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Exploring first integrals of homogeneous Lagrangian systems through nonlocal constants

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Dedicated to Gioele, future explorer of worlds

Abstract. *In this paper we study autonomous systems whose Lagrangian function is the combination of several homogeneous terms with respect to positions and velocities. We show that, assuming certain relations between the degrees of homogeneity of such terms, the systems considered possess (in addition to energy) a further first integral that provides information about their solutions. A new feature of these results is the use of the theory of* nonlocal constants*, which finds useful constants using one-parameter perturbed motions.*

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1. Introduction

First integrals represent a fundamental concept in the study of dynamical systems. They serve various purposes, including acting as significant constants that can (i) yield solutions through quadratures [10] and (ii) confine motion to a limited domain within phase space [15]. Furthermore, in numerous systems, first integrals are often the sole computable indicators of the efficacy of numerical methods employed for their study [14].

Focusing on systems characterized by an *autonomous*[1] Lagrangian function $\mathcal{L}(q,\dot{q})$, we say that for any *Euler-Lagrange* equation of motion of the form

$$\big[\nabla_{\dot{q}}\mathcal{L}(q,\dot{q})\big]^{\cdot} - \nabla_q\mathcal{L}(q,\dot{q}) = 0\,, \qquad q\in\mathbb{R}^n\,, \tag{1}$$

a smooth point-function $I(t,q,\dot{q})$ is a *first integral* if it remains constant when evaluated along any solution $t\mapsto q(t)$ of (1). In this context, the notation $\nabla_q = \partial/\partial q\in\mathbb{R}^n$ and $\nabla_{\dot{q}} = \partial/\partial\dot{q}\in\mathbb{R}^n$ is employed, while the dot signifies total

[1] An autonomous Lagrangian function $\mathcal{L}(q,\dot{q})$ doesn't explicitly depend on time.

derivative with respect to time $t \in \mathbb{R}$. Thus, for a first integral, the condition $\dot{I}|_{q(t)} = 0$ must hold true.

The identification of (non-trivial) first integrals typicailly necessitates mathematical proficiency and extensive analytical effort. Nevertheless, a comprehensive range of techniques for their construction can be found in the literature [11]. Notably, the celebrated *Noether's Theorem* [12] asserts that if the Lagrangian $\mathcal{L}$ exhibits a symmetry, then a corresponding first integral exists. Recently, Gorni and Zampieri [6] have put forth a rethinking of Noether's Theorem by introducing the concept of *nonlocal constants* of motion. These constants are defined as functions that maintain their values along the trajectories of the system described by equation (1), while their specific values at any given time t are determined by the historical trajectory of the motion.

In our investigation of the implications related to this nonlocal framework, we noted that the authors of [6] have developed an new approach that recovers a well-known result by Logan (L) [13], which focuses on deriving energy-dependent first integrals from these nonlocal constants. Specifically, their system was characterized by an autonomous Lagrangian $\mathcal{L}_L = K(\dot{q}) - U(\|q\|)$, where $K(\dot{q}) = \frac{1}{2}\|\dot{q}\|^2$ and $U(\|q\|)$ represents a *homogeneous* potential of degree $\kappa_U = -2$. The first integral they identified was $I_L = \dot{q} \cdot q - 2E_L t$, with E_L the (conserved) energy of the system. For the benefit of a broader audience, it is important to recall that a function $\varphi : x \mapsto \varphi(x)$ is said to be homogeneous of degree κ_φ if $\varphi(sx) = s^{\kappa_\varphi}\varphi(x)$ for any $s \in \mathbb{R}$. Hence, in our case, $U(\|sq\|) = s^{-2}U(\|q\|)$.

In examining Logan's example, we asked ourselves what general property connects the expression of the kinetic term K with the specified degree of homogeneity $\kappa_U = -2$ for U, which is necessary for the system to potentially exhibit a first integral such as I_L. Through this investigation, we found an intriguing derivation demonstrating that when the Lagrangian function is formulated as a combination of homogeneous terms, each subject to specific homogeneity constraints, the presence of a first integral like I_L is readily guaranteed, as outlined in our main result below:

THEOREM 1.1. *Let $\sigma_{1,2,3} \in \{0,1\}$ and consider the class of autonomous Lagrangian systems*

$$\mathcal{L}(q,\dot{q}) = \sigma_1 K(\dot{q}) + \sigma_2\, g(q) f(\dot{q}) - \sigma_3 U(q)\,, \tag{2}$$

where the kinetic term $K(\dot{q})$, the potential $U(q)$, and $g(q), f(\dot{q})$ are non-zero homogeneous functions of degrees κ_K, κ_U and κ_g, κ_f, respectively. If these functions satisfy the following conditions:

$$\begin{cases} \big[\kappa_K - \Upsilon_\kappa(1-\kappa_K)\big]\sigma_1 = 0\,, \\ \big[\kappa_g + \kappa_f - \Upsilon_\kappa(1-\kappa_f)\big]\sigma_2 = 0\,, & \textit{for some } \Upsilon_\kappa \in \mathbb{R}\,, \\ \big[\kappa_U - \Upsilon_\kappa\big]\sigma_3 = 0\,, \end{cases} \tag{3}$$

then, I is constant along the solutions of the Euler-Lagrange equation for (2)*, where*

$$I = \nabla_{\dot{q}}\mathcal{L} \cdot q + \Upsilon_\kappa E t\,, \tag{4}$$

and $E \equiv \nabla_{\dot{q}}\mathcal{L} \cdot \dot{q} - \mathcal{L}$ is the (conserved) energy of the system.

In this context, the coefficients $\sigma_{1,2,3}$ take on values from the set $\{0, 1\}$, thereby determining which terms do not vanish in expression (2). Consequently, it is possible that, depending on the parameterization of $\mathcal{L}$, some conditions in (3) are trivially fulfilled, as the coefficients σ_i can nullify the terms located to the left hand side of the equalities. It is also noteworthy that the condition (3) is not influenced by the specific expression of the functions that parameterize our Lagrangian (2); rather, it is solely determined by their degree of homogeneity.

In practical terms, the procedure for verifying the operational efficiency of our machinery entails simple calculations. Specifically, one must (i) confirm that the given Lagrangian conforms to the structure outlined in (2) with homogeneous terms, (ii) determine the coefficients $\sigma_{1,2,3}$ and the relevant degrees of homogeneity $\{\kappa_i\}$, (iii) resolve the system presented in (3) for the variable Υ_κ, and (iv) substitute the obtained Υ_κ into expression (4).

Example 1.2. Consider the Lagrangian $\mathcal{L} = \lambda\|\dot{q}\|^m - U(\|q\|)$, where λ is a constant in $\mathbb{R}$. It is evident that $\sigma_1 = \sigma_3 = 1$ and $\sigma_2 = 0$, with $\kappa_K = m$. Consequently, from equation (3), the degree of homogeneity κ_U is given by $\kappa_U = \Upsilon_\kappa = \kappa_K/(1-\kappa_K) = m/(1-m)$. In the specific instance of $\mathcal{L}_L = \frac{1}{2}\|\dot{q}\|^2 - U(\|q\|)$ as referenced in [6], we find that $m = 2$ and $\lambda = 1/2$. Therefore, to derive a first integral from our Theorem 1.1, the potential $U(\|q\|)$ must exhibit a homogeneity degree of $\kappa_U = \Upsilon_\kappa = -2$, which aligns with expectations. The corresponding first integral, as expressed in (4), is $I_L = \dot{q} \cdot q - 2E_L t$, indicating its dependence on energy E_L.

Our paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1, with a particular emphasis on the introduction of the theory of non local constants of motion and its application in deriving energy conservation and the existence of the first integral presented in (4). The subsequent section, Section 3, explores various applications of our findings, including specific cases such as (i) the Poincaré half-plane model, (ii) a Painlevé-Gambier equation, (iii) a particle system with Calogero's potential, and (iv) a generalization of Logan's result.

As a final remark, we clarify that throughout this work $\mathbb{R}^n$ will be the usual Euclidean space. Additionally, we will adopt the notation (i) $x \cdot y$ to denote the inner product, and (ii) $\|x\| = \sqrt{x \cdot x}$ to represent the norm within $\mathbb{R}^n$.

2. Proof of Theorem 1.1

To provide a proof of our Theorem 1.1, we employ the framework of nonlocal constants, which identifies valuable constants through one-parameter perturbed motions. The fundamental finding regarding nonlocal constants of motion, as presented in [6], can be stated in a self-sufficient manner, providing all the necessary information for the subsequent discussion.

THEOREM 2.1 (Nonlocal Constants). *Let $t \mapsto q(t)$ be a solution of* (1)*, and let $q_\varepsilon(t)$, $\varepsilon \in \mathbb{R}$, be a one-parameter smooth family such that $q_0(t) = q(t)$. Then, I is constant along the solutions of* (1)*, with*

$$I = \nabla_{\dot q}\mathcal{L}(q,\dot q)\cdot \partial_\varepsilon q_\varepsilon\big|_{\varepsilon=0} - \int_{t_0}^{t} ds\, \partial_\varepsilon \mathcal{L}(q_\varepsilon,\dot q_\varepsilon)\big|_{\varepsilon=0}\,. \tag{5}$$

Proof. Take the time derivative of expression (5) and change the order of differentiation so that the Euler-Lagrange equation (1) can be applied. □

2.1. Energy conservation

The principle of energy conservation serves as a simple example of a conservation law that can be derived from nonlocal constants when the Lagrangian function is autonomous (as it is also well known from the Noetherian approach). Given that our Lagrangian (2) is explicitly independent of time, we introduce the time-shift family defined by $q_\varepsilon = q(t+\varepsilon)$. This leads to the relation $\partial_\varepsilon q_\varepsilon|_{\varepsilon=0} = \dot q$, allowing us to utilize (5) for the computation of the nonlocal constant as follows

$$I = \nabla_{\dot q}\mathcal{L}\cdot \dot q - \int_{t_0}^{t} ds \left(\nabla_q \mathcal{L}\cdot \dot q + \nabla_{\dot q}\mathcal{L}\cdot \ddot q\right). \tag{6}$$

In this context, under the assumption that q_ε is a smooth function, we have used that $\partial_\varepsilon \dot q_\varepsilon|_{\varepsilon=0} = (\partial_\varepsilon q_\varepsilon|_{\varepsilon=0})^{\cdot} = \ddot q$. It is important to recognize that the integrand in (6) represents the total derivative $\dot{\mathcal{L}}$, which leads us to conclude that the energy defined by

$$E \equiv \nabla_{\dot q}\mathcal{L}\cdot \dot q - \mathcal{L}\,, \tag{7}$$

constitutes a conserved quantity within the framework of our system.

2.2. Proof of Theorem 1.1

As outlined in the introduction, our focus is on autonomous Lagrangian functions $\mathcal{L}(\dot q, q)$. By expressing the nonlocal constant (5) in relation to (2), we arrive at the following formulation:

$$I = \nabla_{\dot q}\mathcal{L}(q, \dot q) \cdot \partial_\varepsilon q_\varepsilon\big|_{\varepsilon=0} - \int_{t_0}^{t} \xi \, ds \,. \tag{8}$$

Here, the integrand $\xi = \xi[q(s), \dot q(s)]$ has expression:

$$\xi = \partial_\varepsilon \mathcal{L}_\varepsilon|_{\varepsilon=0} = \nabla_q \mathcal{L} \cdot \partial_\varepsilon q_\varepsilon|_{\varepsilon=0} + \nabla_{\dot q}\mathcal{L} \cdot (\partial_\varepsilon q_\varepsilon|_{\varepsilon=0})^{\cdot} \,. \tag{9}$$

Drawing from the perturbed motions adopted in references [2, 6], we choose our perturbed motion as follows

$$q_\varepsilon(t) = e^{\varepsilon} q\big[h_\varepsilon(t)\big]. \tag{10}$$

In this formulation, $h_\varepsilon(t)$ is a free function dependent on the parameters (ε, t), with the condition that $h_\varepsilon(t)|_{\varepsilon=0} = t$. Consequently, we derive that

$$\partial_\varepsilon q_\varepsilon|_{\varepsilon=0} = q + \dot q \cdot v, \tag{11}$$

where $v = v(t) = \partial_\varepsilon h_\varepsilon|_{\varepsilon=0}$. Usefully, since q_ε is assumed to be a smooth function, we can also write $\partial_\varepsilon \dot q_\varepsilon|_{\varepsilon=0} = (\partial_\varepsilon q_\varepsilon|_{\varepsilon=0})^{\cdot}$.

Utilizing the aforementioned fact in conjunction with the parametrization outlined in equation (2), it can be determined that the integrand represented in expression (9) is expressible as:

$$\begin{aligned}\xi = \dot{\mathcal{L}} v + \big\{&\sigma_1(\nabla_{\dot q}K \cdot \dot q) + \sigma_2\big[(\nabla_q g \cdot q)f + g(\nabla_{\dot q} f \cdot \dot q)\big] \\ &- \sigma_3(\nabla_q U \cdot q) + \big[\sigma_1(\nabla_{\dot q}K \cdot \dot q) + \sigma_2\, g\,(\nabla_{\dot q} f \cdot \dot q)\big]\dot v\big\} \,.\end{aligned} \tag{12}$$

In general, since (12) does not represent a total derivative, the expression (8) formally contains an integral that we aim to eliminate. The situation can be simplified by recognizing that certain terms of (12) can be combined into total derivatives through a suitable selection of the parameter v. Specifically, the configuration of expression (12) indicates that the term enclosed in curly brackets should complete the total derivative $(\mathcal{L}v)^{\cdot} = \dot{\mathcal{L}}v + \mathcal{L}\dot v$. To realize this, we impose the condition

$$\begin{aligned}&\sigma_1(\nabla_{\dot q}K \cdot \dot q) + \sigma_2\big[(\nabla_q g \cdot q)f + g(\nabla_{\dot q} f \cdot \dot q)\big] \\ &\qquad - \sigma_3(\nabla_q U \cdot q) + \big[\sigma_1(\nabla_{\dot q}K \cdot \dot q) + \sigma_2\, g\,(\nabla_{\dot q} f \cdot \dot q)\big]\dot v = \mathcal{L}\dot v \,.\end{aligned} \tag{13}$$

In this way it is immediate to check that $\xi = (\mathcal{L}v)^{\cdot}$. Hence, by using expression (11) the nonlocal constant (8) becomes the conserved first integral

$$\begin{aligned} I &= \nabla_{\dot q}\mathcal{L} \cdot (q + \dot q v) - \mathcal{L}v \,, \\ &= \nabla_{\dot q}\mathcal{L} \cdot q + (\nabla_{\dot q}\mathcal{L} \cdot \dot q - \mathcal{L})v \,, \\ &= \nabla_{\dot q}\mathcal{L} \cdot q + Ev \,. \end{aligned} \tag{14}$$

Here, the E symbol represents the (conserved) energy of the system, as derived in (7) of Section 2.1. It is important to observe that the expression in (14) still depends on the arbitrary function $v(t)$, which we now aim to determine.

By construction, the first order differential equation (13) constraints $v(t)$ as a function of K, f, g and U. So, separating the variables, we get

$$v = \int_{t_0}^{t} \frac{\sigma_1(\nabla_{\dot{q}} K \cdot \dot{q}) + \sigma_2\big[(\nabla_q g \cdot q) f + g(\nabla_{\dot{q}} f \cdot \dot{q})\big] - \sigma_3(\nabla_q U \cdot q)}{\mathcal{L} - \sigma_1(\nabla_{\dot{q}} K \cdot \dot{q}) - \sigma_2\, g\,(\nabla_{\dot{q}} f \cdot \dot{q})}\, d\tau\,. \tag{15}$$

By hypothesis the K, f, g and U functions are assumed to be homogeneous. This assumption leads to the applicability of *Euler's Theorem* for homogeneous functions, which states that $x \cdot \nabla_x \vartheta(x) = \kappa_\vartheta \vartheta(x)$, where ϑ is identified as a homogeneous function of degree κ_ϑ. Consequently, the solution (15) can be rewritten as follows

$$v = \int_{t_0}^{t} \frac{\sigma_1 \kappa_K K + \sigma_2(\kappa_g + \kappa_f) g f - \sigma_3 \kappa_U U}{\sigma_1(1-\kappa_K)K + \sigma_2(1-\kappa_f) g f - \sigma_3 U}\, d\tau\,. \tag{16}$$

When this expression is replaced in the equation (14), it retains its non-local characteristics due to the presence of the integration operator. One may inquire about the specific circumstances under which the expression (16) can be rewritten as a local constant, i.e. as a local function of t that does not require integration over time. A straightforward scenario to consider is when the numerator is directly proportional to the denominator

$$\begin{aligned}\sigma_1 \kappa_K K + \sigma_2(\kappa_g + \kappa_f) g f - \sigma_3 \kappa_U U &= \\ &\Upsilon_\kappa \big[\sigma_1(1-\kappa_K)K + \sigma_2(1-\kappa_f) g f - \sigma_3 U\big]\,. \end{aligned} \tag{17}$$

This relation must hold for all variables K, g, f, U, where Υ_κ represents an arbitrary proportionality constant. By aligning the terms with respect to the basis of σ_i, one derives the conditions outlined in (3).

Consequently, from (16), we arrive at the expression

$$v = \Upsilon_\kappa \int_{t_0}^{t} d\tau = \Upsilon_\kappa t\,, \tag{18}$$

in which t_0 has been set to zero without loss of generality. Hence, by substituting (18) into (14), we ultimately obtain our final result as presented in (4). This concludes our demonstration. □

Remark 2.2. Given that $\partial_\varepsilon h_\varepsilon|_{\varepsilon=0} = v = \Upsilon_\kappa t$, there exist multiple formulations for $h_\varepsilon(t)$ that can fulfill this condition. Among the most fundamental expressions, we mention

$$h_\varepsilon(t) = t e^{\varepsilon \Upsilon_\kappa t} \quad \text{or} \quad h_\varepsilon(t) = (1 + \varepsilon \Upsilon_\kappa) t\,. \tag{19}$$

In this way, the one-parameter family described in (10) will be uniquely determined.

REMARK 2.3. We dedicate a brief remark to double check that I is a first integral for the system described by (2). The calculation is straightforward, considering that

$$\nabla_q \mathcal{L} = \sigma_2 \kappa_g f g - \sigma_3 \kappa_U U \qquad \text{and} \qquad \nabla_{\dot q} \mathcal{L} = \sigma_1 \kappa_K K + \sigma_2 \kappa_f f g \,. \tag{20}$$

Indeed, it suffices to compute the time derivative of I to establish that

$$\begin{aligned} \dot I &= (\nabla_{\dot q}\mathcal{L})^{\cdot} \cdot q + \nabla_{\dot q}\mathcal{L} \cdot \dot q + \Upsilon_\kappa E \,, \\ &= (\nabla_{\dot q}\mathcal{L})^{\cdot} \cdot q + \nabla_{\dot q}\mathcal{L} \cdot \dot q + \Upsilon_\kappa (\nabla_{\dot q}\mathcal{L} \cdot \dot q - \mathcal{L}) \,, \\ &= \big[\kappa_K - \Upsilon_\kappa (1-\kappa_K)\big] K \sigma_1 + \big[\kappa_g + \kappa_f - \Upsilon_\kappa (1-\kappa_f)\big] g f \sigma_2 + \big[\kappa_U - \Upsilon_\kappa\big] U \sigma_3 \,. \end{aligned} \tag{21}$$

In the second step, we utilized the fact that $q(t)$ is a solution to the system, thereby satisfying the Euler-Lagrange equations (1). The final term is evidently zero, as the conditions (3) hold true by assumption. Consequently, I remains constant along the solutions corresponding to the Euler-Lagrange equations of (2).

3. Applications

In the present section we deal with some neat applications of our theorem to well-known systems.

3.1. Poincaré half plane

We consider a model called *Poincaré half plane*, namely the set $q = q_1 \hat e_1 + q_2 \hat e_2 \in \mathbb{R}^2$, $q_2 > 0$, where we call "straight line" any half-circle with center on the x-axis and any vertical half-line. Here, $\hat e_i$ are the basis versors of $\mathbb{R}^2$. Interestingly, the Poincaré half plane model can be formulated in the language of variational mechanics by the following Lagrangian function

$$\mathcal{L} = \frac{\|\dot q\|^2}{2\,(q \cdot \hat e_2)^2} \,. \tag{22}$$

The Euler-Lagrange equation (1) for the Lagrangian function (22) reads

$$\ddot q - 2\,\frac{\dot q \cdot \hat e_2}{q \cdot \hat e_2}\,\dot q + \frac{\|\dot q\|^2}{q \cdot \hat e_2}\,\hat e_2 = 0 \,. \tag{23}$$

In the spirit of Theorem 1.1, after a comparison of expression (22) with our parametrization (2), we identify $\sigma_1 = \sigma_3 = 0$ and $\sigma_2 = 1$. We are attracted to this Lagrangian because the σ_2-contributions $f = \|\dot{q}\|^2$ and $g = (2\, q \cdot \hat{e}_2)^{-1}$, with $\kappa_f = -\kappa_g = 2$, satisfy the system (3) with $\Upsilon_\kappa = 0$ Consequently, this Lagrangian has the following function as immediate conserved quantity (4)

$$I = \frac{\dot{q} \cdot q}{(q \cdot \hat{e}_2)^2} \,. \tag{24}$$

has been discussed from a nonlocal point of view in [7].

3.2. Autonomous Painlevé-Gambier equation XXII

Painlevé-Gambier equations are attracting much attention in last years, since some problems related to their solutions, the Painlevé transcendents, are already under discussion [1, 8]. In particular, it is well known that the only movable singularities of these equations are poles.

Interestingly, the Lagrangian for the autonomous Painlevé-Gambier equation XXII

$$\mathcal{L} = \tfrac{1}{2} q^{-3/2} \dot{q}^2 + 2 q^{-1/2} \,, \tag{25}$$

belongs to the parametrization (2). The Euler-Lagrange equation (1) for the Lagrangian function (25) reads

$$\ddot{q} - \tfrac{3}{4} q^{-1} \dot{q}^2 + 1 = 0 \,. \tag{26}$$

First of all, a comparison between the two Lagrangians yields $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 = 1$. Then, we deduce that $g = \frac{1}{2} q^{-3/2}$, $f = \dot{q}^2$ and $U = -2q^{-1/2}$. Since $\kappa_g = -3/2$ and $\kappa_f = 2$ and $\kappa_U = -1/2$, the second and the third conditions of (3) are satisfied by $\Upsilon_\kappa = -1/2$. Hence from (4) we get the conserved quantity:

$$I = q^{-1/2} \dot{q} - \tfrac{1}{2} t E \,, \tag{27}$$

where $E = \frac{1}{2} q^{-3/2} \dot{q}^2 - 2q^{-1/2}$. Expression (27) is equivalent to a Cauchy problem for a first order differential equation with separated variables. It is easy to see that the general solution of this equation is

$$q(t) = \tfrac{1}{4} \Big\{ \tfrac{1}{4} \bar{E} t^2 + \bar{I} t + \tfrac{1}{\bar{E}} (\bar{I}^2 - 4) \Big\}^2 \,. \tag{28}$$

Here, the $\bar{E}$ and $\bar{I}$ parameters are constant.

3.3. Calogero's potential

In [6] Gorni e Zampieri show that for Lagrangians of the form $\mathcal{L} = \frac{1}{2}\|\dot{q}\|^2 - U(q)$ with a homogeneous potential $U(q)$ of degree -2 the nonlocal constant (5) reduces to a useful point-function of $(\dot{q}, q)$ and t. Inspired by this, we expect that such result must be recovered by our machinery. In fact, since in this case $\sigma_1 = \sigma_3 = 1$ and $\sigma_2 = 0$, from (3) we obtain $\kappa_U = -\kappa_K = -2$.

A famous potential that is homogenous with degree $\kappa_U = -2$ is the so-called *Calogero's inverse-square scattering* potential for the n-body problem [9]. With the kinetic term, the full Lagrangian of the system reads

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n} \|\dot{q}_i\|^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\|q_i - q_j\|^2}\,. \tag{29}$$

Then, the Euler-Lagrange equation (1) for expression (29) is

$$\ddot{q}_i - 2\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\|q_i - q_j\|^3}\frac{q_i - q_j}{\|q_i - q_j\|} = 0 \qquad \text{for} \quad i = 1, ..., n\,. \tag{30}$$

Since $\Upsilon_\kappa = \kappa_U = -2$, Theorem 1.1 tells us that we can obtain a conserved quantity using expression (4), that is

$$I = \sum_{i=1}^{n}\left\{\dot{q}_i q_i - t\left[\|\dot{q}_i\|^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{\|q_i - q_j\|^2}\right]\right\}. \tag{31}$$

3.4. Generalization of Logan's result

Let us reexamine the Logan system as presented in [13], characterized by the Lagrangian $\mathcal{L}_L = \frac{1}{2}\|\dot{q}\|^2 - U(\|q\|)$, where $U(\|q\|)$ represents a homogeneous potential of degree $\kappa_U = -2$. In the introduction of this article, we have applied Theorem 1.1 to establish that, consistent with existing literature, the Lagrangian $\mathcal{L}_L$ possesses a first integral given by $I_L = \dot{q}\cdot q - 2E_L t$ with $\Upsilon_\kappa = -2$.

Now, we will explore a potential generalization of Logan's system, parameterized as follows:

$$\mathcal{L} = \mathcal{L}_L + g(q)f(\dot{q})\,, \tag{32}$$

where g and f are non-zero homogeneous functions of degrees κ_g and κ_f, respectively. According to our theorem, this generalized system retains equivalence to the original Logan case, with the exception of an additional term weighted by $\sigma_2 = 1$. This modification ensures that the first and third conditions of (3) are inherently satisfied, mirroring the scenario in Logan's case with $\Upsilon_\kappa = -2$. Furthermore, it is evident from (3) that if the degrees κ_g and κ_f fulfill the new condition

$$\kappa_g = \kappa_f - 2\,, \tag{33}$$

which constrains only one degree of homogeneity while leaving the other entirely arbitrary, then the generalization expressed in (36) will also possess a first integral in a form analogous to I_L, albeit with a distinct expression for $\nabla_{\dot q}\mathcal{L}$

$$I = (\dot q + g\,\nabla_{\dot q} f)\cdot q - 2\bar E t\,. \tag{34}$$

In the realm of systems belonging to the class of these Lagrangian functions, one could consider a set of n particle interacting through the combination of a Calogero's potential ($\mathcal{L}_L$) and a gravitational (or Coulomb) potential ($\kappa_g = -1$) with the mass $m_i(\dot q)$ (or electrical density $\varrho(\dot q)$) varying linearly with $\dot q$ ($\kappa_g = 1$).

3.4.1. An interesting subcase

It is noteworthy that by selecting a specific subcase of our Lagrangian function (36), we can reformulate expression (34) to resemble the conservation law of an additional first integral.

By integrating (34) with respect to time, we derive the following equation:

$$\left(\frac{1}{2}\|q\|^2 - \frac{1}{2}\bar E t^2 - \bar I t + \int \nabla_{\dot q} f \cdot g q \, dt\right)^{\cdot} = 0\,. \tag{35}$$

It is evident that if the total derivative condition, $\nabla_{\dot q} f\cdot gq = \dot\rho$, holds for some function $\rho = \rho(q)$, then (35) simplifies to a total derivative with respect to time. It is important to note that $\nabla_{\dot q} f\cdot gq$ is independent of $\ddot q$, which implies that ρ cannot depend on $\dot q$. Hence, given that $\dot\rho = \nabla_q\rho\cdot\dot q$ is linear in $\dot q$, the only viable option to fulfill the total derivative condition is to set $f = \frac{1}{2}\|\dot q\|^2$. Given that $\kappa_f = 2$, we can derive from relation (33) that $\kappa_g = 0$. This indicates that any function g satisfying the condition $g(q)q = \nabla_q\Gamma$ for some scalar function $\Gamma = \Gamma(q)$ will complete the total derivative condition.

By consolidating these elements, we find that if

$$\mathcal{L} = \mathcal{L}_L + \frac{1}{2}\frac{\|\dot q\|^2}{\|q\|^2}\,\nabla_q\Gamma(q)\cdot q\,, \tag{36}$$

then $K = \frac{1}{2}\|q\|^2 - \frac{1}{2}\bar E t^2 - \bar I t + \Gamma(q)$, is a first integral for (36). Here, the parameters $\bar E$ and $\bar I$ are constants.

As a consequence, a constraint emerges for the solutions to the corresponding Euler-Lagrange equations, which is solely dependent on time. In fact, from (35) we get

$$\tfrac{1}{2}\|q\|^2 + \Gamma(q) = \tfrac{1}{2}\bar E t^2 + \bar I t + \bar K\,. \tag{37}$$

Interestingly, in the $\mathcal{L} = \mathcal{L}_L$ case, $\Gamma(q) = 0$ and $\|q\| = \sqrt{\bar E t^2 + 2\bar I t + 2\bar K}$. It is noteworthy that this result has been achieved without the necessity of solving the equations of motion.

Statements and Declarations

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