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Sign-changing solutions for (sub)critical problems in higher dimensional spheres

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Dedicated with great esteem and admiration to Enzo Mitidieri on the occasion of his 70th birthday

Abstract. *By using mountain pass arguments and a novel group-theoretical approach, in this paper we study the existence of multiple sequences of nodal solutions with prescribed different symmetries for a wide class of (sub)critical elliptic problems settled on the unit sphere* $(\mathbb{S}^d, h)$, *of dimension* $d \geq 5$, *whose simple prototype is given by the celebrated Yamabe equation on the sphere.*

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1. Introduction

In this short note we are interested on the existence of nodal (sign-changing) solutions for the following nonlinear eigenvalue problem

$$-\Delta_h w + \alpha(\sigma)w = \lambda\beta(\sigma)|w|^{q-2}w, \quad \text{in } \mathbb{S}^d, \tag{1}$$

where $(\mathbb{S}^d, h)$ denotes the unit sphere $\mathbb{S}^d$, with $d \geq 5$, endowed by the standard metric h induced by the natural embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ and Δ_h is the Laplace-Beltrami operator on $(\mathbb{S}^d, h)$ whose expression in local coordinates and standard notations is given by $\Delta_h = h^{ij}(\partial_{ij} - \Gamma_{ij}^k \partial_k)$. Furthermore

$$\alpha, \beta \in \Lambda_+(\mathbb{S}^d) := \left\{ \eta \in L^\infty(\mathbb{S}^d; \mathbb{R}) : \eta \text{ is } O(d+1)\text{-invariant and } \operatorname*{essinf}_{\sigma \in \mathbb{S}^d} \eta(\sigma) > 0 \right\}$$

where $O(d+1)$ denotes the orthogonal group acting on $\mathbb{R}^{d+1}$, λ is a positive real parameter.

Finally, from now on, along the paper we assume that $q \in [1, 2^*]$, where, as usual, $2^* := 2d/(d-2)$ denotes the critical Sobolev exponent. As customary, we say that problem (1) is *subcritical* provided that $q \in [1, 2^*)$ and *critical* if $q = 2^*$. Hence, in our setting, the main problem (1) should be either subcritical or critical. To the best of our knowledge the main results are new in any of the above cases.

Let us first recall that slightly subcritical problems on the round sphere have been studied by F. Robert and J. Vétois [45] obtaining the existence of multiple nodal solutions blowing-up at points, while the general subcritical case has been investigated by H. Brézis and Y. Li [8] and, more recently, by G. Henry and J. Petean along the paper [23].

In the last cited paper the authors also developed a novel technical approach by proving the existence of an infinite number of non constant positive solutions having prescribed level sets in terms of isoparametric hypersurfaces.

On the other hand, a special case of (1) is clearly given by the following celebrated Yamabe-type equation

$$-\Delta_h w + \frac{d(d-2)}{4} w = \frac{d(d-2)}{4} |w|^{\frac{4}{d-2}} w, \quad \text{in } \mathbb{S}^d. \tag{2}$$

A complete classification of all positive solutions of (2) goes back to the result of M. Obata [42]. As far as nodal solutions are concerned, we recall the result of W. Ding [18] on the existence of solutions which are invariant under the action of the Lie group $O(k) \times O(d+1-k)$ for $k = 2, ..., d-1$; see also the quoted papers [4] and [5] due to T. Bartsch and M. Willem.

Unlike what happens with the positive solutions to this problem on the Euclidean sphere, a classification of all the nodal solutions is far for being complete. To this reason, the existence of nodal solutions for critical problems on the round sphere, whose prototype is given by (2), have been studied by several authors. Among others, we mention here just some contributions due to M. Clapp [9], M. Clapp and J.C. Fernández [11], M. del Pino, M. Musso, F. Pacard, and A. Pistoia [15, 16], and M. Musso and J. Wei [41]. For the sake of completeness, we also note that several recent and significant contributions on nodal solutions to the Yamabe problem have been made by J.C. Fernández and J. Petean [19], as well as by M. Clapp, J. Faya, and A. Saldaña [10].

This bibliography does not escape the usual role to be incomplete. Motivated by this wide interest in the current literature, in the spirit of seminal papers of P.-L. Lions [30, 31], the main result reads as follows.

THEOREM 1.1. *Let $(\mathbb{S}^d, h)$ be the unit sphere $\mathbb{S}^d$, with $d \geq 5$, endowed by the standard metric h induced by the natural embedding $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$. Moreover, let $\alpha, \beta \in \Lambda_+(\mathbb{S}^d)$ and let $\lceil \cdot \rceil$ be the integer function, i.e. the largest integer less that or equal to a given real number. Then, for every $\lambda > 0$, equation* (1) *admits at least*

$$s_d := \lceil d/2 \rceil + (-1)^{d+1} - 1$$

sequences of nodal solutions with mutually different symmetric structures.

A meaningful consequence of Theorem 1.1 is the following.

Corollary 1.2. *Assume that $d \geq 5$. Then, the Yamabe-type equation* (2) *admits at least s_d sequences of nodal solutions with mutually different symmetric structures.*

In the critical case, the energy functional $\mathcal{J}_\lambda : H_1^2(\mathbb{S}^d) \to \mathbb{R}$ given by

$$\mathcal{J}_\lambda(w) := \frac{1}{2}\left(\int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} \alpha(\sigma)|w(\sigma)|^2 d\sigma_h\right) - \frac{\lambda}{q}\int_{\mathbb{S}^d} \beta(\sigma)|w(\sigma)|^q d\sigma_h,$$

and associated to problem (1) does not satisfy the usual Palais-Smale condition due to the lack of compactness of the embedding $H_1^2(\mathbb{S}^d) \hookrightarrow L^{2^*}(\mathbb{S}^d)$; see Section 2 for the functional setting involving here. In order to regain some compactness properties, we use here [24, Theorem 3.1] in the more precise form given in Proposition 3.2.

Indeed, for every $i \in J_d := \{1, ..., s_d\}$, the subgroups $G_{d,i}^{\tau_i} := \langle G_{d,i}, \tau_i \rangle$ of the orthogonal group $O(d+1)$, where

$$G_{d,i} := \begin{cases} O(i+1) \times O(d-2i-1) \times O(i+1), & \text{if } i \neq \dfrac{d-1}{2}, \\ O(i+1) \times O(i+1), & \text{if } i = \dfrac{d-1}{2}, \end{cases}$$

and $\tau_i : \mathbb{S}^d \to \mathbb{S}^d$ is the involution function defined by

$$\tau_i(\sigma) := \begin{cases} (\sigma_3, \sigma_2, \sigma_1), & \text{if } i \neq \dfrac{d-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1},\ \sigma_2 \in \mathbb{R}^{d-2i-1}, \\ (\sigma_3, \sigma_1), & \text{if } i = \dfrac{d-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1}, \end{cases}$$

for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^d$, will imply the compactness of the embedding of $G_{d,i}^{\tau_i}$-invariant functions $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ of $H_1^2(\mathbb{S}^d)$ into the Lebesgue spaces $L^q(\mathbb{S}^d)$ whenever $q \in [1, 2_{d-1}^*)$, with $2_{d-1}^* := 2(d-1)/(d-3)$; see [24], [28, Chapter 10], [36, Chapter 6], as well as Section 3.

Now, having such a compactness, the cassical the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Theorem and the principle of symmetric criticality due to Palais (see [44] and Theorem 2.1 in Section 2) applied to the energy functional $\mathcal{J}_\lambda$ will guarantee the existence of a whole sequence of $G_{d,i}^{\tau_i}$-invariant solutions for (1). Finally, the number of s_d sequences of nodal solutions for (1) with mutually different nodal properties will follow by the careful choices of the subgroups $G_{d,i}^{\tau_i} \subset O(d+1)$; see Proposition 3.2 in Section 3.

Furthermore, since the appearance of the celebrated paper of W. Ding [18] on the conformally invariant scalar field equation in $\mathbb{R}^d$, concerning the existence of infinitely many conformally inequivalent changing sign solutions, with finite energy, the method of pulling back the problem into the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$ by means of a stereographic projection and then into its variational formulation has been having a large use in literature for different problems, involving critical nonlinearities in the sense of Sobolev.

Along this direction, a remarkable case of problem (1) is given by the critical equation

$$-\Delta_h w + \frac{(d-1)^2}{4} w = \lambda \beta(\sigma) |w|^{2^*-2} w, \quad \text{in } \mathbb{S}^d. \tag{3}$$

Indeed, by using an appropriate change of coordinates due to M.F. Bidaut-Véron and L. Véron [6], existence results for problem (3) yield the existence of solutions to the following parameterized Emden-Fowler equation

$$-\Delta u = \lambda |x|^{\frac{2}{d-2}} \beta\left(\frac{x}{|x|}\right) |u|^{\frac{4}{d-2}} u, \quad x \in \mathbb{R}^{d+1} \setminus \{0\}. \tag{4}$$

We emphasize that equations of type (4) have been largely studied in the literature. For instance, among others, we just mention here the pioneering papers due to A. Cotsiolis and D. Iliopoulos [13] and J.L. Vázquez and L. Véron [46]; see also A. Cotsiolis and D. Iliopoulos [14].

Thanks to the stereographic projection method developed in the cited papers, existence results for equation (4) have been established recently in [7, 27, 28, 32], as well as [37, 40], via variational methods.

The main result for parameterized Emden-Fowler equations states as follows; see also Remark 4.3.

Corollary 1.3. *Assume that* $d \geq 5$ *and let* $\beta \in \Lambda_+(\mathbb{S}^d)$. *Then, for every* $\lambda > 0$, *equation* (4) *admits at least* s_d *sequences of nodal solutions with mutually different symmetric structures.*

We emphasize that, taking into account the simple nature of the pure (critical) power, Theorem 1.1, and its consequences reported here, cannot be deduced by the results proved in [24]. Indeed, conditions (f_2^0) in [24, Theorem 2.1] and (f_2^∞) in [24, Theorem 2.2] are clearly not verified in our setting.

Furtheromre, for the sake of completeness, we point out that in recent years, singular Riemannian foliations have come to the forefront as a natural and compelling generalization of symmetry in Riemannian geometry, particularly within the setting of manifolds with nonnegative sectional curvature. As extensions of classical notions such as isometric group actions and Riemannian submersions, they provide a versatile framework that captures both geometric and analytic phenomena beyond the reach of traditional symmetry models. Notably, a variety of results that depend solely on the transverse geometry to group orbits extend naturally to this broader context.

Along this direction, the work of D. Corro, J.C. Fernández, and R. Perales [12] stands out as a significant, elegant and general contribution. In particular, the authors constructed an unbounded sequence of nodal, symmetric solutions to the Yamabe problem on the sphere, each invariant along the leaves of a singular Riemannian foliation induced by the orbits of certain subgroups of the orthogonal group $O(d+1)$; see [12, Corollary C].

Their approach, which blends geometric insight with analytic precision, also encompasses a wider class of Yamabe-type equations with symmetric, non-constant coefficients, encompassing equation (1) as a particular case. Their work offers a deep and unifying perspective on how symmetry, in its generalized form, governs the structure of solutions to geometric PDEs.

Building upon this foundational framework, the present work aims to further articulate and enrich the structure of nodal solutions. Unlike the solutions obtained in [12], which share a unified symmetry structure, the solutions we construct exhibit a diversity of symmetry types, each precisely characterized and associated to a separate sequence of nodal solutions.

In this perspective, our results can be naturally reinterpreted within the general framework developed in [12]. While our analysis is restricted to a specific and constrained case within the broader scope of [12], it nonetheless leads to a more detailed formulation of the results in this setting. In particular, it yields a precise enumeration of symmetry-distinct nodal solutions and a more articulated understanding of the corresponding symmetry structures.

The plan of the paper is as follows. In the next section we recall some basic facts on the Sobolev spaces defined on the sphere $\mathbb{S}^d$. In Section 3 we will discuss our abstract group-theoretical arguments while in the last section we are dealing with the proof Theorem 1.1 and its consequences.

The main results of the paper are mainly based on the arguments and ideas contained in [3, 18, 24] as well as [13, 46]; see also the monographs [28, 36]. Related existence and multiplicity results can be found in [26, 29, 33, 34] and [35, 38, 39].

2. Framework and Notations

We start this section with a short list of notions in Riemannian geometry. We refer to Aubin [1, 2] and Hebey [21] for detailed derivations of the geometric quantities, their motivation and further applications; see also the work [1] and the brief introduction on the subject given in [7].

As usual, we denote by $C^\infty(\mathbb{S}^d)$ the space of smooth functions defined on $\mathbb{S}^d$. Let $\alpha \in \Lambda_+(\mathbb{S}^d)$ and put $\|\alpha\|_\infty := \operatorname*{esssup}_{\sigma\in\mathbb{S}^d} \alpha(\sigma)$.

For every $w \in C^\infty(\mathbb{S}^d)$, set

$$\|w\|^2_{H^2_\alpha} := \int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} \alpha(\sigma)|w(\sigma)|^2 d\sigma_h,$$

where ∇w is the covariant derivative of w, and $d\sigma_h$ is the Riemannian measure.

Hence, let

$$\omega_d := \mathrm{Vol}_h(\mathbb{S}^d) = \int_{\mathbb{S}^d} d\sigma_h.$$

The Sobolev space $H^2_\alpha(\mathbb{S}^d)$ is defined as the completion of $C^\infty(\mathbb{S}^d)$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Then $H^2_\alpha(\mathbb{S}^d)$ is a Hilbert space endowed with the inner product

$$\langle v, w\rangle_{H^2_\alpha} := \int_{\mathbb{S}^d} \langle \nabla v(\sigma), \nabla w(\sigma)\rangle_h d\sigma_h + \int_{\mathbb{S}^d} \alpha(\sigma)\langle v(\sigma), w(\sigma)\rangle_h d\sigma_h,$$

for every $v, w \in H^2_\alpha(\mathbb{S}^d)$, where $\langle\cdot,\cdot\rangle_h$ is the inner product on covariant tensor fields associated to h.

Since α is positive, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent with the standard norm

$$\|w\|_{H^2_1} := \left(\int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h\right)^{1/2}.$$

Moreover, if $w \in H^2_\alpha(\mathbb{S}^d)$, the following inequalities hold

$$\min\{1, \operatorname*{essinf}_{\sigma\in\mathbb{S}^d} \alpha(\sigma)^{1/2}\}\|w\|_{H^2_1} \leq \|w\|_{H^2_\alpha} \leq \max\{1, \|\alpha\|_\infty^{1/2}\}\|w\|_{H^2_1}. \tag{5}$$

From the Rellich-Kondrachov theorem (for compact manifolds without boundary) one has

$$H^2_1(\mathbb{S}^d) \hookrightarrow L^q(\mathbb{S}^d),$$

for every $q \in [1, 2d/(d-2)]$. In particular, the embedding is compact whenever $q \in [1, 2d/(d-2))$. Hence, there exists a positive constant S_q such that

$$\|w\|_q \leq S_q\|w\|_{H^2_1}, \qquad \forall\, w \in H^2_1(\mathbb{S}^d), \tag{6}$$

where the norm of the Lebesgue spaces $L^q(\mathbb{S}^d)$ are denoted by $\|\cdot\|_q$, $q \in [1,\infty)$.

Furthermore, for the sake of completeness, we recall that, fixed $\lambda \in \mathbb{R}$, a function $w \in H^2_1(\mathbb{S}^d)$ is a *weak solution* of problem (1) if

$$\int_{\mathbb{S}^d} \langle \nabla w(\sigma), \nabla v(\sigma)\rangle_h + \int_{\mathbb{S}^d} \alpha(\sigma)w(\sigma)v(\sigma)d\sigma_h = \lambda \int_{\mathbb{S}^d} \beta(\sigma)|w(\sigma)|^{q-2}w(\sigma)v(\sigma)d\sigma_h$$

for every $v \in H^2_1(\mathbb{S}^d)$.

Due to the regularity assumptions on the data, the weak solutions of problem (1) are also classical; see, for instance, the paper [27] as well as the books [28, 36].

Finally, let us recall the well known principle of symmetric criticality of R. Palais. A group $(\mathscr{H}, *)$ acts continuously on a real Banach space X by an application $(\tau, u) \mapsto \tau \circledast_{\mathscr{H}} u$ from $\mathscr{H} \times X$ to X if this map itself is continuous on $\mathscr{H} \times X$ and satisfies

(i_1) $id_{\mathscr{H}} \circledast_{\mathscr{H}} u = u$ for every $u \in X$, where $id_{\mathscr{H}} \in \mathscr{H}$ is the identity element of $\mathscr{H}$;

(i_2) $(\tau_1 * \tau_2) \circledast_{\mathscr{H}} u = \tau_1 \circledast_{\mathscr{H}} (\tau_2 \circledast_{\mathscr{H}} u)$ for every $\tau_1, \tau_2 \in \mathscr{H}$ and $u \in X$;

(i_3) $u \mapsto \tau \circledast_{\mathscr{H}} u$ is linear for every $\tau \in \mathscr{H}$.

Set

$$Fix_{\mathscr{H}}(X) := \{u \in X : \tau \circledast_{\mathscr{H}} u = u \text{ for every } \tau \in \mathscr{H}\}.$$

A functional $\mathcal{J} : X \to \mathbb{R}$ is said to be $\mathscr{H}$–invariant if

$$\mathcal{J}(\tau \circledast_{\mathscr{H}} u) = \mathcal{J}(u),$$

for every $u \in X$ and $\tau \in \mathscr{H}$.

With the notation introduced above, the following classical and celebrated result by R.S. Palais [44] holds.

THEOREM 2.1. *Let X be a real Banach space, $\mathscr{H}$ be a compact topological group acting continuously on X by a map $\circledast_{\mathscr{H}} : \mathscr{H} \times X \to X$, and $\mathcal{J} : X \to \mathbb{R}$ be a $\mathscr{H}$–invariant C^1–function. If $u \in Fix_{\mathscr{H}}(X)$ is a critical point of the restriction $\mathcal{J}_{|Fix_{\mathscr{H}}(X)}$, then $u \in X$ is also a critical point of $\mathcal{J}$.*

For details and comments we refer to [36, Appendix A].

3. Group-theoretical arguments

Let $d \geq 5$ and let us define $s_d := \lceil d/2 \rceil + (-1)^{d+1} - 1$, where $\lceil \cdot \rceil$ denotes the integer function. In this section, arguing as in [24], we describe the construction of s_d subspaces $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ of the Sobolev space $H_1^2(\mathbb{S}^d)$ related to certain subgroups $G_{d,i}^{\tau_i}$ of the orthogonal group $O(d+1)$, for every $i \in J_d$.

To this aim, let $i \in J_d = \{1, ..., s_d\}$ and set

$$G_{d,i} := \begin{cases} O(i+1) \times O(d-2i-1) \times O(i+1), & \text{if } i \neq \dfrac{d-1}{2}, \\ O(i+1) \times O(i+1), & \text{if } i = \dfrac{d-1}{2}. \end{cases}$$

Furthermore, $G_{i,j}^d$ denotes the group generated by $G_{d,i}$ and $G_{d,j}$ whenever $i, j \in J_d$ and $i \neq j$. The following result, showed in [24, Proposition 3.2], is crucial in order to prove the geometric shape described in Proposition 3.2.

Now, let $\tau_i : \mathbb{S}^d \to \mathbb{S}^d$ be the involution function associated to $G_{d,i}$ and defined by

$$\tau_i(\sigma) := \begin{cases} (\sigma_3, \sigma_2, \sigma_1), & \text{if } i \neq \dfrac{d-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1},\ \sigma_2 \in \mathbb{R}^{d-2i-1}, \\ (\sigma_3, \sigma_1), & \text{if } i = \dfrac{d-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1}, \end{cases}$$

for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^d$. By construction,

$$\tau_i \notin G_{d,i}, \quad \tau_i G_{d,i} \tau_i^{-1} = G_{d,i} \quad \text{and} \quad \tau_i^2 = \mathrm{id}_{\mathbb{S}^d}.$$

For instance, if $d = 11$, then $s_{11} = 5$ and the groups and the involution functions are described below:

$G_{11,1} = O(2) \times O(8) \times O(2)$,

$$\tau_1(\sigma_1, \sigma_2, \sigma_3) := (\sigma_3, \sigma_2, \sigma_1)$$

for $\sigma_1, \sigma_3 \in \mathbb{R}^2$ and $\sigma_2 \in \mathbb{R}^8$, when $i = 1$;

$G_{11,2} = O(3) \times O(6) \times O(3)$,

$$\tau_2(\sigma_1, \sigma_2, \sigma_3) := (\sigma_3, \sigma_2, \sigma_1)$$

for $\sigma_1, \sigma_3 \in \mathbb{R}^3$ and $\sigma_2 \in \mathbb{R}^6$, when $i = 2$;

$G_{11,3} = O(4) \times O(4) \times O(4)$,

$$\tau_3(\sigma_1, \sigma_2, \sigma_3) := (\sigma_2, \sigma_1, \sigma_3)$$

for $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^4$, when $i = 3$;

$G_{11,4} = O(5) \times O(2) \times O(5)$,

$$\tau_4(\sigma_1, \sigma_2, \sigma_3) := (\sigma_3, \sigma_2, \sigma_1)$$

for $\sigma_1, \sigma_3 \in \mathbb{R}^5$ and $\sigma_2 \in \mathbb{R}^2$, when $i = 4$;

$G_{11,5} = O(6) \times O(6)$,

$$\tau_5(\sigma_1, \sigma_2) := (\sigma_2, \sigma_1)$$

for $\sigma_1, \sigma_2 \in \mathbb{R}^6$, when $i = 5$.

See [28, Chapter 10], as well as [24], for additional comments and remarks.

For every $i \in J_d$ let $\widehat{\circledast}_i$ be an action of the compact group

$$G_{d,i}^{\tau_i} := \langle G_{d,i}, \tau_i \rangle \subset O(d+1) \tag{7}$$

on the Sobolev space $H_1^2(\mathbb{S}^d)$.

More precisely, we consider the action $\widehat{\circledast}_i : G_{d,i}^{\tau_i} \times H_1^2(\mathbb{S}^d) \to H_1^2(\mathbb{S}^d)$, $(\widetilde{g}, w) \mapsto g\widehat{\circledast}_i w$, which is defined pointwise for a.e. $\sigma \in \mathbb{S}^d$ by

$$(g\widehat{\circledast}_i w)(\sigma) := \begin{cases} w(g^{-1}\sigma) & \text{if } g \in G_{d,i} \\ -w(g^{-1}\tau_i^{-1}\sigma) & \text{if } g = \tau_i \tilde{g} \in G_{d,i}^{\tau_i} \setminus G_{d,i},\ \tilde{g} \in G_{d,i}. \end{cases} \tag{8}$$

This can be done by the properties of τ_i. Therefore, $\widehat{\circledast}_i$ is well defined, linear and continuous.

Let us consider for every $i \in J_d$ the subspace $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ of $H_1^2(\mathbb{S}^d)$ given by

$$E_i := H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) = \{w \in H_1^2(\mathbb{S}^d) \,:\, g\widehat{\circledast}_i w = w \text{ for all } g \in G_{d,i}^{\tau_i}\}.$$

Clearly, E_i contains all the functions $u \in H_1^2(\mathbb{S}^d)$, which are symmetric with respect to the action $\widehat{\circledast}_i$ of the compact group $G_{d,i}^{\tau_i}$.

We notice that the spaces E_i are infinite-dimensional. Indeed, by [43] there exist $G_{d,i}$-invariant partitions of unity. Hence, from the fact that the groups $G_{d,i}^{\tau_i}$ are disconnected it follows that there exists a nontrivial continuous homomorphism from $G_{d,i}^{\tau_i}$ to $\mathbb{Z}_2$. According to the remark given in [9] the conclusion is achieved.

We also observe that every $w \in E_i \setminus \{0\}$ has no constant sign. Indeed, $w(\sigma) = -w(\tau_i^{-1}\sigma)$ for every $\sigma \in \mathbb{S}^d$, since w is $G_{d,i}^{\tau_i}$–invariant by (9). The conclusion then follows immediately from the fact that w is not zero.

Moreover, for every $i \in J_d$ we also introduce

$$\mathscr{E}_i := H_{G_{d,i}}(\mathbb{S}^d) = \{w \in H_1^2(\mathbb{S}^d) \,:\, g \circledast_i w = w \text{ for all } g \in G_{d,i}\},$$

where the action $\circledast_i : G_{d,i} \times H_1^2(\mathbb{S}^d) \to H_1^2(\mathbb{S}^d)$ of the compact group $G_{d,i}$ on $H_1^2(\mathbb{S}^d)$, $(g, w) \mapsto g\circledast_i w$, is defined pointwise for a.e. $\sigma \in \mathbb{S}^d$ by

$$(g\circledast_i w)(\sigma) := w(g^{-1}\sigma). \tag{9}$$

That the spaces $\mathscr{E}_i$ are infinite dimensional follows from the fact of the existence of $G_{d,i}$-invariant partitions of the unity and because every group $G_{d,i}$ does not acts transitively on the sphere.

Finally, for the sake of clarity let us recall the following abstract embedding result given by E. Hebey and M. Vaugon in [22]; see also [3, Lemma 3.2].

PROPOSITION 3.1. *Let G be a closed topological subgroup of the isometries group $Isom_h(\mathbb{S}^d)$ and let $\cdot : G \times H_1^2(\mathbb{S}^d) \to H_1^2(\mathbb{S}^d)$, with $d \geq 3$, be the natural action of the topological group G on the Hilbert Sobolev space $H_1^2(\mathbb{S}^d)$. Set*

$$H_G(\mathbb{S}^d) := \{w \in H_1^2(\mathbb{S}^d) \,:\, gw = w \textit{ for all } g \in G\}.$$

Let

$$d_G := \min_{\sigma \in \mathbb{S}^d} \dim(G\sigma)$$

be the minimal dimension of the orbits in $\mathbb{S}^d$, where the orbit $G\sigma$ of an element $\sigma \in \mathbb{S}^d$ is given by

$$G\sigma := \{g\sigma \,:\, \textit{for all } g \in G\},$$

and $g\sigma$ denotes the natural multiplicative action of G over $\mathbb{S}^d$.

Then the Sobolev embedding

$$H_G(\mathbb{S}^d) \hookrightarrow L^q(\mathbb{S}^d)$$

is compact for every $q \in [1, q_G)$*, where*

$$q_G := \begin{cases} \dfrac{2(d-d_G)}{d-d_G-2} & \text{if } d > 2 + d_G \\ +\infty & \text{if } d \leq 2 + d_G. \end{cases}$$

If $d > 2 + d_G$*, then the space* $H_G(\mathbb{S}^d)$ *is continuously embedded in* $L^{q_G}(\mathbb{S}^d)$*.*

We notice that if G is a connected algebraic group, which acts on a variety Y (not necessarily affine), then for each $y \in Y$ the orbit Gy is an irreducible variety, that is Gy is open in its closure. Moreover, its boundary, $\partial Gy = \overline{Gy} \setminus Gy$, is the union of orbits of strictly smaller dimension. Finally, in this case orbits of minimal dimension are closed.

By Proposition 3.1, the next result holds.

Proposition 3.2. *Let* $(\mathbb{S}^d, h)$ *be the unit sphere* $\mathbb{S}^d$*, with* $d \geq 5$*. The following statements hold:*

(i_1) *The Hilbert Sobolev space* $\mathscr{E}_i = H_{G_{d,i}}(\mathbb{S}^d)$ *is compactly embedded into* $L^q(\mathbb{S}^d)$*, whenever* $q \in \left[1, 2^*_{d-1}\right)$*, where*

$$2^*_{d-1} := \frac{2(d-1)}{d-3}.$$

Moreover, for a fixed $i \in J_d$*, one has:*

(i_2) $\mathscr{E}_i \cap \mathscr{E}_j = \{\text{constant functions on } \mathbb{S}^d\}$ *for every* $j \in J_d$*, with* $j \neq i$*;*

(i_3) $E_i \cap E_j = \{0\}$ *for every* $j \in J_d$*, with* $j \neq i$*.*

Proof. In order to prove item (i_1), let us notice that the definition of $G_{d,i}$ shows that the $G_{d,i}$-orbit of every point $\sigma \in \mathbb{S}^d$ has at least dimension 1, i.e., $\dim(G_{d,i}\sigma) \geq 1$ for every $\sigma \in \mathbb{S}^d$. Thus

$$d_{G_{d,i}} := \min\{\dim(G_{d,i}\sigma) : \sigma \in \mathbb{S}^d\} \geq 1.$$

By using Proposition 3.1, we conclude that $H_{G_{d,i}}(\mathbb{S}^d)$ is compactly embedded into $L^q(\mathbb{S}^d)$, whenever $q \in \left[1, 2^*_{d-1}\right)$ as claimed. The rest of the proof is given in [24, Theorem 3.1]. □

We refer to [17, 26, 34] for related topics.

4. A proof of Theorem 1.1 and its consequences

In this section we assume the hypotheses of Theorem 1.1 are fulfilled and let $\lambda > 0$. The energy functional $\mathcal{J}_\lambda : H_1^2(\mathbb{S}^d) \to \mathbb{R}$ associated with problem (1) has the form

$$\mathcal{J}_\lambda(w) := \frac{1}{2}\|w\|^2_{H^2_\alpha} - \frac{\lambda}{q}\int_{\mathbb{S}^d} \beta(\sigma)|w(\sigma)|^q d\sigma_h. \tag{10}$$

Since $\alpha, \beta \in \Lambda_+(\mathbb{S}^d)$, standard arguments ensure that the energy functional $\mathcal{J}_\lambda$ is well-defined, it belongs to $C^1(H_1^2(\mathbb{S}^d), \mathbb{R})$, and its critical points are precisely the solutions of problem (1). Moreover, $\mathcal{J}_\lambda$ is even in $H_1^2(\mathbb{S}^d)$. Furthermore, it easy to check that, for every fixed $i \in J_d$, one has $\mathcal{J}_\lambda(g \widehat{\circledast}_i w) = \mathcal{J}_\lambda(w)$ for every $g \in G^{\tau_i}_{d,i}$ and $w \in H_1^2(\mathbb{S}^d)$, where $\widehat{\circledast}_i$ is defined in (8). In other words, the functional $\mathcal{J}_\lambda$ is $G^{\tau_i}_{d,i}$-invariant on $H_1^2(\mathbb{S}^d)$.

Now, the topological group $G^{\tau_i}_{d,i} \subset O(d+1)$ is compact and the representation map $\widehat{\circledast}_i : G^{\tau_i}_{d,i} \times H_1^2(\mathbb{S}^d) \to H_1^2(\mathbb{S}^d)$ given in (8) is continuous. By the principle of symmetric criticality recalled in Theorem 2.1 the critical points of the restriction $\mathcal{J}_\lambda$ to E_i are also critical points of the energy functional $\mathcal{J}_\lambda$ in $H_1^2(\mathbb{S}^d)$.

Taking into account the above remarks, let us fix $i \in J_d$ and consider the functionals $\Phi_i, \Psi_i : E_i \to \mathbb{R}$ defined by

$$\Phi_i(w) := \frac{1}{2}\|w\|^2_{H^2_\alpha} \quad \text{and} \quad \Psi_i(w) := \frac{1}{q}\int_{\mathbb{S}^d} \beta(\sigma)|w(\sigma)|^q d\sigma_h, \quad w \in E_i. \tag{11}$$

Consequently, the restriction of the functional $\mathcal{J}_\lambda$ to E_i can be written as follows

$$\mathcal{J}^{(i)}_\lambda(w) = \Phi_i(w) - \lambda\Psi_i(w), \quad w \in E_i. \tag{12}$$

By exploiting Proposition 3.2 - Part (i_1), since $2^*_{d-1} > 2^*$, the Hilbert Sobolev space $E_i \subset \mathscr{E}_i$ is compactly embedded into $L^q(\mathbb{S}^d)$, whenever $q \in [1, 2^*]$. Hence, we can apply the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Theorem to $\mathcal{J}^{(i)}_\lambda$ for every $i \in J_d$. Therefore, one can guarantee the existence of the sequences of distinct critical points $\{w_n^{\lambda,i}\}_n \subset E_i$, with $i \in J_d$, of the energy functionals $\mathcal{J}^{(i)}_\lambda$, with $i \in J_d$.

They are also critical points of $\mathcal{J}_\lambda$ due to the principle of symmetric criticality recalled in Theorem 2.1. In view of Proposition 3.2 - Part (i_3), the symmetric structure of the elements in the aforementioned sequences mutually differ. The proof of Theorem 1.1 is complete. □

Remark 4.1. As observed in [27], we notice that, for a fixed parameter $\lambda > 0$, the constant function $w_\lambda(\sigma) = k \in \mathbb{R}$, for every $\sigma \in \mathbb{S}^d$, is a solution of (1) if and only if

$$\alpha(\sigma)k = \lambda\beta(\sigma)|k|^{q-2}k,$$

for almost every $\sigma \in \mathbb{S}^d$. In particular, when $w_\lambda(\sigma) = k \neq 0$, the function $\sigma \mapsto \lambda\beta(\sigma)/\alpha(\sigma)$ is constant. Let us denote this value by $\mu_\lambda > 0$. Thus, nonzero constant solutions of problem (1) appear as fixed points of the function $t \mapsto \mu_\lambda |t|^{q-2}t$. We emphasize that, as a byproduct of Theorem 1.1, the existence of multiple nonconstant solutions of problem (1) can be proved also in the case when $\sigma \mapsto \lambda\beta(\sigma)/\alpha(\sigma)$ is constant for certain $\lambda > 0$, by using variational and group-theoretical arguments. Hence, Corollary 1.2 is an immediate consequence of Theorem 1.1.

Proof of of Corollary 1.3. Let us consider the constant positive function $\alpha(\sigma) := (d-1)^2/4$ for every $\sigma \in \mathbb{S}^d$. Thus, by Theorem 1.1, for every $\lambda > 0$, the following equation

$$-\Delta_h w + \frac{(d-1)^2}{4} w = \lambda\beta(\sigma)|w|^{2^*-2}w, \quad \text{in } \mathbb{S}^d, \tag{13}$$

admits at least τ_d distinct pairs of sign-changing weak solutions distinguished by their symmetry properties. Moreover, due to the regularity assumptions on the data, the weak solutions of (13) are also classical; see, for instance, the paper [27] as well as the books [28, 36]. Now, the solutions of

$$-\Delta u = \lambda |x|^{\frac{2}{d-2}} \beta\left(\frac{x}{|x|}\right) |u|^{\frac{4}{d-2}} u, \quad x \in \mathbb{R}^{d+1} \setminus \{0\}, \tag{14}$$

are being sought in the particular form

$$u(x) = r^{-\frac{d-1}{2}} w(\sigma), \tag{15}$$

where, $(r,\sigma) := (|x|, x/|x|) \in (0,\infty) \times \mathbb{S}^d$ are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$ and w be a smooth function defined on $\mathbb{S}^d$. Throughout (15) equation (14) reduces to (13). Moreover, on account of (15), the elements $u_{i,\lambda}(x) = |x|^{-\frac{d-1}{2}} w_{i,\lambda}(x/|x|)$, $i \in J_d$, are solutions of (14). The conclusion immediately follows. □

Remark 4.2. We point out that the main approach is inspired by the recent paper [25] in which the existence of nodal solutions for the fractional Yamabe problem on Heisenberg groups has been proved in [25, Theorem 1.1] by using a Hebey-Vaugon compactness type result and a group-theoretical construction for suitable subgroups of the classical unitary group.

Remark 4.3. Corollary 1.3 can be done in a more general form. Indeed, a careful analysis of the proof of Corollary 1.3 ensures that for every $\lambda > 0$, the following equation

$$-\Delta u = \lambda |x|^{\frac{(q-2)(d-1)-4}{2}} \beta\left(\frac{x}{|x|}\right) |u|^{q-2} u, \quad x \in \mathbb{R}^{d+1} \setminus \{0\}, \tag{16}$$

admits at least s_d sequences of nodal solutions with mutually different symmetric structures, provided that $q \in [2, 2^*]$. Clearly, equation (16) reduces to (14) by taking $q = 2^*$.

REMARK 4.4. A careful analysis of the proof of Theorem 1.1 ensures that the conclusion of the main result remains valid for the following nonlinear eigenvalue problem

$$-\mathrm{div}(a(\sigma)\nabla w) + b(\sigma)w = \lambda\beta(\sigma)|w|^{q-2}w, \quad \text{in } \mathbb{S}^d, \tag{17}$$

where $a, b, \beta \in C^\infty(\mathbb{S}^d)$ are $O(d+1)$-invariant, with $a > 0$ and

$$-\mathrm{div}(a(\sigma)\nabla) + b(\sigma)$$

coercive; see [11] for related results.

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