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Collapsing of Mean Curvature Flow of Hypersurfaces to Complex Submanifolds

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Abstract. *In this paper, we produce explicit examples of mean curvature flow of* $(2m-1)$*-dimensional submanifolds which converge to* $(2m-2)$*-dimensional submanifolds at a finite time. These examples are a special class of hyperspheres in* $\mathbb{C}^m$ *with a* $U(m)$*-invariant Kähler metrics. We first discuss the mean curvature flow problem and then investigate the type of singularities for them.*

Keywords: Mean Curvature flow, Hypersurfaces, Complex manifolds.
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1. Introduction

Mean curvature flow is a well-known geometric evolution equation for hypersurfaces in which each point moves with a velocity given by the mean curvature vector. If the hypersurface is compact, the short time existence and uniqueness of the mean curvature flow are well-known. In general, it is very hard to find an exact solution of mean curvature flow problem. In fact there are very few explicit examples. Round spheres in Euclidean space are non trivial examples of evolving hypersurface under mean curvature flow which concentrically shrink inward until they collapse at a finite time to a single point. Another instance would be the marriage ring that under mean curvature flow shrinks to a circle. A round cylinder also remains round and finally converges to a line. Mean curvature flow develops singularities if the second fundamental forms of the time dependent immersions become unbounded. It is well-known that mean curvature flow of any closed manifold in Euclidean space develops singularities at a finite time.

The mean curvature flow has first been investigated by Brakke [2]. Later on, Huisken [8] showed that any closed convex hypersurface in Euclidean space shrinks to a round point at a finite time. He then proved [9] that the same holds for hypersurfaces in general Riemannian manifolds satisfying a strong convexity condition which takes into account the geometry of the ambient space. Brakke used geometric measure theory, but Huisken employed a more classical differential geometric approach. In order to describe singularities of the flow, Osher-Sethian introduced a level-set formulation for the mean curvature flow which was investigated later by Evans-Spruck [4, 5, 6, 7] and Chen-Giga-Goto [3]. Ilmanen [10] revealed the relation between the level-set formulation and the geometric measure theory approach.

In this paper, we consider a class of canonical hyperspheres in $\mathbb{C}^m$. We will make an important assumption about the symmetry group, i.e., the Kähler metric on $\mathbb{C}^m \setminus 0$ has $U(m)$ as the group of isometries. We study the mean curvature flow problem for hyperspheres in $Bl_0\mathbb{C}^m$ which reduce to an ordinary differential equation due to invariance of the metric and mean curvature under isometries. In general, it is not easy to compute the second fundamental form to investigate the singularities of different types. We computed all the principal curvatures and observed that near the exceptional divisor, all the principal curvatures vanish except for one direction which goes to infinity. By knowing the principal curvatures, we can compute the mean curvature and also the square of the norm of the second fundamental form. In this work, we demonstrate that a special class of hyperspheres in $\mathbb{C}^m$, endowed with a $U(m)$-invariant Kähler metrics, are specific examples of mean curvature flow of $(2m-1)$-dimensional submanifolds that converge to $(2m-2)$-dimensional submanifolds within a finite time. Initially, we address the mean curvature flow problem and subsequently investigate the nature of singularities associated with these flows. Our main result shows that in these examples, there is a jump in dimension in the MCF problem. A well-known example is the Burns metric on $Bl_0\mathbb{C}^2$, which we will examine in Section 5 to study the mean curvature flow problem and determine the exact time of singularity.

The rest of the paper is organized as follows. Section 2 is devoted to definitions and some well-known results that will be used throughout the paper. Section 3 focuses on the blowup of $\mathbb{C}^m$ at the origin, where we discuss the condition when a $U(m)$-invariant metric on $\mathbb{C}^m \setminus 0$ can be extended to the blowup of $\mathbb{C}^m$ at the origin. In Section 4, we state and prove a result on computing principal curvatures on special cases that leads to the proof of our main theorem. Finally, Section 5 is dedicated to the mean curvature flow in our setting and some examples.

2. Preliminaries

This section is dedicated to recalling the fundamental definitions and key results regarding the mean curvature flow problem, which are essential for our subsequent discussions. We refer the reader to references [11, 12, 14, 18] for further details.

Definition 2.1. *Let $F_0 : \Sigma^m \longrightarrow M^{m+1}$ be a smooth immersion of an m-dimensional manifold. The mean curvature flow of F_0 is a family of smooth immersions $F_t : \Sigma \longrightarrow M^{m+1}$ for $t \in [0, T)$ such that setting $F(p, t) = F_t(p)$*

the map $F : \Sigma \times [0,T) : \Sigma^m \longrightarrow M^{m+1}$ *is a smooth solution of the following system of PDE's*

$$\begin{cases} \frac{\partial}{\partial t} F(p,t) = H(p,t)n(p,t), \\ F(p,0) = F_0(p)\,, \end{cases}$$

where $H(p,t)$ *and* $n(p,t)$ *are respectively the mean curvature and the unit normal of the hypersurface* F_t *at the point* $p \in \Sigma$.

Usually the Riemannian manifold M is called the ambient manifold and the parameter t is considered as time. Minimal submanifolds, i.e. submanifolds with zero mean curvature everywhere, are the stationary solutions of this flow.

There are two important results which we recalled below in the Euclidean case. These results are well-known local theorems that we can apply in the Riemannian case too. Consequently, we utilize them in the proof of our main theorem on the mean curvature flow problem [12, pages 39, 40].

PROPOSITION 2.2. *If the second fundamental form is bounded in the interval* $[0,T)$ *with* $T < +\infty$, *then all its covariant derivatives are also bounded.*

PROPOSITION 2.3. *If the second fundamental form is bounded in the interval* $[0,T)$ *with* $T < +\infty$, *then* T *cannot be a singular time for the mean curvature flow of a compact hypersurface* $F : \Sigma \times [0,T) \longrightarrow \mathbb{R}^{n+1}$.

From these two propositions, we have the following.

REMARK 2.4. The above estimate can be found independent of T and also independent of initial data.

One of the most important problems in studying the mean curvature flow is to understand the possible singularities the flow goes through. We introduce the notion of singularity in mean curvature flow and their types in the following.

DEFINITION 2.5. *If the second fundamental form* $|A|^2$ *blows up at* $t \longrightarrow T$, *then we call* T *a singular time of the flow.*

DEFINITION 2.6. *We say that the flow is developing a type I singularity at time* T *if there exists a constant* $C > 1$ *such that we have the upper bound*

$$max_{p\in\Sigma}|A(p,t)|^2 \leq \frac{C}{T-t}.$$

Otherwise, we say it is a type II *singularity.*

3. Kähler Metrics on the Blowup of $\mathbb{C}^m$ at the Origin

This section is devoted to the construction of the blowup manifold and the necessary condition for extending the Kähler metrics on it. The blowup process is an operation that replaces a point in the complex space $\mathbb{C}^m$ with an exceptional divisor, which is isomorphic to $\mathbb{C}P^{m-1}$. The blowup manifold denoted by $Bl_0\mathbb{C}^m$ can admit a Kähler metric typically constructed by modifying the standard flat metric on $\mathbb{C}^m$, where the blowup of $\mathbb{C}^m$ at the origin is defined as

$$\begin{aligned} Bl_0\mathbb{C}^m &= \{((z_1,\dots,z_m),[t_1,\dots,t_m]) \in \mathbb{C}^m \times \mathbb{C}P^{m-1} : z_i t_j - z_j t_i = 0\} \\ &\subset \mathbb{C}^m \times \mathbb{C}P^{m-1}\,. \end{aligned}$$

For more details about this topics, we refer to [1, 15, 16, 17].

There is a natural projection map $\pi_1 : Bl_0\mathbb{C}^m \to \mathbb{C}^m$ defined by

$$\pi_1((z_1,z_2,\dots,z_m),[t_1,t_2,\dots,z_m]) = (z_1,z_2,\dots,z_m)\,.$$

The inverse image $\pi_1^{-1}(p)$ of $p \in \mathbb{C}^m$ is a line passing the point p.

The **exceptional divisor** E is defined as the inverse image of the origin i.e., $\pi^{-1}(0) = \mathbb{C}P^{m-1}$.

Moreover the map π_1 can be restricted to a biholomorphism

$$\pi_1 : Bl_0\mathbb{C}^m \setminus E \to \mathbb{C}^m \setminus 0.$$

A system of charts that covers the exceptional divisor is given as follows: for every $i = 1,2,\dots,m$,

$$U_i = \{((z_1,\dots,z_m),[t_1,\dots,t_m]) : t_i \neq 0, z_j = z_i t_j\}\,.$$

The coordinate map $\Phi_i : U_i \to \mathbb{C}^m$ is defined as

$$((z_1,\dots,z_m),[t_1,\dots,t_m]) \to \left(z_i, \frac{t_1}{t_i},\dots,\frac{t_{i-1}}{t_i},\frac{t_{i+1}}{t_i},\dots,\frac{t_m}{t_i}\right),$$

with inverse map $\Phi_i^{-1} : \mathbb{C}^m \to U_i$

$$(z_1,\dots,z_m) \to ((z_1 z_i, z_2 z_i,\dots,z_i,\dots,z_i z_m),[z_1,\dots,z_{i-1},1,z_{i+1},\dots,z_m]). \tag{1}$$

For every $i = 1,2,\dots,m$, the chart U_i intersects the exceptional divisor E:

$$E \cap U_i = \{z_i = 0\}\,.$$

We now take the smooth $(1,1)$-form on $\mathbb{C}^m \setminus 0$ given by

$$\omega = \sqrt{-1}\partial\bar{\partial}\log(S),$$

where $S = \sum_{i=1}^m |z_i|^2$.

The pull back $\pi_1^*\omega$ of the smooth form $\omega = \sqrt{-1}\partial\bar\partial\log(S)$ on $\mathbb{C}^m \setminus 0$ extends to the Fubini Study metric on the exceptional divisor $E = \mathbb{C}P^{m-1}$, which is given in local coordinates (1) as follows

$$\begin{aligned}\pi_1^*\omega &= \partial\bar\partial\log(|z_i|^2(|z_1^2| + |z_2|^2\cdots + |z_{i-1}|^2 + 1 + |z_{i+1}|^2 + \cdots + |z_m|^2)\\ &= \partial\bar\partial\log(|z_1|^2 + |z_2|^2\cdots + |z_{i-1}|^2 + 1 + |z_{i+1}|^2 + \cdots + |z_m|^2). \qquad (2)\end{aligned}$$

One can easily see that (2) is the Fubini Study metric on the exceptional divisor E in homogeneous coordinates $[z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_m]$.

For a smooth function $g : \mathbb{C}^m \to \mathbb{R}_+$ depending on $S = \sum_{i=1}^m |z_i|^2$, the smooth form

$$\omega = \sqrt{-1}\partial\bar\partial f(S) = \sqrt{-1}\partial\bar\partial(\log S + g(S))\,, \qquad (3)$$

gives a Kähler metric on $\mathbb{C}^m \setminus \{0\}$ if and only if $\frac{1}{S} + g_S > 0$ and $g_S + S g_{SS} > 0$. The next proposition explains the necessary and sufficient condition for the Kähler form (3) on $\mathbb{C}^m \setminus 0$ to be extended to $Bl_0\mathbb{C}^m$.

Proposition 3.1. *The smooth form $\omega = \sqrt{-1}\partial\bar\partial(\log S + g(S))$ on $\mathbb{C}^m \setminus \{0\}$ extends to Kähler metric on $Bl_0\mathbb{C}^m$ if and only if $g_S(0) > 0$, $\frac{1}{S} + g_S > 0$ and $g_S + S g_{SS} > 0$.*

Proof. For the sake of simplicity, we only prove the case when $m = 2$. The general case follows from the same argument, we leave the details for briefness.

Given the projection map

$$\pi_1 : Bl_0\mathbb{C}^2 \to \mathbb{C}^2,$$

on the chart U_1 we have $S = |z_1|^2(1 + |z_2|^2)$ and $E \cap U_1 = \{z_1 = 0\}$. The pull back of the Kähler metric (3) to $Bl_0\mathbb{C}^2$ is given in coordinates (1) by

$$\pi_1^*\omega = \begin{bmatrix}(1 + |z_2|^2)(g_S + S g_{SS}) & z_1\bar z_2(g_S + S g_{SS})\\ z_2\bar z_1(g_S + S g_{SS}) & |z_1|^2(g_S + |z_1|^2|z_2|^2 g_{SS}) + \frac{1}{1+|z_2|^2}\end{bmatrix}.$$

The restriction of $\pi_1^*\omega$ to the exceptional divisor E is:

$$\pi_1^*\omega|_E = \begin{bmatrix}(1 + |z_2|^2)g_S(0) & 0\\ 0 & \frac{1}{1+|z_2|^2}\end{bmatrix}.$$

Clearly $\pi_1^*\omega|_E$ is positive definite if and only if $g_S(0) > 0$.

In the same way on U_2, the pull back

$$\pi_1^*\omega = \begin{bmatrix}\frac{1}{1+|z_1|^2} + |z_2|^2(g_S + |z_1|^2|z_2|^2 g_{SS}) & z_1\bar z_2(g_S + S g_{SS})\\ z_2\bar z_1(g_S + S g_{SS}) & (1 + |z_1|^2)(g_S + S g_{SS})\end{bmatrix},$$

can be restricted to the exceptional divisor as follows:

$$\pi_1^*\omega|_E = \begin{bmatrix} \frac{1}{1+|z_1|^2} & 0 \\ 0 & (1+|z_1|^2)g_S \end{bmatrix}.$$

Clearly, $\pi_1^*\omega|_E$ is positive definite if and only if $g_S(0) > 0$. □

REMARK 3.2. If $g_S(0) = 0$, then $\pi_1^*\omega|_E$ defines a metric only along the exceptional divisor. Therefore the condition $g_S(0) \neq 0$ guarantees the non degeneracy of the metric orthogonal to the exceptional divisor. The other two conditions $\frac{1}{S} + g_S > 0$ and $g_S + Sg_{SS} > 0$ are considered because ω must be a Kähler metric on $\mathbb{C}^m \setminus 0$.

4. Principal Curvatures of Hyperspheres

In this section, we compute the second fundamental form for hyperspheres under special conditions. In order to investigate the mean curvature flow for our examples, we need to know the principal curvatures which are the eigenvalues of the second fundamental form.

Let Σ be an d-dimensional smooth submanifold in an $d+1$-dimensional manifold M and g be the Riemannian metric on M with Levi–Civita connection ∇.

DEFINITION 4.1. *The second fundamental form of Σ is defined by*

$$\Pi_n(X, X) = g\left(\nabla_X(X), n\right), \tag{4}$$

where $X \in T_pM$ and $n \in (T_p\Sigma)^\perp$.

LEMMA 4.2. *Suppose X and n are local vector fields on M such that*

1. *$||X||_g^2$ and $||n||_g^2$ are constants,*
2. *for all $p \in \Sigma$, $X(p) \in T_p\Sigma$ and $n(p) \in (T_p\Sigma)^\perp$.*

Then

$$\Pi_n(X, X) = -g([X, n], X).$$

Proof. We have

$$\begin{aligned}\Pi_n(X, X) &= g\left(\nabla_X(X), \eta\right) = -g(\nabla_X(n), X) = -g([X, n] + \nabla_n(X), X) \\ &= -g([X, n], X) + \frac{1}{2}n(||X||_g) = -g([X, n], X),\end{aligned}$$

which completes the proof. □

We now state and prove the main result of this section, and calculate the second fundamental form for hyperspheres with some particular assumptions. In this result, the ambient space is $M = \mathbb{C}^m \setminus 0$, so, we can consider the Euclidean metric on M.

PROPOSITION 4.3. *Suppose that g_0 and g are Euclidean and Riemannian metrics on M respectively. Let $e_1, ..., e_{d+1}$ be orthonormal local vector fields for M with respect to g_0 i.e., $g_0(e_i, e_j) = \delta_{ij}$. $\Sigma \subset M$ is an m-dimensional submanifold such that for each $p \in \Sigma$ we have $e_{d+1}(p) \perp T_p\Sigma$. Let $n = e_{d+1}$ and A, η, μ be local functions on M such that their restrictions on Σ are constants. We have the following conditions:*

1. $g(e_{d+1}, e_{d+1}) = A^2$, $g(e_d, e_d) = \mu^2$;
2. $g(e_i, e_i) = \eta^2 \quad$ *if* $1 \le i \le d-1$;
3. $g(e_i, e_j) = 0 \quad \forall i \ne j$;
4. $[e_d, e_{d+1}] \in \mathbb{R}\langle e_d, e_{d+1}\rangle$.

Now if $\Pi_\Sigma(g_0) = \tau g_0$ for some $\tau \in \mathbb{R}$, then

$$\Pi_g(e_i, e_j) = \begin{bmatrix} (\eta^2 A^{-1}\tau + \eta A^{-1}\nabla_n \eta) I_{m-1} & 0 \\ 0 & \mu^2 A^{-1}\tau + \mu A^{-1}\nabla_n \mu \end{bmatrix}.$$

Proof. We fix some notations which will be used in the proof.

Let $[n, e_i] = \sum_{j=1}^{d+1} a_{ij} e_j$ for $1 \le i \le d$. Then,

- $a_{ii} = g_0([n, e_i], e_i) = -g_0([e_i, n], e_i) = \Pi_{g_0}(e_1, e_1) = \tau$.
- $0 = 2\Pi_{g_0}(e_i, e_j) = g_0([e_i, n], e_j) + g_0([e_j, n], e_i) = a_{ij} + a_{ji}$.

Notice that $\{\eta^{-1}e_1, ..., \eta^{-1}e_{d-1}, \mu^{-1}e_d, A^{-1}n\}$ is an orthonormal frame for the metric g. We prove the proposition in the following steps.

Step 1: By Lemma 4.2, we have

$$\begin{aligned}
\Pi(\eta^{-1}e_1, \eta^{-1}e_1) &= -g([\eta^{-1}e_1, A^{-1}n], \eta^{-1}e_1) \\
&= -g(\eta^{-1}A^{-1}[e_1, n] - A^{-1}\nabla_n(\eta^{-1}e_1), \eta^{-1}e_1) \\
&= -\eta^{-1}A^{-1}g([e_1, n], \eta^{-1}e_1) + \eta^{-1}A^{-1}\nabla_n(\eta^{-1})g(e_1, e_1) \\
&= -\eta^{-2}A^{-1}g([e_1, n], e_1) + A^{-1}\eta\nabla_n(\eta^{-1}) \\
&= -\eta^{-2}A^{-1}g([e_1, n], e_1) - A^{-1}\eta^{-1}\nabla_n(\eta) \\
&= -A^{-1}(\tau + \eta^{-1}\nabla_n\eta)\,,
\end{aligned}$$

where we employed the property of Lie bracket and the fact that $\nabla_{e_1} A^{-1} = 0$ on Σ. For the last step we use the following relation

$$g([e_1, n], e_1) = a_{11} g_{11} = \tau\eta^2.$$

Step 2: The same calculation shows that

$$\Pi(\eta^{-1}e_i, \eta^{-1}e_i) = A^{-1}(\tau + \eta^{-1}\nabla_n\eta),$$

for $1 \le i \le d-1$, and

$$\Pi(\mu^{-1}e_d, \mu^{-1}e_d) = g([\mu^{-1}e_d, A^{-1}n], \mu^{-1}e_d).$$

Similarly to **Step 1**, we have

$$\Pi(\mu^{-1}e_d, \mu^{-1}e_d) = A^{-1}(\tau + \mu^{-1}\nabla_n \mu).$$

Now similar to the last calculation, we get $\Pi(e_i, e_j) = 0$ for each $1 \leq i < j \leq d-1$.

In the next step we show that $\Pi(\eta^{-1}e_1, \mu^{-1}e_d) = 0$.

Step 3: Since

$$2\Pi(\eta^{-1}e_1, \mu^{-1}e_d) = g([\eta^{-1}e_1, n], \mu^{-1}e_d) + g([\mu^{-1}e_d, n], \eta^{-1}e_1)\,,$$

we have

$$[\eta^{-1}e_1, n] = \eta^{-1}[e_1, n] - \nabla_n \eta^{-1}e_1 = \eta^{-1}\Sigma a_{ij}e_j - \nabla_n \eta^{-1}e_1\,,$$

and

$$[\mu^{-1}e_d, n] = \mu^{-1}[e_d, n] - \nabla_n \mu^{-1}e_d = \mu^{-1}\Sigma a_{dj}e_j - \nabla_n \mu^{-1}e_d\,.$$

Hence, we obtain

$$\begin{aligned}
2\Pi(\eta^{-1}e_1, \mu^{-1}e_d) &= \mu^{-1}\eta^{-1}g([e_1, n], e_d) - \mu^{-1}\nabla_n \eta^{-1}g(e_1, e_d) \\
&\quad + \mu^{-1}\eta^{-1}g([e_d, n], e_1) - \eta^{-1}\nabla_n \mu^{-1}g(e_d, e_1) \\
&= \mu^{-1}\eta^{-1}g([e_1, n], e_d) + g([e_d, n], e_1)) \\
&= \mu^{-1}\eta^{-1}(\Sigma a_{ij}e_j, e_d) + g(\Sigma a_{dj}e_j, e_1)) \\
&= \mu^{-1}\eta^{-1}(a_{1d}g(e_d, e_d) + a_{d1}g(e_1, e_1)) \\
&= \mu^{-1}\eta^{-1}a_{d1}(-g(e_d, e_d) + g(e_1, e_1)).
\end{aligned}$$

Since $[e_d, n] \in span\langle e_d, n\rangle$, so $a_{d1} = ... = a_{d-1} = 0$, and thus we conclude that

$$\Pi(\eta^{-1}e_1, \mu^{-1}e_d) = 0\,,$$

which completes the proof. □

We consider $\mathbb{C}^m \setminus 0$ with Kähler metric $g = \partial\overline{\partial} f(S) = (f_S\delta_{ij} + f_{SS}\bar{z}_i z_j)dz_i \wedge d\bar{z}_j$, $\Sigma = \{(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m : S = R^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_m|^2\} \subset \mathbb{C}^m$, the normal vector n and $J(n) = in$ and moreover an orthonormal basis $e_1, ..., e_{2m-2}$ for $\langle n, J(n)\rangle^\perp$. Let $e_{2m-1} = J(n)$ and $e_{2m} = n$, the metric g is written by:

$$\begin{bmatrix} f_S I_{2m-2} & 0 \\ 0 & (f_S + f_{SS}S)I_2 \end{bmatrix}.$$

THEOREM 4.4. *The principal curvatures of the family* $\Sigma_S^{2m-1} \subset \mathbb{C}^m \setminus 0$ *with a* $U(m)$*-invariant Kähler metric* $\omega = \sqrt{-1}\partial\bar{\partial}f(S)$ *are as follows:*

$$\lambda_1 = \lambda_2 = ... = \lambda_{2m-2} = -\frac{\sqrt{f_S + f_{SS}S}}{f_S\sqrt{S}}, \ \lambda_{2m-1} = -\frac{f_S + 3Sf_{SS} + S^2 f_{SSS}}{(f_S + f_{SS}S)^{\frac{3}{2}}\sqrt{S}}.$$

where $S = \Sigma_{i=1}^m |z_i|^2$.

Proof. In the setting of the Proposition 4.3, we have $\Sigma = S^{2m-1}(r)$ and $M = \mathbb{C}^m \setminus 0$. Furthermore, we have $A^2 = \mu^2 = f_S + f_{SS}S$ and $\eta^2 = f_S$. Additionally we get $\eta^{-1}\nabla_n \eta = \frac{S}{f_S} f_{SS}$ and $\mu^{-1}\nabla_n \mu = \frac{\sqrt{S}}{\mu^2}(2f_{SS} + Sf_{SSS})$.

Now by computing $g^{-1}\Pi(g)$, we obtain the following principal curvatures

$$\lambda_1 = ... = \lambda_{2m-2} = -\frac{\sqrt{f_S + Sf_{SS}}}{f_S\sqrt{S}}, \ \lambda_{2m-1} = -\frac{(f_S + 3Sf_{SS} + S^2 f_{SSS})}{(f_S + f_{SS}S)^{\frac{3}{2}}\sqrt{S}},$$

which was required. □

5. Mean Curvature Flow

In this section, we prove our main result, presenting the mean curvature flow with initial data given by a special class of hyperspheres in $\mathbb{C}^m$ with a $U(m)$-invariant Kähler metric. In the following lemma, we compute the mean curvature, which is the sum of the eigenvalues of the second fundamental form.

LEMMA 5.1. *The mean curvature of the family* $\Sigma_S^{2m-1} \subset \mathbb{C}^m \setminus 0$ *with* $U(m)$*-invariant Kähler metric* $\omega = \sqrt{-1}\partial\bar{\partial}f(S)$ *is given as follows:*

$$H(S) = -\frac{(2m-2)(f_S + Sf_{SS})^2 + f_S(f_{SSS}S^2 + 3Sf_{SS} + f_S)}{(2m-1)(f_S + Sf_{SS})^{\frac{3}{2}}\sqrt{S}f_S}.$$

In the following two lemmas, we compute the square of the norm of the second fundamental form to determine whether the mean curvature flow contains a singularity or not. These lemmas are essential for proving our main result.

LEMMA 5.2. *Let* A *be the second fundamental form of the family of* $\Sigma_S^{2m-1} \subset \mathbb{C}^m \setminus 0$ *with* $U(m)$*-invariant Kähler metric* $\omega = \sqrt{-1}\partial\bar{\partial}f(S)$*. Then the square of its norm,* $|A|^2$ *is as follows:*

$$\frac{(2m-2)(f_S + f_{SS}S)^4 + f_S^2(f_S + 3Sf_{SS} + S^2 f_{SSS})^2}{f_S^2(f_S + f_{SS}S)^3 S}.$$

Proof. The principal curvatures for the hyperspheres are:

$$\lambda_1 = \lambda_2 = ... = \lambda_{2m-2} = -\frac{\sqrt{f_S + f_{SS}S}}{f_S\sqrt{S}}, \ \lambda_{2m-1} = -\frac{f_S + 3Sf_{SS} + S^2 f_{SSS}}{(f_S + f_{SS}S)^{\frac{3}{2}}\sqrt{S}}.$$

Now we can compute $|A|^2$ as follows:

$$\begin{aligned} |A|^2 &= \lambda_1^2 + \lambda_2^2 + ... + \lambda_{2m-1}^2 = (2m-2)\lambda_1^2 + \lambda_{2m-1}^2 \\ &= \frac{(2m-2)(f_S + f_{SS}S)^4 + f_S^2(f_S + 3Sf_{SS} + S^2 f_{SSS})^2}{f_S^2(f_S + f_{SS}S)^3 S}. \end{aligned}$$

□

LEMMA 5.3. *For each g with the following conditions,*

$$g_S(0) > 0, \ \frac{1}{S} + g_S > 0, \ \textit{and } g_S + Sg_{SS} > 0,$$

$|A|^2$ blows up only at $S = 0$.

Proof. We have

$$|A|^2 = \frac{(2m-2)(g_S + g_{SS}S)^4 + (\frac{1}{S} + g_S)^2(g_S + 3Sg_{SS} + S^2 g_{SSS})^2}{(\frac{1}{S} + g_S)^2(g_S + g_{SS}S)^3 S}.$$

We know that g is a smooth function and does not blowup. When $S = 0$, the numerator is always positive by the above conditions of g. Thus the singularity only happens when $S = 0$. □

Now we prove the main result of this paper in which we investigate the mean curvature flow for our setting. The examples presented in this work illustrate specific examples of mean curvature flow of submanifolds converging to submanifolds of one lower dimension at a finite time.

THEOREM 5.4. *Consider $\mathbb{C}^m \setminus 0$ with a $U(m)$-invariant Kähler metric $\omega = \sqrt{-1}\partial\bar{\partial}f(S)$, where $f(S) = \log S + g(S)$ and g is an analytic function with the following conditions:*

$$g_S(0) > 0, \ \frac{1}{S} + g_S > 0, \ \textit{and } g_S + Sg_{SS} > 0.$$

There exists $\epsilon > 0$ such that if $R_o < \epsilon$, we can choose one hypersphere with radius R_0 in such a way that the mean curvature flow with initial condition $\Sigma_{R(0)} = \Sigma_{R_0}$ converges to the exceptional divisor at a finite time and we have a singularity of Type I.

Proof. The mean curvature flow problem for the hyperspheres Σ_S is the following ordinary differential equation

$$\frac{dR(t)}{dt} = H(R(t))\,.$$

We can choose $\epsilon > 0$ such that if we start the flow with the initial data $R(0) = R_0 < \epsilon$, the mean curvature does not vanish and is negative. In Lemma 5.3, we observe that there is only one singularity at $R(t) = 0$. Therefore, the time of singularity (T_{sing}) happens whenever $R(t) = 0$. This means that if the flow starts at $t = 0$, then $|A|^2$ is bounded for all $t \in [0, T_{sing})$. By the expression in Lemma 5.1, we can write the mean curvature flow problem as

$$\frac{dR(t)}{dt} = \frac{1}{R^\alpha(t)} K(R(t))\,,$$

for some $\alpha > 0$, where $K(R(t))$ is an analytic function without singularity and its Taylor series near $R(t) = 0$ is given by

$$K(R(t)) = \sum_{n=0}^{\infty} \frac{K^n(0)}{n!} R^n(t)\,.$$

We thus obtain

$$R^\alpha(t)\frac{dR(t)}{dt} = \sum_{n=0}^{\infty} \frac{K^n(0)}{n!} R^n(t)\,,$$

and by applying integral, we have

$$\frac{1}{\alpha+1} R^{\alpha+1}(t) = K(0)t + \sum_{n=1}^{\infty} \frac{K^n(0)}{(n+1)!} R^{n+1}(t) + C\,,$$

for some constant C. Moreover, with initial condition $R(0) = R_0$ we get $C = \frac{R_0^{\alpha+1}}{\alpha+1} - \sum_{n=1}^{\infty} \frac{K^n(0)}{(n+1)!} R_0^{n+1}$. Since we have the singularity only at $R(t) = 0$, so

$$T_{sing} = \frac{1}{K(0)}(\sum_{n=1}^{\infty} \frac{K^n(0)}{(n+1)!} R_0^{n+1} - \frac{R_0^{\alpha+1}}{\alpha+1}).$$

We can easily conclude that the time of singularity is finite. By Lemma 5.3, we deduce that there is only one singularity at $R(t) = 0$ and that $|A|^2$ is bounded and does not blow up before $R(t) = 0$. We can now employ Propositions 2.2 and 2.3, and conclude that the flow does not stop (i.e., keep restarting) and converges to $R(t) = 0$, which is the exceptional divisor in $Bl_0\mathbb{C}^m$. Moreover, We can write the square of the norm of the second fundamental form as

$|A|^2 = \frac{W(R(t))}{R^2(t)}$, where $W(R(t))$ is an analytic function without singularity. Its Taylor series then near $R(t) = 0$ is

$$W(R(t)) = \sum_{n=0}^{\infty} \frac{W^n(0)}{n!} R^n(t)\,.$$

Clearly, we have

$$|A|^2 = \frac{W^0(0)}{R^2(t)} + \frac{W^1(0)}{R(t)} + \frac{W^2(0)}{2} + \sum_{n=3}^{\infty} \frac{W^n(0)}{n!} R^{n-2}(t),$$

and thus we get

$$\begin{aligned}\lim_{t\to\ T_{sing}} (T_{sing}-t)|A|^2 &= \lim_{t\to\ T_{sing}} (T_{sing}-t)\frac{W^0(0)}{R^2(t)} + \lim_{t\to\ T_{sing}} (T_{sing}-t)\frac{W^1(0)}{R(t)} \\ &+ \lim_{t\to\ T_{sing}} (T_{sing}-t)\frac{W^2(0)}{2} + \lim_{t\to\ T_{sing}} (T_{sing}-t)\sum_{n=3}^{\infty} \frac{W^n(0)}{n!} R^{n-2}(t).\end{aligned}$$

But $R(t)$ goes to zero as t goes to T_{sing}, so we can see that

$$\lim_{t\to\ T_{sing}} (T_{sing}-t)\frac{W^2(0)}{2} = \lim_{t\to\ T_{sing}} (T_{sing}-t)\sum_{n=3}^{\infty} \frac{W^n(0)}{n!} R^{n-2}(t) = 0.$$

Since $\frac{dR(t)}{dt} = H(R(t))$, we have

$$R'(T_{sing}) = \frac{dR(t)}{dt}|_{t=T_{sing}} = H(R(T_{sing})) = H(0) = \infty.$$

By using L'Hôpital's rule, we conclude that

$$\lim_{t\to\ T_{sing}} (T_{sing}-t)\frac{W^1(0)}{R(t)} = \lim_{t\to\ T_{sing}} \frac{-W^1(0)}{R'(t)} < \infty.$$

We also compute that $\lim_{t\to\ T_{sing}} R(t)H(R(t)) \neq 0$. Again by using L'Hôpital's rule we see that

$$\lim_{t\to\ T_{sing}} (T_{sing}-t)\frac{W^0(0)}{R^2(t)} < \infty,$$

and consequently,

$$\lim_{t\to\ T_{sing}} (T_{sing}-t) max|A|^2 < \infty.$$

Thus the singularity is of Type I. $\square$

The assumption of analyticity in the above Theorem is not restrictive. Many interesting Kähler metrics are analytic. For example, as proved by Hopf and Morrey constant scalar curvature Kähler metrics satisfy this hyphothesis [13].

REMARK 5.5. We observe that when $S(t) \to 0$, then $\lambda_1 = ... = \lambda_{2m-2} \to 0$ and $\lambda_{2m-1} \to \infty$. This means that when $S(t) \to 0$, one of the principal directions collapses and the hypersphere converges to the exceptional divisor, which is holomorphic submanifold of $Bl_0\mathbb{C}^m$. Since holomorphic submanifolds of complex manifolds are minimal, so one would naturally expect that the principal curvature vanishes there.

In some examples we can estimate ϵ as $+\infty$ including the Burns metric. Example 5.6 provides an instance of the mean curvature flow problem for the Burns metric.

EXAMPLE 5.6. Consider $Bl_0\mathbb{C}^2$ with the Burns metric $\omega = \sqrt{-1}\partial\bar{\partial}(\log(S)+S)$. We can choose an arbitrary hypersphere Σ_{R_0} as initial condition for the mean curvature flow. The mean curvature flow of the hypersphere converges to S^2 at a finite time and we have the singularity of Type I.

Proof. The mean curvature flow problem for the hyperspheres Σ_S is the following ODE

$$\frac{dR(t)}{dt} = H(R(t)).$$

Now the principal curvatures of Σ_S with Burns metric are

$$\lambda_1 = \lambda_2 = \frac{-R}{(R^2+1)}, \quad \lambda_3 = \frac{-1}{R}.$$

Moreover, the mean curvature of these families and $|A|^2$ are given by

$$H(R(t)) = \frac{-1}{3}\frac{3R^2(t)+1}{R(t)(R^2(t)+1)}, \ |A|^2 = \frac{2R^4(t)+(R^2(t)+1)^2}{R^2(t)(R^2(t)+1)^2}.$$

Therefore the mean curvature problem is equivalent to

$$\frac{dR(t)}{dt} = \frac{-1}{3}\frac{3R^2(t)+1}{R(t)(R^2(t)+1)}.$$

The solution of the equation with initial data $R(0) = R_0$ would be

$$\frac{R^2(t)}{2} + \frac{1}{3}\log(3R^2(t)+1) = -t + c.$$

Thus, $|A|^2$ blows up only when $R(t) = 0$. With the initial condition $R(0) = R_0$ we get the time of singularity as follows

$$T_{sing} = \frac{R_0^2}{2} + \frac{1}{3}\log(3R_0^2+1).$$

The time of singularity is finite and the flow exists for all $t \in [0, T_{sing})$. We can also check that there exists a positive constant C such that $|A|^2 < C/|T_{sing} - t|$. Hence, the singularity is of Type I. $\square$

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