to build our examples, we start by describing certain (branched) covers of surfaces. As in [2], the 3-manifolds and orbifolds we are looking for will be then obtained as mapping tori of pseudo-Anosov maps defined on the surfaces considered. The maps will be chosen so that they "commute" with the covering projections, indeed all maps will be lifts of a single Anosov map defined on the common quotient of all surface covers, that is a torus in this construction.

## 2. Surface covers

Given two integers $n > m \geq 2$ we wish to construct two non-equivalent $2mn$-sheeted covers from the (closed, connected, orientable) surface $S_g$ of genus $g = 6nm - n - m + 1$ onto the surface of genus 2, branched over five points, with orders of ramification $(2, 2, 2m, 2m, n)$ for the first cover and $(2, 2, 2n, 2n, m)$ for the second one. We shall see the bases of the two covers as two 2-orbifolds: $\Sigma_2(2, 2, 2m, 2m, n)$ and $\Sigma_2(2, 2, 2n, 2n, m)$. Both orbifolds are orbifold double covers of the torus $T(2, 2, 2m, 2n)$ with four cone points of orders $(2, 2, 2m, 2n)$.

In order to make the construction of the different coverings easier to understand we will separate it in two different steps. In step one we will construct an $mn$-sheeted branched cover from $S_g$ to the orbifold $\Sigma_5(m, m, n, n)$ of genus five with four cone points of orders $(m, m, n, n)$. In step two we will show that the orbifold $\Sigma_5(m, m, n, n)$ double covers both $\Sigma_2(2, 2, 2m, 2m, n)$ and $\Sigma_2(2, 2, m, 2n, 2n)$. Using the fact that the deck transformations of the covering constructed commute, we will also see that $\Sigma_2(2, 2, 2m, 2m, n)$ and $\Sigma_2(2, 2, m, 2n, 2n)$ have a common quotient, which is the $\mathbb{Z}/2 \times \mathbb{Z}/2$ quotient of $\Sigma_5(m, m, n, n)$. This quotient will be $T(2, 2, 2m, 2n)$.

The branched covers just discussed can be summarised in the following commuting diagram of covers:

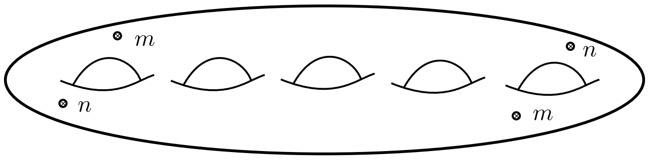


Figure 1: The orbifold $\Sigma_5(m,m,n,n)$.

$$\begin{array}{ccc} & S_g & \\ & \downarrow & \\ & \Sigma_5(m,m,n,n) & \\ \swarrow & & \searrow \\ \Sigma_2(2,2,2m,2m,n) & & \Sigma_2(2,2,m,2n,2n) \\ \searrow & & \swarrow \\ & T(2,2,2m,2n) & \end{array}$$

Note that all covers associated to arrows appearing in the diagram are regular. The group of deck transformations associated to $S_g \longrightarrow \Sigma_5(m,m,n,n)$ is isomorphic to $\mathbb{Z}/m\times\mathbb{Z}/n$, the other being double covers, as already observed.

## 2.1. Step one

In this part we start by constructing a cover of order $mn$ from the surface of genus $(m-1)(n-1)$ onto the 2-sphere with four branch points, two of order $n$ and two of order $m$. Having fixed an integer $h \geq 0$ we will then adapt the construction in order to have a cover from the surface of genus $g = hmn + (m-1)(n-1)$ onto the surface of genus $h$ again with four branch points, two of order $n$ and two of order $m$. Observe that for the case we are interested in we have $h = 5$ and $g = 6nm - n - m + 1 = 5mn + (m-1)(n-1)$.

To construct the covering we want, we will find a surface of genus $(m-1)(n-1)$ admitting a symmetry of type $\mathbb{Z}/m \times \mathbb{Z}/n$ where the generators of both cyclic subgroups have fixed points belonging to precisely two orbits of the $\mathbb{Z}/m\times\mathbb{Z}/n$-action. A simple way to build a surface having prescribed symmetry is to use a symmetric graph and see the surface as the boundary of a regular neighbourhood of a standard embedding of the graph in 3-space (as was done in [2]).

We will build a graph embedded in the 3-sphere. To make things more explicit, it is convenient to see the 3-sphere $\mathbf{S}^3 \subset \mathbb{C}^2$ as the set of points $(z_1, z_2)$ such that $|z_1|^2 + |z_2|^2 = 1$. Note that $\mathbf{S}^3$ admits a $\mathbb{Z}/m\times\mathbb{Z}/n$-action defined on the generators as $(z_1, z_2) \mapsto (e^{2i\pi/m}z_1, z_2)$ and $(z_1, z_2) \mapsto (z_1, e^{2i\pi/n}z_2)$. Consider now the following sets of points in the 3-sphere: $A = \{p_k = (e^{2ik\pi/m}, 0) \mid k = 0, \ldots, m-1\}$ and $B = \{q_l = (0, e^{2il\pi/n}) \mid l = 0, \ldots, n-1\}$. Observe that both sets are invariant by the $\mathbb{Z}/m \times \mathbb{Z}/n$-action by construction. The graph we are interested in is the complete bipartite graph with sets of vertices $A$ and $B$ and set of edges $\{e_{kl} : k = 0, \ldots, m-1; l = 0, \ldots, n-1\}$ where $e_{kl} = \{(\cos t\, p_k + \sin t\, q_l) \mid 0 < t < \pi/2\}$. It is clear that the graph is embedded in $\mathbf{S}^3$ in a $\mathbb{Z}/m \times \mathbb{Z}/n$-equivariant way. Figure 2 shows the case $n = 3$ and $m = 4$, where the solid tori $\{(z_1, z_2) \in \mathbf{S}^3 \mid |z_2|^2 \leq 1\}$ and $\{(z_1, z_2) \in \mathbf{S}^3 \mid |z_2|^2 \geq 1\}$ have their boundaries identified, and their cores are the components of a Hopf link. Figure 3 shows the whole graph for $n = 2$ and $m = 3$.

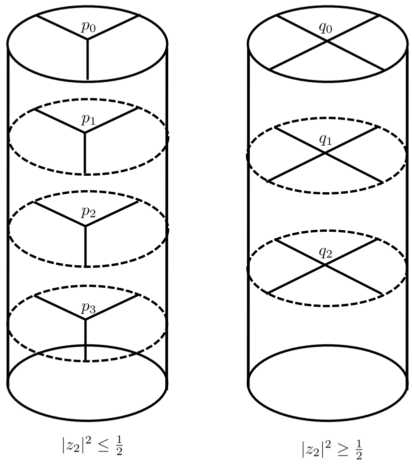


Figure 2: A local picture of the $(n, m)$-complete bipartite graph with vertices on the Hopf link for $n = 3$ and $m = 4$.

We then obtain the desired surface by taking the boundary of a sufficiently small regular and equivariant neighbourhood of the graph. Note that the Euler characteristic of our graph is $m + n - nm$, so it follows immediately that the genus of the boundary surface is $mn - m - n + 1$. We wish to stress that the fixed-point set of the cyclic subgroup of order $m$ acting on the 3-sphere is the circle of equation $z_1 = 0$ containing the set $B$ and, similarly, the fixed-point set of the cyclic subgroup of order $n$ is the circle of equation $z_2 = 0$ containing the set $A$. As a consequence, nearby each point of $A$ (respectively $B$) the cyclic group of order $n$ (respectively $m$) has two fixed points on the surface. Since the $\mathbb{Z}/m \times \mathbb{Z}/n$-action is freely transitive on the edges of the graph, it is not hard to see that the quotient of this surface by the action is a sphere with two cone points of order $m$ and two of order $n$.

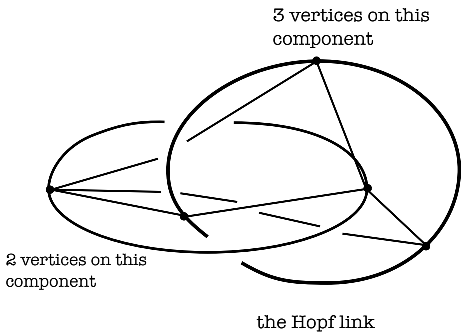


Figure 3: A global picture of the $(n, m)$-complete bipartite graph with vertices on the Hopf link for $n = 2$ and $m = 3$.

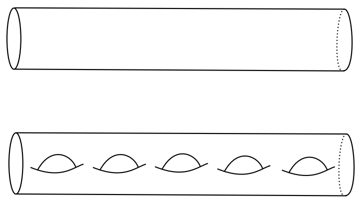

Figure 4: A tube and a surface of genus $h = 5$ that replaces it in the construction.

To construct a covering with the same type of action from the surface of genus $hmn+(m-1)(n-1)$ onto a surface of genus $h$, we start by observing that the surface we have just built can be decomposed into $n$ $m$-holed spheres, $m$ $n$-holed spheres and $mn$ tubes each attached on one side to some hole of a sphere of the first type and on the other to some hole of a sphere of the second type. To generalise the construction it suffices to replace each tube with a surface of genus $h$ with two holes (see Figure 4): the boundary components of the surface are attached as the boundary components of the original tubes were. It is obvious that one can carry out this construction in a $\mathbb{Z}/m \times \mathbb{Z}/n$-equivariant way.

## 2.2. Step two

For the remaining coverings we follow the same strategy: we start by considering a symmetric graph embedded in $\mathbb{R}^3$. Consider the unit circle $C_0$ in the plane

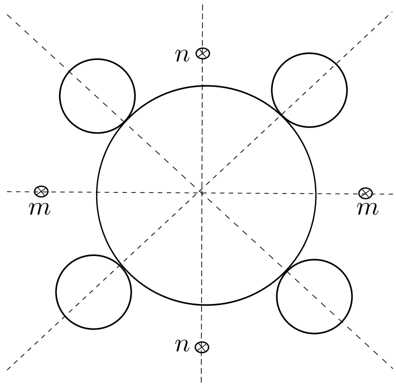


Figure 5: The circle pattern in the $x_3 = 0$-plane and the position of the cone points of orders $n$ and $m$ on the genus-5 surface boundary of a regular neighbourhood (not shown in the picture).

of equation $x_3 = 0$. In the same plane, consider four circles $C_i$, $i = 1, 2, 3, 4$, of centres $(1, 1, 0)$, $(-1, 1, 0)$, $(-1, -1, 0)$, and $(1, -1, 0)$ respectively, all of radius $\sqrt{2} - 1$. The set $\Gamma = \cup_{i=0}^{4} C_i$ is a graph in the plane of equation $x_3 = 0$. By construction, $\Gamma$ is invariant by the action of the $\pi$-rotations about the $x_1$ and $x_2$ axes. Once more, consider a small and invariant regular neighbourhood of $\Gamma$: its boundary is clearly a surface of genus 5. We wish to identify this surface with the quotient $\Sigma_5(m, m, n, n)$ of the cover constructed in Step one. We do so by imposing that the outermost points of intersection of the surface with the $x_1$ axis are the two cone points of order $m$ and the outermost points of intersection of the surface with the $x_2$ axis are two cone points of order $n$. It is now straightforward to realise that the quotient of $\Sigma_5(m, m, n, n)$ by the $\pi$-rotation about the $x_1$ (respectively $x_2$) axis is the orbifold $\Sigma_2(2, 2, 2m, 2m, n)$ (respectively $\Sigma_2(2, 2, 2n, 2n, m)$). The two $\pi$-rotations commute and generate a Klein group of order four. The quotient of $\Sigma_5(m, m, n, n)$ by its action is $T(2, 2, 2m, 2n)$.

Remark that, although the two double branched covers from the surface of genus five to that of genus two are equivalent, the $2mn$-sheeted branched coverings $S_g \longrightarrow \Sigma_2(2, 2, 2m, 2m, n)$ and $S_g \longrightarrow \Sigma_2(2, 2, 2n, 2n, m)$ are not, for the orders of ramification are different. Notice that the two double branched covers are equivalent since the surface of genus five admits a cyclic symmetry of order four (a $\pi/2$-rotation about the $x_3$-axis) which conjugates the two covering involutions. This symmetry preserves the marked points, although not their orders and generates, together with the Klein group, a dihedral group of order eight.

## 3. Mapping tori

We wish now to construct 3-manifolds as mapping tori of homeomorphisms defined on the surfaces built in the previous section. We want, moreover, that the mapping tori fulfill some extra requirements.

LEMMA 3.1. *Consider the five surfaces $S_g$, $\Sigma_5(m,m,n,n)$, $\Sigma_2(2,2,2m,2m,n)$, $\Sigma_2(2,2,2n,2n,m)$, and $T(2,2,2m,2n)$, and branched coverings constructed in the previous section. For each homeomorphism $\phi$ of the torus fixing four points (corresponding to the four cone points of $T(2,2,2m,2m)$), it is possible to choose a power $\psi = \phi^k$ and a homeomorphism of each of the four remaining surfaces in such a way that each of the constructed branched coverings of surfaces induces a branched covering of the corresponding mapping torus.*

*Moreover, there exist infinite order homeomorphisms $\phi$ such that the following extra properties hold.*

1. *The mapping tori associated to the homeomorphisms of $\Sigma_2(2,2,2m,2m,n)$ and $\Sigma_2(2,2,2n,2n,m)$ are homeomorphic.*
2. *The cone points of $\Sigma_2(2,2,2m,2m,n)$ and $\Sigma_2(2,2,2n,2n,m)$ are fixed by the chosen homeomorphisms and they close up to equivalent five-component links in the mapping tori.*

*Proof.* We start by explaining what it means for a regular (branched) covering of a surface to induce a regular (branched) covering of a mapping torus of a homeomorphism of the surface. Observe the following basic fact. If $G$ is a finite group of homeomorphisms acting on a surface $S$, the map $S \longrightarrow S/G$ is a regular, possibly branched, cover. $G$ induces an action on the product $S \times [0,1]$ which is trivial on the interval $[0,1]$. Given a homeomorphism of $S$, this latter action induces an action on the mapping torus of the homeomorphism (and thus a regular (branched) covering) provided the homeomorphism centralises $G$.

Keeping the above in mind, since we are dealing with finite covers there is a power of $\phi$ that lifts to all covers and, up to choosing possibly a further power $\psi = \phi^k$, we can even ensure that the fibres of the cone points are fixed pointwise by the lift. The lifts commute with the actions of the goups of deck transformations by construction and are the homeomorphisms of the four other surfaces that we need to choose. Note that if $\phi$ is of infinite order, so will be $\psi$ and all its lifts. If $\phi$ is of finite order one can choose $\psi$, as well as all its lifts, to be the identity.

To ensure that the two extra conditions hold, we need the lifts of $\psi$ to $\Sigma_2(2,2,2m,2m,n)$ and $\Sigma_2(2,2,m,2n,2n)$ to be conjugate, and the conjugation to map the cone points of the first orbifold to those of the second one.

This may not be achieved for any arbitrary infinite order $\phi$. We thus provide a construction allowing to build suitable homeomorphisms $\phi$. We start by taking a further quotient of our surfaces. We point out that this quotient will not be the base space of an orbifold covering with total space $S_g$. Consider $T(2,2,2m,2m)$. If we forget the orders of the cone points we see that the symmetry of order four of the surface of genus five induces an elliptic involution of $T(2,2,2m,2m)$ which exchanges the two cone points of order two and those of orders $2m$ and $2n$. Observe that this elliptic involution is also induced by any of the two $\pi$-rotations about the diagonals $x_1 = \pm x_2$ in the $x_3 = 0$ plane of the previous construction (see Figure 5).

Let $\varphi$ be an infinite order homeomorphism $\varphi$ of the 2-sphere quotient of the torus by the action of the elliptic involution. We require moreover that $\varphi$ fixes six points: two points $a$ and $b$ corresponding to the orbits of the cone points of $T(2,2,2m,2m)$ and four others corresponding to the orbits of the fixed-points of the elliptic involution. Up to passing to a power, we can assume that $\varphi$ lifts to a homeomorphism $\phi$ of the covering torus which fixes each point $a_1, a_2$ in the fibre of $a$ and $b_1, b_2$ in the fibre of $b$. We identify this torus with $T(2,2,2m,2n)$ in such a way that $a_1$ and $a_2$ are cone points of order 2, $b_1$ of order $2m$, and $b_2$ of order $2n$. We also require that $\{a_i, b_i\}$ are the images of the fixed points of the $\pi$-rotation about the axis $x_i$, $i = 1, 2$.

With this choice of $\phi$, the mapping tori associated to the lifts of $\psi$ discussed above satisfy the desired requirements. □

The diagram of surface orbifold covers gives thus rise to a diagram of induced 3-dimensional orbifold covers between mapping tori:

$$
\begin{array}{ccccc}
 & & M & & \\
 & & \downarrow & & \\
 & & N(m,m,n,n) & & \\
 & \swarrow & & \searrow & \\
O(2,2,n,2m,2m) & & & & O(2,2,m,2n,2n) \\
 & \searrow & & \swarrow & \\
 & & Q(2,2,2m,2m) & &
\end{array}
$$

We stress again that here the orbifolds $O(2,2,n,2m,2m)$ and $O(2,2,2n,2n,m)$ have the same topological type, that is are homeomorphic as manifolds and their singular sets are equivalent links.

## 4. Hyperbolic structures

The mapping tori constructed in Lemma 3.1 satisfy the covering conditions of the statement of Theorem 1.1. To finish the proof of the theorem we still have to prove that each of the mapping tori can be chosen to admit a hyperbolic structure and that there are infinitely many pairwise non isometric manifolds with this property.

Thurston's hyperbolization theorem for manifolds that fibre over the circle (see [4]) ensures that the mapping tori constructed in Lemma 3.1 are hyperbolic manifolds if and only if they are mapping tori of pseudo-Anosov maps. In what follows we show how one can choose the homeomorphisms in Lemma 3.1 to be pseudo-Anosov.

For this to be the case it suffices to choose the homeomorphism $\psi$ described in the previous section to be Anosov. This is a standard fact that can be derived from [1, Exposé 13] as discussed in detail in [2, Section 3.1].

It is possible to exhibit an Anosov map satisfying the desired properties $\psi$ must have even without considering the quotient by the elliptic involution. Indeed, each linear Anosov map commutes with the standard elliptic involution corresponding to minus the identity; in particular, the elliptic involution preserves the fixed points of the Anosov map. Now, to our purposes it is sufficient that the linear Anosov we choose has at least four fixed points that are exchanged in pairs by the hyperelliptic involution. Indeed, we only need to identify these two orbits with the set of cone points of order two, and of order $2m$ and $2n$ respectively. Since Anosov maps have periodic orbits of arbitrarily large orders (see for instance [5]), up to perhaps taking a power, we can assume that our linear Anosov has at least eight fixed points. Now the standard elliptic involution has precisely four fixed points, all other points belonging to orbits with two elements. We can thus conclude that the Anosov map has at least four fixed points that are exchanged in pairs by the standard elliptic involution.

To finish the proof of Theorem 1.1, it is enough to construct infinitely many non isometric hyperbolic manifolds obtained as mapping tori of pseudo-Anosov maps.

Note that all finite groups of deck transformations can be realised as isometry groups for the hyperbolic structure of the 3-manifolds or orbifolds on which they act by Mostow's rigidity [3], so the quotients are hyperbolic orbifolds. Now, since the orders of the groups acting are arbitrarily large, we see that we must have infinitely many non isometric manifolds $M$ having two non equivalent branched covers on the same link.

Also, by taking powers of $\psi$ we obtain infinitely many commensurable examples and infinitely many links. Indeed, for a fixed choice of $n > m \geq 2$, and for a fixed link $L$, the set of possible volumes of orbifolds with singular set $L$ and ramifications orders of the form $(2, 2, 2n, 2n, m)$ or $(2, 2, 2m, 2m, n)$ must be finite. However, by taking mapping tori associated to powers of $\psi$ the volume must grow, because they cover each other.

Note that the five-component links in $|O|$ are fibred and hyperbolic. Indeed, the pseudo-Anosov maps defined over the surface of genus 2 with five cone points restrict to pseudo-Anosov maps of the five-punctured genus-2 surface obtained by removing the cone points.

This ends the proof of Theorem 1.1.


## Acknowledgements

The authors would like to thank the anonymous referee for his/her careful reading and accurate comments. A. Salgueiro was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.


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Received December 13, 2019
Revised June 22, 2020
Accepted June 23, 2020