

On Hardy's Inequality for Geometric Sums and Series

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ABSTRACT. *We prove the inequalities*

$$\sum_{j=1}^n \left(\frac{1}{j} \sum_{k=1}^j x^{k-1} \right)^3 < \alpha \sum_{j=1}^n (x^{j-1})^3 \quad (0 < x < 1, n \geq 1)$$

and

$$\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{k=1}^j x^{k-1} \right)^3 < a \sum_{j=1}^{\infty} (x^{j-1})^3 \quad (0 < x < 1),$$

where the best possible constants are given by

$$\alpha = a = \frac{27}{2} \log(3) - 18 \log(2) = 2.35461....$$

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1. Introduction and statement of the main results

I. The discrete form of Hardy's inequality states that if $p > 1$ and $a_j \geq 0$ ($j \geq 1$), then

$$\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{k=1}^j a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{j=1}^{\infty} a_j^p. \quad (1)$$

G.H. Hardy [1] proved (1) with the factor $(p^2/(p-1))^p$ in 1920. In a private communication, E. Landau informed him that (1) holds and that the factor $(p/(p-1))^p$ is best possible; see Hardy [2]. Moreover, Landau [6] offered the following counterpart of (1) for finite sums:

$$\sum_{j=1}^n \left(\frac{1}{j} \sum_{k=1}^j a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{j=1}^n a_j^p, \quad (2)$$

valid for $p > 1$, $a_j \geq 0$ ($1 \leq j \leq n$) and all $n \geq 1$. This improves a result of Hardy [1] who found a short proof for (2) again with the factor $(p^2/(1-p))^p$.

Both inequalities attracted the attention of many mathematicians who found numerous generalizations, refinements, integral analogues and remarkable variants of (1) and (2). For detailed information on this subject we refer to the paper [4] which describes the scientific developments of (1) and its integral version and to the excellent monograph [5] (with about 300 references) of A. Kufner, L. Maligranda and L.-E. Persson.

II. In 1986, P.F. Renaud [7] proved the following reverse of (1). If $p > 1$ and $(a_j)_{j \geq 1}$ is a positive decreasing sequence, then

$$\zeta(p) \sum_{j=1}^{\infty} a_j^p \leq \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{k=1}^j a_k \right)^p,$$

where ζ denotes the Riemann zeta function. The factor $\zeta(p)$ is best possible. For finite sums we obtain

$$\sum_{j=1}^n a_j^p \leq \sum_{j=1}^n \left(\frac{1}{j} \sum_{k=1}^j a_k \right)^p.$$

This is valid for $p > 0$ and for any positive decreasing sequence $(a_j)_{j \geq 1}$.

III. The work on this paper was motivated by Problem 6663 published in 1991 by W. Janous and the editors in “The American Mathematical Monthly”.

Show that

$$\sum_{j=1}^n \left(\frac{1}{j} \sum_{k=1}^j x^{k-1} \right)^2 < 4 \log(2) \sum_{j=1}^n (x^{j-1})^2 \quad (3)$$

for $0 < x < 1$ and all positive integers n ; also show that the constant $4 \log(2)$ is best possible; see [3].

Inequality (3) refines (2) in the case of geometric sums and $p = 2$. In view of (2) and (3) it is natural to ask for the best possible factor $c(p)$ such that

$$\sum_{j=1}^n \left(\frac{1}{j} \sum_{k=1}^j x^{k-1} \right)^p \leq c(p) \sum_{j=1}^n (x^{j-1})^p$$

holds for $x \in (0, 1)$ and $n \geq 1$, where $p > 1$ is a fixed real number. Here, we consider the special case $p = 3$. The following theorem offers a cubic companion to (3).

THEOREM 1.1. *For all real numbers $x \in (0, 1)$ and natural numbers $n \geq 1$, we have*

$$\sum_{j=1}^n \left(\frac{1}{j} \sum_{k=1}^j x^{k-1} \right)^3 < \alpha \sum_{j=1}^n (x^{j-1})^3 \quad (4)$$

with the best possible constant factor

$$\alpha = \frac{27}{2} \log(3) - 18 \log(2) = 2.35461....$$

Our second result provides a counterpart of (4) for series. In particular, we obtain an improvement of (1) in the case $a_k = x^{k-1}$, $p = 3$.

THEOREM 1.2. *For all real numbers $x \in (0, 1)$, we have*

$$\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{k=1}^j x^{k-1} \right)^3 < a \sum_{j=1}^{\infty} (x^{j-1})^3 \quad (5)$$

with the best possible constant factor

$$a = \frac{27}{2} \log(3) - 18 \log(2). \quad (6)$$

IV. In the next section, we present notations and in Section 3, we collect eleven lemmas. With the help of the lemmas we are able to offer short proofs for both theorems. These proofs are given in Section 4.

The numerical calculations have been carried out by using the computer software Maple 13.

2. Notation

The classical polylogarithm function is defined by

$$\text{Li}_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m},$$

where $m \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| < 1$. This function plays an important role in various fields, like, for example, algebraic K-theory and hyperbolic geometry. Here, we present five functions which are defined in terms of the di- and

trilogarithm, Li_2 and Li_3 .

$$\begin{aligned} g(x) &= 3\text{Li}_2(x) + 3\text{Li}_2(x^3), \\ h(x) &= 6\text{Li}_2(x^2) - 3\log(1-x) + 12\log(1-x^2) - 9\log(1-x^3), \\ z_1(t) &= \text{Li}_2(1-t) - \frac{1}{2}\text{Li}_2((1-t)^2), \\ z_2(t) &= \text{Li}_2((1-t)^2) - \frac{2}{3}\text{Li}_2((1-t)^3), \\ \phi(x) &= \zeta(3) - 3\text{Li}_3(x) + 3\text{Li}_3(x^2) - \text{Li}_3(x^3). \end{aligned}$$

Moreover, we need the following functions:

$$\begin{aligned} \theta(t) &= \frac{4-t}{t(2-t)}\phi(1-t) + \phi'(1-t), \\ P(x) &= \frac{1+x}{(1-x)^2}\phi(x), \quad Q(x) = \frac{1-x^3}{(1-x)^3}\phi(x), \\ A_1(t) &= \frac{\pi^2}{12} - \log(2)t + \left(\frac{1}{4} - \frac{1}{2}\log(2)\right)t^2, \\ A_2(t) &= \frac{\pi^2}{18} + 2\log(2/3)t + \left(\frac{1}{2} + \log(2/3)\right)t^2, \\ B(t) &= (9\log(3) - 12\log(2))t + \left(\frac{9}{2}\log(3) - 6\log(2) - \frac{3}{2}\right)t^2, \\ S_n(x) &= \frac{1-x^3}{1-x^{3n}} \sum_{j=1}^n \left(\frac{1-x^j}{j(1-x)}\right)^3. \end{aligned}$$

3. Lemmas

We collect some properties of the functions which are defined in the previous section.

LEMMA 3.1. *The functions g and h are increasing on $(0, 1)$.*

Proof. Let $x \in (0, 1)$. Then

$$\frac{1}{3}g'(x) = \text{Li}'_2(x) + 3x^2\text{Li}'_2(x^3) = -\frac{1}{x}(\log(1-x) + 3\log(1-x^3)) > 0.$$

We have

$$xh'(x) = -12\log(1-x^2) - \frac{3x(2x^2+5x-1)}{(1+x)(x^2+x+1)} = h_1(x), \text{ say.}$$

It follows that

$$h_1'(x) = \frac{h_2(x)}{(1-x)(1+x)^2(x^2+x+1)^2}$$

with $h_2(x) = 24x^6 + 69x^5 + 153x^4 + 144x^3 + 48x^2 - 9x + 3$.

Since

$$h_2(x) > \frac{27}{4}x^2 - 9x + 3 = 3(3x/2 - 1)^2 \geq 0,$$

we obtain $h_1'(x) > 0$ and $h_1(x) > h_1(0) = 0$. This yields $h'(x) > 0$. $\square$

LEMMA 3.2. *For $x \in (0, 0.1]$, we have $h''(x) < 0$.*

Proof. Let $x \in (0, 0.1]$. Then

$$h''(x) = \frac{12 \log(1-x^2)}{x^2} + h_3(x)$$

with

$$h_3(x) = \frac{3(6x^5 + 16x^4 + 47x^3 + 51x^2 + 23x + 1)}{(1+x)^2(1-x^3)(x^2+x+1)}.$$

We have

$$\frac{12 \log(1-x^2)}{x^2} \leq -12$$

and

$$9 - h_3(x) = \frac{-3h_4(x)}{(1+x)^2(1-x^3)(x^2+x+1)},$$

where $h_4(x) = 3x^7 + 9x^6 + 18x^5 + 22x^4 + 41x^3 + 39x^2 + 14x - 2$.

The function h_4 is increasing with $h_4(0.1) = -0.16\dots$. It follows that $h_3(x) < 9$. Hence, $h''(x) < -12 + 9 = -3$. $\square$

LEMMA 3.3. *For $x \in (0, 1)$, we have $\phi(x) > 0$, $\phi'(x) < 0$, $\phi''(x) > 0$.*

Proof. We have

$$x^2\phi''(x) = g(x) - h(x) = \phi_1(x), \text{ say.}$$

To prove that ϕ_1 is positive on $(0, 1)$ we consider two cases.

Case 1. $0 < x \leq 0.1$. Since $g''(x) \geq 0$, we obtain from Lemma 3.2

$$\phi_1''(x) = g''(x) - h''(x) > 0.$$

Hence $\phi_1'(x) > \phi_1'(0) = 0$ and $\phi_1(x) > \phi_1(0) = 0$.

Case 2. $0.1 \leq x < 1$. Let $0.1 \leq r \leq x \leq s \leq 1$. Applying Lemma 3.1 gives

$$\phi_1(x) \geq g(r) - h(s) = \phi_2(r, s), \text{ say.}$$

We have

$$[0.1, 1] = \bigcup_{k=0}^{89} \left[0.1 + \frac{k}{100}, 0.1 + \frac{k+1}{100} \right].$$

By direct computation we obtain

$$\phi_2 \left(0.1 + \frac{k}{100}, 0.1 + \frac{k+1}{100} \right) > 0 \quad \text{for } k = 0, 1, \dots, 89.$$

This implies that ϕ_1 is positive on $[0.1, 1]$. We obtain for $x \in (0, 1)$,

$$\phi''(x) > 0, \quad \phi'(x) < \phi'(1) = 0, \quad \phi(x) > \phi(1) = 0. \quad \square$$

LEMMA 3.4. *For $t \in (0, 0.1]$, we have $z_1^{(4)}(t) < 0 < z_2^{(4)}(t)$.*

Proof. Let $t \in (0, 0.1]$. We have

$$-\frac{1}{6}(1-t)^4 z_1^{(4)}(t) = \log(2-t) + \frac{11t^3 - 48t^2 + 69t - 32}{6(2-t)^3} = \lambda_1(t), \text{ say.}$$

Then

$$\lambda_1'(t) = -\frac{(1-t)^3}{(2-t)^4} < 0 \quad \text{and} \quad \lambda_1(t) \geq \lambda_1(0.1) = 0.020\dots$$

This leads to $z_1^{(4)}(t) < 0$.

We have

$$-z_2^{(4)}(t) = \frac{\lambda_2(t) + \lambda_3(t)}{(1-t)^4(2-t)^3(t^2-3t+3)^3}$$

with

$$\lambda_2(t) = 12(2-t)^3(t^2-3t+3)^3 \log \frac{t^2-3t+3}{2-t}$$

and

$$\begin{aligned} \lambda_3(t) = & 22t^9 - 330t^8 + 2136t^7 - 7910t^6 + 18588t^5 - 28878t^4 \\ & + 29764t^3 - 19674t^2 + 7578t - 1296. \end{aligned}$$

λ_2 is the product of positive decreasing functions. It follows that λ_2 is decreasing. Next, we apply Sturm's theorem to determine the number of distinct real roots of an algebraic polynomial located in an interval; see van der Waerden [8, Section 79].

It turns out that λ'_3 has no zero on $[0, 0.1]$ with $\lambda'_3(0) = 7578$. This implies that λ_3 is increasing. Let $0 \leq r \leq t \leq s \leq 0.1$. We obtain

$$\lambda_2(t) + \lambda_3(t) \leq \lambda_2(r) + \lambda_3(s) = \lambda_4(r, s), \text{ say.}$$

Since

$$\lambda_4(0, 0.03) < 0, \quad \lambda_4(0.03, 0.05) < 0, \quad \lambda_4(0.05, 0.07) < 0, \quad \lambda_4(0.07, 0.1) < 0,$$

we conclude that $\lambda_2 + \lambda_3$ is negative on $[0, 0.1]$. Thus $z_2^{(4)}(t) > 0$. $\square$

LEMMA 3.5. *For $t \in (0, 0.1]$, we have*

$$A_1(t) - 0.0256t^3 \leq z_1(t) \leq A_1(t) - 0.0227t^3, \quad (7)$$

$$A_2(t) + 0.0908t^3 \leq z_2(t) \leq A_2(t) + 0.112t^3. \quad (8)$$

Proof. Let

$$H_1(t) = A_1(t) - \alpha_1 t^3 - z_1(t), \quad \alpha_1 = -\frac{1}{6}z_1'''(0) = 0.0227....$$

We obtain $H_1'''(t) = -6\alpha_1 - z_1'''(t)$, so that Lemma 3.4 implies that H_1''' is strictly increasing on $(0, 0.1]$. Since

$$H_1(0) = H_1'(0) = H_1''(0) = H_1'''(0) = 0,$$

we conclude that H_1 is positive on $(0, 0.1]$. This leads to the second inequality in (7).

Let

$$H_2(t) = -A_1(t) + \alpha_2 t^3 + z_1(t), \quad \alpha_2 = -\frac{1}{6}z_1'''(0.1) = 0.02559....$$

Since z_1''' is decreasing, we get $H_2'''(t) = 6\alpha_2 + z_1'''(t) \geq H_2'''(0.1) = 0$. Using $H_2(0) = H_2'(0) = H_2''(0) = 0$ yields $H_2(t) \geq 0$ for $t \in (0, 0.1]$. This gives the first inequality in (7).

Let

$$H_3(t) = A_2(t) + \alpha_3 t^3 - z_2(t), \quad \alpha_3 = \frac{1}{6}z_2'''(0.1) = 0.11191....$$

We apply Lemma 3.4 and obtain

$$H_3'''(t) = 6\alpha_3 - z_2'''(t) \geq H_3'''(0.1) = 6\alpha_3 - z_2'''(0.1) = 0.$$

Since $H_3(0) = H_3'(0) = H_3''(0) = 0$, we conclude that $H_3(t) \geq 0$ which leads to the second inequality in (8).

Let

$$H_4(t) = -A_2(t) - \alpha_4 t^3 + z_2(t), \quad \alpha_4 = \frac{1}{6} z_2'''(0) = 0.090801....$$

We obtain $H_4'''(t) = -6\alpha_4 + z_2'''(t) > H_4'''(0) = 0$, so that $H_4(0) = H_4'(0) = H_4''(0) = 0$, leads to $H_4(t) > 0$. This yields the first inequality in (8). $\square$

LEMMA 3.6. *For $t \in (0, 0.1]$, we have*

$$\frac{B(t) - 0.5808t^3}{1-t} \leq -\phi'(1-t) \leq \frac{B(t) - 0.4767t^3}{1-t}.$$

Proof. Let $t \in (0, 0.1]$. We have

$$-(1-t)\phi'(1-t) = 3z_1(t) - \frac{9}{2}z_2(t).$$

Applying Lemma 3.5 gives

$$\begin{aligned} -(1-t)\phi'(1-t) &\leq 3(A_1(t) - 0.0227t^3) - \frac{9}{2}(A_2(t) + 0.0908t^3) \\ &= B(t) - 0.4767t^3 \end{aligned}$$

and

$$\begin{aligned} -(1-t)\phi'(1-t) &\geq 3(A_1(t) - 0.0256t^3) - \frac{9}{2}(A_2(t) + 0.112t^3) \\ &= B(t) - 0.5808t^3. \end{aligned}$$

$\square$

LEMMA 3.7. *For $t \in (0, 0.1]$, we have*

$$\left(\frac{9}{2}\log(3) - 6\log(2)\right)t^2 + \left(\frac{9}{2}\log(3) - 6\log(2) - \frac{1}{2}\right)t^3 < \phi(1-t).$$

Proof. We define

$$R(x) = (9\log(3) - 12\log(2))x + \left(\frac{27}{2}\log(3) - 18\log(2) - \frac{3}{2}\right)x^2.$$

Since

$$\frac{B(x) - 0.5808x^3}{1-x} - R(x) = \frac{9r_0x^3}{1250(1-x)},$$

$$r_0 = 1875\log(3) - 2500\log(2) - 289 = 38.03...,$$

we conclude from Lemma 3.6 that for $x \in (0, 0.1]$, $R(x) < -\phi'(1-x)$.

Let $t \in (0, 0.1]$. Then

$$\begin{aligned} &\left(\frac{9}{2}\log(3) - 6\log(2)\right)t^2 + \left(\frac{9}{2}\log(3) - 6\log(2) - \frac{1}{2}\right)t^3 \\ &= \int_0^t R(x)dx < -\int_0^t \phi'(1-x)dx = \phi(1-t). \end{aligned}$$

$\square$

LEMMA 3.8. For $t \in (0, 1)$, we have $\theta(t) > 0$.

Proof. We consider two cases.

Case 1. $0 < t \leq 0.1$. Applying Lemma 3.6 and Lemma 3.7 gives

$$\theta(t) \geq \frac{4-t}{t(2-t)} \left[\left(\frac{9}{2} \log(3) - 6 \log(2) \right) t^2 + \left(\frac{9}{2} \log(3) - 6 \log(2) - \frac{1}{2} \right) t^3 \right] - \frac{B(t) - 0.4767t^3}{1-t} = \frac{t^2 C(t)}{(1-t)(2-t)}$$

with

$$\begin{aligned} C(t) &= \gamma_1 - \gamma_2 t - \gamma_3 t^2, \\ \gamma_1 &= 6 \log(2) - \frac{9}{2} \log(3) + 1 = 0.215..., \\ \gamma_2 &= \frac{27}{2} \log(3) - 18 \log(2) - \frac{9767}{5000} = 0.401..., \\ \gamma_3 &= 6 \log(2) - \frac{9}{2} \log(3) + \frac{9767}{10000} = 0.191.... \end{aligned}$$

Since $C(t) > 0.2 - 0.5t - 0.2t^2 > 0$, we conclude that θ is positive on $(0, 0.1]$.

Case 2. $0.1 \leq t < 1$. We have

$$t\theta(t) = \eta_1(t) - \eta_2(t),$$

where

$$\eta_1(t) = \frac{4-t}{2-t} \phi(1-t) \quad \text{and} \quad \eta_2(t) = t(-\phi'(1-t)).$$

From Lemma 3.3 we conclude that η_1 and η_2 are the product of two positive increasing functions. This implies that η_1 and η_2 are increasing on $[0.1, 1)$. Let $0.1 \leq r \leq t \leq s \leq 1$. It follows that

$$\eta_1(t) - \eta_2(t) \geq \eta_1(r) - \eta_2(s) = \eta_3(r, s), \text{ say.}$$

Using

$$\eta_3 \left(0.1 + \frac{k}{3000}, 0.1 + \frac{k+1}{3000} \right) > 0 \quad \text{for } k = 0, 1, 2, \dots, 2699$$

and

$$[0.1, 1] = \bigcup_{k=0}^{2699} \left[0.1 + \frac{k}{3000}, 0.1 + \frac{k+1}{3000} \right]$$

we obtain that $\eta_1(t) - \eta_2(t)$ is positive for $t \in [0.1, 1)$. □

LEMMA 3.9. *The function P is positive and strictly increasing on $(0, 1)$.*

Proof. Let $x \in (0, 1)$. We have $P'(x) = P_1(x)P_2(x)$ with

$$P_1(x) = \frac{1}{(1-x)^3} \quad \text{and} \quad P_2(x) = (x+3)\phi(x) + (1-x^2)\phi'(x) = (1-x^2)\theta(1-x).$$

Applying Lemma 3.8 gives that P_2 is positive on $(0, 1)$. It follows that $P'(x) > 0$ and $P(x) > P(0) = \zeta(3)$. $\square$

LEMMA 3.10. *The function Q is strictly increasing on $(0, 1)$ with*

$$\lim_{x \rightarrow 1} Q(x) = \frac{27}{2} \log(3) - 18 \log(2).$$

Proof. We have

$$Q(x) = \left(1 + \frac{x^2}{1+x}\right) P(x).$$

From Lemma 3.9 we obtain that Q is strictly increasing on $(0, 1)$. We have

$$Q(x) = (1+x+x^2) \frac{\phi(x)}{(1-x)^2}, \quad \phi(1) = \phi'(1) = 0, \quad \phi''(1) = 9 \log(3) - 12 \log(2).$$

It follows that

$$\lim_{x \rightarrow 1} \frac{\phi(x)}{(1-x)^2} = \lim_{x \rightarrow 1} \frac{\phi'(x)}{2(x-1)} = \frac{\phi''(1)}{2}.$$

Thus

$$\lim_{x \rightarrow 1} Q(x) = 3 \cdot \frac{\phi''(1)}{2} = \frac{27}{2} \log(3) - 18 \log(2). \quad \square$$

LEMMA 3.11. *Let $0 < x < 1$ and $n \geq 1$. Then*

$$S_n(x) < S_{n+1}(x). \quad (9)$$

Proof. Let $x \in (0, 1)$ and $n \geq 1$. Inequality (9) is equivalent to

$$\frac{x^{3n}(1-x^3)}{1-x^{3n}} \sum_{j=1}^n c_j(x) < c_{n+1}(x)$$

with

$$c_j(x) = \left(\frac{1-x^j}{j(1-x)} \right)^3.$$

Since $0 < c_j(x) \leq 1$ ($j \geq 1$), we obtain

$$\sum_{j=1}^n c_j(x) \leq n.$$

It follows that

$$\frac{x^{3n}(1-x^3)}{1-x^{3n}} \sum_{j=1}^n c_j(x) \leq n \frac{x^{3n}(1-x^3)}{1-x^{3n}}.$$

Thus it suffices to show that

$$n \frac{x^{3n}(1-x^3)}{1-x^{3n}} < \left(\frac{1-x^{n+1}}{(n+1)(1-x)} \right)^3. \quad (10)$$

Using the arithmetic mean - geometric mean inequality gives

$$\frac{1-x^{n+1}}{(n+1)(1-x)} = \frac{1}{n+1} \sum_{k=0}^n x^k > \left(\prod_{k=0}^n x^k \right)^{1/(n+1)} = x^{n/2}$$

and

$$\frac{1-x^{3n}}{n(1-x^3)} = \frac{1}{n} \sum_{k=0}^{n-1} x^{3k} \geq \left(\prod_{k=0}^{n-1} x^{3k} \right)^{1/n} = x^{3(n-1)/2}.$$

It follows that

$$\left(\frac{1-x^{n+1}}{(n+1)(1-x)} \right)^3 \frac{1-x^{3n}}{n(1-x^3)} > x^{3n/2} x^{3(n-1)/2} = x^{3n-3/2} > x^{3n}.$$

This implies (10). $\square$

4. Proofs of the theorems

Proof of Theorem 1.1. Let $x \in (0, 1)$ and $n \geq 1$. We have

$$\lim_{n \rightarrow \infty} S_n(x) = (1-x^3) \sum_{j=1}^{\infty} \left(\frac{1-x^j}{j(1-x)} \right)^3 = Q(x). \quad (11)$$

Applying Lemma 3.11, (11) and Lemma 3.10 gives

$$S_n(x) < Q(x) < Q(1) = \frac{27}{2} \log(3) - 18 \log(2).$$

This settles (4). Moreover, we have

$$\lim_{x \rightarrow 1} \lim_{n \rightarrow \infty} S_n(x) = \lim_{x \rightarrow 1} Q(x) = Q(1) = \frac{27}{2} \log(3) - 18 \log(2).$$

This limit relation shows that the given upper bound is best possible. $\square$

Proof of Theorem 1.2. Let $x \in (0, 1)$. Using Lemma 3.10 yields

$$Q(x) < Q(1) = \frac{27}{2} \log(3) - 18 \log(2).$$

This shows that (5) is valid for all $x \in (0, 1)$ and that the constant factor a given in (6) is sharp. $\square$

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