Rend. Istit. Mat. Univ. Trieste
Vol. 57 (2025), Art. No. 23, 27 pages
DOI: 10.13137/2464-8728/37467

Coincidence point results in partial b-metric spaces via tri-simulation function and digraph with an application

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Abstract. *In this article, we introduce the concept of generalized* (α, T)*-*G*-contractive mappings in partial* b*-metric spaces endowed with a digraph* G *and obtain a new coincidence point and common fixed point result for a pair of self mappings satisfying such contractive condition. Our main result will extend and unify several known results in the existing literature and also brings some new results as consequences. Finally, we give an application of our main result to obtain a unique solution of an integral equation.*

Keywords: Tri-simulation function, digraph, weakly compatible maps, point of coincidence.
MS Classification 2020: 54H25, 47H10.

1. Introduction

It is well known that the Banach contraction theorem [6] in complete metric spaces is an important and useful tool in modern analysis. It has many applications in different fields of mathematics and applied sciences. Several authors have successfully generalized this famous theorem in different directions. There exist a lot of generalizations of the notion of metric spaces such as b-metric space, introduced by Bakhtin [5], partial metric space by Matthews [24], and dislocated metric space by Hitzler et al. [20]. In [30], S. Shukla introduced the concept of a partial b-metric as a generalization of the notions of b-metric and partial metric and established some fixed point results in such spaces.

Coincidence point and common fixed point results for a pair of mappings satisfying some contractive type conditions in various spaces have been studied extensively by many researchers. In recent investigations, the study of fixed point theory via simulation functions takes a vital role in many aspects. In 2015, Khojasteh et al. [22] initiated the idea of $\mathcal{Z}$-contraction by using a simulation function and generalized the Banach contraction theorem by combining various types of nonlinear contractions. Afterwards, Argoubi et al. [4] and

Roldán et al. [28] modified the existing idea of simulation functions in different ways and established some common fixed point results utilizing this modified class of simulation functions. In [10], S. H. Cho introduced the concept of $\mathscr{L}$-contractions and unified some existing metric fixed point results. Very recently, Gubran et al. [19] introduced a new simulation function involving three variables, called a tri-simulation function which is also designed to unify several known contractions. The study of fixed point theory combining a graph is a new development in the domain of single valued and multi valued fixed point theory. Echenique [13] studied fixed point theory by using graphs and then Espinola and Kirk [14] applied fixed point results in graph theory. Motivated by the idea given in [15, 19, 22] and some recent works on partial b-metric and b-metric spaces with a graph (see [2, 3, 7, 16, 23, 25, 26, 27, 29]), we reformulated some important coincidence point and common fixed point results in partial b-metric spaces endowed with a digraph by using tri-simulation functions. Also, we construct some non trivial examples to examine the strength of the hypotheses of our main result.

2. Some Basic Concepts

In this section we recall some basic notations, definitions and necessary results that will be needed in the sequel.

Definition 2.1 ([11]). *Let X be a nonempty set and $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leq b\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is valuable to note that the family of b-metric spaces is effectively larger than that of the ordinary metric spaces.

Definition 2.2 ([24]). *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:*

(p_1) $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$;

(p_2) $p(x, x) \leq p(x, y)$;

(p_3) $p(x, y) = p(y, x)$;

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

Example 2.3 ([24]). Let $X = [0, \infty)$ and let $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a partial metric space but p is not a metric on X.

Definition 2.4 ([30]). *A partial b-metric on a nonempty set X is a function $p_b : X \times X \to \mathbb{R}^+$ such that for some real number $b \geq 1$ and all $x, y, z \in X$:*

(p_{b1}) $p_b(x, x) = p_b(y, y) = p_b(x, y) \Longleftrightarrow x = y$;

(p_{b2}) $p_b(x, x) \leq p_b(x, y)$;

(p_{b3}) $p_b(x, y) = p_b(y, x)$;

(p_{b4}) $p_b(x, y) \leq b[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The pair (X, p_b) is called a partial b-metric space. The number b is called the coefficient of (X, p_b).

Remark 2.5 ([30]). In a partial b-metric space (X, p_b) if $x, y \in X$ and $p_b(x, y) = 0$, then $x = y$, but the converse may not be true.

It is clear that every partial metric space is a partial b-metric space with the coefficient $b = 1$ and every b-metric space is also a partial b-metric space with the same coefficient b. However, the reverse implications need not hold true, in general.

Example 2.6 ([30]). Let $X = \mathbb{R}^+$, $p > 1$ a constant, and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x, y) = [\max\{x, y\}]^p + |x - y|^p, \ \forall x, y \in X.$$

Then (X, p_b) is a partial b-metric space with coefficient $b = 2^p$, but it is neither a partial metric space nor a b-metric space.

Example 2.7 ([30]). Let (X, p) be a partial metric space and define $p_b(x, y) = (p(x, y))^p$, where $p \geq 1$ is a real number. Then p_b is a partial b-metric with coefficient $b = 2^{p-1}$.

Let (X, p_b) be a partial b-metric space. For each $x \in X$ and for each $\epsilon > 0$, put $B(x, \epsilon) = \{y \in X : p_b(x, y) < p_b(x, x) + \epsilon\}$. Let $\mathscr{B} = \{B(x, \epsilon) : x \in X \text{ and } \epsilon > 0\}$. Ge and Lin [17] proved that $\mathscr{B}$ is not a base for any topology on X. However, they proved that $\mathscr{B}$ is a subbase for some topology τ on X such that (X, τ) is a T_0-space.

Proposition 2.8 ([17]). *Let (X, p_b) be a partial b-metric space and (x_n) be a sequence in X. If (x_n) converges to $x \in X$ with respect to τ, then $\lim\limits_{n\to\infty} p_b(x_n, x) = p_b(x, x)$.*

The above proposition cannot be reversed (see [17]).

DEFINITION 2.9 ([30]). *Let (X, p_b) be a partial b-metric space with coefficient $b \geq 1$ and let (x_n) be a sequence in X. Then*

(i) (x_n) *converges to a point* $x \in X$ *if* $\lim_{n\to\infty} p_b(x_n, x) = p_b(x, x)$. *This will be denoted as* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$.

(ii) (x_n) *is called a Cauchy sequence if* $\lim_{n,m\to\infty} p_b(x_n, x_m)$ *exists and is finite.*

(iii) (X, p_b) *is said to be complete if every Cauchy sequence* (x_n) *in* X*, there exists* $x \in X$ *such that* $\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x, x)$.

DEFINITION 2.10 ([12]). *A sequence (x_n) in a partial b-metric space (X, p_b) is called 0-Cauchy if*

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = 0.$$

The space (X, p_b) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p_b(x, x) = 0$, i.e., $\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x, x) = 0$.

LEMMA 2.11 ([12]). *If (X, p_b) is complete, then it is 0-complete.*

The converse assertion of the above lemma may not hold, in general. The following example supports this fact.

EXAMPLE 2.12. The space $X = [0, \infty) \cap \mathbb{Q}$ with $p_b(x, y) = \max\{x, y\}$ is a 0-complete partial b-metric space with coefficient $b = 1$, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p_b), but it is not a 0-Cauchy sequence.

REMARK 2.13 ([30]). In a partial b-metric space (X, p_b), the limit of a convergent sequence need not be unique.

DEFINITION 2.14. *A sequence (x_n) in a partial b-metric space (X, p_b) is said to be bounded if the set $\{p_b(x_n, x_m) : n, m \in \mathbb{N}\}$ of real numbers is bounded in $\mathbb{R}$, that is, there exists $M > 0$ such that $p_b(x_n, x_m) \leq M$ for all $n, m \in \mathbb{N}$.*

DEFINITION 2.15 ([1]). *Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.*

DEFINITION 2.16 ([21]). *The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:*

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.17 ([1]). *Let S and T be weakly compatible self mappings of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Definition 2.18 ([19]). *Let $T : [0,\infty) \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping. Then T is called a tri-simulation function if it satisfies the following conditions:*

($T1$) *$T(z, y, x) < x - yz$ for all x, $y > 0$, $z \geq 0$;*

($T2$) *if (z_n), (y_n) and (x_n) are sequences in $(0,\infty)$ such that $y_n < x_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} z_n \geq 1$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, then*

$$\limsup_{n\to\infty} T(z_n, y_n, x_n) < 0.$$

The set of all tri-simulation functions is denoted by $\mathcal{T}$.

We now give some examples of tri-simulation functions.

Example 2.19 ([19]). Let $T(z, y, x) = \lambda x - yz$ for all x, y, $z \in [0,\infty)$, where $\lambda \in [0, 1)$. Then $T \in \mathcal{T}$.

Example 2.20 ([19]). Let $T(z, y, x) = x - \psi(x) - yz$ for all x, y, $z \in [0,\infty)$, where $\psi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous function such that $\psi(t) = 0$ if and only if $t = 0$. Then $T \in \mathcal{T}$.

Example 2.21 ([19]). Let $T(z, y, x) = \frac{x}{x+1} - yz$ for all x, y, $z \in [0,\infty)$. Then $T \in \mathcal{T}$.

Example 2.22 ([19]). Let $T(z, y, x) = \psi(x) - yz$ for all x, y, $z \in [0,\infty)$, where $\psi : [0,\infty) \to [0,\infty)$ is Matkowski function, i.e., non-decreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Observe that, $\psi(t) < t$ for all $t > 0$. Then $T \in \mathcal{T}$.

Example 2.23 ([19]). Let $T(z, y, x) = \psi(x) - z\phi(y)$ for all x, y, $z \in [0,\infty)$, where ϕ, $\psi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$. Then $T \in \mathcal{T}$.

We now assign a digraph in partial b-metric spaces (X, p_b) as follows.

Let (X, p_b) be a partial b-metric space and let $\Delta = \{(x, x) : x \in X\}$. We consider a digraph G whose vertex set $V(G)$ coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We also assume that G has no parallel edges. Under these assumptions, we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [8, 9, 18].

DEFINITION 2.24. *Let (X, p_b) be a partial b-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a digraph. A mapping $f : X \to X$ is called a Banach G-contraction or simply G-contraction if there exists $k \in (0, \frac{1}{b})$ such that*

$$p_b(fx, fy) \leq k\, p_b(x, y)$$

for all x, $y \in X$ with $(x, y) \in E(G)$.

Any Banach contraction is a G_0-contraction, where the graph G_0 is defined by $E(G_0) = X \times X$. But it is valuable to note that a Banach G-contraction need not be a Banach contraction (see Remark 3.9).

REMARK 2.25. If f is a G-contraction, then f is both a G^{-1}-contraction and a $\tilde{G}$-contraction.

3. Main Result

In this section we assume that (X, p_b) is a partial b-metric space with the coefficient $b \geq 1$ and $G = (V(G), E(G))$ is a reflexive digraph which has no parallel edges. Let f, $g : (X, p_b) \to (X, p_b)$ be two mappings. We use the following notations:

$$M_{fg}(x, y) := \max\left\{p_b(gx, gy), p_b(gx, fx), \frac{p_b(gy, fy)}{2}, \frac{p_b(gx, fy) + p_b(gy, fx)}{2b}\right\}$$

for all x, $y \in X$.

If $g = I$, the identity map on X, we denote $M_f(x, y) := M_{fg}(x, y)$.

DEFINITION 3.1. *Let f, $g : (X, p_b) \to (X, p_b)$ be two mappings. Then, the mapping f is called a generalized (α, T)-G-contractive w.r.t. the mapping g if there exist two functions $T \in \mathcal{T}$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$T\big(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)\big) \geq 0 \tag{1}$$

for all x, $y \in X$ with $(gx, gy) \in E(\tilde{G})$ and $\alpha(gx, gy) \geq 1$.

Taking $g = I$, the above definition gives the following definition.

DEFINITION 3.2. *The mapping $f : (X, p_b) \to (X, p_b)$ is called a generalized (α, T)-G-contractive if there exist two functions $T \in \mathcal{T}$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$T\big(\alpha(x, y), bp_b(fx, fy), M_f(x, y)\big) \geq 0$$

for all x, $y \in X$ with $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geq 1$.

Taking $G = G_0$ in Definition 3.1, we get the following.

DEFINITION 3.3. *Let f, $g : (X, p_b) \to (X, p_b)$ be two mappings. Then, the mapping f is called a generalized (α, T)-contractive w.r.t. the mapping g if there exist two functions $T \in \mathcal{T}$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$T\big(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)\big) \geq 0$$

for all x, $y \in X$ and $\alpha(gx, gy) \geq 1$.

Let f, $g : X \to X$ be such that $f(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $gx_1 = fx_0$. Continuing in this way, we can construct a sequence (gx_n) in $g(X)$ such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$.

DEFINITION 3.4. *Let the mappings f, $g : X \to X$ be such that $f(X) \subseteq g(X)$ and let $\alpha : X \times X \to [0, \infty)$ be another mapping. We define $C_{fg}^{\alpha G}$ the set of all elements x_0 of X such that for all m, $n = 0, 1, 2, \cdots$, $(gx_n, gx_m) \in E(\tilde{G})$ and $\alpha(gx_n, gx_m) \geq 1$, for every sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$.*

Taking $g = I$, we denote $C_f^{\alpha G} := C_{fg}^{\alpha G}$.

Taking $G = G_0$, $C_{fg}^{\alpha G}$ becomes C_{fg}^{α} which is the collection of all elements x_0 of X such that for all m, $n = 0, 1, 2, \cdots$, $\alpha(gx_n, gx_m) \geq 1$, for every sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$.

Before presenting our main result, we state a property of the graph G, call it Property $(*)$.

PROPERTY $(*)$: If (gx_k) is a sequence in (X, p_b) such that $p_b(gx_k, x) \to 0$, $(gx_k, gx_{k+1}) \in E(\tilde{G})$ and $\alpha(gx_k, gx_{k+1}) \geq 1$ for all $k \geq 1$, then there exists a subsequence (gx_{k_i}) of (gx_k) such that $(gx_{k_i}, x) \in E(\tilde{G})$ and $\alpha(gx_{k_i}, x) \geq 1$ for all $i \geq 1$.

Taking $g = I$, the above property reduces to Property $(*')$:

PROPERTY $(*')$: If (x_k) is a sequence in a partial b-metric space (X, p_b) such that $p_b(x_k, x) \to 0$, $(x_k, x_{k+1}) \in E(\tilde{G})$ and $\alpha(x_k, x_{k+1}) \geq 1$ for all $k \geq 1$, then there exists a subsequence (x_{k_i}) of (x_k) such that $(x_{k_i}, x) \in E(\tilde{G})$ and $\alpha(x_{k_i}, x) \geq 1$ for all $i \geq 1$.

Taking $G = G_0$ in Property $(*)$, we get the following property:

PROPERTY $(\dagger)$: If (gx_k) is a sequence in (X, p_b) such that $p_b(gx_k, x) \to 0$ and $\alpha(gx_k, gx_{k+1}) \geq 1$ for all $k \geq 1$, then there exists a subsequence (gx_{k_i}) of (gx_k) such that $\alpha(gx_{k_i}, x) \geq 1$ for all $i \geq 1$.

If $(X, p_b, \preceq)$ is a partially ordered partial b-metric space, then by taking $\alpha(x, y) = 1$ for all x, $y \in X$ and $G = G_2$, where the graph G_2 is defined by

$E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$, the Property $(\acute{*})$ reduces to the Property (‡) which can be stated as follows:

PROPERTY (‡): If (x_k) is a sequence in a partially ordered partial b-metric space $(X, p_b, \preceq)$ such that $p_b(x_k, x) \to 0$ and x_k, x_{k+1} are comparable for all $k \geq 1$, then there exists a subsequence (x_{k_i}) of (x_k) such that x_{k_i}, x are comparable for all $i \geq 1$.

We now present our main result.

THEOREM 3.5. *Let (X, p_b) be a partial b-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a digraph. Let the mappings f, $g : X \to X$ be such that $p_b(fx, fy) > 0$ implies that $p_b(gx, gy) > 0$. Suppose that f is generalized (α, T)-G-contractive w.r.t. the mapping g. Suppose also that $f(X) \subseteq g(X)$, $g(X)$ is a 0-complete subspace of X and the graph G has the Property $(*)$. Then f and g have a point of coincidence u(say) in $g(X)$ with $p_b(u, u) = 0$ if $C_{fg}^{\alpha G} \neq \emptyset$.*

Moreover, f and g have a unique point of coincidence in $g(X)$ if the graph G has the following property:

$(**)$ If x, y are points of coincidence of f and g in $g(X)$, then $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geq 1$.

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. Suppose that $C_{fg}^{\alpha G} \neq \emptyset$. We choose an $x_0 \in C_{fg}^{\alpha G}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence (gx_n) in X such that $gx_n = fx_{n-1}$, for $n \in \mathbb{N}$ and $(gx_n, gx_m) \in E(\tilde{G})$ and $\alpha(gx_n, gx_m) \geq 1$ for m, $n = 0, 1, 2, \cdots$.

We assume that $gx_n \neq gx_{n-1}$ for every $n \in \mathbb{N}$. In fact, if $gx_n = gx_{n-1}$ for some $n \in \mathbb{N}$ then $gx_n = fx_{n-1} = gx_{n-1}$ which implies that gx_n is a point of coincidence of f and g.

We now prove that $\lim\limits_{n\to\infty} p_b(gx_{n-1}, gx_n) = 0$.

First we note that for all $n \in \mathbb{N}$, $(gx_{n-1}, gx_n) \in E(\tilde{G})$, $\alpha(gx_{n-1}, gx_n) \geq 1$ and $p_b(fx_{n-1}, fx_n) > 0$, $p_b(gx_{n-1}, gx_n) > 0$. Therefore, it follows from conditions (1) and $(T1)$ that

$$\begin{aligned} 0 &\leq T\big(\alpha(gx_{n-1}, gx_n), bp_b(fx_{n-1}, fx_n), M_{fg}(x_{n-1}, x_n)\big) \\ &< M_{fg}(x_{n-1}, x_n) - b\alpha(gx_{n-1}, gx_n)p_b(fx_{n-1}, fx_n), \end{aligned} \tag{2}$$

where

$$M_{fg}(x_{n-1},x_n) = \max \left\{ \begin{array}{l} p_b(gx_{n-1},gx_n), p_b(gx_{n-1},gx_n), \\ \frac{p_b(gx_n,gx_{n+1})}{2}, \frac{p_b(gx_{n-1},gx_{n+1})+p_b(gx_n,gx_n)}{2b} \end{array} \right\}$$
$$\leq \max \left\{ \begin{array}{l} p_b(gx_{n-1},gx_n), p_b(gx_n,gx_{n+1}), \\ \frac{p_b(gx_{n-1},gx_n)+p_b(gx_n,gx_{n+1})}{2} \end{array} \right\}$$
$$= \max \{p_b(gx_{n-1},gx_n), p_b(gx_n,gx_{n+1})\}.$$

It now follows from condition (2) that, for all $n = 1, 2, \cdots,$

$$\begin{aligned} p_b(gx_n,gx_{n+1}) &\leq b\alpha(gx_{n-1},gx_n)p_b(gx_n,gx_{n+1}) \\ &< M_{fg}(x_{n-1},x_n) \\ &\leq \max\{p_b(gx_{n-1},gx_n), p_b(gx_n,gx_{n+1})\}. \end{aligned} \tag{3}$$

If $\max\{p_b(gx_{n-1},gx_n), p_b(gx_n,gx_{n+1})\} = p_b(gx_n,gx_{n+1})$, then by using (3), we get

$$p_b(gx_n,gx_{n+1}) < p_b(gx_n,gx_{n+1}),$$

which is a contradiction. Therefore,

$$\max\{p_b(gx_{n-1},gx_n), p_b(gx_n,gx_{n+1})\} = p_b(gx_{n-1},gx_n).$$

Hence from condition (3), we can compute that

$$\begin{aligned} p_b(gx_n,gx_{n+1}) &\leq bp_b(gx_n,gx_{n+1}) \\ &\leq b\alpha(gx_{n-1},gx_n)p_b(gx_n,gx_{n+1}) \\ &< M_{fg}(x_{n-1},x_n) \\ &\leq p_b(gx_{n-1},gx_n). \end{aligned} \tag{4}$$

Hence, we conclude that $(p_b(gx_{n-1},gx_n))$ is a decreasing sequence of positive real numbers, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p_b(gx_{n-1},gx_n) = r.$$

We shall show that $r = 0$. Assume that $r > 0$. Then by taking limit as $n \to \infty$, it follows from condition (4) that

$$\lim_{n\to\infty} bp_b(gx_n,gx_{n+1}) = r,$$

$$\lim_{n\to\infty} M_{fg}(x_{n-1},x_n) = r$$

and

$$\lim_{n\to\infty} \alpha(gx_{n-1},gx_n) = 1.$$

Let $z_n := \alpha(gx_{n-1}, gx_n)$, $y_n := bp_b(gx_n, gx_{n+1})$ and $t_n := M_{fg}(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Then, $y_n < t_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = 1$, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} t_n = r > 0$. From $(T2)$, we obtain

$$0 \leq \limsup_{n\to\infty} T(z_n, y_n, t_n) < 0,$$

which is a contradiction. This implies that for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} p_b(gx_{n-1}, gx_n) = 0. \tag{5}$$

We now proceed to show that (gx_n) is a bounded sequence in (X, p_b). To obtain this assertion, we suppose that the sequence (gx_n) is not bounded. Then there exists a subsequence (gx_{n_k}) of (gx_n) such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the smallest integer satisfying

$$p_b(gx_{n_{k+1}}, gx_{n_k}) > 1 \quad \text{and} \quad p_b(gx_l, gx_{n_k}) \leq 1$$

for all $n_k \leq l \leq n_{k+1} - 1$.

We note that $(gx_{n_{k+1}-1}, gx_{n_k-1}) \in E(\tilde{G})$, $\alpha(gx_{n_{k+1}-1}, gx_{n_k-1}) \geq 1$ and

$$\begin{aligned} p_b(gx_{n_{k+1}}, gx_{n_k}) > 1 &\Rightarrow p_b(fx_{n_{k+1}-1}, fx_{n_k-1}) > 0 \\ &\Rightarrow p_b(gx_{n_{k+1}-1}, gx_{n_k-1}) > 0 \\ &\Rightarrow M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}) > 0. \end{aligned}$$

Using conditions (1) and $(T1)$, we obtain

$$\begin{aligned} 0 &\leq T\big(\alpha(gx_{n_{k+1}-1}, gx_{n_k-1}), bp_b(gx_{n_{k+1}}, gx_{n_k}), M_{fg}(x_{n_{k+1}-1}, x_{n_k-1})\big) \\ &< M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}) - b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}). \end{aligned}$$

That is,

$$b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}) < M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}), \tag{6}$$

where

$$\begin{aligned} &M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}) \\ &= \max\left\{ \begin{array}{l} p_b(gx_{n_{k+1}-1}, gx_{n_k-1}), p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}}), \\ \frac{p_b(gx_{n_k-1}, gx_{n_k})}{2}, \frac{p_b(gx_{n_{k+1}-1}, gx_{n_k}) + p_b(gx_{n_k-1}, gx_{n_{k+1}})}{2b} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{l} p_b(gx_{n_{k+1}-1}, gx_{n_k-1}), p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}}), \\ \frac{p_b(gx_{n_k-1}, gx_{n_k})}{2}, \\ \frac{2bp_b(gx_{n_{k+1}-1}, gx_{n_k-1}) + bp_b(gx_{n_k-1}, gx_{n_k}) + bp_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2b} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}}), \\ p_b(gx_{n_{k+1}-1}, gx_{n_k-1}) + \frac{p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2} \end{array} \right\}. \end{aligned}$$

We now compute that

$$\begin{aligned}
&p_b(gx_{n_{k+1}-1}, gx_{n_k-1}) + \frac{p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2} \\
&\le bp_b(gx_{n_{k+1}-1}, gx_{n_k}) + bp_b(gx_{n_k}, gx_{n_k-1}) \\
&\qquad + \frac{p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2} \\
&\le b + bp_b(gx_{n_k}, gx_{n_k-1}) + \frac{p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2} \\
&= b + \frac{(2b+1)p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}) \\
&\quad \le \max\left\{ \begin{array}{l} p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}}), \\ b + \frac{(2b+1)p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2} \end{array} \right\}.
\end{aligned}$$

Hence, it follows from condition (6) that

$$\begin{aligned}
b &< bp_b(gx_{n_{k+1}}, gx_{n_k}) \\
&\le b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}) \\
&< M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}) \\
&\le \max\left\{ \begin{array}{l} p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}}), \\ b + \frac{(2b+1)p_b(gx_{n_k-1}, gx_{n_k}) + p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}})}{2} \end{array} \right\}.
\end{aligned}$$

Taking limit as $k \to \infty$ and using condition (5), we get

$$\lim_{k\to\infty} M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}) = b, \tag{7}$$

$$\lim_{k\to\infty} bp_b(gx_{n_{k+1}}, gx_{n_k}) = b \tag{8}$$

and

$$\lim_{k\to\infty} \alpha(gx_{n_{k+1}-1}, gx_{n_k-1}) = 1. \tag{9}$$

Let

$$\begin{aligned}
W_k &:= \alpha(gx_{n_{k+1}-1}, gx_{n_k-1}), \\
V_k &:= bp_b(gx_{n_{k+1}}, gx_{n_k}), \\
U_k &:= M_{fg}(x_{n_{k+1}-1}, x_{n_k-1}).
\end{aligned}$$

Then $V_k < U_k$ for all $k \in \mathbb{N}$. By using conditions (9), (7) and (8), we get $\lim_{k\to\infty} W_k = 1$ and $\lim_{k\to\infty} U_k = \lim_{k\to\infty} V_k = b > 0$. By condition $(T2)$, we obtain that

$$0 \leq \limsup_{k\to\infty} T\big(W_k, V_k, U_k\big) < 0,$$

which is a contradiction. This ensures that the sequence (gx_n) cannot have any unbounded subsequence. Thus (gx_n) is a bounded sequence in (X, p_b).

Now, we shall show that (gx_n) is a 0-Cauchy sequence. Let

$$R_n = \sup\{p_b(gx_i, gx_j) > 0 : i,\, j \geq n\}, n \in \mathbb{N}.$$

Since the sequence (gx_n) is bounded, $R_n < +\infty$ for every $n \in \mathbb{N}$. But (R_n) being a decreasing sequence of positive real numbers, there exists $R \geq 0$ such that

$$\lim_{n\to\infty} R_n = R. \tag{10}$$

We assume that $R > 0$. Then by the definition of R_n, it follows that for every natural number k, there exist $n_k,\, m_k \in \mathbb{N}$ such that $m_k,\, n_k \geq k$, $p_b(gx_{m_k}, gx_{n_k}) > 0$ and

$$R_k - \frac{1}{k} < p_b(gx_{m_k}, gx_{n_k}) \leq R_k.$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} p_b(gx_{m_k}, gx_{n_k}) = R > 0. \tag{11}$$

We note that for every $k \in \mathbb{N}$,

$$\begin{aligned} p_b(gx_{m_k}, gx_{n_k}) > 0 &\Rightarrow p_b(fx_{m_k-1}, fx_{n_k-1}) > 0 \\ &\Rightarrow p_b(gx_{m_k-1}, gx_{n_k-1}) > 0 \\ &\Rightarrow M_{fg}(x_{m_k-1}, x_{n_k-1}) > 0 \end{aligned}$$

and $\alpha(gx_{m_k-1}, gx_{n_k-1}) \geq 1,\ (gx_{m_k-1}, gx_{n_k-1}) \in E(\tilde{G})$.

Using conditions (1) and $(T1)$, we get

$$\begin{aligned} 0 &\leq T\big(\alpha(gx_{m_k-1}, gx_{n_k-1}),\, bp_b(gx_{m_k}, gx_{n_k}), M_{fg}(x_{m_k-1}, x_{n_k-1})\big) \\ &< M_{fg}(x_{m_k-1}, x_{n_k-1}) - b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}). \end{aligned}$$

That is,

$$b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}) < M_{fg}(x_{m_k-1}, x_{n_k-1}), \tag{12}$$

where

$$
\begin{aligned}
&M_{fg}(x_{m_k-1}, x_{n_k-1}) \\
&= \max\left\{\begin{array}{l} p_b(gx_{m_k-1}, gx_{n_k-1}), p_b(gx_{m_k-1}, gx_{m_k}), \\ \frac{p_b(gx_{n_k-1}, gx_{n_k})}{2}, \frac{p_b(gx_{m_k-1}, gx_{n_k})+p_b(gx_{n_k-1}, gx_{m_k})}{2b} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} p_b(gx_{m_k-1}, gx_{n_k-1}), p_b(gx_{m_k-1}, gx_{m_k}), \\ \frac{p_b(gx_{n_k-1}, gx_{n_k})}{2}, \\ \frac{2bp_b(gx_{m_k-1}, gx_{n_k-1})+bp_b(gx_{n_k-1}, gx_{n_k})+bp_b(gx_{m_k-1}, gx_{m_k})}{2b} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} p_b(gx_{m_k-1}, gx_{m_k}), \\ p_b(gx_{m_k-1}, gx_{n_k-1}) + \frac{p_b(gx_{n_k-1}, gx_{n_k})+p_b(gx_{m_k-1}, gx_{m_k})}{2} \end{array}\right\}. \\
&\leq \max\left\{\begin{array}{l} p_b(gx_{m_k-1}, gx_{m_k}), \\ R_{k-1} + \frac{p_b(gx_{n_k-1}, gx_{n_k})+p_b(gx_{m_k-1}, gx_{m_k})}{2} \end{array}\right\}.
\end{aligned}
$$

By using the definition of R_n, it follows from condition (12) that

$$
\begin{aligned}
p_b(gx_{m_k}, gx_{n_k}) &\leq bp_b(gx_{m_k}, gx_{n_k}) \\
&\leq b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}) \\
&< M_{fg}(x_{m_k-1}, x_{n_k-1}) \\
&\leq \max\left\{\begin{array}{l} p_b(gx_{m_k-1}, gx_{m_k}), \\ R_{k-1} + \frac{p_b(gx_{n_k-1}, gx_{n_k})+p_b(gx_{m_k-1}, gx_{m_k})}{2} \end{array}\right\}. \qquad (13)
\end{aligned}
$$

Taking limit as $k \to \infty$ and using conditions (10) and (11), we obtain that

$$\lim_{k\to\infty} M_{fg}(x_{m_k-1}, x_{n_k-1}) = \lim_{k\to\infty} bp_b(gx_{m_k}, gx_{n_k}) = R.$$

Also, condition (13) ensures that $\lim_{k\to\infty} \alpha(gx_{m_k-1}, gx_{n_k-1}) = 1$.

Let

$$
\begin{aligned}
\mathcal{Z}_k &:= \alpha(gx_{m_k-1}, gx_{n_k-1}), \\
\mathcal{Y}_k &:= bp_b(gx_{m_k}, gx_{n_k}) \\
\mathcal{X}_k &:= M_{fg}(x_{m_k-1}, x_{n_k-1}).
\end{aligned}
$$

Then $\mathcal{Y}_k < \mathcal{X}_k$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} \mathcal{Z}_k = 1$, $\lim_{k\to\infty} \mathcal{Y}_k = \lim_{k\to\infty} \mathcal{X}_k = R > 0$. It now follows from condition $(T2)$ that

$$0 \leq \limsup_{k\to\infty} T\big(\mathcal{Z}_k, \mathcal{Y}_k, \mathcal{X}_k\big) < 0,$$

which is a contradiction and so we have $R = 0$. Hence, we deduce that

$$\lim_{n,m\to\infty} p_b(gx_n, gx_m) = 0.$$

Therefore, (gx_n) is a 0-Cauchy sequence in $g(X)$. As $g(X)$ is 0-complete, there exists an $u = g\nu \in g(X)$ for some $\nu \in X$ such that $gx_n \to u$ and $p_b(u,u) = 0$.

Therefore,

$$\lim_{n,m\to\infty} p_b(gx_n, gx_m) = \lim_{n\to\infty} p_b(gx_n, u) = p_b(u,u) = 0.$$

By Property $(*)$ there is a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, g\nu) \in E(\tilde{G})$ and $\alpha(gx_{n_i}, g\nu) \geq 1$ for all $i \geq 1$.

Next, we shall show that f and g have a point of coincidence in $g(X)$. We note that

$$M_{fg}(x_{n_i}, \nu) = \max\left\{\begin{array}{c} p_b(gx_{n_i}, g\nu), p_b(gx_{n_i}, gx_{n_i+1}), \\ \frac{p_b(g\nu, f\nu)}{2}, \frac{p_b(gx_{n_i}, f\nu) + p_b(g\nu, gx_{n_i+1})}{2b} \end{array}\right\}.$$

If $M_{fg}(x_{n_i}, \nu) = 0$, then $\frac{p_b(g\nu, f\nu)}{2} = 0$ and hence $g\nu = f\nu = u$. So we assume that $M_{fg}(x_{n_i}, \nu) > 0$.

If $p_b(fx_{n_i}, f\nu) > 0$, then we obtain from condition (1) that

$$\begin{aligned} 0 &\leq T\big(\alpha(gx_{n_i}, g\nu), bp_b(fx_{n_i}, f\nu), M_{fg}(x_{n_i}, \nu)\big) \\ &< M_{fg}(x_{n_i}, \nu) - b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu). \end{aligned}$$

That is,

$$b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu) < M_{fg}(x_{n_i}, \nu),$$

which gives that

$$bp_b(fx_{n_i}, f\nu) \leq b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu) < M_{fg}(x_{n_i}, \nu).$$

Moreover, if $p_b(fx_{n_i}, f\nu) = 0$, then

$$0 = bp_b(fx_{n_i}, f\nu) < M_{fg}(x_{n_i}, \nu).$$

Therefore,

$$bp_b(fx_{n_i}, f\nu) < M_{fg}(x_{n_i}, \nu) \text{ for all } i \in \mathbb{N}.$$

Now,

$$\begin{aligned} 0 &\leq p_b(f\nu, g\nu) \\ &\leq bp_b(f\nu, fx_{n_i}) + bp_b(fx_{n_i}, g\nu) - bp_b(fx_{n_i}, fx_{n_i}) \\ &< M_{fg}(x_{n_i}, \nu) + bp_b(gx_{n_i+1}, g\nu) \\ &\leq \max\left\{\begin{array}{c} p_b(gx_{n_i}, g\nu), p_b(gx_{n_i}, gx_{n_i+1}), \frac{p_b(g\nu, f\nu)}{2}, \\ \frac{bp_b(gx_{n_i}, g\nu) + bp_b(g\nu, f\nu) + p_b(g\nu, gx_{n_i+1})}{2b} \end{array}\right\} \\ &\quad + bp_b(gx_{n_i+1}, g\nu). \end{aligned}$$

Taking limit as $i \to \infty$, we obtain that

$$0 \leq p_b(f\nu, g\nu) \leq \frac{p_b(g\nu, f\nu)}{2}.$$

This implies that $p_b(f\nu, g\nu) = 0$ and so, $f\nu = g\nu = u$. Therefore, u is a point of coincidence of f and g.

For uniqueness, we assume that there exists $u^* \in X$ such that $fx = gx = u^*$ for some $x \in X$ with $p_b(u^*, u^*) = 0$ and $u \neq u^*$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$ and $\alpha(u, u^*) \geq 1$. Then,

$$\begin{aligned} 0 &\leq T\big(\alpha(g\nu, gx), bp_b(f\nu, fx), M_{fg}(\nu, x)\big) \\ &= T\big(\alpha(u, u^*), bp_b(u, u^*), M_{fg}(\nu, x)\big) \\ &< M_{fg}(\nu, x) - b\alpha(u, u^*)p_b(u, u^*). \end{aligned}$$

That is, $b\alpha(u, u^*)p_b(u, u^*) < M_{fg}(\nu, x) = p_b(u, u^*)$, since

$$\begin{aligned} M_{fg}(\nu, x) &= \max\left\{ \begin{array}{l} p_b(g\nu, gx), p_b(g\nu, f\nu), \\ \frac{p_b(gx, fx)}{2}, \frac{p_b(g\nu, fx) + p_b(gx, f\nu)}{2b} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} p_b(u, u^*), p_b(u, u), \frac{p_b(u^*, u^*)}{2}, \\ \frac{p_b(u, u^*) + p_b(u^*, u)}{2b} \end{array} \right\} \\ &= p_b(u, u^*). \end{aligned}$$

This gives that $bp_b(u, u^*) \leq b\alpha(u, u^*)p_b(u, u^*) < p_b(u, u^*)$, a contradiction. Hence, $u = u^*$. Therefore, f and g have a unique point of coincidence in $g(X)$.

If f and g are weakly compatible, then by Proposition 2.17, f and g have a unique common fixed point in $g(X)$. □

We give some examples to illustrate our main result.

Example 3.6. Let $X = [0, \infty)$ and define $p_b : X \times X \to \mathbb{R}^+$ by $p_b(x, y) = [\max\{x, y\}]^3 + |x - y|^3$ for all $x, y \in X$. Then (X, p_b) is a 0-complete partial b-metric space with the coefficient $b = 8$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{2^n}) : n = 1, 2, 3, \cdots\}$.

Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{5}, & \text{if } x \neq \frac{1}{3}, \\ 1, & \text{if } x = \frac{1}{3} \end{cases}$$

and $gx = 4x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and $p_b(fx, fy) > 0 \Rightarrow p_b(gx, gy) > 0$.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x,\, y \in [0, 1], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

The point $x_0 := 0$ belongs to $C_{fg}^{\alpha G}$.

Let $T : [0, \infty) \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $T(z, y, x) = \frac{1}{2}x - yz$ for all $x,\, y,\, z \geq 0$.

For $x = 0,\ y = \frac{1}{2^{n+2}},\ n \in \mathbb{N}$, we have $gx = 0,\ gy = \frac{1}{2^n},\ fx = 0,\ fy = \frac{1}{5 \cdot 2^{n+2}}$ and so $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) = 1$. We now compute that $p_b(fx, fy) = p_b(0, \frac{1}{5 \cdot 2^{n+2}}) = \frac{2}{125 \cdot 2^{3n+6}}$, $p_b(gx, gy) = p_b(0, \frac{1}{2^n}) = \frac{2}{2^{3n}}$, $p_b(gx, fx) = 0$,

$$\begin{aligned} p_b(gy, fy) &= p_b\left(\frac{1}{2^n}, \frac{1}{5 \cdot 2^{n+2}}\right) \\ &= \frac{1}{2^{3n}} + \frac{1}{2^{3n}}\left(1 - \frac{1}{20}\right)^3 = \frac{1}{2^{3n}}\left(1 + \left(\frac{19}{20}\right)^3\right) < \frac{2}{2^{3n}}, \end{aligned}$$

$p_b(gx, fy) = p_b(0, \frac{1}{5 \cdot 2^{n+2}}) = \frac{2}{125 \cdot 2^{3n+6}}$, $p_b(gy, fx) = p_b(\frac{1}{2^n}, 0) = \frac{2}{2^{3n}}$.

Now,

$$\frac{p_b(gx, fy) + p_b(gy, fx)}{2b} = \frac{1}{2^{3n}}\left(\frac{1}{125 \cdot 2^9} + \frac{1}{2^3}\right) < \frac{1}{2^{3n}}.$$

Thus, we obtain that $M_{fg}(x, y) = \frac{2}{2^{3n}}$.

Therefore,

$$\begin{aligned} &T(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)) \\ &\quad = \frac{1}{2} \cdot \frac{2}{2^{3n}} - 8 \cdot \frac{2}{125 \cdot 2^{3n+6}} = \frac{1}{2^{3n}}\left(1 - \frac{1}{500}\right) > 0. \end{aligned}$$

Moreover, for $0 \leq x = y \leq \frac{1}{4}$, we have $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) = 1$ and

$$p_b(fx, fy) = p_b\left(\frac{x}{5}, \frac{x}{5}\right) = \frac{x^3}{125},\ p_b(gx, gy) = p_b(4x, 4x) = 64x^3,$$

$$p_b(gx, fx) = p_b(gy, fy) = p_b(4x, \frac{x}{5}) = 64x^3 + \left(\frac{19x}{5}\right)^3,$$

$$\frac{p_b(gx, fy) + p_b(gy, fx)}{2b} = \frac{p_b(gx, fx)}{b} = \frac{1}{8}\left[64x^3 + \left(\frac{19x}{5}\right)^3\right].$$

It now follows from the above computation that

$$M_{fg}(x, y) = 64x^3 + \left(\frac{19x}{5}\right)^3.$$

In this case, we have

$$\begin{aligned} T(\alpha(gx,gy), bp_b(fx,fy), M_{fg}(x,y)) &= \frac{1}{2}\left[64x^3 + \left(\frac{19x}{5}\right)^3\right] - 8.\frac{x^3}{125} \\ &= 8x^3\left(4 - \frac{1}{125}\right) + \frac{1}{2}\left(\frac{19x}{5}\right)^3 \geq 0. \end{aligned}$$

Therefore,

$$T(\alpha(gx,gy), bp_b(fx,fy), M_{fg}(x,y)) \geq 0$$

for all x, $y \in X$ with $(gx,gy) \in E(\tilde{G})$ and $\alpha(gx,gy) \geq 1$.

Any sequence (gx_n) with the property $p_b(gx_n, x) \to 0$, $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \geq 1$ must be either the zero sequence or a sequence of the following form

$$gx_n = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{1}{2^n}, & \text{if } n \text{ is even} \end{cases}$$

where the words 'odd' and 'even' are interchangeable. Also, $p_b(gx_n, x) \to 0$ ensures that $x = 0$ and consequently, it follows that Property $(*)$ holds true. Moreover, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.5 and 0 is the unique common fixed point of f and g in $g(X)$ with $p_b(0,0) = 0$.

REMARK 3.7. We see that in the above example, f is not a Banach G-contraction. Because, if we take $x = y = \frac{1}{3}$, then $(x,y) \in E(G)$ and

$$p_b(fx,fy) = p_b(1,1) = 1 = 27 \cdot \frac{1}{27} > k\, p_b(x,y)$$

for any $k \in (0, \frac{1}{b})$.

We now give an example in favour of the Property $(*)$ of the graph G in Theorem 3.5.

EXAMPLE 3.8. Let $X = [0,\infty)$ and define $p_b : X \times X \to \mathbb{R}^+$ by

$$p_b(x,y) = |x-y|^3 \text{ for all } x,\ y \in X.$$

Then (X, p_b) is a 0-complete partial b-metric space with the coefficient $b = 4$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x,y) : x, y \in (0,1] \text{ and } x \leq y\}$.

Let f, $g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{9}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0 \end{cases}$$

and $gx = \frac{x}{3}$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and $p_b(fx, fy) > 0 \Rightarrow p_b(gx, gy) > 0$.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x,\, y \in [0, 1], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

The point $x_0 := \frac{1}{3}$ belongs to $C_{fg}^{\alpha G}$.

Let $T : [0, \infty) \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $T(z, y, x) = \frac{5}{9}x - yz$ for all $x,\, y,\, z \geq 0$.

For $x,\, y \in (0, 3]$, we have $(gx, gy) \in E(\tilde{G})$ and $\alpha(gx, gy) = 1$. Then,

$$p_b(fx, fy) = \frac{1}{9^3}|x - y|^3,\ p_b(gx, gy) = \frac{1}{27}|x - y|^3.$$

Therefore,

$$\begin{aligned} T(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)) &= \frac{5}{9}M_{fg}(x, y) - bp_b(fx, fy) \\ &\geq \frac{5}{9}p_b(gx, gy) - 4p_b(fx, fy) \\ &= \frac{5}{9 \times 27}|x - y|^3 - \frac{4}{9^3}|x - y|^3 = \frac{11}{729}|x - y|^3 \geq 0. \end{aligned}$$

Thus,

$$T(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)) \geq 0$$

for all $x,\, y \in X$ with $(gx, gy) \in E(\tilde{G})$ and $\alpha(gx, gy) \geq 1$.

We now show that the graph G does not satisfy the Property $(*)$. For $x_n = \frac{3}{n}$, $gx_n = \frac{1}{n}$ and hence $p_b(gx_n, 0) \to 0$. Also, $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\alpha(gx_n, gx_{n+1}) = 1$ for all $n \in \mathbb{N}$. But there exists no subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, 0) \in E(\tilde{G})$. It is obvious that f and g have no point of coincidence in $g(X)$. This proves that Theorem 3.5 remains invalid without the Property $(*)$ of the graph G.

Remark 3.9. In Example 3.8, f is not a Banach contraction. For instance, if we take $x = 0$, $y = 1$, then

$$p_b(fx, fy) = p_b\left(1, \frac{1}{9}\right) = \left(\frac{8}{9}\right)^3 = \left(\frac{8}{9}\right)^3 p_b(x, y) > k\, p_b(x, y)$$

for any $k \in (0, \frac{1}{b})$.

However, f is a Banach G-contraction since for all $x,\, y \in X$ with $(x, y) \in E(G)$, we have

$$p_b(fx, fy) = \frac{1}{729}\, p_b(x, y),$$

where $\frac{1}{729} \in (0, \frac{1}{b})$.

We now examine the role of the weak compatibility condition in Theorem 3.5. In fact, we shall show that this condition is necessary in Theorem 3.5 to obtain a common fixed point.

EXAMPLE 3.10. Let $X = \mathbb{R}$ and define $p_b : X \times X \to \mathbb{R}^+$ by $p_b(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, p_b) is a 0-complete partial b-metric space with the coefficient $b = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : x, y \in [0, 1] \text{ and } x \leq y\}$.

Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{4}, & \text{if } x \neq \frac{2}{5}, \\ 1, & \text{if } x = \frac{2}{5} \end{cases}$$

and $gx = 3x - 11$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and $p_b(fx, fy) > 0 \Rightarrow p_b(gx, gy) > 0$.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

The point $x_0 := 4$ belongs to $C_{fg}^{\alpha G}$.

Let $T : [0, \infty) \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $T(z, y, x) = \frac{1}{4}x - yz$ for all $x, y, z \geq 0$.

If $\frac{11}{3} \leq x, y \leq 4$, then $(gx, gy) \in E(\tilde{G})$ and $\alpha(gx, gy) = 1$. We now compute that $p_b(fx, fy) = p_b(\frac{x}{4}, \frac{y}{4}) = \frac{1}{16}|x - y|^2$, $p_b(gx, gy) = 9|x - y|^2$.

Therefore,

$$\begin{aligned} T(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)) &= \frac{1}{4}M_{fg}(x, y) - bp_b(fx, fy) \\ \geq \frac{1}{4}p_b(gx, gy) - 2p_b(fx, fy) &= \frac{9}{4}|x - y|^2 - \frac{1}{8}|x - y|^2 = \frac{17}{8}|x - y|^2 \geq 0. \end{aligned}$$

Thus,

$$T(\alpha(gx, gy), bp_b(fx, fy), M_{fg}(x, y)) \geq 0$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$ and $\alpha(gx, gy) \geq 1$.

It is obvious that any sequence (gx_n) with $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \geq 1$ must be a sequence in $[0, 1]$. Also, $p_b(gx_n, x) \to 0 \Rightarrow |gx_n - x| \to 0 \Rightarrow x \in [0, 1]$ which proves that the graph G has the Property $(*)$.

We note that $f(4) = g(4) = 1$ but $g(f(4)) \neq f(g(4))$ which ensures that f and g are not weakly compatible. However, all other conditions of Theorem 3.5 are fulfilled. We observe that 1 is the unique point of coincidence of f and g without being any common fixed point.

4. Some Consequences of the Main Result

In this section our aim is to present some important coincidence point and fixed point results which will justify the extension of our main result.

THEOREM 4.1. *Let (X, p_b) be a 0-complete partial b-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a digraph. Let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that f is generalized (α, T)-G-contractive and the graph G has the Property $(\acute{*})$. Then f has a fixed point u(say) in X with $p_b(u, u) = 0$ if $C_f^{\alpha G} \neq \emptyset$.*

Moreover, f has a unique fixed point in X if the graph G has the following property:

$(\acute{**})$ *If x, y are fixed points of f in X, then $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geq 1$.*

Proof. The proof follows from Theorem 3.5 by taking $g = I$, the identity map on X. □

THEOREM 4.2. *Let (X, p_b) be a partial b-metric space with the coefficient $b \geq 1$ and let f, $g : X \to X$ be mappings with the property that $p_b(fx, fy) > 0$ implies that $p_b(gx, gy) > 0$. Suppose that f is generalized (α, T)-contractive w.r.t. the mapping g. Suppose also that $f(X) \subseteq g(X)$, $g(X)$ is a 0-complete subspace of X and α has the Property (†). Then f and g have a point of coincidence u(say) in $g(X)$ with $p_b(u, u) = 0$ if $C_{fg}^{\alpha} \neq \emptyset$.*

Moreover, f and g have a unique point of coincidence in $g(X)$ if α has the following property:

If x, y are points of coincidence of f and g in $g(X)$, then $\alpha(x, y) \geq 1$.

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof can be obtained from Theorem 3.5 by considering $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

THEOREM 4.3. *Let (X, p_b) be a 0-complete partial b-metric space and let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that there exists $T \in \mathcal{T}$ such that*

$$T\big(1, bp_b(fx, fy), M_f(x, y)\big) \geq 0$$

for all x, $y \in X$. Then f has a unique fixed point u(say) in X with $p_b(u, u) = 0$.

Proof. The proof follows from Theorem 3.5 by taking $g = I$, $G = G_0$ and $\alpha(x, y) = 1$ for all x, $y \in X$. □

THEOREM 4.4. *Let (X, p_b) be a 0-complete partial b-metric space and let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that f satisfies the following condition:*

$$bp_b(fx, fy) \leq \psi(M_f(x, y)) \text{ for all } x,\, y \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then f has a unique fixed point u(say) in X with $p_b(u, u) = 0$.

Proof. The proof follows directly from Theorem 4.3 and by taking T as in Example 2.22. □

THEOREM 4.5. *Let (X, d) be a complete b-metric space with the coefficient $b \geq 1$ and let $f : X \to X$ be a mapping satisfying the following condition:*

$$d(fx, fy) \leq \beta M_f(x, y) \text{ for all } x,\, y \in X,$$

where $\beta \in [0, \frac{1}{b})$ is a constant. Then f has a unique fixed point in X.

Proof. The proof follows directly from Theorem 4.3 and by taking T as in Example 2.19 and $p_b = d$. □

We now present the b-metric version of Banach contraction theorem.

THEOREM 4.6. *Let (X, d) be a complete b-metric space with the coefficient $b \geq 1$ and let $f : X \to X$ be a mapping satisfying $d(fx, fy) \leq \beta d(x, y)$ for all $x,\, y \in X$, where $\beta \in [0, \frac{1}{b})$ is a constant. Then f has a unique fixed point in X.*

Proof. Since $d(x, y) \leq M_f(x, y)$, the conclusion of the theorem follows directly from Theorem 4.5. □

REMARK 4.7. Theorem 4.6 shows that our main result is a generalization of the well known Banach contraction theorem.

The following is the b-metric version of Fisher's theorem.

THEOREM 4.8. *Let (X, d) be a complete b-metric space with the coefficient $b \geq 1$ and let $f : X \to X$ be a mapping satisfying*

$$d(fx, fy) \leq k[d(fx, y) + d(fy, x)]$$

for all $x,\, y \in X$, where $k \in [0, \frac{1}{2b^2})$ is a constant. Then f has a unique fixed point in X.

Proof. Since

$$d(fx, fy) \leq k[d(fx, y) + d(fy, x)] = 2kb\,\frac{d(fx, y) + d(fy, x)}{2b} \leq \beta M_f(x, y)\,,$$

where $\beta = 2kb \in [0, \frac{1}{b})$, the conclusion of the theorem follows directly from Theorem 4.5. □

We now present analogue of Kannan's fixed point theorem in b-metric spaces.

THEOREM 4.9. *Let (X, d) be a complete b-metric space with the coefficient $b \geq 1$ and let $f : X \to X$ be a mapping satisfying*

$$d(fx, fy) \leq a_1\, d(fx, x) + a_2\, d(fy, y)$$

for all $x,\ y \in X$, where $a_1,\ a_2 \geq 0$ with $a_1 + 2a_2 < \frac{1}{b}$. Then f has a unique fixed point in X.

Proof. Since $d(fx, fy) \leq a_1\, d(fx, x) + a_2\, d(fy, y) \leq (a_1 + 2a_2)M_f(x, y) = \beta M_f(x, y)$, where $\beta = (a_1 + 2a_2) \in [0, \frac{1}{b})$, the conclusion of the theorem follows directly from Theorem 4.5. □

THEOREM 4.10. *Let (X, p_b) be a 0-complete partial b-metric space and let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that f satisfies the following condition:*

$$bp_b(fx, fy) \leq \frac{M_f(x, y)}{M_f(x, y) + 1} \text{ for all } x,\ y \in X.$$

Then f has a unique fixed point u(say) in X with $p_b(u, u) = 0$.

Proof. The proof follows directly from Theorem 4.3 and by taking T as in Example 2.21. □

THEOREM 4.11. *Let (X, d) be a complete b-metric space with the coefficient $b \geq 1$ and let $f : X \to X$ be a mapping satisfying the following condition:*

$$bd(fx, fy) \leq M_f(x, y) - \psi(M_f(x, y)) \text{ for all } x,\ y \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function such that $\psi(t) = 0$ if and only if $t = 0$. Then f has a unique fixed point in X.

Proof. The proof follows directly from Theorem 4.3 and by taking T as in Example 2.20 and $p_b = d$. □

THEOREM 4.12. *Let (X, d) be a complete b-metric space endowed with a partial ordering $\preceq$ and let $f : X \to X$ be a mapping. Suppose that there exists $T \in \mathcal{T}$ such that*

$$T(1, bd(fx, fy), M_f(x, y)) \geq 0$$

for all comparable elements x, $y \in X$. Suppose also that the triple $(X, d, \preceq)$ has the Property (‡). If there exists $x_0 \in X$ such that x_n, x_m are comparable for all n, $m = 0, 1, 2, \cdots$, where $x_n = fx_{n-1}$, $\forall n \in \mathbb{N}$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:

If x, y are fixed points of f in X, then x and y are comparable.

Proof. The proof can be obtained from Theorem 3.5 by taking $p_b = d$, $g = I$, $\alpha(x, y) = 1$ for all x, $y \in X$ and $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

THEOREM 4.13. *Let (X, p_b) be a 0-complete partial b-metric space and let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that f satisfies the following condition:*

$$\phi(bp_b(fx, fy)) \leq \psi(M_f(x, y)) \text{ for all } x,\ y \in X,$$

where ϕ, $\psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$. Then f has a unique fixed point u(say) in X with $p_b(u, u) = 0$.

Proof. The proof follows directly from Theorem 4.3 and by taking T as in Example 2.23. □

REMARK 4.14. It is valuable to note that several well known but important fixed point results in metric, partial metric and b-metric spaces can be obtained by suitable choices of tri-simulation function $T \in \mathcal{T}$, digraph G and the function α.

5. An Application to the Integral Equation

In this section we utilize Theorem 3.5 to obtain a unique solution of the following integral equation

$$x(t) = \int_0^{\xi} F(t, r, x(r))dr, \tag{14}$$

where $\xi > 0$, $F : [0, \xi] \times [0, \xi] \times \mathbb{R} \to \mathbb{R}$, $x : [0, \xi] \to \mathbb{R}$ are continuous functions and the integral is taken in Riemann sense.

Let $X = C[0,\xi]$ be the set of all real valued continuous functions defined on $[0,\xi]$. We define $p_b : X \times X \to \mathbb{R}^+$ by

$$p_b(x,y) = \sup_{0\leq t\leq \xi} |x(t)-y(t)|^p \text{ for all } x,\, y \in X,$$

where $p > 1$. Then it is easy to verify that (X, p_b) is a 0-complete partial b-metric space with the coefficient $b = 2^{p-1}$. In the next theorem X represents the above partial b-metric space and $N_F(x,y)(t)$ is defined as follows:

$$N_F(x,y)(t) = \max\left\{\begin{array}{l} |x(t)-y(t)|, \left|\int_0^\xi F(t,r,x(r))dr - x(t)\right|, \\ \frac{\left|\int_0^\xi F(t,r,y(r))dr - y(t)\right|}{2^{\frac{1}{p}}}, \\ \left(\frac{\left|\int_0^\xi F(t,r,x(r))dr - y(t)\right|^p + \left|\int_0^\xi F(t,r,y(r))dr - x(t)\right|^p}{2b}\right)^{\frac{1}{p}} \end{array}\right\},$$

where $x,\, y \in X$ and $t \in [0,\xi]$.

THEOREM 5.1. *Suppose that $X = C[0,\xi]$ and the following hypotheses hold:*

(*i*) $F : [0,\xi]\times[0,\xi]\times\mathbb{R} \to \mathbb{R}$ *is continuous;*

(*ii*) *for all* $t,\, r \in [0,\xi]$, *there exists a continuous function* $\eta : [0,\xi]\times[0,\xi] \to \mathbb{R}$ *such that*

$$|F(t,r,x(r)) - F(t,r,y(r))| \leq \beta^{\frac{1}{p}}\eta(t,r)N_F(x,y)(t) \text{ for all } x, y \in X \quad (15)$$

and

$$\sup_{0\leq t\leq\xi}\int_0^\xi \eta(t,r)dr \leq 1, \quad (16)$$

where $0 \leq \beta < \frac{1}{b}$.

Then the integral equation (14) *has a unique solution in* X.

Proof. Let $f : X \to X$ be defined by $(fx)(t) = \int_0^\xi F(t,r,x(r))dr$ for all $x \in X$ and for all $t \in [0,\xi]$. Then the existence of a solution to the integral equation (14) is equivalent to the existence of a fixed point of f. We now compute that

$$N_F(x,y)(t) = \max\left\{\begin{array}{l} |x(t)-y(t)|, |(fx)(t)-x(t)|, \\ \frac{|(fy)(t)-y(t)|}{2^{\frac{1}{p}}}, \\ \left(\frac{|(fx)(t)-y(t)|^p + |(fy)(t)-x(t)|^p}{2b}\right)^{\frac{1}{p}} \end{array}\right\}.$$

Therefore,

$$(N_F(x,y)(t))^p = \max\left\{\begin{array}{l} |x(t)-y(t)|^p, |(fx)(t)-x(t)|^p, \\ \frac{|(fy)(t)-y(t)|^p}{2}, \\ \left(\frac{|(fx)(t)-y(t)|^p+|(fy)(t)-x(t)|^p}{2b}\right) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{l} p_b(x,y), p_b(fx,x), \frac{p_b(fy,y)}{2}, \\ \frac{p_b(fx,y)+p_b(fy,x)}{2b} \end{array}\right\} = M_f(x,y). \tag{17}$$

Utilizing conditions (15), (16) and (17), for all x, $y \in X$ and $t \in [0,\xi]$, we obtain that

$$\begin{aligned} |(fx)(t)-(fy)(t)|^p &= \left|\int_0^\xi (F(t,r,x(r))-F(t,r,y(r)))dr\right|^p \\ &\leq \left(\int_0^\xi |F(t,r,x(r))-F(t,r,y(r))|dr\right)^p \\ &\leq \left(\int_0^\xi \beta^{\frac{1}{p}}\eta(t,r)N_F(x,y)(t)dr\right)^p \\ &= \beta\ (N_F(x,y)(t))^p \left(\int_0^\xi \eta(t,r)dr\right)^p \\ &\leq \beta\, M_f(x,y)\left(\int_0^\xi \eta(t,r)dr\right)^p \leq \beta\, M_f(x,y). \end{aligned}$$

Therefore,

$$p_b(fx,fy) = \sup_{0\leq t\leq \xi} |(fx)(t)-(fy)(t)|^p \leq \beta\, M_f(x,y) \text{ for all } x,\, y \in X, \tag{18}$$

where $0 \leq \beta < \frac{1}{b}$.

We note that $p_b(fx,fy) > 0$ implies that $p_b(x,y) > 0$. Let us consider the tri-simulation function $T : [0,\infty)\times[0,\infty)\times[0,\infty) \to \mathbb{R}$ defined by $T(z,y,x) = \lambda x - yz$ for all x, y, $z \in [0,\infty)$, where $\lambda = \beta b \in [0,1)$.

Let us take $g = I$, the identity map on X, $G = G_0$, where G_0 is the complete graph $(X, X\times X)$ and $\alpha(x,y) = 1$ for all x, $y \in X$. It now follows from condition (18) that

$$T\left(\alpha(gx,gy), bp_b(fx,fy), M_f(x,y)\right) = \lambda M_f(x,y) - bp_b(fx,fy) \geq 0$$

for all x, $y \in X$.

Thus all the hypotheses of Theorem 3.5 holds good and hence f has a unique fixed point x (say) in X. This means that x is the unique solution for the integral equation (14). $\square$

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Received October 24, 2024
Revised May 25, 2025
Accepted June 18, 2025