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Theory of the (m, σ)-general functions over infinite-dimensional Banach spaces

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Abstract. *In this paper, we introduce some functions, called (m, σ)-general, that generalize the (m, σ)-standard functions and are defined in the infinite-dimensional Banach space E_I of the bounded real sequences $\{x_n\}_{n \in I}$, for some subset I of $\mathbf{N}^*$. Moreover, we recall the main results about the differentiation theory over E_I, and we expose some properties of the (m, σ)-general functions. Finally, we study the linear (m, σ)-general functions, by introducing a theory that generalizes the standard theory of the $m \times m$ matrices.*

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1. Introduction

In this paper, we generalize the results of the articles [3] and [4], where, for any subset I of $\mathbf{N}^*$, we define the Banach space $E_I \subset \mathbf{R}^I$ of the bounded real sequences $\{x_n\}_{n \in I}$, the σ-algebra $\mathcal{B}_I$ given by the restriction to E_I of $\mathcal{B}^{(I)}$ (defined as the product indexed by I of the same Borel σ-algebra $\mathcal{B}$ on $\mathbf{R}$), and a class of functions over an open subset of E_I, with values on E_I, called (m, σ)-standard. The properties of these functions generalize the analogous ones of the standard finite-dimensional diffeomorphisms; moreover, these functions are introduced in order to provide a change of variables' formula for the integration of the measurable real functions on $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$. For any strictly positive integer k, this integration is obtained by using an infinite-dimensional measure $\lambda_{N,a,v}^{(k,I)}$, over the measurable space $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$, that in the case $I = \{1, ..., k\}$ coincides with the k-dimensional Lebesgue measure on $\mathbf{R}^k$.

In the mathematical literature, some articles introduced infinite-dimensional measures analogue of the Lebesgue one (see for example the paper of Léandre [8], in the context of the noncommutative geometry, that one of Tsilevich et al. [10], which studies a family of σ-finite measures on $\mathbf{R}^+$, and that one of Baker [5], which defines a measure on $\mathbf{R}^{\mathbf{N}^*}$ that is not σ-finite).

In the paper [3], we define the linear (m,σ)-standard functions. The motivation of this paper follows from the natural extension to the infinite-dimensional case of the results of the article [2], where we estimate the rate of convergence of some Markov chains in $[0,p)^k$ to a uniform random vector. In order to consider the analogue random elements in $[0,p)^{\mathbf{N}^*}$, it is necessary to overcome some difficulties: for example, the lack of a change of variables formula for the integration in the subsets of $\mathbf{R}^{\mathbf{N}^*}$. A related problem is studied in the paper of Accardi et al [1], where the authors describe the transformations of generalized measures on locally convex spaces under smooth transformations of these spaces. In the paper [4], we expose a differentiation theory for the functions over an open subset of E_I, and in particular we define the functions C^1 and the diffeomorphisms; moreover, we remove the assumption of linearity for the (m,σ)-standard functions, and we present a change of variables' formula for the integration of the measurable real functions on $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$; this change of variables is defined by the (m,σ)-standard diffeomorphisms, with further properties. This result agrees with the analogous finite-dimensional result.

In this paper, we introduce a class of functions, called (m,σ)-general, that generalizes the set of the (m,σ)-standard functions given in [4]. In Section 2, we recall the main results about the differentiation theory over the infinite-dimensional Banach space E_I. Moreover, we expose some properties of the (m,σ)-general functions. In Section 3, we study the linear (m,σ)-general functions and we expose a theory that generalizes the standard theory of the $m\times m$ matrices and the results about the linear (m,σ)-standard functions, given in [3]. The main result is the definition of the determinant of a linear (m,σ)-general function, as the limit of a sequence of the determinants of some standard matrices (Theorem 3.6 and Definition 3.7). Moreover, we study some properties of this determinant, and we provide an example (Example 3.19). In Section 4, we expose some ideas for further study in the probability theory.

2. Theory of the (m,σ)-general functions

Let $I\neq\emptyset$ be a set and let $k\in\mathbf{N}^*$; indicate by τ, by $\tau^{(k)}$, by $\tau^{(I)}$, by $\mathcal{B}$, by $\mathcal{B}^{(k)}$, by $\mathcal{B}^{(I)}$, and by *Leb*, respectively, the euclidean topology on $\mathbf{R}$, the euclidean topology on $\mathbf{R}^k$, the topology $\bigotimes_{i\in I}\tau$, the Borel σ-algebra on $\mathbf{R}$, the Borel σ-algebra on $\mathbf{R}^k$, the σ-algebra $\bigotimes_{i\in I}\mathcal{B}$, and the Lebesgue measure on $\mathbf{R}$. Moreover, for any set $A\subset\mathbf{R}$, indicate by $\mathcal{B}(A)$ the σ-algebra induced by $\mathcal{B}$ on A, and by $\tau(A)$ the topology induced by τ on A; analogously, for any set $A\subset\mathbf{R}^I$, define the σ-algebra $\mathcal{B}^{(I)}(A)$ and the topology $\tau^{(I)}(A)$. Finally, if $S=\prod_{i\in I}S_i$ is a Cartesian product, for any $(x_i : i\in I)\in S$ and for any $\emptyset\neq H\subset I$, define

$x_H = (x_i : i \in H) \in \prod_{i\in H} S_i$, and define the projection $\pi_{I,H}$ on $\prod_{i\in H} S_i$ as the function $\pi_{I,H} : S \longrightarrow \prod_{i\in H} S_i$ given by $\pi_{I,H}(x_I) = x_H$.

Henceforth, we will suppose that I, J are sets such that $\emptyset \neq I, J \subset \mathbf{N}^*$; moreover, for any $k \in \mathbf{N}^*$, we will indicate by I_k the set of the first k elements of I (with the natural order and with the convention $I_k = I$ if $|I| < k$); furthermore, for any $i \in I$, set $|i| = |I \cap (0, i]|$. Analogously, define J_k and $|j|$, for any $k \in \mathbf{N}^*$ and for any $j \in J$.

Definition 2.1. *For any set $I \neq \emptyset$, define the function $\|\cdot\|_I : \mathbf{R}^I \longrightarrow [0, +\infty]$ by*

$$\|x\|_I = \sup_{i\in I}|x_i|,\ \forall\, x = (x_i : i \in I) \in \mathbf{R}^I,$$

and define the vector space

$$E_I = \{x \in \mathbf{R}^I : \|x\|_I < +\infty\}.$$

Moreover, indicate by $\mathcal{B}_I$ the σ-algebra $\mathcal{B}^{(I)}(E_I)$, by τ_I the topology $\tau^{(I)}(E_I)$, and by $\tau_{\|\cdot\|_I}$ the topology induced on E_I by the the distance $d : E_I \times E_I \longrightarrow [0, +\infty)$ defined by $d(x, y) = \|x - y\|_I$, $\forall\, x, y \in E_I$; furthermore, for any set $A \subset E_I$, indicate by $\tau_{\|\cdot\|_I}(A)$ the topology induced by $\tau_{\|\cdot\|_I}$ on A. Finally, for any $x_0 \in E_I$ and for any $\delta > 0$, indicate by $B(x_0, \delta)$ the set $\{x \in E_I : \|x - x_0\|_I < \delta\}$.

Remark 2.2: For any $A \subset E_I$, one has $\tau^{(I)}(A) \subset \tau_{\|\cdot\|_I}(A)$; moreover, E_I is a Banach space, with the norm $\|\cdot\|_I$.

Proof. The proof that $\tau^{(I)}(A) \subset \tau_{\|\cdot\|_I}(A)$, $\forall\, A \subset E_I$, follows from the definitions of $\tau^{(I)}$ and $\tau_{\|\cdot\|_I}$; moreover, the proof that E_I is a Banach space can be found, for example, in [3] (Remark 2). □

The following concept generalizes the definition 6 in [3] (see also the theory in the Lang's book [7] and that in the Weidmann's book [11]).

Definition 2.3. *Let $A = (a_{ij})_{i\in I, j\in J}$ be a real matrix $I \times J$ (eventually infinite); then, define the linear function $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow \mathbf{R}^I$, and write $x \longrightarrow Ax$, in the following manner:*

$$(Ax)_i = \sum_{j\in J} a_{ij} x_j,\ \forall\, x \in E_J,\ \forall\, i \in I, \tag{1}$$

on condition that, for any $i \in I$, the sum in (1) converges to a real number. In *particular, if* $|I| = |J|$, *indicate by* $\mathbf{I}_{I,J} = \left(\overline{\delta}_{ij}\right)_{i\in I, j\in J}$ *the real matrix defined by*

$$\overline{\delta}_{ij} = \begin{cases} 1 & \text{if } |i| = |j| \\ 0 & \text{otherwise} \end{cases},$$

and call $\overline{\delta}_{ij}$ *generalized Kronecker symbol. Moreover, indicate by* $A^{(L,N)}$ *the real matrix* $(a_{ij})_{i\in L, j\in N}$, *for any* $L \subset I$, *for any* $N \subset J$, *and indicate by* ${}^tA = (b_{ji})_{j\in J, i\in I} : E_I \longrightarrow \mathbf{R}^J$ *the linear function defined by* $b_{ji} = a_{ij}$, *for any* $j \in J$ *and for any* $i \in I$. *Furthermore, if* $I = J$ *and* $A = {}^tA$, *we say that* A *is a symmetric function. Finally, if* $B = (b_{jk})_{j\in J, k\in K}$ *is a real matrix* $J \times K$, *define the* $I \times K$ *real matrix* $AB = ((AB)_{ik})_{i\in I, k\in K}$ *by*

$$(AB)_{ik} = \sum_{j\in J} a_{ij} b_{jk}, \tag{2}$$

on condition that, for any $i \in I$ *and for any* $k \in K$, *the sum in (2) converges to a real number.*

Proposition 2.4. *Let* $A = (a_{ij})_{i\in I, j\in J}$ *be a real matrix* $I \times J$; *then:*

1. *The linear function* $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow \mathbf{R}^I$ *given by (1) is defined if and only if, for any* $i \in I$, $\sum_{j\in J} |a_{ij}| < +\infty$.
2. *One has* $A(E_J) \subset E_I$ *if and only if* A *is continuous and if and only if* $\sup_{i\in I} \sum_{j\in J} |a_{ij}| < +\infty$; *moreover,* $\|A\| = \sup_{i\in I} \sum_{j\in J} |a_{ij}|$.
3. *If* $B = (b_{jk})_{j\in J, k\in K} : E_K \longrightarrow E_J$ *is a linear function, then the linear function* $A \circ B : E_K \longrightarrow \mathbf{R}^I$ *is defined by the real matrix* AB.

Proof. The proofs of points 1 and 2 are analogous to the proof of Proposition 7 in [3]. Moreover, the proof of point 3 is analogous to that one true in the particular case $|I|, |J|, |K| < +\infty$ (see, e.g., the Lang's book [7]). □

The following definitions and results (from Definition 2.5 to Proposition 2.19) can be found in [4] and generalize the differentiation theory in the finite case (see, e.g., the Lang's book [6]).

Definition 2.5. *Let* $U \in \tau_{\|\cdot\|_J}$; *a function* $\varphi : U \subset E_J \longrightarrow E_I$ *is called differentiable in* $x_0 \in U$ *if there exists a linear and continuous function* $A : E_J \longrightarrow E_I$ *defined by a real matrix* $A = (a_{ij})_{i\in I, j\in J}$, *and one has*

$$\lim_{h\to 0} \frac{\|\varphi(x_0 + h) - \varphi(x_0) - Ah\|_I}{\|h\|_J} = 0. \tag{3}$$

If φ is differentiable in x_0 for any $x_0 \in U$, φ is called differentiable in U. The function A is called differential of the function φ in x_0, and it is indicated by the symbol $d\varphi(x_0)$.

REMARK 2.6: Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi, \psi : U \subset E_J \longrightarrow E_I$ be differentiable functions in $x_0 \in U$; then, for any $\alpha, \beta \in \mathbf{R}$, the function $\alpha\varphi + \beta\psi$ is differentiable in x_0, and $d(\alpha\varphi + \beta\psi)(x_0) = \alpha d\varphi(x_0) + \beta d\psi(x_0)$.

REMARK 2.7: A linear and continuous function $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$, defined by

$$(Ax)_i = \sum_{j\in J} a_{ij} x_j,\ \forall\, x \in E_J,\ \forall\, i \in I,$$

is differentiable and $d\varphi(x_0) = A$, for any $x_0 \in E_J$.

REMARK 2.8: Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function differentiable in $x_0 \in U$; then, for any $i \in I$, the component $\varphi_i : U \longrightarrow \mathbf{R}$ is differentiable in x_0, and $d\varphi_i(x_0)$ is the matrix A_i given by the i-th row of $A = d\varphi(x_0)$. Moreover, if $|I| < +\infty$ and $\varphi_i : U \subset E_J \longrightarrow \mathbf{R}$ is differentiable in x_0, for any $i \in I$, then $\varphi : U \subset E_J \longrightarrow E_I$ is differentiable in x_0.

REMARK 2.9: Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function differentiable in $x_0 \in U$; then, φ is continuous in x_0.

DEFINITION 2.10. *Let $U \in \tau_{\|\cdot\|_J}$, let $v \in E_J$ such that $\|v\|_J = 1$ and let a function $\varphi : U \subset E_J \longrightarrow \mathbf{R}^I$; for any $i \in I$, the function φ_i is called differentiable in $x_0 \in U$ in the direction v if there exists the limit*

$$\lim_{t\to 0} \frac{\varphi_i(x_0 + tv) - \varphi_i(x_0)}{t}.$$

This limit is indicated by $\frac{\partial \varphi_i}{\partial v}(x_0)$, and it is called derivative of φ_i in x_0 in the direction v. If, for some $j \in J$, one has $v = e_j$, where $(e_j)_k = \delta_{jk}$, for any $k \in J$, indicate $\frac{\partial \varphi_i}{\partial v}(x_0)$ by $\frac{\partial \varphi_i}{\partial x_j}(x_0)$, and call it partial derivative of φ_i in x_0, with respect to x_j. Moreover, if there exists the linear function defined by the matrix $J_\varphi(x_0) = \Big((J_\varphi(x_0))_{ij}\Big)_{i\in I, j\in J} : E_J \longrightarrow \mathbf{R}^I$, where $(J_\varphi(x_0))_{ij} = \frac{\partial \varphi_i}{\partial x_j}(x_0)$, for any $i \in I$, $j \in J$, then $J_\varphi(x_0)$ is called Jacobian matrix of the function φ in x_0.

REMARK 2.11: Let $U \in \tau_{\|\cdot\|_J}$ and suppose that a function $\varphi : U \subset E_J \longrightarrow E_I$ is differentiable in $x_0 \in U$; then, for any $v \in E_J$ such that $\|v\|_J = 1$ and for any $i \in I$, the function $\varphi_i : U \subset E_J \longrightarrow \mathbf{R}$ is differentiable in x_0 in the direction v, and one has

$$\frac{\partial \varphi_i}{\partial v}(x_0) = d\varphi_i(x_0)v.$$

Corollary 2.12. *Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function differentiable in $x_0 \in U$; then, there exists the function $J_\varphi(x_0) : E_J \longrightarrow \mathbf{R}^I$, and it is continuous; moreover, for any $h \in E_J$, one has $d\varphi(x_0)(h) = J_\varphi(x_0)h$.*

Theorem 2.13. *Let $U \in \tau_{\|\cdot\|_J}$, let $\varphi : U \subset E_J \longrightarrow E_I$ be a function differentiable in $x_0 \in U$, let $V \in \tau_{\|\cdot\|_I}$ such that $V \supset \varphi(U)$, and let $\psi : V \subset E_I \longrightarrow E_H$ a function differentiable in $y_0 = \varphi(x_0)$. Then, the function $\psi \circ \varphi$ is differentiable in x_0, and one has $d(\psi \circ \varphi)(x_0) = d\psi(y_0) \circ d\varphi(x_0)$.*

Definition 2.14. *Let $U \in \tau_{\|\cdot\|_J}$, let $i, j \in J$ and let $\varphi : U \subset E_J \longrightarrow \mathbf{R}$ be a function differentiable in $x_0 \in U$ with respect to x_i, such that the function $\frac{\partial \varphi}{\partial x_i}$ is differentiable in x_0 with respect to x_j. Indicate $\frac{\partial}{\partial x_j}\left(\frac{\partial \varphi}{\partial x_i}\right)(x_0)$ by $\frac{\partial^2 \varphi}{\partial x_j \partial x_i}(x_0)$ and call it second partial derivative of φ in x_0 with respect to x_i and x_j. If $i = j$, it is indicated by $\frac{\partial^2 \varphi}{\partial x_i^2}(x_0)$. Analogously, for any $k \in \mathbf{N}^*$ and for any $j_1, ..., j_k \in J$, define $\frac{\partial^k \varphi}{\partial x_{j_k} ... \partial x_{j_1}}(x_0)$ and call it k-th partial derivative of φ in x_0 with respect to $x_{j_1}, ... x_{j_k}$.*

Definition 2.15. *Let $U \in \tau_{\|\cdot\|_J}$ and let $k \in \mathbf{N}^*$; a function $\varphi : U \subset E_J \longrightarrow E_I$ is called C^k in $x_0 \in U$ if, in a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0, for any $i \in I$ and for any $j_1, ..., j_k \in J$, there exists the function defined by $x \longrightarrow \frac{\partial^k \varphi_i}{\partial x_{j_k} ... \partial x_{j_1}}(x)$, and this function is continuous in x_0; φ is called C^k in U if, for any $x_0 \in U$, φ is C^k in x_0. Moreover, φ is called strongly C^1 in $x_0 \in U$ if, in a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0, there exists the function defined by $x \longrightarrow J_\varphi(x)$, this function is continuous in x_0, and one has $\|J_\varphi(x_0)\| < +\infty$. Finally, φ is called strongly C^1 in U if, for any $x_0 \in U$, φ is strongly C^1 in x_0.*

Definition 2.16. *Let $U \in \tau_{\|\cdot\|_J}$ and let $V \in \tau_{\|\cdot\|_I}$; a function $\varphi : U \subset E_J \longrightarrow V \subset E_I$ is called diffeomorphism if φ is bijective and C^1 in U, and the function $\varphi^{-1} : V \subset E_I \longrightarrow U \subset E_J$ is C^1 in V.*

Remark 2.17: Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function C^1 in $x_0 \in U$, where $|I| < +\infty$, $|J| < +\infty$, then φ is strongly C^1 in x_0.

Theorem 2.18. *Let $U \in \tau_{\|\cdot\|_J}$, let $\varphi : U \subset E_J \longrightarrow \mathbf{R}$ be a function C^k in $x_0 \in U$, let $i_1, ..., i_k \in J$, and let $j_1, ..., j_k \in J$ be a permutation of $i_1, ..., i_k$. Then, one has*

$$\frac{\partial^k \varphi}{\partial x_{i_1} ... \partial x_{i_k}}(x_0) = \frac{\partial^k \varphi}{\partial x_{j_1} ... \partial x_{j_k}}(x_0).$$

Proposition 2.19. *Let* $U = \left(\prod_{j\in J} A_j\right) \cap E_J \in \tau_{\|\cdot\|_J}$*, where* $A_j \in \tau$*, for any* $j \in J$*, and let* $\varphi : U \subset E_J \longrightarrow E_I$ *be a function* C^1 *in* $x_0 \in U$*, such that*

$$\varphi_i(x) = \sum_{j\in J} \varphi_{ij}(x_j),\ \forall\, x = (x_j : j \in J) \in U,\ \forall\, i \in I, \tag{4}$$

where $\varphi_{ij} : A_j \longrightarrow \mathbf{R}$*, for any* $i \in I$ *and for any* $j \in J$*; moreover, suppose that, in a neighbourhood* $V \in \tau_{\|\cdot\|_J}(U)$ *of* x_0*, there exists the function defined by* $x \longrightarrow J_\varphi(x)$ *and one has* $\sup_{x\in V} \|J_\varphi(x)\| < +\infty$*. Then,* φ *is continuous in* x_0*; in particular, if* φ *is strongly* C^1 *in* x_0 *and* $|I| < +\infty$*,* φ *is differentiable in* x_0*.*

Definition 2.20. *Let* $m \in \mathbf{N}^*$ *and let* $U = \left(U^{(m)} \times \prod_{j\in J\setminus J_m} A_j\right) \cap E_J \in \tau_{\|\cdot\|_J}$*, where* $U^{(m)} \in \tau^{(m)}$*,* $A_j \in \tau$*, for any* $j \in J\setminus J_m$*. A function* $\varphi : U \subset E_J \longrightarrow E_I$ *is called* m*-general if, for any* $i \in I$ *and for any* $j \in J\setminus J_m$*, there exist some functions* $\varphi_i^{(I,m)} : U^{(m)} \longrightarrow \mathbf{R}$ *and* $\varphi_{ij} : A_j \longrightarrow \mathbf{R}$ *such that*

$$\varphi_i(x) = \varphi_i^{(I,m)}(x_{J_m}) + \sum_{j\in J\setminus J_m} \varphi_{ij}(x_j),\ \forall\, x \in U.$$

Moreover, for any $\emptyset \neq L \subset I$ *and for any* $J_m \subset N \subset J$*, indicate by* $\varphi^{(L,N)}$ *the function* $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ *defined by*

$$\varphi_i^{(L,N)}(x_N) = \varphi_i^{(I,m)}(x_{J_m}) + \sum_{j\in N\setminus J_m} \varphi_{ij}(x_j),\ \forall\, x_N \in \pi_{J,N}(U),\ \forall\, i \in L. \tag{5}$$

Furthermore, for any $\emptyset \neq L \subset I$ *and for any* $\emptyset \neq N \subset J\setminus J_m$*, indicate by* $\varphi^{(L,N)}$ *the function* $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ *given by*

$$\varphi_i^{(L,N)}(x_N) = \sum_{j\in N} \varphi_{ij}(x_j),\ \forall\, x_N \in \pi_{J,N}(U),\ \forall\, i \in L. \tag{6}$$

In particular, suppose that $m = 1$*; then, let* $j \in J$ *such that* $\{j\} = J_1$ *and indicate* $U^{(1)}$ *by* A_j *and* $\varphi_i^{(I,1)}$ *by* φ_{ij}*, for any* $i \in I$*; moreover, for any* $\emptyset \neq L \subset I$ *and for any* $\emptyset \neq N \subset J$*, indicate by* $\varphi^{(L,N)}$ *the function* $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ *defined by formula (6).*

Furthermore, for any $l, n \in \mathbf{N}^*$*, indicate* $\varphi^{(I_l,N)}$ *by* $\varphi^{(l,N)}$*,* $\varphi^{(L,J_n)}$ *by* $\varphi^{(L,n)}$*, and* $\varphi^{(I_l,J_n)}$ *by* $\varphi^{(l,n)}$*.*

Definition 2.21. *Let* $m \in \mathbf{N}^*$, *let* $U = \left(U^{(m)} \times \prod_{j \in J \backslash J_m} A_j \right) \cap E_J \in \tau_{\|\cdot\|_J}$, *where* $U^{(m)} \in \tau^{(m)}$, $A_j \in \tau$, *for any* $j \in J \backslash J_m$, *and let* $\sigma : I \backslash I_m \longrightarrow J \backslash J_m$ *be an increasing function; a function* $\varphi : U \subset E_J \longrightarrow E_I$ m*-general and such that* $|J| = |I|$ *is called* (m, σ)*-general if:*

1. $\forall\, i \in I \backslash I_m$, $\forall\, j \in J \backslash (J_m \cup \{\sigma(i)\})$, $\forall\, t \in A_j$, *one has* $\varphi_{ij}(t) = 0$*; moreover*
$$\varphi^{(I \backslash I_m, J \backslash J_m)} \left(\pi_{J, J \backslash J_m}(U) \right) \subset E_{I \backslash I_m}.$$
2. $\forall\, i \in I \backslash I_m$, $\forall\, x \in U$, *there exists* $J_{\varphi_i}(x) : E_J \longrightarrow \mathbf{R}$*; moreover,* $\forall\, x_{J_m} \in U^{(m)}$, *one has* $\sum_{i \in I \backslash I_m} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| < +\infty$.
3. $\forall\, i \in I \backslash I_m$, *the function* $\varphi_{i, \sigma(i)} : A_{\sigma(i)} \longrightarrow \mathbf{R}$ *is constant or injective; moreover,* $\forall\, x_{\sigma(I \backslash I_m)} \in \prod_{j \in \sigma(I \backslash I_m)} A_j$, *one has* $\sup_{i \in I \backslash I_m} \left| \varphi'_{i, \sigma(i)} \left(x_{\sigma(i)} \right) \right| < +\infty$ *and* $\inf_{i \in \mathcal{I}_\varphi} \left| \varphi'_{i, \sigma(i)} \left(x_{\sigma(i)} \right) \right| > 0$, *where* $\mathcal{I}_\varphi = \{ i \in I \backslash I_m : \varphi_{i, \sigma(i)} \text{ is injective} \}$.
4. *If, for some* $h \in \mathbf{N}$, $h \geq m$, *one has* $|\sigma(i)| = |i|$, $\forall\, i \in I \backslash I_h$, *then,* $\forall\, x_{\sigma(I \backslash I_m)} \in \prod_{j \in \sigma(I \backslash I_m)} A_j$, *there exists* $\prod_{i \in \mathcal{I}_\varphi} \varphi'_{i, \sigma(i)} \left(x_{\sigma(i)} \right) \in \mathbf{R}^*$.

Moreover, set
$$\mathcal{A} = \mathcal{A}(\varphi) = \{ h \in \mathbf{N},\, h \geq m : |\sigma(i)| = |i|,\, \forall\, i \in I \backslash I_h \}.$$

If the sequence $\left\{ J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\}_{i \in I \backslash I_m}$ *converges uniformly on* $U^{(m)}$ *to the matrix* $(0 \ldots 0)$ *and there exists* $a \in \mathbf{R}$ *such that, for any* $\varepsilon > 0$, *there exists* $i_0 \in \mathbf{N}$, $i_0 \geq m$, *such that, for any* $i \in \mathcal{I}_\varphi \cap (I \backslash I_{i_0})$ *and for any* $t \in A_{\sigma(i)}$, *one has* $\left| \varphi'_{i, \sigma(i)}(t) - a \right| < \varepsilon$, *then* φ *is called strongly* (m, σ)*-general.*

Furthermore, for any $I_m \subset L \subset I$ *and for any* $J_m \subset N \subset J$, *define the function* $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow \mathbf{R}^I$ *in the following manner:*
$$\overline{\varphi}_i^{(L,N)}(x) = \begin{cases} \varphi_i^{(L,N)}(x_N) & \forall\, i \in I_m,\, \forall\, x \in U \\ \varphi_i(x) & \forall\, i \in L \backslash I_m,\, \forall\, x \in U \\ \varphi_{i, \sigma(i)}(x_{\sigma(i)}) & \forall\, i \in I \backslash L,\, \forall\, x \in U \end{cases}.$$

Finally, for any $l, n \in \mathbf{N}$, $l, n \geq m$, *indicate* $\overline{\varphi}^{(I_l, N)}$ *by* $\overline{\varphi}^{(l,N)}$, $\overline{\varphi}^{(L, J_n)}$ *by* $\overline{\varphi}^{(L,n)}$, $\overline{\varphi}^{(I_l, J_n)}$ *by* $\overline{\varphi}^{(l,n)}$, *and* $\overline{\varphi}^{(m,m)}$ *by* $\overline{\varphi}$.

Definition 2.22. *A function $\varphi : U \subset E_J \longrightarrow E_I$ (m,σ)-general is called (m,σ)-standard (or (m,σ) of the first type) if, for any $i \in I \backslash I_m$ and for any $x_{J_m} \in U^{(m)}$, one has $\varphi_i^{(I,m)}(x_{J_m}) = 0$. Moreover, a function $\varphi : U \subset E_J \longrightarrow E_I$ (m,σ)-standard and strongly (m,σ)-general is called strongly (m,σ)-standard (see also Definition 28 in [4]).*

Remark 2.23: Let $\varphi : U \subset E_J \longrightarrow E_I$ be a m-general function; then:

1. Let $\emptyset \neq L \subset I$ and let $J_m \subset N \subset J$ such that $\varphi^{(L,N)}(\pi_{J,N}(U)) \subset E_L$; then, for any $n \in \mathbf{N}$, $n \geq m$, the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow E_L$ is n-general.
2. Let $\emptyset \neq L \subset I$ and let $\emptyset \neq N \subset J \backslash J_m$ such that $\varphi^{(L,N)}(\pi_{J,N}(U)) \subset E_L$; then, for any $n \in \mathbf{N}^*$, the function $\varphi^{(L,N)}(\pi_{J,N}(U)) \longrightarrow E_L$ is n-general.
3. If $m = 1$, let $\emptyset \neq L \subset I$ and let $\emptyset \neq N \subset J$ such that $\varphi^{(L,N)}(\pi_{J,N}(U)) \subset E_L$; then, for any $n \in \mathbf{N}^*$, the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow E_L$ is n-general.

Proof. The proof follows from the definition of $\varphi^{(L,N)}$. □

Proposition 2.24. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function; then:*

1. *σ is bijective if and only if $|\sigma(i)| = |i|$, $\forall\, i \in I \backslash I_m$.*
2. *$\prod_{j \in J \backslash J_m} A_j \subset E_{J \backslash J_m}$ if and only if there exist $a \in \mathbf{R}^+$ and $m_0 \in \mathbf{N}$, $m_0 \geq m$, such that, for any $j \in J \backslash J_{m_0}$, one has $A_j \subset (-a, a)$.*
3. *Let $I_m \subset L \subset I$ and let $J_m \subset N \subset J$; then, one has $\varphi^{(L,N)}(\pi_{J,N}(U)) \subset E_L$ and $\overline{\varphi}^{(L,N)}(U) \subset E_I$; moreover, the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow E_I$ is (m,σ)-general.*
4. *For any $x \in U$, there exists the function $J_{\varphi^{(I \backslash I_m, J)}}(x) : E_J \longrightarrow E_{I \backslash I_m}$, and it is continuous.*
5. *If, for any $j \in J \backslash J_m$ and for any $t \in A_j$, one has $\sum_{i \in I \backslash I_m} \left|\varphi'_{i,j}(t)\right| < +\infty$, then, for any $n \in \mathbf{N}$, $n \geq m$, φ is (n,ξ)-general, where the increasing function $\xi : I \backslash I_n \longrightarrow J \backslash J_n$ is defined by:*

$$\xi(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \in J \backslash J_n \\ \min(J \backslash J_n) & \text{if } \sigma(i) \notin J \backslash J_n \end{cases}, \forall\, i \in I \backslash I_n. \tag{7}$$

6. *Suppose that* σ *is injective; moreover, for any* $I_m \subset L \subset I$ *such that* $|L| < +\infty$ *and for any* $J_m \subset N \subset J$*, let* $\widehat{m} = |\max L| \in \mathbf{N} \backslash \{0, ...m-1\}$*; then, for any* $n \in \mathbf{N}$*,* $n \geq \widehat{m}$*, the function* $\overline{\varphi}^{(L,N)}$ *is* $\left(n, \sigma|_{I \backslash I_n}\right)$*-standard.*

Proof.

1. The proof follows from the fact that σ is increasing.
2. The proof follows from the definition of $E_{J \backslash J_m}$.
3. $\forall\, x \in \pi_{J,N}(U)$, let $y \in U$ such that $y_N = x$; then, $\forall\, i \in L \backslash I_m$, we have $\varphi_i^{(L,m)}(x_{J_m}) = \varphi_i(y) - \varphi_{i,\sigma(i)}\left(y_{\sigma(i)}\right)$, and so
$$\sup_{i \in L \backslash I_m} \left|\varphi_i^{(L,m)}(x_{J_m})\right| \leq \sup_{i \in L \backslash I_m} |\varphi_i(y)| + \sup_{i \in L \backslash I_m} \left|\varphi_{i,\sigma(i)}\left(y_{\sigma(i)}\right)\right| < +\infty;$$
then, we obtain
$$\sup_{i \in L \backslash I_m} \left|\varphi_i^{(L,N)}(x)\right| \leq \sup_{i \in L \backslash I_m} \left|\varphi_i^{(L,m)}(x_{J_m})\right| + \sup_{i \in L \backslash I_m} \left|\varphi_{i,\sigma(i)}\left(y_{\sigma(i)}\right)\right| < +\infty,$$
from which $\varphi^{(L,N)}\left(\pi_{J,N}(U)\right) \subset E_L$. Moreover, $\forall\, z \in U$, $\forall\, i \in I \backslash I_m$, we have
$$\left|\overline{\varphi}_i^{(L,N)}(z)\right| \leq \left|\varphi_i^{(I,m)}(z_{J_m})\right| + \left|\varphi_{i,\sigma(i)}\left(z_{\sigma(i)}\right)\right|,$$
and so $\sup_{i \in I \backslash I_m} \left|\overline{\varphi}_i^{(L,N)}(z)\right| < +\infty$; then, $\overline{\varphi}^{(L,N)}(U) \subset E_I$. Finally, from the definition of $\overline{\varphi}^{(L,N)}$, the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow E_I$ is (m, σ)-general.
4. $\forall\, x \in U$, $\forall\, i \in I \backslash I_m$, we have
$$\|J_{\varphi_i}(x)\| = \left\|J_{\varphi_i^{(I,m)}}(x_{J_m})\right\| + \left|\varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right|;$$
furthermore, since $\sum_{i \in I \backslash I_m} \left\|J_{\varphi_i^{(I,m)}}(x_{J_m})\right\| < +\infty$, we have
$$\sup_{i \in I \backslash I_m} \left\|J_{\varphi_i^{(I,m)}}(x_{J_m})\right\| < +\infty,$$
and so
$$\sup_{i \in I \backslash I_m} \|J_{\varphi_i}(x)\| \leq \sup_{i \in I \backslash I_m} \left\|J_{\varphi_i^{(I,m)}}(x_{J_m})\right\| + \sup_{i \in I \backslash I_m} \left|\varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right| < +\infty;$$
then, from Proposition 2.4, there exists the function $J_{\varphi^{(I \backslash I_m, J)}}(x) : E_J \longrightarrow E_{I \backslash I_m}$, and it is continuous.

5. $\forall\, n \in \mathbf{N}$, $n \geq m$, and $\forall\, x_{J_n} \in \pi_{J,J_n}(U)$, we have

$$\sum_{i \in I\backslash I_n} \left\| J_{\varphi_i^{(I,n)}}(x_{J_n}) \right\|$$
$$= \sum_{i \in I\backslash I_n} \left\| J_{\varphi_i^{(I,m)}}(x_{J_m}) \right\| + \sum_{j \in J_n \backslash J_m} \left(\sum_{i \in I\backslash I_n} \left| \varphi'_{i,j}(x_j) \right| \right) < +\infty;$$

then, by Definition 2.21 and by definition of ξ, φ is (n,ξ)-general.

6. From points 3 and 5 and since σ is injective, $\forall\, n \in \mathbf{N}$, $n \geq \widehat{m}$, $\overline{\varphi}^{(L,N)}$ is $\left(n, \sigma|_{I\backslash I_n}\right)$-general; moreover, since σ is increasing, $\forall\, i \in I\backslash I_n$ and $\forall\, x_{J_n} \in \pi_{J,J_n}(U)$, we have $\varphi_i^{(I,n)}(x_{J_n}) = 0$; then, we have the statement.

□

Remark 2.25: Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function such that $U^{(m)} = \prod_{j \in J_m} A_j$, where $A_j \in \tau$, for any $j \in J_m$, and

$$\varphi_i^{(I,m)}(x_{J_m}) = \sum_{j \in J_m} \varphi_{ij}(x_j),\ \forall\, x_{J_m} \in U^{(m)},\ \forall\, i \in I,$$

where $\varphi_{ij} : A_j \longrightarrow \mathbf{R}$, for any $i \in I$ and for any $j \in J_m$; moreover, suppose that, for any $j \in J_m$, for any $t \in A_j$, one has $\sup_{i \in I\backslash I_m} |\varphi_{i,j}(t)| < +\infty$, and, for any $j \in J\backslash J_m$, for any $t \in A_j$, one has $\sum_{i \in I\backslash I_m} \left|\varphi'_{i,j}(t)\right| < +\infty$; furthermore, let $\emptyset \neq L \subset I$ and let $\emptyset \neq N \subset J$ such that $|I\backslash L| = |J\backslash N| < +\infty$. Then, for any $n \in \mathbf{N}$, $n \geq m$, the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow \mathbf{R}^L$ is (n,ρ)-general, where the function $\rho : L\backslash L_n \longrightarrow N\backslash N_n$ is defined by

$$\rho(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \in N\backslash N_n \\ \min\{j > \sigma(i) : j \in N\backslash N_n\} & \text{if } \sigma(i) \notin N\backslash N_n \end{cases},\ \forall\, i \in L\backslash L_n.$$

Proof. We have $|L| = |N|$; moreover, $\forall\, n \in \mathbf{N}$, $n \geq m$, $\forall\, i \in L\backslash L_n$ and $\forall\, x \in \pi_{J,N}(U)$, let $y \in U$ such that $y_N = x$; we have

$$|\varphi_i(x)| \leq \sum_{j \in N \cap J_m} |\varphi_{i,j}(x_j)| + \left|\varphi_{i,\sigma(i)}\left(y_{\sigma(i)}\right)\right|$$
$$\Rightarrow \|\varphi(x)\|_{L\backslash L_n} \leq \sum_{j \in N \cap J_m} \sup_{i \in L\backslash I_n} |\varphi_{i,j}(x_j)| + \sup_{i \in L\backslash I_n} \left|\varphi_{i,\sigma(i)}\left(y_{\sigma(i)}\right)\right| < +\infty,$$

from which $\varphi\left(\pi_{J,N}(U)\right) \subset E_L$. Analogously, $\forall\, n \in \mathbf{N}$, $n \geq m$, and $\forall\, x_{N_n} \in \pi_{J,N_n}(U)$, we have

$$\sum_{i\in L\backslash L_n} \left\| J_{\varphi_i^{(L,N_n)}}(x_{N_n}) \right\|$$
$$= \sum_{i\in L\backslash L_n} \left\| J_{\varphi_i^{(L,N_n\cap J_m)}}(x_{N_m\cap J_m}) \right\| + \sum_{j\in N_n\backslash J_m} \left(\sum_{i\in L\backslash L_n} \left|\varphi'_{i,j}\left(x_j\right)\right| \right) < +\infty;$$

then, by definition of ρ, $\varphi^{(L,N)}$ is (n,ρ)-general. □

Proposition 2.26. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function such that there exists $m_0 \in \mathbf{N}$, $m_0 \geq m$, such that, for any $j \in J\backslash J_{m_0}$, A_j is bounded; moreover, suppose that $\sigma\left(I\backslash I_m\right) \cap \left(J\backslash J_{m_0}\right) \neq \emptyset$ and, for any $i \in I\backslash I_m$, $\varphi_i^{(I,m)}$ is bounded; then, there exists $m_1 \in \mathbf{N}$, $m_1 \geq m$, such that, for any $i \in I\backslash I_{m_1}$, φ_i is bounded. In particular, if $|I| = +\infty$, φ is not surjective.*

Proof. Let $j_0 = \min\left(\sigma\left(I\backslash I_m\right) \cap \left(J\backslash J_{m_0}\right)\right)$, let $i_0 = \min\left(\sigma^{-1}\left(j_0\right)\right) \in I$, let $\widehat{m} = |i_0| - 1$ and let $\mathcal{H} = \left\{i \in I\backslash I_{\widehat{m}} : \varphi_{i,\sigma(i)} \text{ is not bounded}\right\}$; we have $|\mathcal{H}| < +\infty$; indeed, $\forall\, i \in \mathcal{H}$, the set $A_{\sigma(i)}$ is bounded, and so there exists $t_i \in A_{\sigma(i)}$ such that $\left|\varphi'_{i,\sigma(i)}(t_i)\right| > |i|$; then, $\forall\, x_{\sigma(I\backslash I_m)} \in \prod_{j\in\sigma(I\backslash I_m)} A_j$ such that $\left(x_{\sigma(i)} : i \in \mathcal{H}\right) = \left(t_i : i \in \mathcal{H}\right)$, by supposing by contradiction $|\mathcal{H}| = +\infty$, we would obtain

$$\sup_{i\in I\backslash I_m} \left|\varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right| \geq \sup_{i\in\mathcal{H}} \left|\varphi'_{i,\sigma(i)}\left(x_{\sigma(i)}\right)\right| = \sup_{i\in\mathcal{H}} \left|\varphi'_{i,\sigma(i)}\left(t_i\right)\right| = +\infty$$

(a contradiction). Then, there exists $m_1 \in \mathbf{N}$, $m_1 \geq m$, such that, $\forall\, i \in I\backslash I_{m_1}$, $\varphi_{i,\sigma(i)}$ is bounded, and so φ_i is bounded. In particular, $\forall\, i \in I\backslash I_{m_1}$, φ_i is not surjective; then, if $|I| = +\infty$, φ is not surjective. □

Proposition 2.27. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function such that $\varphi_{ij}(x_j) = 0$, for any $i \in I_m$, for any $j \in J\backslash J_m$ and for any $x_j \in A_j$; then:*

1. *If the functions $\varphi_{i,\sigma(i)}$, for any $i \in I\backslash I_m$, and $\varphi^{(m,m)}$ are injective, and σ is surjective, then φ is injective.*
2. *If the functions $\varphi_{i,\sigma(i)}$, for any $i \in I\backslash I_m$, and $\varphi^{(m,m)}$ are surjective, and σ is injective, then φ is surjective.*

Proof.

1. Let $x,\ y \in U$ be such that $\varphi(x) = \varphi(y)$; we have $\varphi^{(m,m)}(x_{J_m}) = (\varphi(x))_{I_m} = (\varphi(y))_{I_m} = \varphi^{(m,m)}(y_{J_m})$; then, if $\varphi^{(m,m)}$ is injective, we have $x_{J_m} = y_{J_m}$; moreover, $\forall\, i \in I \backslash I_m$:
$$\begin{aligned}&\varphi^{(\{i\},m)}(x_{J_m}) + \varphi_{i,\sigma(i)}(x_{\sigma(i)})\\ &\qquad = \varphi_i(x) = \varphi_i(y) = \varphi^{(\{i\},m)}(y_{J_m}) + \varphi_{i,\sigma(i)}(y_{\sigma(i)}),\end{aligned}$$
from which $\varphi_{i,\sigma(i)}(x_{\sigma(i)}) = \varphi_{i,\sigma(i)}(y_{\sigma(i)})$; then, if $\varphi_{i,\sigma(i)}$ is injective, we have $x_{\sigma(i)} = y_{\sigma(i)}$; finally, if σ is surjective, we obtain $x_{J\backslash J_m} = y_{J\backslash J_m}$, and so $x = y$; then, φ is injective.

2. Let $y \in E_I$; moreover, if the functions $\varphi_{i,\sigma(i)}$, for any $i \in I\backslash I_m$, and $\varphi^{(m,m)}$ are surjective, and σ is injective, define $x \in U^{(m)} \times \prod_{j \in J\backslash J_m} A_j$ in the following manner:
$$\begin{aligned}x_{J_m} &= \left(\varphi^{(m,m)}\right)^{-1}(y_{I_m}) \in U^{(m)},\\ x_j &= \varphi^{-1}_{\sigma^{-1}(j),j}(z_i) \in A_j,\ \forall\, j \in \sigma\left(I\backslash I_m\right),\\ x_j &= 0,\ \forall\, j \in J\backslash\sigma\left(I\backslash I_m\right),\end{aligned}$$
where
$$z_i = y_i - \varphi_i^{(I,m)}(x_{J_m}),\ \forall\, i \in I\backslash I_m. \tag{8}$$
Let $x_0 = (x_{0,j} : j \in J) \in U$; $\forall\, i \in I\backslash I_m$, we have
$$\begin{aligned}\left|x_{\sigma(i)}\right| &= \left|\varphi^{-1}_{i,\sigma(i)}(z_i) - x_{0,\sigma(i)} + x_{0,\sigma(i)}\right|\\ &\le \left|\varphi^{-1}_{i,\sigma(i)}(z_i) - \varphi^{-1}_{i,\sigma(i)}(\varphi_{i,\sigma(i)}(x_{0,\sigma(i)}))\right| + \left|x_{0,\sigma(i)}\right|;\end{aligned} \tag{9}$$
moreover, the function $\varphi^{-1}_{i,\sigma(i)} : \mathbf{R} \longrightarrow A_{\sigma(i)}$ is derivable, and
$$\left(\varphi^{-1}_{i,\sigma(i)}\right)'(t) = \frac{1}{\varphi'_{i,\sigma(i)}(\varphi^{-1}_{i,\sigma(i)}(t))} \in \mathbf{R}^*,\ \forall\, i \in I\backslash I_m,\ \forall\, t \in \mathbf{R}; \tag{10}$$
then, the Lagrange theorem implies that, for some
$$\xi_i \in \left(\min\{z_i, \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\}, \max\{z_i, \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}\}\right),$$
we have
$$\begin{aligned}&\left|\varphi^{-1}_{i,\sigma(i)}(z_i) - \varphi^{-1}_{i,\sigma(i)}(\varphi_{i,\sigma(i)}(x_{0,\sigma(i)}))\right|\\ &\qquad = \left|\left(\varphi^{-1}_{i,\sigma(i)}\right)'(\xi_i)\right| \left|z_i - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right|;\end{aligned}$$

thus, from (9) and (10), we obtain

$$\left|x_{\sigma(i)}\right| \leq \frac{\left|z_i - \varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right|}{\left|\varphi'_{i,\sigma(i)}(\varphi^{-1}_{i,\sigma(i)}(\xi_i))\right|} + \left|x_{0,\sigma(i)}\right|. \tag{11}$$

Furthermore, from point 3 of Proposition 2.24, we have $\varphi^{(I,m)}(U^{(m)}) \subset E_I$, and so, from (8), we have

$$\|z\|_{I\setminus I_m} \leq \|y\|_{I\setminus I_m} + \sup_{i\in I\setminus I_m} \left|\varphi_i^{(I,m)}(x_{J_m})\right| < +\infty, \tag{12}$$

and analogously

$$\begin{aligned} \varphi_{i,\sigma(i)}(x_{0,\sigma(i)}) &= \varphi_i(x_0) - \varphi_i^{(I,m)}\left((x_0)_{J_m}\right), \ \forall\, i \in I\setminus I_m \\ \Longrightarrow \sup_{i\in I\setminus I_m} &\left|\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right| \\ &\leq \|\varphi(x_0)\|_{I\setminus I_m} + \sup_{i\in I\setminus I_m}\left|\varphi_i^{(I,m)}\left((x_0)_{J_m}\right)\right| < +\infty. \end{aligned} \tag{13}$$

Moreover, we have $\inf_{i\in\mathcal{I}_\varphi}\left|\varphi'_{i,\sigma(i)}(\varphi^{-1}_{i,\sigma(i)}(\xi_i)\right| > 0$; furthermore, since, $\forall\, i \in I\setminus I_m$, $\varphi_{i,\sigma(i)}$ is surjective, then $\varphi_{i,\sigma(i)}$ is injective too, and so $\mathcal{I}_\varphi = I\setminus I_m$; then, there exists $c \in \mathbf{R}^+$ such that $\sup_{i\in I\setminus I_m}\left|\varphi'_{i,\sigma(i)}(\varphi^{-1}_{i,\sigma(i)}(\xi_i))\right|^{-1} \leq c$, and so formulas (11), (12) and (13) imply

$$\sup_{i\in I\setminus I_m}\left|x_{\sigma(i)}\right| \leq c\left(\|z\|_{I\setminus I_m} + \sup_{i\in I\setminus I_m}\left|\varphi_{i,\sigma(i)}(x_{0,\sigma(i)})\right|\right) + \|x_0\|_J < +\infty;$$

then, we have $x \in E_J$, from which $x \in U$. Finally, it is easy to prove that $\varphi(x) = y$, and so φ is surjective.

□

Proposition 2.28. *Let $m \in \mathbf{N}^*$, let $\emptyset \neq L \subset I$, let $J_m \subset N \subset J$ and let $\varphi : U \subset E_J \longrightarrow E_I$ be a function m-general and C^1 in $x_0 = (x_{0,j} : j \in J) \in U$; then:*

1. *If $\varphi^{(L,N)}(\pi_{J,N}(U)) \subset E_L$, then the function $\varphi^{(L,N)} : \pi_{J,N}(U) \longrightarrow E_L$ is C^1 in $(x_{0,j} : j \in N)$.*
2. *If φ is (m,σ)-general and $I_m \subset L$, then the function $\overline{\varphi}^{(L,N)} : U \subset E_J \longrightarrow E_I$ is C^1 in x_0.*

3. *If φ is (m,σ)-general, $I_m \subset L$ and $|N| < +\infty$, then there exists the function $J_{\overline{\varphi}^{(L,N)}}(x_0) : E_J \longrightarrow E_I$, and it is continuous.*

4. *If φ is strongly (m,σ)-general, $I_m \subset L$ and $|N| < +\infty$, then $\overline{\varphi}^{(L,N)}$ is differentiable in x_0.*

5. *If φ is strongly C^1 in x_0 and strongly (m,σ)-general, then φ is differentiable in x_0.*

Proof.

1. By assumption, there exists a neighbourhood $V = \prod_{j\in J} V_j \in \tau_{\|\cdot\|_J}(U)$ of x_0 such that, $\forall\, i \in I$, $\forall\, j \in J$, there exists the function $x \longrightarrow \frac{\partial \varphi_i(x)}{\partial x_j}$ on V, and this function is continuous in x_0; then, $\forall\, x \in \prod_{j\in N} V_j$, let $\overline{x} = (\overline{x}_j : j \in J) \in V$ such that $(\overline{x}_j : j \in N) = x$; since φ is a m-general function, $\forall\, i \in L$, $\forall\, j \in N$, we have
$$\frac{\partial \varphi_i^{(L,N)}(x)}{\partial x_j} = \frac{\partial \varphi_i(\overline{x})}{\partial x_j},$$
from which $\varphi^{(L,N)}$ is C^1 in $(x_{0,j} : j \in N)$.

2. Let $V \in \tau_{\|\cdot\|_J}(U)$ be the neighbourhood of x_0 defined in the proof of point 1; if φ is (m,σ)-general and $I_m \subset L$, $\forall\, x \in V$, we have
$$\frac{\partial \overline{\varphi}_i^{(L,N)}(x)}{\partial x_j} = \begin{cases} \frac{\partial \varphi_i(x)}{\partial x_j} & \text{if } (i,j) \notin (I_m \times (J\backslash N)) \cup ((I\backslash L) \times J_m) \\ 0 & \text{if } (i,j) \in (I_m \times (J\backslash N)) \cup ((I\backslash L) \times J_m) \end{cases},$$
and so $\overline{\varphi}^{(L,N)}$ is C^1 in x_0.

3. If φ is C^1 in x_0 and (m,σ)-general, $I_m \subset L$ and $|N| < +\infty$, then, from point 2, $\forall\, i \in I_m$, the function $\overline{\varphi}_i^{(L,N)} : U \subset E_J \longrightarrow \mathbf{R}$ is C^1 in x_0 and depends only on a finite number of variables; then, we have $\left\| J_{\overline{\varphi}_i^{(L,N)}}(x_0) \right\| < +\infty$; moreover, $\forall\, i \in I\backslash I_m$, we have
$$\left\| J_{\overline{\varphi}_i^{(L,N)}}(x_0) \right\| \le \| J_{\varphi_i}(x_0) \| \,;$$
then, from point 4 of Proposition 2.24:
$$\sup_{i\in I\backslash I_m} \left\| J_{\overline{\varphi}_i^{(L,N)}}(x_0) \right\| \le \sup_{i\in I\backslash I_m} \| J_{\varphi_i}(x_0) \| < +\infty;$$
then, from Proposition 2.4, there exists the function $J_{\overline{\varphi}^{(L,N)}}(x_0) : E_J \longrightarrow E_I$, and it is continuous.

4. If φ is strongly (m,σ)-general, there exists $a \in \mathbf{R}$ such that, $\forall \varepsilon > 0$, there exists $\widehat{i} \in \mathbf{N}$, $\widehat{i} \geq m$, such that

$$\left\| J_{\varphi_i^{(I,m)}}\left(x_{J_m}\right)\right\| < \frac{\varepsilon}{4}, \forall i \in I \backslash I_{\widehat{i}}, \forall x_{J_m} \in U^{(m)};$$
$$\left|\varphi'_{i,\sigma(i)}\left(t\right) - a\right| < \frac{\varepsilon}{4}, \forall i \in \mathcal{I}_{\varphi} \cap I \backslash I_{\widehat{i}}, \forall t \in A_{\sigma(i)}. \tag{14}$$

Moreover, if $I_m \subset L$ and $|N| < +\infty$, $\forall i \in I$, the function $\overline{\varphi}_i^{(L,N)} : U \subset E_J \longrightarrow \mathbf{R}$ is C^1 in x_0 and depends only on a finite number of variables; then, $\overline{\varphi}_i^{(L,N)}$ is differentiable in x_0, and so there exists a neighbourhood $D = \prod_{j \in J} D_j \in \tau_{\|\cdot\|_J}(U)$ of x_0, where D_j is an open interval, $\forall j \in J$, such that, $\forall x = (x_j : j \in J) \in D \backslash \{x_0\}$, we have

$$\sup_{i \in I_{\widehat{i}}} \frac{\left|\overline{\varphi}_i^{(L,N)}(x) - \overline{\varphi}_i^{(L,N)}(x_0) - J_{\overline{\varphi}_i^{(L,N)}}(x_0)(x - x_0)\right|}{\|x - x_0\|_J} < \varepsilon. \tag{15}$$

Observe that, $\forall i \in \left(I \backslash I_{\widehat{i}}\right) \backslash L$, $\forall y = (y_j : j \in J) \in U$, we have $\overline{\varphi}_i^{(L,N)}(y) = \varphi_{i,\sigma(i)}(y_{\sigma(i)})$; moreover, $\varphi_{i,\sigma(i)}$ is derivable in $A_{\sigma(i)}$ and so, from the Lagrange theorem, $\forall x \in D \backslash \{x_0\}$, there exists $\theta_i \in (\min\{x_{0,\sigma(i)}, x_{\sigma(i)}\}, \max\{x_{0,\sigma(i)}, x_{\sigma(i)}\})$ such that

$$\varphi_{i,\sigma(i)}\left(x_{\sigma(i)}\right) - \varphi_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) = \varphi'_{i,\sigma(i)}\left(\theta_i\right)\left(x_{\sigma(i)} - x_{0,\sigma(i)}\right),$$

from which

$$\begin{aligned}
&\frac{\left|\overline{\varphi}_i^{(L,N)}(x) - \overline{\varphi}_i^{(L,N)}(x_0) - J_{\overline{\varphi}_i^{(L,N)}}(x_0)(x - x_0)\right|}{\|x - x_0\|_J} \\
&= \frac{\left|\varphi_{i,\sigma(i)}\left(x_{\sigma(i)}\right) - \varphi_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) - \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right)\left(x_{\sigma(i)} - x_{0,\sigma(i)}\right)\right|}{\|x - x_0\|_J} \\
&= \frac{\left|\varphi'_{i,\sigma(i)}\left(\theta_i\right) - \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right)\right|\left|x_{\sigma(i)} - x_{0,\sigma(i)}\right|}{\|x - x_0\|_J} \\
&\leq \left(\left|\varphi'_{i,\sigma(i)}\left(\theta_i\right) - a\right| + \left|\varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) - a\right|\right) 1_{\mathcal{I}_{\varphi}}(i) < \frac{\varepsilon}{2}.
\end{aligned} \tag{16}$$

Conversely, $\forall i \in \left(I \backslash I_{\widehat{i}}\right) \cap L$, $\forall y = (y_j : j \in J) \in U$, we have $\overline{\varphi}_i^{(L,N)}(y) = \varphi_i(y)$; moreover, from point 3 of Proposition 2.24 and from point 1, $\varphi_i^{(I,m)}$ is C^1 in $(x_0)_{J_m}$ and so $\varphi_i^{(I,m)}$ is C^1 in a neighbourhood $M = \prod_{j \in J_m} M_j \in \tau_{\|\cdot\|_{J_m}}(U^{(m)})$ of $(x_0)_{J_m}$ such that M_j is an open interval,

$\forall j \in J_m$, and $M \subset \prod_{j\in J_m} D_j$; then, from the Taylor theorem, $\forall x \in \left(M \times \prod_{j\in J\setminus J_m} D_j\right)\setminus\{x_0\}$, there exists $\xi_{J_m} \in (M\setminus\{(x_0)_{J_m}\})$ such that

$$\varphi_i^{(I,m)}(x_{J_m}) - \varphi_i^{(I,m)}((x_0)_{J_m}) = J_{\varphi_i^{(I,m)}}(\xi_{J_m})(x_{J_m} - (x_0)_{J_m}),$$

and so

$$\begin{aligned}
&\frac{\left|\overline{\varphi}_i^{(L,N)}(x) - \overline{\varphi}_i^{(L,N)}(x_0) - J_{\overline{\varphi}_i^{(L,N)}}(x_0)(x-x_0)\right|}{\|x-x_0\|_J}\\
&= \frac{|\varphi_i(x) - \varphi_i(x_0) - J_{\varphi_i}(x_0)(x-x_0)|}{\|x-x_0\|_J}\\
&\leq \frac{\left|\varphi_i^{(I,m)}(x_{J_m}) - \varphi_i^{(I,m)}((x_0)_{J_m}) - J_{\varphi_i^{(I,m)}}((x_0)_{J_m})(x_{J_m}-(x_0)_{J_m}))\right|}{\|x-x_0\|_J}\\
&\quad + \frac{\left|\varphi_{i,\sigma(i)}\left(x_{\sigma(i)}\right) - \varphi_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) - \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right)\left(x_{\sigma(i)} - x_{0,\sigma(i)}\right)\right|}{\|x-x_0\|_J}\\
&\leq \frac{\left\|J_{\varphi_i^{(I,m)}}(\xi_{J_m}) - J_{\varphi_i^{(I,m)}}((x_0)_{J_m})\right\| \|(x_{J_m} - (x_0)_{J_m})\|_{J_m}}{\|x-x_0\|_J}\\
&\quad + \frac{\left|\varphi'_{i,\sigma(i)}(\theta_i) - \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right)\right| \left|x_{\sigma(i)} - x_{0,\sigma(i)}\right|}{\|x-x_0\|_J}\\
&\leq \left\|J_{\varphi_i^{(I,m)}}(\xi_{J_m}) - J_{\varphi_i^{(I,m)}}((x_0)_{J_m})\right\| + \left|\varphi'_{i,\sigma(i)}(\theta_i) - \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right)\right|\\
&\leq \left\|J_{\varphi_i^{(I,m)}}(\xi_{J_m})\right\| + \left\|J_{\varphi_i^{(I,m)}}((x_0)_{J_m})\right\|\\
&\quad + \left(\left|\varphi'_{i,\sigma(i)}(\theta_i) - a\right| + \left|\varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) - a\right|\right) 1_{\mathcal{I}_\varphi}(i) < \varepsilon. \qquad (17)
\end{aligned}$$

Then, from (15), (16) and (17), $\forall x \in \left(M \times \prod_{j\in J\setminus J_m} D_j\right)\setminus\{x_0\}$, we have

$$\frac{\left\|\overline{\varphi}^{(L,N)}(x) - \overline{\varphi}^{(L,N)}(x_0) - J_{\overline{\varphi}^{(L,N)}}(x_0)(x-x_0)\right\|_I}{\|x-x_0\|_J} < \varepsilon; \qquad (18)$$

thus, $\overline{\varphi}^{(L,N)}$ is differentiable in x_0.

5. If φ is strongly C^1 in x_0 and (m,σ)-general, the function $\psi = \varphi - \overline{\varphi}^{(I,m)}$:

$U \subset E_J \longrightarrow E_I$ given by

$$\psi_i(x) = \begin{cases} \sum_{j \in J \setminus J_m} \varphi_{ij}(x_j) & \forall\, i \in I_m,\, \forall\, x \in U \\ 0 & \forall\, i \in I \setminus I_m,\, \forall\, x \in U \end{cases} \tag{19}$$

is strongly C^1 in x_0, and so it is differentiable in x_0 from Proposition 2.19, since $|I_m| < +\infty$; then, if φ is strongly (m, σ)-general, from point 4 $\overline{\varphi}^{(I,m)}$ is differentiable in x_0, and so this is true for $\varphi = \psi + \overline{\varphi}^{(I,m)}$ too, from Remark 2.6.

□

Proposition 2.29. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a function C^1 and m-general; then, $\varphi : (U, \mathcal{B}^{(J)}(U)) \longrightarrow \left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$ is measurable.*

Proof. From point 1 of Proposition 2.28, $\forall\, i \in I$ and $\forall\, n \in \mathbf{N}$, $n \geq m$, the function $\varphi^{(\{i\},n)} : \pi_{J,J_n}(U) \longrightarrow \mathbf{R}$ is C^1; thus, $\forall\, C \in \tau$, we have $\left(\varphi^{(\{i\},n)}\right)^{-1}(C) \in \tau^{(n)}(\pi_{J,J_n}(U)) \subset \mathcal{B}^{(n)}(\pi_{J,J_n}(U))$; then, since $\sigma(\tau) = \mathcal{B}$, $\forall\, C \in \mathcal{B}$, we obtain $\left(\varphi^{(\{i\},n)}\right)^{-1}(C) \in \mathcal{B}^{(n)}(\pi_{J,J_n}(U))$. Moreover, $\forall\, i \in I$, consider the function $\widehat{\varphi}^{(\{i\},n)} : U \longrightarrow \mathbf{R}$ defined by

$$\widehat{\varphi}^{(\{i\},n)}(x) = \varphi^{(\{i\},n)}(x_{J_n}),\ \forall\, x \in U;$$

$\forall\, C \in \mathcal{B}$, we have

$$\left(\widehat{\varphi}^{(\{i\},n)}\right)^{-1}(C) = \left(\varphi^{(\{i\},n)}\right)^{-1}(C) \times \pi_{J,J \setminus J_n}(U) \in \mathcal{B}^{(J)}(U),$$

and so $\widehat{\varphi}^{(\{i\},n)}$ is $\left(\mathcal{B}^{(J)}(U), \mathcal{B}\right)$-measurable; then, since $\lim_{n \longrightarrow +\infty} \widehat{\varphi}^{(\{i\},n)} = \varphi_i$, the function φ_i is $\left(\mathcal{B}^{(J)}(U), \mathcal{B}\right)$-measurable too. Furthermore, let

$$\Sigma(I) = \left\{ B = \prod_{i \in I} B_i : B_i \in \mathcal{B},\, \forall\, i \in I \right\};$$

$\forall\, B = \prod_{i \in I} B_i \in \Sigma(I)$, we have

$$\varphi^{-1}(B) = \bigcap_{i \in I} (\varphi_i)^{-1}(B_i) \in \mathcal{B}^{(J)}(U).$$

Finally, since $\sigma(\Sigma(I)) = \mathcal{B}^{(I)}$, $\forall\, B \in \mathcal{B}^{(I)}$, we obtain $\varphi^{-1}(B) \in \mathcal{B}^{(J)}(U)$. □

3. Linear (m,σ)-general functions

DEFINITION 3.1. *Let $A=(a_{ij})_{i\in I,j\in J}: E_J \longrightarrow E_I$ be a linear (m,σ)-general function; $\forall\, i\in I\backslash I_m$, set $\lambda_i=\lambda_i(A)=a_{i,\sigma(i)}$.*

REMARK 3.2: For any $m\in\mathbf{N}^*$, a linear function $A=(a_{ij})_{i\in I,j\in J}: E_J \longrightarrow E_I$ is m-general; moreover, if $|J|=|I|$ and $\sigma: I\backslash I_m \longrightarrow J\backslash J_m$ is an increasing function, A is (m,σ)-general if and only if:

1. $\forall\, i\in I\backslash I_m$, $\forall\, j\in J\backslash\left(J_m\cup\{\sigma(i)\}\right)$, one has $a_{ij}=0$.
2. $\forall\, j\in J_m$, $\sum_{i\in I\backslash I_m}|a_{ij}|<+\infty$; moreover, one has $\sup_{i\in I\backslash I_m}|\lambda_i|<+\infty$ and $\inf_{i\in I\backslash I_m:\lambda_i\neq 0}|\lambda_i|>0$.
3. If $\mathcal{A}\neq\emptyset$, there exists $\prod_{i\in I\backslash I_m:\lambda_i\neq 0}\lambda_i\in\mathbf{R}^*$.

Furthermore, A is strongly (m,σ)-general if and only if A is (m,σ)-general and there exists $a\in\mathbf{R}$ such that the sequence $\{\lambda_i\}_{i\in I\backslash I_m:\lambda_i\neq 0}$ converges to a.

Finally, A is (m,σ)-standard if and only if A is (m,σ)-general and $a_{ij}=0$, for any $i\in I\backslash I_m$, for any $j\in J_m$.

COROLLARY 3.3. *Let $A=(a_{ij})_{i\in I,j\in J}: E_J \longrightarrow E_I$ be a linear function; then, $A:(E_J,\mathcal{B}_J)\longrightarrow\left(\mathbf{R}^I,\mathcal{B}^{(I)}\right)$ is measurable.*

Proof. The statement follows from Remark 3.2 and Proposition 2.29. □

PROPOSITION 3.4. *Let $A=(a_{ij})_{i\in I,j\in J}: E_J \longrightarrow E_I$ be a linear (m,σ)-general function. Then:*

1. *A is continuous.*
2. *Let $\mathcal{C}=\left\{h\in\mathbf{N},\, h\geq m : \sigma|_{I\backslash I_h} \text{ is injective}\right\}$; if $\mathcal{C}\neq\emptyset$, by setting $\widetilde{m}=\min\mathcal{C}$, let $i_{\widetilde{m}}\in I$ such that $|i_{\widetilde{m}}|=\widetilde{m}$ and let*

$$\widetilde{\widetilde{m}}=\begin{cases}\min\{\widetilde{m},|\sigma\left(i_{\widetilde{m}}\right)|\} & \text{if } \widetilde{m}>m\\ m & \text{if } \widetilde{m}=m\end{cases}; \tag{20}$$

then, for any $n\in\mathbf{N}$, $n\geq\widetilde{\widetilde{m}}$, the linear function ${}^tA: E_I\longrightarrow\mathbf{R}^J$ is (n,τ)-general, where $\tau: J\backslash J_n\longrightarrow I\backslash I_n$ is the increasing function defined by

$$\tau(j)=\min\left\{\ \sigma^{-1}(k): k\geq j,\, k\in\sigma\left(I\backslash I_n\right)\ \right\},\ \forall\, j\in J\backslash J_n. \tag{21}$$

Proof.

1. Since $A\left(E_J\right) \subset E_I$, the statement follows from Proposition 2.4.
2. We have

$$\sup_{j \in J} \sum_{i \in I} \left| \left({}^t A\right)_{ji} \right| = \sup_{j \in J} \sum_{i \in I} \left|a_{ij}\right|$$
$$= \sup \left\{ \sup_{j \in J_m} \sum_{i \in I} \left|a_{ij}\right|, \sup_{j \in J_{\widetilde{\widetilde{m}}} \backslash J_m} \sum_{i \in I} \left|a_{ij}\right|, \sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \sum_{i \in I} \left|a_{ij}\right| \right\}. \quad (22)$$

Moreover, from point 2 of Remark 3.2, we have $\sup_{j \in J_m} \sum_{i \in I} \left|a_{ij}\right| < +\infty$; furthermore, by definition of $\widetilde{m}$ and $\widetilde{\widetilde{m}}$, $\forall\, j \in J_{\widetilde{\widetilde{m}}} \backslash J_m$, we have $\sum_{i \in I} \left|a_{ij}\right| = \sum_{i \in I_{\widetilde{m}+1}} \left|a_{ij}\right| < +\infty$; finally, observe that

$$\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \sum_{i \in I} \left|a_{ij}\right| \leq \sum_{i \in I} \left(\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \left|a_{ij}\right| \right)$$
$$= \sum_{i \in I_{\widetilde{\widetilde{m}}}} \left(\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \left|a_{ij}\right| \right) + \sum_{i \in I \backslash I_{\widetilde{\widetilde{m}}}} \left(\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \left|a_{ij}\right| \right)$$
$$\leq \sum_{i \in I_{\widetilde{\widetilde{m}}}} \left(\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \left|a_{ij}\right| \right) + \sup_{i \in I \backslash I_m} \left|\lambda_i\right|. \quad (23)$$

From Proposition 2.4, $\forall\, i \in I_{\widetilde{\widetilde{m}}}$, we have $\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \left|a_{ij}\right| \leq \sum_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \left|a_{ij}\right| < +\infty$; moreover, we have $\sup_{i \in I \backslash I_m} \left|\lambda_i\right| < +\infty$; then, from (23), we obtain $\sup_{j \in J \backslash J_{\widetilde{\widetilde{m}}}} \sum_{i \in I} \left|a_{ij}\right| < +\infty$, from which $\sup_{j \in J} \sum_{i \in I} \left| \left({}^t A\right)_{ji} \right| < +\infty$, from formula (22), and so ${}^t A(E_I) \subset E_J$ from Proposition 2.4. Finally, from Remark 3.2, $\forall\, n \in \mathbf{N}$, $n \geq \widetilde{\widetilde{m}}$, the function ${}^t A : E_I \longrightarrow E_J$ is (n, τ)-general, where $\tau : J \backslash J_n \longrightarrow I \backslash I_n$ is the increasing function defined by

$$\tau(j) = \min \left\{ \ \sigma^{-1}(k) : k \geq j,\, k \in \sigma\left(I \backslash I_n\right) \ \right\}, \forall\, j \in J \backslash J_n.$$

□

Henceforth, we will suppose that $|I| = +\infty$.

Definition 3.5. *Let* $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ *be a linear* (m, σ)*-general function; indicate by* $N(A) \in \{0, 1, ..., m\}$ *the number of zero columns of the matrix* $A^{(I \backslash I_m, J_m)}$.

Theorem 3.6. *Let* $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ *be a linear* (m, σ)*-general function; then, the sequence* $\left\{\det A^{(n,n)}\right\}_{n \geq m}$ *converges to a real number. Moreover, if* $\mathcal{A} \neq \emptyset$*, by setting* $\overline{m} = \min \mathcal{A}$*, we have*

$$\lim_{n \longrightarrow +\infty} \det A^{(n,n)} = \sum_{p \in I \backslash I_{\overline{m}}} \left(\prod_{q \in I \backslash I_{|p|}} \lambda_q \right) \sum_{j \in J_m} a_{p,j} \left(cof A^{(|p|,|p|)} \right)_{p,j} + \det A^{(\overline{m},\overline{m})} \left(\prod_{q \in I \backslash I_{\overline{m}}} \lambda_q \right). \quad (24)$$

Conversely, if $\mathcal{A} = \emptyset$*, we have* $\lim_{n \longrightarrow +\infty} \det A^{(n,n)} = 0$.

Proof. $\forall\, l \in \mathbf{Z}$, set $\mathcal{D}_l = \mathcal{D}_l(A) = \{h \in \mathbf{N},\, h \geq m : |\sigma(i)| = |i| + l,\, \forall\, i \in I \backslash I_h\}$; moreover, if $\mathcal{D}_l \neq \emptyset$, set $\overline{m}_l = \min \mathcal{D}_l$; furthermore, set $\mathcal{D} = \mathcal{D}(A) = \bigcup_{l \in \mathbf{Z}} \mathcal{D}_l$. If there exists $l \in \mathbf{N}$ such that $\mathcal{D}_l \neq \emptyset$, we will prove the statement by recursion on $N(A) = k \in \{0, 1, ..., m\}$. Suppose that $N(A) = 0$ and observe that, if $\mathcal{A} \neq \emptyset$, we have $\overline{m}_0 = \overline{m}$, since $\mathcal{D}_0 = \mathcal{A}$; then, $\forall\, n \in \mathbf{N}$, $n > \overline{m}_l$, we have

$$\det A^{(n,n)} = \begin{cases} \det A^{(\overline{m},\overline{m})} \left(\prod_{q \in I_n \backslash I_{\overline{m}}} \lambda_q \right) & \text{if } l = 0 \\ 0 & \text{if } l \in \mathbf{N}^* \end{cases},$$

from which

$$\lim_{n \longrightarrow +\infty} \det A^{(n,n)} = \begin{cases} \det A^{(\overline{m},\overline{m})} \left(\prod_{q \in I \backslash I_{\overline{m}}} \lambda_q \right) \in \mathbf{R} & \text{if } l = 0 \\ 0 & \text{if } l \in \mathbf{N}^* \end{cases};$$

then, since we have $a_{p,j} = 0$, $\forall\, p \in I \backslash I_{\overline{m}}$, $\forall\, j \in J_m$, the statement is true. Suppose that the statement is true for $N(A) = k$, where $0 \leq k \leq m - 1$, and suppose that $N(A) = k + 1$; $\forall\, n \in \mathbf{N}$, $n > \overline{m}_l$, let $i_n \in I$ such that $|i_n| = n$; we have

$$\det A^{(n,n)} = \sum_{j \in J_n} a_{i_n,j} \left(cof A^{(n,n)} \right)_{i_n,j}; \quad (25)$$

moreover, let $\{j_1, ..., j_{k+1}\} \subset J_m$ such that $a_{i_n,j} = 0$, $\forall\, j \in J_m \backslash \{j_1, ..., j_{k+1}\}$.

If $l=0$, from (25), we have

$$\det A^{(n,n)}=\sum_{h=1}^{k+1}a_{i_n,j_h}\left(cofA^{(n,n)}\right)_{i_n,j_h}+\lambda_{i_n}\det A^{(n-1,n-1)};$$

then, by induction on n, we obtain

$$\det A^{(n,n)}=a_n+\det A^{(\overline{m},\overline{m})}\left(\prod_{q\in I_n\backslash I_{\overline{m}}}\lambda_q\right),\ \forall\, n>\overline{m}, \tag{26}$$

where

$$a_n=\sum_{p\in I_n\backslash I_{\overline{m}}}\left(\prod_{q\in I_n\backslash I_{|p|}}\lambda_q\right)\sum_{h=1}^{k+1}a_{p,j_h}\left(cofA^{(|p|,|p|)}\right)_{p,j_h}. \tag{27}$$

Moreover, $\forall\, h=1,...,k+1$, $\forall\, p\in I\backslash I_{\overline{m}}$, we have

$$\left|\left(cofA^{(|p|,|p|)}\right)_{p,j_h}\right|=\left|\det A^{(I_{|p|-1},I_{|p|}\backslash\{j_h\})}\right|=\left|\det B_{j_h,p}^{(|p|-1,|p|-1)}\right|, \tag{28}$$

where $B_{j_h,p}:E_J\longrightarrow E_I$ is the linear function obtained by exchanging the $|j_h|$-th column of A for the $|p|$-th column of A; furthermore

$$\begin{aligned}\left|\det B_{j_h,p}^{(|p|-1,|p|-1)}\right| &= \left|\sum_{i\in I_m}a_{i,p}\left(cofB_{j_h,p}^{(|p|-1,|p|-1)}\right)_{i,j_h}\right|\\ &\leq \sum_{i\in I_m}|a_{i,p}|\left|\det\left(A^{(I\backslash\{i\},J\backslash\{j_h\})}\right)^{(|p|-2,|p|-2)}\right|. \end{aligned}\tag{29}$$

Observe that, $\forall\, i\in I_m$, $A^{(I\backslash\{i\},J\backslash\{j_h\})}:E_{J\backslash\{j_h\}}\longrightarrow E_{I\backslash\{i\}}$ is a linear $(m-1,\sigma)$-general function such that $\mathcal{D}_0\left(A^{(I\backslash\{i\},J\backslash\{j_h\})}\right)\neq\emptyset$, $N\left(A^{(I\backslash\{i\},J\backslash\{j_h\})}\right)=k$; then, from the recursive assumption, there exists

$$\lim_{|p|\longrightarrow+\infty}\det\left(A^{(I\backslash\{i\},J\backslash\{j_h\})}\right)^{(|p|-2,|p|-2)}\in\mathbf{R},$$

and so

$$\lim_{|p|\longrightarrow+\infty}\sum_{i\in I_m}|a_{i,p}|\left|\det\left(A^{(I\backslash\{i\},J\backslash\{j_h\})}\right)^{(|p|-2,|p|-2)}\right|=0,\ \forall\, h=1,...,k+1;$$

consequently, from (28) and (29), there exists $b\in\mathbf{R}^+$ such that

$$\sup\left\{\left|\left(cofA^{(|p|,|p|)}\right)_{p,j_h}\right|:h\in\{1,...,k+1\},p\in I\backslash I_{\overline{m}}\right\}\leq b. \tag{30}$$

Moreover, since $\prod_{q\in I\setminus I_m:\lambda_q\neq 0}\lambda_q\in\mathbf{R}^*$, we have $\prod_{q\in I\setminus I_m}\overline{\lambda}_q\equiv c\in\mathbf{R}^+$, where

$$\overline{\lambda}_q=\begin{cases}1 & \text{if } \lambda_q=0\\ \frac{1}{|\lambda_q|} & \text{if } 0<|\lambda_q|<1\\ |\lambda_q| & \text{if } |\lambda_q|\geq 1\end{cases},$$

and so

$$\left|\prod_{q\in H}\lambda_q\right|\leq c,\ \forall\, H\subset I\setminus I_{\overline{m}}. \tag{31}$$

Observe that

$$\lim_{n\longrightarrow+\infty}\det A^{(\overline{m},\overline{m})}\left(\prod_{q\in I_n\setminus I_{\overline{m}}}\lambda_q\right)=\det A^{(\overline{m},\overline{m})}\left(\prod_{q\in I\setminus I_{\overline{m}}}\lambda_q\right)\in\mathbf{R}; \tag{32}$$

moreover, set

$$a=\sum_{p\in I\setminus I_{\overline{m}}}\left(\prod_{q\in I\setminus I_{|p|}}\lambda_q\right)\sum_{h=1}^{k+1}a_{p,j_h}\left(cofA^{(|p|,|p|)}\right)_{p,j_h}; \tag{33}$$

then, $\forall\, n>\overline{m}$, we have

$$\begin{aligned}a-a_n=&\sum_{p\in I\setminus I_n}\left(\prod_{q\in I\setminus I_{|p|}}\lambda_q\right)\sum_{h=1}^{k+1}a_{p,j_h}\left(cofA^{(|p|,|p|)}\right)_{p,j_h}\\&+\sum_{p\in I_n\setminus I_{\overline{m}}}\left(\prod_{q\in I_n\setminus I_{|p|}}\lambda_q\right)\left(\left(\prod_{r\in I\setminus I_n}\lambda_r\right)-1\right)\sum_{h=1}^{k+1}a_{p,j_h}\left(cofA^{(|p|,|p|)}\right)_{p,j_h}.\end{aligned} \tag{34}$$

If there exists $n_0\in\mathbf{N}$, $n_0\geq\overline{m}$, such that $\lambda_q\neq 0\ \forall\, q\in I\setminus I_{n_0}$, we have $\prod_{q\in I\setminus I_{n_0}}\lambda_q\in\mathbf{R}^*$; then $\forall\,\varepsilon\in\mathbf{R}^+$, there exists $n_1\in\mathbf{N}$, $n_1\geq n_0$, such that, $\forall\, n\in\mathbf{N}$, $n>n_1$, we have $\left|\left(\prod_{r\in I\setminus I_n}\lambda_r\right)-1\right|<\varepsilon$; thus, from formulas (34), (30) and (31), we obtain

$$|a-a_n|\leq bc\sum_{p\in I\setminus I_n}\sum_{h=1}^{k+1}|a_{p,j_h}|+bc\varepsilon\sum_{p\in I_n\setminus I_{\overline{m}}}\sum_{h=1}^{k+1}|a_{p,j_h}|,\ \forall\, n>n_1. \tag{35}$$

Finally, there exists $d \in \mathbf{R}^+$ such that $\sum_{p \in I \setminus I_{\overline{m}}} \sum_{h=1}^{k+1} |a_{p,j_h}| \leq d$, and so there exists $n_2 \in \mathbf{N}$, $n_2 \geq n_1$, such that, $\forall\, n \in \mathbf{N}$, $n \geq n_2$, we have $\sum_{p \in I \setminus I_n} \sum_{h=1}^{k+1} |a_{p,j_h}| < \varepsilon$; then, from formula (35), we obtain

$$|a - a_n| \leq bc\varepsilon + bcd\varepsilon = bc\,(1+d)\,\varepsilon,\ \forall\, n \geq n_2.$$

Then, from (26) and (32), we have

$$\begin{aligned}
&\lim_{n \longrightarrow +\infty} \det A^{(n,n)} \\
&= \sum_{p \in I \setminus I_{\overline{m}}} \left(\prod_{q \in I \setminus I_{|p|}} \lambda_q \right) \sum_{h=1}^{k+1} a_{p,j_h} \left(cof A^{(|p|,|p|)} \right)_{p,j_h} + \det A^{(\overline{m},\overline{m})} \left(\prod_{q \in I \setminus I_{\overline{m}}} \lambda_q \right) \\
&= \sum_{p \in I \setminus I_{\overline{m}}} \left(\prod_{q \in I \setminus I_{|p|}} \lambda_q \right) \sum_{j \in J_m} a_{p,j} \left(cof A^{(|p|,|p|)} \right)_{p,j} + \det A^{(\overline{m},\overline{m})} \left(\prod_{q \in I \setminus I_{\overline{m}}} \lambda_q \right) \in \mathbf{R}.
\end{aligned}$$

Moreover, suppose that σ is bijective and there exists a subsequence $\{\lambda_{q_t}\}_{t \in \mathbf{N}} \subset \{\lambda_q\}_{q \in I \setminus I_{\overline{m}} : \lambda_q = 0}$; then, from formulas (27) and (33), $\forall\, t \in \mathbf{N}$, $\forall\, n \geq |q_t|$, we obtain

$$\begin{aligned}
a - a_n = &\sum_{p \in I \setminus I_n} \left(\prod_{q \in I \setminus I_{|p|}} \lambda_q \right) \sum_{h=1}^{k+1} a_{p,j_h} \left(cof A^{(|p|,|p|)} \right)_{p,j_h} \\
&- \sum_{p \in I_n \setminus I_{\overline{m}}} \left(\prod_{q \in I_n \setminus I_{|p|}} \lambda_q \right) \sum_{h=1}^{k+1} a_{p,j_h} \left(cof A^{(|p|,|p|)} \right)_{p,j_h} \\
= &- \sum_{p \in I_n \setminus I_{|q_t|-1}} \left(\prod_{q \in I_n \setminus I_{|p|}} \lambda_q \right) \sum_{h=1}^{k+1} a_{p,j_h} \left(cof A^{(|p|,|p|)} \right)_{p,j_h}. \qquad (36)
\end{aligned}$$

Thus, from formulas (30), (31) and (36):

$$|a - a_n| \leq bc \sum_{p \in I_n \setminus I_{|q_t|-1}} \sum_{h=1}^{k+1} |a_{p,j_h}|\,,\ \forall\, t \in \mathbf{N},\ \forall\, n \geq |q_t|\,. \qquad (37)$$

Finally, $\forall\, \varepsilon \in \mathbf{R}^+$, there exists $t \in \mathbf{N}$ such that $\sum_{p \in I_n \setminus I_{|q_t|-1}} \sum_{h=1}^{k+1} |a_{p,j_h}| < \varepsilon$, $\forall\, n \geq |q_t|$; then, from (37), we obtain

$$|a - a_n| \leq bc\varepsilon,\ \forall\, n \geq |q_t|\,.$$

Thus, from (26) and (32), we have formula (24).

Moreover, if $l \in \mathbf{N}^*$, from (25) we have

$$\det A^{(n,n)} = \sum_{h=1}^{k+1} a_{i_n,j_h} \left(cof A^{(n,n)} \right)_{i_n,j_h}, \; \forall\, n > \overline{m}_l; \tag{38}$$

moreover, $\forall\, h = 1, ..., k+1$, we have

$$\left| \left(cof A^{(n,n)} \right)_{i_n,j_h} \right| = \left| \det A^{(I_{n-1}, I_n \backslash \{j_h\})} \right| = \left| \det \left(A^{(I,J\backslash\{j_h\})} \right)^{(n-1,n-1)} \right|. \tag{39}$$

Observe that $A^{(I,J\backslash\{j_h\})} : E_{J\backslash\{j_h\}} \longrightarrow E_I$ is a linear (m,τ)-general function, where $\tau : I\backslash I_m \longrightarrow J\backslash J_{m+1}$ is the function defined by $\tau(i) = \sigma(i)$, $\forall\, i \in I\backslash I_m$; moreover, $\mathcal{D}_{l-1}\left(A^{(I,J\backslash\{j_h\})}\right) \neq \emptyset$, $l-1 \in \mathbf{N}$, $N\left(A^{(I,J\backslash\{j_h\})}\right) = k$; then, from the recursive assumption, there exists $\lim\limits_{n\longrightarrow+\infty} \det \left(A^{(I,J\backslash\{j_h\})}\right)^{(n-1,n-1)} \in \mathbf{R}$, and so

$$\lim_{n\longrightarrow+\infty} |a_{i_n,j_h}| \left| \det \left(A^{(I,J\backslash\{j_h\})} \right)^{(n-1,n-1)} \right| = 0, \; \forall\, h = 1, ..., k+1;$$

consequently, from (38) and (39), we obtain $\lim\limits_{n\longrightarrow+\infty} \det A^{(n,n)} = 0$.

Furthermore, suppose that there exists $l \in \mathbf{Z}^-$ such that $\mathcal{D}_l \neq \emptyset$; since the function $\sigma|_{I\backslash I_{\overline{m}_l}}$ is injective, from Proposition 3.4, the linear function ${}^tA : E_I \longrightarrow E_J$ is $(\overline{m}_l, \tau)$-general, where $\tau : J\backslash J_{\overline{m}_l} \longrightarrow I\backslash I_{\overline{m}_l}$ is the increasing function defined by $\tau(j) = \sigma^{-1}(j)$, $\forall\, j \in J\backslash J_{\overline{m}_l}$; moreover, we have $\mathcal{D}_{-l}\left({}^tA\right) \neq \emptyset$, $-l \in \mathbf{N}^*$; then, from the previous arguments, we obtain

$$\lim_{n\longrightarrow+\infty} \det A^{(n,n)} = \lim_{n\longrightarrow+\infty} {}^tA^{(n,n)} = 0.$$

Finally, if $\mathcal{D} = \emptyset$, we have

$$|\{i \in I\backslash I_m : \sigma(i) = \sigma(h), \text{ fore some } h \in I\backslash I_m, h < i\}| = +\infty$$

or $|(J\backslash J_m)\backslash\sigma(I\backslash I_m)| = +\infty$; then, the rows or the columns of the matrix $A^{(n,n)}$ are linearly dependent, for n sufficiently large, and so we have $\det A^{(n,n)} = 0$, from which $\lim\limits_{n\longrightarrow+\infty} \det A^{(n,n)} = 0$. □

Definition 3.7. *Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function; define the determinant of A, and call it $\det A$, the real number*

$$\det A = \lim_{n\longrightarrow+\infty} \det A^{(n,n)}.$$

Corollary 3.8. *Let* $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ *be a linear* (m,σ)*-general function such that* $a_{ij} = 0$, $\forall\, i \in I_m$, $\forall\, j \in J\backslash J_m$*, or* A *is* (m,σ)*-standard. Then, if* σ *is bijective, we have*

$$\det A = \det A^{(m,m)} \prod_{i\in I\backslash I_m} \lambda_i.$$

Conversely, if σ *is not bijective, we have* $\det A = 0$*. In particular, if* $A = \mathbf{I}_{I,J}$*, we have* $\det A = 1$*.*

Proof. If σ is bijective, $\forall\, i \in I\backslash I_m$, we have $|\sigma(i)| = |i|$; then, $\forall\, n \in \mathbf{N}$, $n \geq m$, we have

$$\det A^{(n,n)} = \det A^{(m,m)} \prod_{i\in I_n\backslash I_m} \lambda_i,$$

from which

$$\det A = \lim_{n\longrightarrow +\infty} \det A^{(n,n)} = \det A^{(m,m)} \prod_{i\in I\backslash I_m} \lambda_i.$$

Moreover, suppose that $\mathcal{A} \neq \emptyset$ but σ is not bijective, and set $\overline{m} = \min \mathcal{A}$; by definition of $\overline{m}$, we have $\overline{m} > m$ and the matrix $A^{(\overline{m},\overline{m})}$ is not invertible; then, $\forall\, n \in \mathbf{N}$, $n \geq \overline{m}$, we obtain

$$\det A^{(n,n)} = \det A^{(\overline{m},\overline{m})} \prod_{p\in I_n\backslash I_{\overline{m}}} \lambda_p = 0,$$

and so $\det A = \lim_{n\longrightarrow +\infty} \det A^{(n,n)} = 0$. Finally, if $\mathcal{A} = \emptyset$, from Theorem 3.6 we have $\det A = 0$ again. In particular, if $A = \mathbf{I}_{I,J}$, then A is $(1,\sigma)$-standard, where $A^{(1,1)} = (1)$, $\lambda_i = 1$, $\forall\, i \in I\backslash I_1$, and σ is bijective; then, $\det A = 1$. □

Proposition 3.9. *Let* $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ *be a linear* (m,σ)*-general function such that* $a_{ij} = 0$, $\forall\, i \in I_m$, $\forall\, j \in J\backslash J_m$*, or* A *is* (m,σ)*-standard; then:*

1. *One has* $\det A \neq 0$ *if and only if* $A^{(m,m)}$ *is invertible,* $\lambda_i \neq 0$*, for any* $i \in I\backslash I_m$*, and* σ *is bijective.*
2. *If* $a_{ij} = 0$, $\forall\, i \in I_m$, $\forall\, j \in J\backslash J_m$*, and* $\det A \neq 0$*, then* A *is bijective.*
3. *If* A *is* (m,σ)*-standard, then one has* $\det A \neq 0$ *if and only if* A *is bijective.*

Proof.

1. If σ is bijective, from Corollary 3.8, we have

$$\det A = \det A^{(m,m)} \prod_{i \in I \setminus I_m} \lambda_i.$$

Moreover, if $A^{(m,m)}$ is invertible and $\lambda_i \neq 0$, $\forall\, i \in I \setminus I_m$, we have $\det A^{(m,m)} \neq 0$, $\prod_{i \in I \setminus I_m} \lambda_i = \prod_{i \in I \setminus I_m : \lambda_i \neq 0} \lambda_i \in \mathbf{R}^*$, and so $\det A \neq 0$.

Conversely, if $\det A \neq 0$, from Corollary 3.8, σ is bijective, and so

$$\det A^{(m,m)} \prod_{i \in I \setminus I_m} \lambda_i = \det A \neq 0;$$

then, $A^{(m,m)}$ is invertible and $\lambda_i \neq 0$, $\forall\, i \in I \setminus I_m$.

2. If $a_{ij} = 0$, $\forall\, i \in I_m$, $\forall\, j \in J \setminus J_m$, and $\det A \neq 0$, from point 1 and Proposition 2.27, we obtain that A is bijective.

3. The statement follows from Proposition 10 and Remark 14 in [3].

□

Proposition 3.10. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function such that $\left\{ h \in \mathbf{N},\, h \geq m : \sigma|_{I \setminus I_h} \text{ is injective} \right\} \neq \emptyset$; then, $\det A = \det {}^t A$.*

Proof. Since $\left\{ h \in \mathbf{N},\, h \geq m : \sigma|_{I \setminus I_h} \text{ is injective} \right\} \neq \emptyset$, from Proposition 3.4, the function ${}^t A : E_I \longrightarrow E_J$ is $\left(\widetilde{\widetilde{m}}, \tau\right)$-general, where $\widetilde{\widetilde{m}} \in \mathbf{N}^*$ is defined by formula (20), and the function $\tau : J \setminus J_{\widetilde{\widetilde{m}}} \longrightarrow I \setminus I_{\widetilde{\widetilde{m}}}$ is given by

$$\tau(j) = \min \left\{ \ \sigma^{-1}(k) : k \geq j,\, k \in \sigma\left(I \setminus I_{\widetilde{\widetilde{m}}}\right) \ \right\}, \forall\, j \in J \setminus J_{\widetilde{\widetilde{m}}}.$$

Then, we have

$$\begin{aligned} \det A &= \lim_{n \longrightarrow +\infty} \det A^{(n,n)} \\ &= \lim_{n \longrightarrow +\infty} \det {}^t\left(A^{(n,n)}\right) = \lim_{n \longrightarrow +\infty} \det \left({}^t A\right)^{(n,n)} = \det {}^t A. \end{aligned}$$

□

Proposition 3.11. *Let $A = (a_{ij})_{i \in I, j \in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function such that $\sum_{i \in I \setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J \setminus J_m$; moreover, let $s, t \in \mathbf{N}^*$, $s < t$, let $p = \max\{t, m\}$ and let $i_t \in I$ such that $|i_t| = t$; then:*

1. *If there exist* $u = (u_j : j \in J) \in E_J$, $v = (v_j : j \in J) \in E_J$, $c_1, c_2 \in \mathbf{R}$ *such that* $\sum_{j\in J} |u_j| < +\infty$, $\sum_{j\in J} |v_j| < +\infty$, $a_{i_t,j} = c_1 u_j + c_2 v_j$, *for any* $j \in J$, *by indicating by* $U = (u_{ij})_{i\in I, j\in J}$ *and* $V = (v_{ij})_{i\in I, j\in J}$ *the linear functions obtained by substituting the* t*-th row of* A *for* u *and* v*, respectively, then* U *and* V *are* (p,ξ)*-general, where the increasing function* $\xi : I\backslash I_p \longrightarrow J\backslash J_p$ *is defined by*

$$\xi(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \in J\backslash J_p \\ \min(J\backslash J_p) & \text{if } \sigma(i) \notin J\backslash J_p \end{cases}, \ \forall\, i \in I\backslash I_p; \tag{40}$$

   moreover, one has $\det A = c_1 \det U + c_2 \det V$.

2. *If* $B = (b_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ *is the linear function obtained by exchanging the* s*-th row of* A *for the* t*-th row of* A*, then* B *is* (p,ξ)*-general and one has* $\det B = -\det A$.

3. *If* $C = (c_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ *is the linear function obtained by substituting the* t*-th row of* A *for the* s*-th row of* A*, or the* s*-th one for the* t*-th one, then* C *is* (p,ξ)*-general and one has* $\det C = 0$.

Proof.

1. Since $\sum_{i\in I\backslash I_m} |a_{i,j}| < +\infty$, $\forall\, j \in J\backslash J_m$, we have $\sum_{i\in I\backslash I_m} |u_{ij}| < +\infty$, $\sum_{i\in I\backslash I_m} |v_{ij}| < +\infty$, $\forall\, j \in J\backslash J_m$; then, from point 5 of Proposition 2.24, the functions U and V are (p,ξ)-general. Moreover, $\forall\, n \in \mathbf{N}^*$, we have $\det A^{(n,n)} = c_1 \det U^{(n,n)} + c_2 \det V^{(n,n)}$, from which

$$\det A = \lim_{n\longrightarrow+\infty} \det A^{(n,n)} = \lim_{n\longrightarrow+\infty} \left(c_1 \det U^{(n,n)} + c_2 \det V^{(n,n)}\right)$$
$$= c_1 \det U + c_2 \det V.$$

2. By proceeding as in the proof of point 1, we can prove that B is (p,ξ)-general; moreover, $\forall\, n \in \mathbf{N}$, $n \geq p$, $B^{(n,n)}$ is the matrix obtained by exchanging the s-th row of $A^{(n,n)}$ for the t-th row of $A^{(n,n)}$; then, one has $\det B^{(n,n)} = -\det A^{(n,n)}$, from which

$$\det B = \lim_{n\longrightarrow+\infty} \det B^{(n,n)} = -\lim_{n\longrightarrow+\infty} \det A^{(n,n)} = -\det A.$$

3. By proceeding as in the proof of point 1, we can prove that C is (p,ξ)-general; moreover, since the s-th row of C and the t-th row of C are equals, by exchanging these rows among themselves we obtain again the matrix C; then, from point 2, we have $\det C = -\det C$, from which $\det C = 0$.

□

PROPOSITION 3.12. *Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function such that $\sum_{i\in I\setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J\setminus J_m$; moreover, let $s,t \in \mathbf{N}^*$, $s < t$, let $p = \max\{t,m\}$, let $j_t \in J$ such that $|j_t| = t$, and let the function $\xi : I\setminus I_p \longrightarrow J\setminus J_p$ defined by (40); then:*

1. *If there exist $u = (u_i : i \in I) \in E_I$, $v = (v_i : i \in I) \in E_I$, $c_1, c_2 \in \mathbf{R}$ such that $\sum_{i\in I} |u_i| < +\infty$, $\sum_{i\in I} |v_i| < +\infty$, $a_{i,j_t} = c_1 u_i + c_2 v_i$, for any $i \in I$, by indicating by $U = (u_{ij})_{i\in I, j\in J}$ and $V = (v_{ij})_{i\in I, j\in J}$ the linear functions obtained by substituting the t-th column of A for u and v, respectively, then U and V are (p,ξ)-general and one has* $\det A = c_1 \det U + c_2 \det V$.

2. *If $B = (b_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ is the linear function obtained by exchanging the s-th column of A for the t-th column of A, then B is (p,ξ)-general and one has* $\det B = -\det A$.

3. *If $C = (c_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ is the linear function obtained by substituting the t-th column of A for the s-th column of A, or the s-th one for the t-th one, then C is (p,ξ)-general and one has* $\det C = 0$.

Proof. The proof is analogous to that one of Proposition 3.11. □

PROPOSITION 3.13. *Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function such that $\sum_{i\in I\setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J\setminus J_m$. If the dimension of the vector space generated by the rows or the columns of A is finite, then* $\det A = 0$.

Proof. Suppose that the dimension of the vector space generated by the rows of A is finite; then, there exist n rows $v^{(1)}$,...,$v^{(n)}$ of A, where $v^{(k)} = \left(v_j^{(k)} : j \in J\right)$, $\forall\, k \in \{1,...,n\}$, such that, if $v = (v_j : j \in J)$ is as row of A, there exist c_1,...,$c_n \in \mathbf{R}$ such that $v = c_1 v^{(1)} + ... + c_n v^{(n)}$. From Proposition 3.11, by indicating by V_k, $\forall\, k \in \{1,...,n\}$, the linear function obtained by substituting the row v of A for $v^{(k)}$, by recursion we have $\det A = c_1 \det V_1 + ... + c_n \det V_n$; moreover, V_k has two rows equals to $v^{(k)}$, and so $\det V_k = 0$, $\forall\, k \in \{1,...,n\}$; then, $\det A = 0$. Analogously, if the dimension of the vector space generated by the columns of A is finite, from Proposition 3.12 we obtain $\det A = 0$. □

REMARK 3.14: Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m, σ)-general function such that $\sum_{i\in I\setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J\setminus J_m$. Then, for any $n \in \mathbf{N}$, $n \geq m$, for any $\emptyset \neq L \subset I$ and for any $\emptyset \neq N \subset J$ such that $|I\setminus L| = |J\setminus N| < +\infty$, the linear function $A^{(L,N)} : E_N \longrightarrow E_L$ is (n, ρ)-general, where the function $\rho : L\setminus L_n \longrightarrow N\setminus N_n$ is defined by

$$\rho(i) = \begin{cases} \sigma(i) & \text{if } \sigma(i) \in N\setminus N_n \\ \min\{j > \sigma(i) : j \in N\setminus N_n\} & \text{if } \sigma(i) \notin N\setminus N_n \end{cases}, \forall\, i \in L\setminus L_n.$$

Proof. The proof follows from Remark 2.25. □

DEFINITION 3.15. *Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m, σ)-general function such that $\sum_{i\in I\setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J\setminus J_m$; define the $I \times J$ matrix $cof A$ by*

$$(cof A)_{ij} = (-1)^{|i|+|j|} \det\left(A^{(I\setminus\{i\}, J\setminus\{j\})}\right), \forall\, i \in I, \forall\, j \in J.$$

PROPOSITION 3.16. *Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m, σ)-general function such that $\sum_{i\in I\setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J\setminus J_m$; moreover, suppose that $a_{ij} = 0$, $\forall\, i \in I_m$, $\forall\, j \in J\setminus J_m$, or A is (m, σ)-standard; then, one has:*

$$\det A = \sum_{t\in J} a_{it}(cof A)_{it}, \forall\, i \in I; \tag{41}$$

$$\det A = \sum_{s\in I} a_{sj}(cof A)_{sj}, \forall\, j \in J. \tag{42}$$

Proof. Suppose that $\mathcal{A} \neq \emptyset$ and set $\overline{m} = \min \mathcal{A}$; $\forall\, i \in I$, $\forall\, j \in J$ and $\forall\, n \in \mathbf{N}$, $n \geq \max\{|i|, |j|, \overline{m}\}$, we have

$$\det A = \det A^{(n,n)} \prod_{p\in I\setminus I_n} \lambda_p, \tag{43}$$

from which

$$\det A = \sum_{t\in J_n} a_{it}(cof A^{(n,n)})_{it} \prod_{p\in I\setminus I_n} \lambda_p = \sum_{t\in J_n} a_{it}(cof A)_{it};$$

then

$$\det A = \lim_{n\longrightarrow+\infty} \sum_{t\in J_n} a_{it}(cof A)_{it} = \sum_{t\in J} a_{it}(cof A)_{it}.$$

Analogously, from formula (43), we have

$$\det A = \sum_{s\in I_n} a_{sj}(cofA^{(n,n)})_{sj} \prod_{p\in I\setminus I_n} \lambda_p = \sum_{s\in I_n} a_{sj}(cofA)_{sj},$$

and so

$$\det A = \sum_{s\in I} a_{sj}(cofA)_{sj}.$$

Conversely, if $\mathcal{A} = \emptyset$, $\forall\, s \in I$, $\forall\, t \in J$, we have $\mathcal{A}\left(A^{(I\setminus\{s\},J\setminus\{t\})}\right) = \emptyset$; then, from Theorem 3.6, we obtain $\det A = \det\left(A^{(I\setminus\{s\},I\setminus\{t\})}\right) = 0$, and so $(cofA)_{st} = 0$; then:

$$\det A = 0 = \sum_{t\in J} a_{it}(cofA)_{it},\ \forall\, i \in I;$$
$$\det A = 0 = \sum_{s\in I} a_{sj}(cofA)_{sj},\ \forall\, j \in J.$$

□

Corollary 3.17. *Let $A = (a_{ij})_{i\in I, j\in J} : E_J \longrightarrow E_I$ be a linear (m,σ)-general function such that $\sum_{i\in I\setminus I_m} |a_{i,j}| < +\infty$, for any $j \in J\setminus J_m$; moreover, suppose that $a_{ij} = 0$, $\forall\, i \in I_m$, $\forall\, j \in J\setminus J_m$, or A is (m,σ)-standard; then:*

1. *One has*
$$A\ {}^{t}(cofA) = (\det A)\mathbf{I}_{I,I}; \tag{44}$$
moreover, if A is bijective, the linear functions $A^{-1} : E_I \longrightarrow E_J$ and ${}^{t}(cofA) : E_I \longrightarrow E_J$ are continuous.

2. *If A is bijective, then one has $\det A \neq 0$ if and only if $cofA \neq 0$; moreover, in this case*
$$A^{-1} = \frac{1}{\det A}\ {}^{t}(cofA)\ . \tag{45}$$

3. *If A is (m,σ)-standard and bijective, then A^{-1} is $\left(m,\sigma^{-1}\right)$-standard.*

Proof.

1. From formula (41), we have

$$\sum_{t\in J} a_{it}(cofA)_{it} = \det A,\ \forall\, i \in I.$$

Moreover, we have

$$\sum_{t\in J} a_{it}(cofA)_{jt} = 0,\ \forall\, i, j \in I,\ i \neq j; \tag{46}$$

in fact, from formula (41) and Proposition 3.11, the left side of (46) is equal to $\det C$, where C is the (p, ξ)-general function obtained by substituting the j-th row of A for the i-th row of A, $p = \max\{|i|, |j|, m\}$, and the increasing function $\xi : I \backslash I_p \longrightarrow J \backslash J_p$ is defined by (40); then, from Proposition 3.11, we have $\det C = 0$. This implies that

$$\sum_{t\in J} a_{it}(cofA)_{jt} = (\det A)\delta_{ij},\ \forall\, i, j \in I,$$

where δ_{ij} is the Kronecker symbol, and so formula (44) follows, since the functions δ_{ij} and $\overline{\delta}_{ij}$ coincide on $I \times I$. Moreover, suppose that A is bijective; since A is continuous from Proposition 3.4, then the linear function $A^{-1} : E_I \longrightarrow E_J$ is continuous (see, e.g., the theory in Weidmann's book [11]); furthermore, from formula (44), we have

$$^{t}\,(cofA) = (\det A)A^{-1},$$

and so the linear function $^{t}\,(cofA) : E_I \longrightarrow E_J$ is continuous too.

2. If A is bijective, from formula (44) we have $\det A = 0$ if and only if $cofA = 0$, and so $\det A \neq 0$ if and only if $cofA \neq 0$; moreover, in this case, from formula (44) we obtain formula (45).

3. If A is (m, σ)-standard and bijective, from Proposition 3.9, we have $\det A \neq 0$, $\lambda_i \neq 0$, $\forall\, i \in I \backslash I_m$, and σ is bijective; moreover, $\forall\, y \in E_I$, we have $A\left(A^{-1}y\right) = y$, from which

$$\left(A^{-1}y\right)_i = \frac{y_i}{\lambda_i},\ \forall\, i \in I \backslash I_m; \tag{47}$$

furthermore, we have $\left\{i \in I \backslash I_m : (\lambda_i)^{-1} \neq 0\right\} = I \backslash I_m$, from which

$$\prod_{i\in I\backslash I_m : (\lambda_i)^{-1}\neq 0} (\lambda_i)^{-1} = \left(\prod_{i\in I\backslash I_m} \lambda_i\right)^{-1} = \left(\prod_{i\in I\backslash I_m : \lambda_i\neq 0} \lambda_i\right)^{-1} \in \mathbf{R}^*;$$

then, we obtain $\sup_{i\in I\backslash I_m} \left|(\lambda_i)^{-1}\right| < +\infty$ and $\inf_{i\in I\backslash I_m : (\lambda_i)^{-1}\neq 0} \left|(\lambda_i)^{-1}\right| > 0$. Finally, from formula (47) and since the linear function $A^{-1} : E_I \longrightarrow E_J$ is given by formula (45), then A^{-1} is $\left(m, \sigma^{-1}\right)$-standard, with $\lambda_i\left(A^{-1}\right) = (\lambda_i)^{-1}$, $\forall\, i \in I \backslash I_m$.

□

PROPOSITION 3.18. *Let $\varphi : U \subset E_J \longrightarrow E_I$ be a (m,σ)-general function and let $x_0 = (x_{0,j} : j \in J) \in U$ such that there exists the function $J_\varphi(x_0) : E_J \longrightarrow E_I$; then, $J_\varphi(x_0)$ is a linear (m,σ)-general function; moreover, for any $n \in \mathbf{N}$, $n \geq m$, there exists the linear (m,σ)-general function $J_{\overline{\varphi}^{(n,n)}}(x_0) : E_J \longrightarrow E_I$, and one has*

$$\det J_\varphi(x_0) = \lim_{n\to+\infty} \det J_{\overline{\varphi}^{(n,n)}}(x_0).$$

Proof. Since φ is (m,σ)-general, from Remark 3.2, the linear function $J_\varphi(x_0)$ is (m,σ)-general; moreover, $\forall\, n \in \mathbf{N}$, $n \geq m$, from Proposition 2.4, there exists the linear function $J_{\overline{\varphi}^{(n,n)}}(x_0) : E_J \longrightarrow E_I$, and it is (m,σ)-general, from Remark 3.2; furthermore, we have $\mathcal{A}(J_\varphi(x_0)) = \mathcal{A}\left(J_{\overline{\varphi}^{(n,n)}}(x_0)\right)$.

If $\mathcal{A}(J_\varphi(x_0)) \neq \emptyset$, set $\overline{m} = \min \mathcal{A}(J_\varphi(x_0))$; $\forall\, n \geq \overline{m}$, we have

$$\det J_{\overline{\varphi}^{(n,n)}}(x_0) = \det J_{\varphi^{(n,n)}}(x_{0,j} : j \in J_n) \prod_{i \in I\setminus I_n} \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right); \tag{48}$$

if $|(I\setminus I_m)\setminus \mathcal{I}_\varphi| < +\infty$, set $i_0 = \max((I\setminus I_m)\setminus\mathcal{I}_\varphi)$ and $\widehat{m} = \max\{\overline{m}, |i_0|\}$; since $\prod_{i\in I\setminus I_{\widehat{m}}} \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) \in \mathbf{R}^*$, we have $\lim_{n\to+\infty} \prod_{i\in I\setminus I_n} \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right) = 1$; then, from (48) and Theorem 3.6, we obtain

$$\lim_{n\to+\infty} \det J_{\overline{\varphi}^{(n,n)}}(x_0) = \lim_{n\to+\infty} \det J_{\varphi^{(n,n)}}(x_{0,j} : j \in J_n) = \det J_\varphi(x_0);$$

conversely, suppose that $|(I\setminus I_m)\setminus\mathcal{I}_\varphi| = +\infty$; for n sufficiently large, we have $\det J_{\varphi^{(n,n)}}(x_{0,j} : j \in J_n) = 0$, from which

$$\begin{aligned}\det J_\varphi(x_0) &= \lim_{n\to+\infty} \det J_{\varphi^{(n,n)}}(x_{0,j} : j \in J_n) = 0\\ &= \lim_{n\to+\infty} \det J_{\varphi^{(n,n)}}(x_{0,j} : j \in J_n) \prod_{i\in I\setminus I_n} \varphi'_{i,\sigma(i)}\left(x_{0,\sigma(i)}\right)\\ &= \lim_{n\to+\infty} \det J_{\overline{\varphi}^{(n,n)}}(x_0).\end{aligned}$$

Moreover, if $\mathcal{A}(J_\varphi(x_0)) = \emptyset$, $\forall\, n \in \mathbf{N}$, $n \geq m$, we have $\mathcal{A}\left(J_{\overline{\varphi}^{(n,n)}}(x_0)\right) = \emptyset$, and so

$$\det J_\varphi(x_0) = 0 = \lim_{n\to+\infty} \det J_{\overline{\varphi}^{(n,n)}}(x_0).$$

□

Example 3.19: Consider the linear function $A = (a_{ij})_{i,j\in\mathbf{N}^*} : E_{\mathbf{N}^*} \longrightarrow E_{\mathbf{N}^*}$ given by

$$(Ax)_i = \begin{cases} \sum\limits_{j\in\mathbf{N}^*} 2^{-j}x_j & \text{if } i=1 \\ x_1 + \sum\limits_{j\in\mathbf{N}^*} 2^{-j}x_j & \text{if } i=2 \\ 2^{-i}x_1 + 2^{2^{-i}} & \text{if } i \in \mathbf{N}^*\backslash\{1,2\} \end{cases}, \forall\, x = (x_j : j \in \mathbf{N}^*) \in E_{\mathbf{N}^*}.$$

Then, A is a strongly (m,σ)-general function, where $I = J = \mathbf{N}^*$, $m = 2$, $I_m = J_m = \{1,2\}$, σ is the function given by $\sigma(i) = i$, $\forall\, i \in \mathbf{N}^*\backslash\{1,2\}$, and $\mathcal{A} = \mathbf{N}^*\backslash\{1\} \neq \emptyset$; moreover, we have $\lambda_i = 2^{2^{-i}}$, $\forall\, i \in \mathbf{N}^*\backslash\{1,2\}$.

In order to calculate $\det A$, observe that $A^{(\{2\},\mathbf{N}^*)} = u + v$, where $u = A^{(\{1\},\mathbf{N}^*)} \in E_{\mathbf{N}^*}$, and $v = (v_j : j \in \mathbf{N}^*) \in E_{\mathbf{N}^*}$, where $v_j = \delta_{j1}$, $\forall\, j \in \mathbf{N}^*$. Then, from Proposition 3.11, we have $\det A = \det U + \det V$, where $U = (u_{ij})_{i,j\in\mathbf{N}^*}$ and $V = (v_{ij})_{i,j\in\mathbf{N}^*}$ are the linear functions obtained by substituting the second row of A by u and v, respectively; moreover, since $U^{(\{1\},\mathbf{N}^*)} = U^{(\{2\},\mathbf{N}^*)}$, we have $\det U = 0$, from which

$$\det A = \det V = \lim_{n\longrightarrow+\infty} \det V^{(n,n)}. \tag{49}$$

Finally, $\forall\, n \in \mathbf{N}^*\backslash\{1,2\}$, we have

$$\begin{aligned} \det V^{(n,n)} &= (-1)^{n+1}2^{-n} \det V^{(n-1,\{2,\ldots,n\})} + 2^{2^{-n}} \det V^{(n-1,n-1)} \\ &= 2^{2^{-n}} \det V^{(n-1,n-1)}, \end{aligned} \tag{50}$$

since the second row of $V^{(n-1,\{2,\ldots,n\})}$ is zero, and so $\det V^{(n-1,\{2,\ldots,n\})} = 0$. Then, by recursion, from (50) we obtain

$$\det V^{(n,n)} = \det V^{(2,2)} \prod_{j=3}^{n} 2^{2^{-n}},$$

and so formula (49) implies

$$\det A = \lim_{n\longrightarrow+\infty} \det V^{(2,2)} \prod_{j=3}^{n} 2^{2^{-n}} = \det V^{(2,2)} 2^{\sum\limits_{j=3}^{+\infty} 2^{-n}} = -\frac{1}{4}\sqrt[4]{2}.$$

4. Problems for further study

A natural extension of this paper and of the paper [4] is the generalization of the change of variables' formula for the integration of the measurable real functions on $\left(\mathbf{R}^I, \mathcal{B}^{(I)}\right)$, by substituting the (m,σ)-standard functions for the (m,σ)-general functions.

Moreover, a natural application of this paper, in the probabilistic framework, is the development of the theory of the infinite-dimensional continuous random elements, defined in the paper [3]. In particular, we can prove the formula of the density of such random elements composed with the (m,σ)-general functions, with further properties. Consequently, it is possible to introduce many random elements that generalize the well known continuous random vectors in $\mathbf{R}^m$ (for example, the Beta random elements in E_I defined by the (m,σ)-general matrices), and to develop some theoretical results and some applications in the statistical inference. It is possible also to define a convolution between the laws of two independent and infinite-dimensional continuous random elements, as in the finite case.

Furthermore, we can generalize the paper [2] by considering the recursion $\{X_n\}_{n\in\mathbf{N}}$ on $[0,p)^{\mathbf{N}^*}$ defined by

$$X_{n+1} = AX_n + B_n \pmod p,$$

where $X_0 = x_0 \in E_I$, A is a bijective, linear, integer and (m,σ)-general function, $p \in \mathbf{R}^+$, and $\{B_n\}_{n\in\mathbf{N}}$ is a sequence of independent and identically distributed random elements on E_I. Our target is to prove that, with some assumptions on the law of B_n, the sequence $\{X_n\}_{n\in\mathbf{N}}$ converges with geometric rate to a random element with law $\bigotimes_{i\in\mathbf{N}^*}\left(\frac{1}{p}Leb\Big|_{\mathcal{B}([0,p))}\right)$. Moreover, we wish to quantify the rate of convergence in terms of A, p, m, and the law of B_n.

Finally, in the statistical mechanics, it is possible to describe the systems of smooth hard particles, by using the Boltzmann equation or the more general Master kinetic equation, described for example in the paper [9]. In order to study the evolution of these systems, we can consider the model of countable particles, such that their joint infinite-dimensional density can be determined by composing a particular random element with a (m,σ)-general function.

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