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On some "sporadic" moduli spaces of Ulrich bundles on some 3-fold scrolls over $\mathbb{F}_0$

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Dedicated to the memory of Gianfranco Casnati

Abstract. *We investigate the existence of some* sporadic *rank-$r \geqslant 1$ Ulrich vector bundles on suitable 3-fold scrolls X over the Hirzebruch surface $\mathbb{F}_0$, which arise as tautological embeddings of projectivization of very-ample vector bundles on $\mathbb{F}_0$ that are* uniform *in the sense of Brosius and Aprodu–Brinzanescu, cf. [11] and [4] respectively.*
Such Ulrich bundles arise as deformations of "iterative" extensions by means of sporadic *Ulrich line bundles which have been contructed in our former paper [30] (where instead higher-rank* sporadic *bundles were not investigated therein). We explicitely describe irreducible components of the corresponding* sporadic *moduli spaces of rank $r \geqslant 1$ vector bundles which are Ulrich with respect to the tautological polarization on X. In some cases, such irreducible components turn out to be a singleton, in some other such components are generically smooth, whose positive dimension has been computed and whose general point turns out to be a slope-stable, indecomposable vector bundle.*

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Introduction

Let X be a smooth, irreducible projective variety of dimension $n \geqslant 1$ and let $\mathcal{O}_X(H)$ be a very ample line bundle on X. A vector bundle $\mathcal{U}$ on X is said to be *Ulrich with respect to $\mathcal{O}_X(H)$* if it satisfies suitable cohomological conditions involving some multiples of the polarization $\mathcal{O}_X(H)$ (cf. e.g. Definition 1.1 below and [9, Theorem 2.3] for equivalent conditions).

Ulrich bundles originally appeared in Commutative Algebra, in the paper [40] by B. Ulrich, as bundles enjoying suitable extremal cohomological properties. After that, attention on Ulrich bundles entered in the realm of Algebraic Geometry with the paper [28] where, among other things, the authors compute the Chow form of a projective variety X under the assumption that X supports Ulrich bundles of some rank r.

In recent years there has been a huge amount of work on Ulrich bundles, investigating several questions on such bundles like e.g. their existence, the minimal rank for an Ulrich vector bundle supported by the pair $(X, \mathcal{O}_X(H))$ (such a minimal rank is also called the *Ulrich complexity* of $(X, \mathcal{O}_X(H))$ and denoted by $uc_{\mathcal{O}_X(H)}(X)$), as well as their *stability*, their moduli space structure, etcetera (for nice surveys the reader is referred to e.g. [23, 26]). A lot has been proved for some specific classes of projective varieties, e.g. curves (cf. e.g. [27]), Segre and Veronese varieties (cf. e.g. [39]), Grassmann varieties (cf. e.g. [25]), varieties of minimal degree (cf. e.g. [6]), hypersurfaces (cf. e.g. [7, 8]), several classes of surfaces (cf. e.g. [13, 14, 15]), of threefolds (cf. e.g. [2, 3, 9, 16, 17, 18, 19, 20, 21, 22, 12, 23]) and even of special classes of higher dimensional varieties (c.f. e.g. [29, 34]; cf. also [26] for a general overview on known results on Ulrich bundles). However, even in the case of surfaces, there are still several open questions to be answered in their full generality (for example, either classification of arithmetically Cohen-Macaulay varieties as being of *finite, tame* or *wild Ulrich type*, as in [27, 29], or classification of moduli spaces, as in [12, 35], parametrizing isomorphim classes of stable Ulrich vector bundles, or the birational geometry of such moduli spaces, as e.g. in [24]).

In our former paper [30], we focused on 3-fold scrolls $X = X_e$ arising as embedding, via very-ample tautological line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$, of projective bundles $\mathbb{P}(\mathcal{E}_e)$, where $\mathcal{E}_e$ are very-ample rank-2 vector bundles on $\mathbb{F}_e$ with Chern classes $c_1(\mathcal{E}_e)$ numerically equivalent to $3C_e + b_e f$ and $c_2(\mathcal{E}_e) = k_e$, where C_e and f generators of $\mathrm{Num}(\mathbb{F}_e)$ and where b_e and k_e are integers satisfying some natural numerical conditions (cf. Assumptions 1.7 below). In this set-up, one gets 3-fold scrolls $X_e \subset \mathbb{P}^{n_e}$, with $n_e := 4b_e - k_e - 6e + 4$, which are non–degenerate, non-special, of degree $d_e := \deg(X_e) = 6b_e - 9e - k_e$ and sectional genus $g_e := 2b_e - 3e - 2$, whose hyperplane section line bundle is denoted by $\xi_e := \mathcal{O}_{X_e}(1)$ and we studied the behaviour of such 3-fold scrolls (X_e, ξ_e) in terms of some Ulrich bundles they can support.

A reason for such interest came from the fact that the existence of Ulrich bundles on geometrically ruled surfaces has been considered in [5, 1, 13] while in [31] the existence of Ulrich bundles of rank 1 and 2 on low-degree smooth 3–fold scrolls over a surface was investigated and, among such 3–folds, there are scrolls over $\mathbb{F}_e$ with $e = 0, 1$.

In the above set-up, among other things, in [30] we proved the following:

Theorem A ([30, Main Theorem and Main Corollary]). *For any integer $e \geqslant 0$, consider the Hirzebruch surface $\mathbb{F}_e$ and let $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ denote the line bundle $\alpha C_e + \beta f$ on $\mathbb{F}_e$, where C_e and f are generators of* $\mathrm{Num}(\mathbb{F}_e)$. *Let $(X_e, \xi_e) \cong$*

$(\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ be a 3-fold scroll over $\mathbb{F}_e$, where $\mathcal{E}_e$ is as in Assumptions 1.7 and where $\varphi : X_e \to \mathbb{F}_e$ denotes the scroll map. Then:

(a) X_e does not support any Ulrich line bundle w.r.t. the tautological polarization ξ_e unless $e = 0$. In this latter case, the unique Ulrich line bundles on X_0 are the following:

(i) $L_1 := \xi_0 + \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, -1)$ and $L_2 := \xi_0 + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, b_0 - 1)$;

(ii) for any integer $t \geqslant 1$, $M_1 := 2\xi_0 + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, -t-1)$ and $M_2 := \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1)$, which only occur for $b_0 = 2t, k_0 = 3t$.

(b) Set $e = 0$ and let $r \geqslant 2$ be any integer. Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X_0 which are Ulrich w.r.t. ξ_0 and with first Chern class

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi_0 + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, b_0 - 3) + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_0 - 2)\right), \\ \hfill \text{if } r \text{ is odd}, \\ r\xi_0 + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r}{2}, \frac{r}{2}(b_0 - 2)), \hfill \text{if } r \text{ is even}. \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(6b_0 - 4), & \text{if } r \text{ is odd}, \\ \frac{r^2}{4}(6b_0 - 4) + 1, & \text{if } r \text{ is even}. \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ_0 given by $\mu(\mathcal{U}_r) = 8b_0 - k_0 - 3$. If moreover $r = 2$, then $\mathcal{U}_2$ is also special *(cf. Def. 1.3 above).*

(c) When $e > 0$, let $r \geqslant 2$ be any integer. Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X_e which are Ulrich w.r.t. ξ_e and with first Chern class

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi_e + \varphi^*\mathcal{O}_{\mathbb{F}_e}(3, b_e - 3) + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{r-3}{2}(b_e - e - 2)\right), \\ \hfill \text{if } r \text{ is odd}, \\ r\xi_e + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), \hfill \text{if } r \text{ is even}. \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), \\ \hfill \text{if } r \text{ is odd}, \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, \hfill \text{if } r \text{ is even}. \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ *corresponds to a slope-stable vector bundle, of slope w.r.t.* ξ_e *given by* $\mu(\mathcal{U}_r) = 8b_e - k_e - 12e - 3$*. If moreover* $r = 2$*, then* $\mathcal{U}_2$ *is also special.*

(d) In particular,

(i) when $e = 0$*, the* Ulrich complexity *of* X_0 *w.r.t.* ξ_0 *is* $uc_{\xi_0}(X_0) = 1$*; moreover* X_0 *supports slope-stable vector bundles of any rank* $r \geqslant 1$ *which are Ulrich w.r.t.* ξ_0*, i.e. there are no slope-stable-Ulrich-rank gaps on* X_0 *w.r.t. the chosen Chern class;*

(ii) when otherwise $e > 0$*, the* Ulrich complexity *of* X_e *w.r.t.* ξ_e *is* $uc_{\xi_e}(X_e) = 2$*; nonetheless* X_e *supports slope-stable vector bundles of any rank* $r \geqslant 2$ *which are Ulrich w.r.t.* ξ_e*, i.e. the only slope-stable-Ulrich-rank gap w.r.t. the chosen Chern class is* $r = 1$*.*

Part (a) of Theorem A above highlights in particular that, when $e = 0$, 3-fold scrolls X_0 support Ulrich line bundles, namely M_1 and M_2, which have a certain *sporadic behavior* as they actually exist only for pairs $(b_0, k_0) = (2t, 3t)$, for any integer $t \geqslant 1$ where, we recall that the integer pair (b_0, k_0) comes from $c_1(\mathcal{E}_0) = 3C_0 + b_0 f$, $c_2(\mathcal{E}_0) = k_0$ (cf. also (3), (6) below) whereas line bundles L_1 and L_2 actually exist for all pairs (b_0, k_0) associated to $\mathcal{E}_0$ very-ample on $\mathbb{F}_0$. For this reason, M_1 and M_2 will be called *sporadic* Ulrich line bundles on X_0 whereas L_1 and L_2 *non-sporadic* Ulrich line bundles.

We want to stress that parts (b) and (d)-(i) of Theorem A above have been proved in [30] via *iterative constructions* of Ulrich vector bundles $\mathcal{U}_r$, of any rank $r \geqslant 2$, with the use of deformations of non-trivial extensions involving only *non-sporadic* Ulrich line bundles L_1 and L_2 as in part (a)-(i) of Theorem A.

Our goal in the present paper is to focus on moduli spaces of Ulrich vector bundles on (X_0, ξ_0) which arise from *sporadic/mixed cases*; precisely, we are interested in:

- understanding what type of moduli spaces of rank-$r \geqslant 2$ vector bundles on X_0, which are Ulrich w.r.t. ξ_0, arise from iterative constructions by means of either *sporadic pairs* (M_1, M_2) or even *mixed pairs* (L_i, M_j), $1 \leqslant i, j \leqslant 2$, as in Theorem A-(a),
- computing what kind of Chern classes are determined by these types of constructions,
- proving, for any $r \geqslant 2$, the existence of an irreducible component $\mathcal{M}(r)$ of any such a moduli space and deducing whether such a component can be (generically) smooth,
- computing $\dim(\mathcal{M}(r))$,

- establishing slope-stability for the bundle $\mathcal{U}_r$ corresponding to a general point $[\mathcal{U}_r]$ of any such a component $\mathcal{M}(r)$, and
- understanding whether there exists some slope-stable-Ulrich rank gap w.r.t. the chosen Chern classes.

Throughout this work we will be therefore concerned with the case $e = 0$, with 3-fold scrolls arising from bundles $\mathcal{E}_0$ as in Assumptions 1.7 over $\mathbb{F}_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$. Using only *sporadic pairs*, we prove the following:

THEOREM B (Sporadic cases (cf. Theorem 3.5)). *Let $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ be a 3-fold scroll over $\mathbb{F}_0$, with $\mathcal{E} = \mathcal{E}_0$ satisfying Assumptions 1.7. Let $\varphi : X \to \mathbb{F}_0$ be the scroll map and F be the φ-fiber. Let $r \geqslant 1$ be any integer.*

Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X which are Ulrich w.r.t. ξ and with Chern classes

$$c_1(\mathcal{U}_r) := \begin{cases} (r+1)\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0, -2t) + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{(r-3)}{2}, r(t-1)\right), & \textit{if } r \textit{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{r}{2}, r(t-1)\right), & \textit{if } r \textit{ is even,} \end{cases}$$

$$c_2(\mathcal{U}_r) = \begin{cases} \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(2r^2-2, (2t-1)r^2 - 2t + 1\right) - \frac{(r-1)(2rt+r+14t-3)}{2} F, \\ \qquad\qquad \textit{if } r \geqslant 3 \textit{ is odd,} \\ \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(2r^2-2r, r(2rt - r - t + 1)\right) - \frac{r(2rt+r+t-1)}{2} F, \\ \qquad\qquad \textit{if } r \textit{ is even,} \end{cases}$$

$$c_3(\mathcal{U}_r) = \begin{cases} 4r^3t - 2r^3 - 8r^2t + 4r^2 - 4rt + 2r + 8t - 4, & \textit{if } r \geqslant 3 \textit{ is odd,} \\ 4r^3t - 2r^3 - 10r^2t + 6r^2 + 4rt - 4r, & \textit{if } r \geqslant 4 \textit{ is even,} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(8t-4), & \textit{if } r \textit{ is odd,} \\ \frac{r^2}{4}(8t-4) + 1, & \textit{if } r \textit{ is even,} \end{cases}$$

with $t \geqslant 1$. The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 13t - 3$.

In particular, there are no slope-stable-Ulrich-rank gaps on X w.r.t. the chosen Chern classes.

When otherwise *mixed pairs* are considered on the one hand we show that, in some cases, there are 0-dimensional modular components consisting only of one point which corresponds to a (S-equivalence class of a) polystable bundle (cf. Theorem 2.3-(1) and (4)); on the other, using similar strategy as that used to prove Theorem B, one can get existence of several *extra* positive-dimensional, sporadic modular components which are different from those in Theorem B.

Due to the high number of possible pairings at any rank-r step, we will limit here to state detailed results for some significant examples of *extra*, positive-dimensional sporadic modular components of Ulrich bundles on X, arising via the use of some specific *mixed pairs*. In particular we have the following:

THEOREM C (Mixed cases). *Let $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ be a 3-fold scroll over $\mathbb{F}_0$, with $\mathcal{E} = \mathcal{E}_0$ satisfying Assumptions 1.7. Let $\varphi : X \to \mathbb{F}_0$ be the scroll map and F be the φ-fiber. Let $r \geqslant 1$ be any integer. Then (X, ξ) supports several* extra *positive-dimensional, sporadic modular components parametrizing rank-r vector bundles $\mathcal{U}_r$ on X which are Ulrich w.r.t. ξ, with given Chern classes.*

In particular, for Chern class

$$c_1 := \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{(r-3)}{2}, (r-1)(t-1)+1\right), & \textit{if } r \textit{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{r}{2}, (r-2)(t-1)\right), & \textit{if } r \textit{ is even,} \end{cases}$$

the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X which are Ulrich w.r.t. ξ and with first Chern class as above is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{r^2}{4}(10t-5) + r - 2t, & \textit{if } r \textit{ is even,} \\ t(r^2+7r-12) - \frac{1}{2}(4r^2 - 11r + 13), & \textit{if } r \textit{ is odd,} \end{cases}$$

with $t \geqslant 1$.

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 13t - 3$. In particular, there are no slope-stable-Ulrich-rank gaps on X w.r.t. the chosen Chern classes.

For a proof of the previous result, the reader is referred to Theorem 2.3 and to § 4.

The paper consists of four sections. In Section 1 we recall some generalities on Ulrich bundles on projective varieties as well as some other preliminaries necessary to properly define 3-fold scrolls (X, ξ) which are the core of the paper. Section 2 focuses on moduli spaces of rank-2 Ulrich vector bundles obtained via both *sporadic* (cf. Subsect. 2.1) and *mixed extensions* (cf. Subsect. 2.2). Section 3 deals with the general case of higher rank $r \geqslant 3$, obtained via inductive processes, extensions, deformations and modular theory dealing with *sporadic pairs*. Finally in Section 4 we briefly discuss some results which can be obtained, following same strategies as in Section 3, using *mixed pairs*.

Notation and terminology

We work throughout over the field $\mathbb{C}$ of complex numbers. All schemes will be endowed with the Zariski topology. By *variety* we mean an integral algebraic scheme. We say that a property holds for a *general* point of a variety V if it holds for any point in a Zariski open non–empty subset of V. We will interchangeably use the terms *rank-r vector bundle* on a variety V and *rank-r locally free sheaf* on V; in particular for the case $r = 1$, that is line bundles (equiv. invertible sheaves), to ease notation and if no confusion arises, we sometimes identify line bundles with Cartier divisors interchangeably using additive notation instead of multiplicative notation and tensor products. Thus, if L and M are line bundles on V, the *dual* of L will be denoted by either $L^\vee$, or L^{-1} or even $-L$, $L^{\otimes n}$ will be sometimes denoted with nL as well as $L \otimes M$ with $L + M$.

If $\mathcal{P}$ is either a *parameter space* of a flat family of geometric objects $\mathcal{E}$ defined on V (e.g. vector bundles, extensions, etc.) or a *moduli space* parametrizing geometric objects modulo a given equivalence relation, we will denote by $[\mathcal{E}]$ the parameter point (resp., the moduli point) corresponding to the geometric object $\mathcal{E}$ (resp., associated to the equivalence class of $\mathcal{E}$). For further non-reminded terminology, we refer the reader to [33].

In the sequel, we will focus on smooth, irreducible, projective 3-folds and the following notation will be used throughout this work.

- X is a smooth, irreducible, projective variety of dimension 3 (or simply a 3-*fold*);
- $\chi(\mathcal{F}) = \sum_{i=0}^{3}(-1)^i h^i(X, \mathcal{F})$ denotes the Euler characteristic of $\mathcal{F}$, where $\mathcal{F}$ is a vector bundle of rank $r \geqslant 1$ on X;
- ω_X denotes the canonical bundle of X as well as K_X denotes a canonical divisor;
- $c_i = c_i(X)$ denotes the i^{th}-Chern class of X, $0 \leqslant i \leqslant 3$;
- $d = \deg X = L^3$ denotes the degree of X in its embedding given by a very-ample line bundle L on X;
- $g = g(X)$, denotes the sectional genus of (X, L) defined by $2g - 2 := (K_X + 2L)L^2$;
- if S is a smooth surface, $\equiv$ will denote the numerical equivalence of divisors on S whereas $\sim$ their linear equivalence.

1. Preliminaries

For the reader convenience we recall some general facts that we will use in the sequel.

DEFINITION 1.1. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, projective variety of dimension n and let H be a hyperplane section of X. A vector bundle $\mathcal{U}$ on X is said to be* Ulrich *with respect to $\mathcal{O}_X(H)$ if*

$$H^i(X,\mathcal{U}(-jH)) = 0 \quad \textit{for all} \quad 0 \leqslant i \leqslant n \;\; \textit{and} \quad 1 \leqslant j \leqslant n.$$

REMARK 1.2. (i) If X supports Ulrich bundles w.r.t. $\mathcal{O}_X(H)$ then one sets $uc_H(X)$, called the *Ulrich complexity of X w.r.t. $\mathcal{O}_X(H)$*, to be the minimum rank among possible Ulrich vector bundles w.r.t. $\mathcal{O}_X(H)$ on X.
(ii) If $\mathcal{U}$ is a vector bundle on X, which is Ulrich w.r.t. $\mathcal{O}_X(H)$, then $\mathcal{U}' := \mathcal{U}^\vee(K_X + (n+1)H)$ is a vector bundle of the same rank of $\mathcal{U}$, which is also Ulrich w.r.t. $\mathcal{O}_X(H)$. The vector bundle $\mathcal{U}'$ is called the *Ulrich dual* of $\mathcal{U}$. From this we see that, if Ulrich bundles of some given rank $r \geqslant 1$ on X do exist, then they come in pairs.

DEFINITION 1.3. *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, projective variety of dimension n and let H denote a hyperplane section of X. Let $\mathcal{U}$ be a rank-2 vector bundle on X which is* Ulrich *with respect to $\mathcal{O}_X(H)$. Then $\mathcal{U}$ is said to be* special *if $c_1(\mathcal{U}) = K_X + (n+1)H$.*

Notice that, because $\mathcal{U}$ in Definition 1.3 is of rank 2, then $\mathcal{U}^\vee \cong \mathcal{U}(-c_1(\mathcal{U}))$; therefore for a rank-2 Ulrich bundle $\mathcal{U}$ being *special* is equivalent to $\mathcal{U}$ being isomorphic to its Ulrich dual bundle.

We now briefly remind well-known facts concerning (semi)stability and slope-(semi)stability properties of Ulrich bundles (cf. [12, Definition 2.7]). Let $\mathcal{V}$ be a vector bundle on X; recall that $\mathcal{V}$ is said to be *semistable* if for every non-zero coherent subsheaf $\mathcal{K} \subset \mathcal{V}$, with $0 < \mathrm{rk}(\mathcal{K}) :=$ rank of $\mathcal{K} < \mathrm{rk}(\mathcal{V})$, the inequality $\frac{P_{\mathcal{K}}}{\mathrm{rk}(\mathcal{K})} \leqslant \frac{P_{\mathcal{V}}}{\mathrm{rk}(\mathcal{V})}$ holds true, where $P_{\mathcal{K}}$ and $P_{\mathcal{V}}$ are their Hilbert polynomials. Furthermore, $\mathcal{V}$ is said to be *stable* if the strict inequality above holds.

Recall that the *slope* of a vector bundle $\mathcal{V}$, w.r.t. the very ample polarization $\mathcal{O}_X(H)$, is defined to be $\mu(\mathcal{V}) := \frac{c_1(\mathcal{V})\cdot H^{n-1}}{\mathrm{rk}(\mathcal{V})}$; then $\mathcal{V}$ is said to be *μ-semistable*, or even *slope-semistable* (w.r.t. $\mathcal{O}_X(H)$), if for every non-zero coherent subsheaf $\mathcal{K} \subset \mathcal{V}$ with $0 < \mathrm{rk}(\mathcal{K}) < \mathrm{rk}(\mathcal{V})$, one has $\mu(\mathcal{K}) \leqslant \mu(\mathcal{V})$, whereas $\mathcal{V}$ is said to be *μ-stable*, or *slope-stable*, if the strict inequality holds.

The two definitions of (semi)stability are in general related as follows (cf. e.g. [12, § 2]):

slope-stability $\Rightarrow$ stability $\Rightarrow$ semistability $\Rightarrow$ slope-semistability.

When the bundle in question is in particular Ulrich w.r.t. $\mathcal{O}_X(H)$, and in this case we denote it by $\mathcal{U}$ to remind Ulrichness, then one has

$$\mu(\mathcal{U}) = \deg(X) + g - 1, \tag{1}$$

where $\deg(X)$ is the degree of X in the embedding given by $\mathcal{O}_X(H)$ and where g is the sectional genus of $(X, \mathcal{O}_X(H))$ (see e.g. [26, Proposition 3.2.5]), and the following more precise situation holds:

THEOREM 1.4. *(cf. [12, Theorem 2.9]) Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, projective variety of dimension n and let H be a hyperplane section of X. Let $\mathcal{U}$ be a rank-r vector bundle on X which is Ulrich w.r.t. $\mathcal{O}_X(H)$. Then:*
(a) $\mathcal{U}$ is semistable, so also slope-semistable;
(b) If $0 \to \mathcal{F} \to \mathcal{U} \to \mathcal{G} \to 0$ is an exact sequence of coherent sheaves with $\mathcal{G}$ torsion-free, and $\mu(\mathcal{F}) = \mu(\mathcal{U})$, then $\mathcal{F}$ and $\mathcal{G}$ are both vector bundles which are Ulrich w.r.t. $\mathcal{O}_X(H)$.
(c) If $\mathcal{U}$ is stable then it is also slope-stable. In particular, the notions of stability and slope-stability coincide for Ulrich bundles.

We like to point out that the property of being Ulrich in an irreducible, flat family of vector bundles is an open condition; indeed if the bundle $\mathcal{U}$ is a deformation of an Ulrich vector bundle $\widetilde{\mathcal{U}}$ then $\mathcal{U}$ is also Ulrich as the cohomology vanishings of $\widetilde{\mathcal{U}}(-j)$, for $1 \leqslant j \leqslant n$, imply (by semi–continuity in the irreducible flat family) the cohomology vanishings of $\mathcal{U}(-j)$.

We also like to remark that because Ulrich bundles are semistable, then any family of Ulrich bundles with given rank and Chern classes is bounded (see for instance [38]). In this situation, if bundles in a bounded family are *simple*, i.e. $\mathrm{End}(\mathcal{U}) \cong \mathbb{C}$, one has:

PROPOSITION 1.5. (*see [12, Proposition 2.10]*) *On a non-singular projective variety X, any bounded family of simple bundles $\mathcal{E}$ with given rank and Chern classes, satisfying $H^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 0$ admits a* smooth modular family.

The existence of smooth modular families of simple vector bundles, along with the fact that the property of being Ulrich in an irreducible flat family of vector bundles is an open condition, will help us in performing *recursive constructions* of Ulrich bundles in any possible rank $r \geqslant 1$ on the projective varieties we are dealing with, proving also slope-stability for the general bundle parametrized by such an irreducible family.

Let us now introduce projective varieties we are interested in, which will be the support of Ulrich bundles we are going to contruct and study.

DEFINITION 1.6. *Let X be a smooth, irreducible, projective variety of dimension 3, or simply a* 3-fold, *and let L be an ample line bundle on X. The pair (X, L) is said to be a* scroll *over a normal variety Y if there exist an ample line bundle M on Y and a surjective morphism $\varphi : X \to Y$ with connected fibers such that $K_X + (4 - \dim(Y))L = \varphi^*(M)$.*

If, in particular, in the above definition Y is a smooth surface and (X, L) is a scroll over Y, then (see [10, Proposition 14.1.3]) $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \varphi_*(L)$

is a vector bundle on Y and $L \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the *tautological line bundle* on $\mathbb{P}(\mathcal{E})$. Moreover, if $S \in |L|$ is a smooth divisor, then (see e.g. [10, Theorem 11.1.2]) S turns out to be isomorphic to the blow-up of the base surface Y along a reduced zero-dimensional scheme which is an element of $c_2(\mathcal{E})$. If we denote with d the *degree* of the pair (X, L), namely the degree of the image $\Phi_L(X) \subset \mathbb{P}^n$ of X via the complete linear system $|L|$, $n := h^0(X, L) - 1$, then one has

$$d := L^3 = c_1^2(\mathcal{E}) - c_2(\mathcal{E}). \tag{2}$$

In this paper we will be concerned with the case in which the base surface Y of the scroll X, as in Definition 1.6, is the Hirzebruch surface $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, with $e \geqslant 0$ an integer. We let $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$ be the natural projection; it is well known (cf. e.g. [33, V, Proposition 2.3]) that $\mathrm{Num}(\mathbb{F}_e) = \mathbb{Z}[C_e] \oplus \mathbb{Z}[f]$, where: $f := \pi_e^*(p)$, for any $p \in \mathbb{P}^1$, whereas C_e denotes either the unique section corresponding to the morphism of vector bundles on $\mathbb{P}^1$ $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \to \to \mathcal{O}_{\mathbb{P}^1}(-e)$, when $e > 0$, or the fiber of the other ruling different from that induced by f, when otherwise $e = 0$. In particular

$$C_e^2 = -e,\ f^2 = 0,\ C_e f = 1.$$

Let $\mathcal{E}_e$ be a rank-2 vector bundle over $\mathbb{F}_e$ and let $c_i(\mathcal{E}_e)$ be its i^{th}-Chern class, $0 \leqslant i \leqslant 2$. Then $c_1(\mathcal{E}_e) \equiv aC_e + bf$, for some $a, b \in \mathbb{Z}$, and $c_2(\mathcal{E}_e) \in \mathbb{Z}$. For the line bundle $\mathcal{L} \equiv \alpha C_e + \beta f$ we will also use notation $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$. Throughout this paper, we will consider the following:

Assumptions 1.7. *Let $e \geqslant 0$, b_e, k_e be integers such that*

$$b_e - e < k_e < 2b_e - 4e, \tag{3}$$

and let $\mathcal{E}_e$ be a rank-2 vector bundle over $\mathbb{F}_e$, with

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f \ \text{ and } \ c_2(\mathcal{E}_e) = k_e,$$

which fits in the exact sequence

$$0 \to A_e \to \mathcal{E}_e \to B_e \to 0, \tag{4}$$

where A_e and B_e are line bundles on $\mathbb{F}_e$ such that

$$A_e \equiv 2C_e + (2b_e - k_e - 2e)f \ \text{ and } \ B_e \equiv C_e + (k_e - b_e + 2e)f \tag{5}$$

From (4), in particular, one has $c_1(\mathcal{E}_e) = A_e + B_e$ and $c_2(\mathcal{E}_e) = A_e B_e$.

As observed in [30, Remark 1.8], condition (3) and the fact that k_e must be an integer give together that

$$b_e \geqslant 3e + 2; \tag{6}$$

furthermore from (4), any bundle $\mathcal{E}_e$ fitting in this exact sequence turns out to be such that $h^i(\mathcal{E}_e) = 0$, for any $i \geqslant 1$, in particular it is *non-special*, moreover it is *very ample*, i.e. $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$ is a very ample line bundle on $\mathbb{P}(\mathcal{E}_e)$, and *uniform* in the sense of [4, 11].

Therefore, if we let $\varphi : \mathbb{P}(\mathcal{E}_e) \to \mathbb{F}_e$ to be the *scroll map* as in Definition 1.6, the pair $(X, L) := (\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ is a 3-*fold scroll over* $\mathbb{F}_e$ and $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ gives rise to an embedding

$$\Phi_e := \Phi_{|\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)|} : \mathbb{P}(\mathcal{E}_e) \hookrightarrow X_e \subset \mathbb{P}^{n_e}, \tag{7}$$

where $X_e := \Phi_e(\mathbb{P}(\mathcal{E}_e)) \subset \mathbb{P}^{n_e}$, $n_e := h^0(\mathcal{E}_e) - 1 = 4b_e - k_e - 6e + 4$, is a smooth, irreducible, non-degenerate and non-special 3-fold scroll of degree $d_e := 6b_e - 9e - k_e$ and sectional genus $g_e := 2b_e - 3e - 2$. We denote by $\xi_e := \mathcal{O}_{X_e}(H)$, where H a hyperplane section of $X_e \subset \mathbb{P}^{n_e}$, and call ξ_e the *tautological polarization* of X_e, as $(X_e, \xi_e) \cong (\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

In this set-up, in [30] we proved Theorem A as in Introduction; as already mentioned therein, Theorem A-(a) highlights that, when $e = 0$, 3-fold scrolls X_0 support *sporadic* Ulrich line bundles M_1 and M_2, which actually exist only for $(b_0, k_0) = (2t, 3t)$, for $t \geqslant 1$ an integer, where (b_0, k_0) are s.t. $c_1(\mathcal{E}_0) = 3C_0 + b_0 f$, $c_2(\mathcal{E}_0) = k_0$, with b_0, k_0 as in (3), (6). We want to stress that part (b) and (d)-(i) of Theorem A have been proved in [30] using *iterative constructions* of vector bundles $\mathcal{U}_r$ of any rank $r \geqslant 2$, which are Ulrich w.r.t. ξ_0, obtained via deformations, extensions and *modular families* (as in Proposition 1.5) arising from *non-sporadic* line bundles L_1 and L_2 as in part (a)-(i) of Theorem A.

The present paper is therefore focused on *sporadic cases* as well as on *mixed cases*; as explained in the Introduction.

Thus, throughout this work we will be concerned with the case $e = 0$ and with 3-fold scrolls arising from bundles $\mathcal{E}_0$ as in Assumptions 1.7 over $\mathbb{F}_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2})$. To ease notation, taking into account (6), from now on we will simply set

$$\mathcal{E} := \mathcal{E}_0,\ X := X_0,\ \xi := \xi_0,\ b := b_0 = 2t,\ k := k_0 = 3t, \tag{8}$$

for any integer $t \geqslant 1$, so that $X \subset \mathbb{P}^n$, where $n := n_0 = 5t + 4$, is a smooth, irreducible, non-special and linearly normal 3-fold scroll over $\mathbb{F}_0$, of degree $d := d_0 = 9t$, sectional genus $g := g_0 = 4t - 2$ and whose tautological polarization is $\xi = \mathcal{O}_X(1)$.

2. Rank-2 Ulrich bundles arising from sporadic and mixed extensions

From (8), consider any 3-fold scroll (X, ξ), $X \subset \mathbb{P}^n$, arising from $\mathcal{E}$ satisfying Assumptions 1.7. The fiber of the natural scroll map $\varphi : X \cong \mathbb{P}(\mathcal{E}) \to \mathbb{F}_0$ will always be denoted by F. Take further line bundles as in Theorem A-(a).

Let us consider first the two *sporadic* Ulrich line bundles M_1 and M_2 on X of Theorem A-(a-ii), which only occur for $(b,k) = (2t, 3t)$, where $t \geqslant 1$ any integer. From [30, § 3.1-Case M] we know that

$$\dim(\mathrm{Ext}^1(M_2, M_1)) = 6t - 3 \geqslant 3.$$

Thus there are non-trivial extensions

$$0 \to M_1 \to \mathcal{F} \to M_2 \to 0 \tag{9}$$

of M_2 by M_1, i.e. $\mathcal{F}$ is a rank-2 vector bundle on X which corresponds to a non-zero vector of the vector space $\mathrm{Ext}^1(M_2, M_1)$. Similarly,

$$\dim(\mathrm{Ext}^1(M_1, M_2)) = 2t + 1 \geqslant 3,$$

so there are also non-trivial extensions

$$0 \to M_2 \to \mathcal{F}' \to M_1 \to 0 \tag{10}$$

of M_1 by M_2.

Observe that the dimensions of the above family of extensions are different positive integers unless $t = 1$. We like to point out that in such a case, i.e. when $t = 1$, then $b = 2$, i.e. $c_1(\mathcal{E}) = 3C_0 + 2f$, so being $\mathcal{E}$ very-ample with $c_1(\mathcal{E})^2 = 12$, by [37, Corollary 2.6-(ii)] it follows that $\mathcal{E} = \mathcal{O}_{\mathbb{F}_0}(1,1) \oplus \mathcal{O}_{\mathbb{F}_0}(1,2)$. Moreover since on $\mathbb{F}_0$ one can exchange the two distinct rulings C_0 and f, then $c_1(\mathcal{E}) = 3C_0 + 2f = 2C_0 + 3f$, is the same. In such case $X \subset \mathbb{P}^9$ has degree 9, sectional genus 2 and it is isomorphic to $\mathbb{P}^1 \times \mathbb{F}_1$ (see [36, (3.12), (3.4)], or [32, Prop. (3.1)]).

Turning back to the general case, notice that the vector bundles $\mathcal{F}$ and $\mathcal{F}'$ are both rank-2 vector bundles which are Ulrich w.r.t. ξ, such that

$$\begin{aligned} c_1(\mathcal{F}) \;&=\; c_1(\mathcal{F}') \;= c_1(M_1) + c_1(M_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2), \\ c_2(\mathcal{F}) \;&=\; c_2(\mathcal{F}') \;= c_1(M_1) \cdot c_1(M_2) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2) - (5t+1)F. \end{aligned} \tag{11}$$

Moreover, by (1), one has

$$\mu(M_1) = \mu(M_2) = \mu(\mathcal{F}) = \mu(\mathcal{F}') = d + g - 1 = 13t - 3. \tag{12}$$

From Theorem 1.4, $\mathcal{F}$ and $\mathcal{F}'$ are both strictly semistable, so they are S-equivalent (in the GIT sense, c.f. e.g. [12, p. 1250083-9] and [30, Remark ;3.2, Claim 3.3]) to $M_1 \oplus M_2$, i.e. they give rise to a point in the moduli space $\mathcal{M}^{ss}(2; c_1,\ c_2)$, where

$$c_1 := 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2),\ c_2 := \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2) - (5t+1)F,$$

parametrizing (S-equivalence classes of) rank-2 semistable sheaves with the given Chern classes.

If instead one considers *mixed extensions*, namely using both line bundles of type L_i as in Theorem A-(a-i), $1 \leqslant i \leqslant 2$, (which are *non-sporadic* on X as they exist for any pair (b, k) satisfying (3) and (6)) and line-bundles of type M_j, with $1 \leqslant j \leqslant 2$, one has non–trivial extensions only in the following cases (cf. computations as in [30, § 3.1]):

Cases 2.1. Non-zero extension spaces are:

(1) $\mathrm{Ext}^1(L_1, M_1)$, where $\dim(\mathrm{Ext}^1(L_1, M_1)) = 1$ and non-trivial extensions $\mathcal{F}_1$ are such that

$$c_1(\mathcal{F}_1) = c_1(L_1) + c_1(M_1) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, -t-2),$$
$$c_2(\mathcal{F}_1) = c_1(L_1) \cdot c_1(M_1) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(9, 3t-3) - (8t+1)F;$$

(2) $\mathrm{Ext}^1(M_2, L_1)$, where $\dim(\mathrm{Ext}^1(M_2, L_1)) = 10t - 5 \geqslant 5$ and non-trivial extensions are such that

$$c_1(\mathcal{F}_2) = c_1(L_1) + c_1(M_2) = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 3t-2)),$$
$$c_2(\mathcal{F}_2) = c_1(L_1) \cdot c_1(M_2) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1) + (6t-4)F;$$

(3) $\mathrm{Ext}^1(L_2, M_1)$, where $\dim(\mathrm{Ext}^1(L_2, M_1)) = 10t - 5 \geqslant 5$ and non-trivial extensions $\mathcal{F}_3$ are such that

$$c_1(\mathcal{F}_3) = c_1(L_2) + c_1(M_1) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-2, t-2),$$
$$c_2(\mathcal{F}_3) = c_1(L_2) \cdot c_1(M_1) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, 7t-3) + (2-7t)F;$$

(4) $\mathrm{Ext}^1(M_2, L_2)$, where $\dim(\mathrm{Ext}^1(M_2, L_2)) = 1$ and non-trivial extensions $\mathcal{F}_4$ with

$$c_1(\mathcal{F}_4) = c_1(L_2) + c_1(M_2) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 5t-2),$$
$$c_2(\mathcal{F}_4) = c_1(L_2) \cdot c_1(M_2) = \xi\varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1) + (t-1)F.$$

Reasoning as for *sporadic* extensions (9) and (10) and looking at Chern-classes computations above, one determines different moduli spaces $\mathcal{M}^{ss}(2; c_1,\ c_2)$, according to the chosen Chern classes c_1 and c_2.

2.1. Rank-two Ulrich bundles arising from sporadic extensions

Here we focus on *sporadic* extensions (9) and (10), i.e. extensions arising from the two *sporadic* line bundles M_1 and M_2 as in Theorem A-(a-ii). We prove the following result.

THEOREM 2.2. *Let* $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ *be a 3-fold scroll over* $\mathbb{F}_0$, *with* $\mathcal{E} = \mathcal{E}_0$ *as in Assumptions 1.7. Let* $\varphi : X \to \mathbb{F}_0$ *be the scroll map and* F *be the* φ*-fiber. Then, for any* $t \geqslant 1$, *the moduli space of rank-2 vector bundles* $\mathcal{U}$ *on* X, *which are Ulrich w.r.t.* ξ *and with Chern classes*

$$c_1(\mathcal{U}) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2) \text{ and } c_2(\mathcal{U}) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2) - (5t+1)F, \tag{13}$$

is not empty and it contains a generically smooth irreducible component $\mathcal{M} := \mathcal{M}(2)$, *of dimension*

$$\dim(\mathcal{M}) = 8t - 3,$$

whose general point $[\mathcal{U}] \in \mathcal{M}$ *corresponds to a special and slope-stable vector bundle, of slope* $\mu(\mathcal{U}) = 13t - 3$ *w.r.t.* ξ.

Furthermore, deformations of Ulrich bundles $\mathcal{F}$ *and* $\mathcal{F}'$ *as in* (9) *and* (10) *belong to the aforementioned component* $\mathcal{M}$.

Proof. The proof is similar to that of [30, Theorem 3.1]. For the reader's convenience we will recall here main arguments for the proof.

We consider bundles $\mathcal{F}$ arising from non–trivial extensions (9), where $\dim(\mathrm{Ext}^1(M_2, M_1)) = 6t - 3 \geqslant 3$. Any such $\mathcal{F}$ is Ulrich w.r.t. ξ, thus, from (1), one has $\mu(\mathcal{F}) = 13t - 3$.

Since, for the same reason, $\mu(M_1) = \mu(M_2) = 13t - 3$ and since furthermore M_1 and M_2 are both slope–stable, of the same slope w.r.t. ξ and non–isomorphic, by [12, Lemma 4.2], any such bundle $\mathcal{F}$ is simple, i.e. $h^0(\mathcal{F} \otimes \mathcal{F}^\vee) = 1$, in particular it is indecomposable.

With the use of (9) and its dual sequence, one can easily show that $h^2(\mathcal{F} \otimes \mathcal{F}^\vee) = 0 = h^3(\mathcal{F} \otimes \mathcal{F}^\vee)$ and that $\chi(\mathcal{F} \otimes \mathcal{F}^\vee) = -8t + 4$. Indeed, tensoring (9) with $\mathcal{F}^\vee$ one gets

$$0 \to M_1 \otimes \mathcal{F}^\vee \to \mathcal{F} \otimes \mathcal{F}^\vee \to M_2 \otimes \mathcal{F}^\vee \to 0; \tag{14}$$

moreover, dualizing (9) gives

$$0 \to M_2^\vee \to \mathcal{F}^\vee \to M_1^\vee \to 0 \tag{15}$$

which, if tensored by M_1 and M_2, respectively, gives

$$0 \to M_2^\vee \otimes M_1 (= 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -4t)) \to M_1 \otimes \mathcal{F}^\vee \to \mathcal{O}_X \to 0 \tag{16}$$

$$0 \to \mathcal{O}_X \to M_2 \otimes \mathcal{F}^\vee \to M_2 \otimes M_1^\vee (= -2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, 4t)) \to 0 \tag{17}$$

Because $\mathcal{F}$ is simple, then $h^0(X, \mathcal{F} \otimes \mathcal{F}^\vee) = 1$; the other cohomology vector-spaces
$H^i(X, \mathcal{F} \otimes \mathcal{F}^\vee)$, $1 \leqslant i \leqslant 3$, can be easily computed from the cohomology

sequence associated with (16) and (17). Indeed, $h^i(\mathcal{O}_X) = 0$ if $i \geqslant 1$ and $h^0(\mathcal{O}_X) = 1$, whereas

$$\begin{aligned} H^i(X, -2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3,4t)) &\cong H^{3-i}(X, \varphi^*\mathcal{O}_{\mathbb{F}_0}(-2,-2-2t)) \\ &\cong H^{3-i}(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(-2,-2-2t)) \\ &\cong H^{i-1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2t)) = \begin{cases} 0 & \text{if } i = 0,2,3 \\ 2t+1 & \text{if } i = 1 \end{cases} \end{aligned} \tag{18}$$

and, as it was done in [30, Case M]

$$H^i(2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,-4t) = \begin{cases} 0 & \text{if } i = 0,2,3 \\ 6t-3 & \text{if } i = 1 \end{cases} \tag{19}$$

From the previous computations and (14), it thus follows that $h^2(\mathcal{F}\otimes\mathcal{F}^\vee) = 0 = h^3(\mathcal{F}\otimes\mathcal{F}^\vee)$. Once again from (14), one has that

$$\chi(\mathcal{F}\otimes\mathcal{F}^\vee) = \chi(M_1\otimes\mathcal{F}^\vee) + \chi(M_2\otimes\mathcal{F}^\vee) = -8t+4,$$

and thus, from the previous vanishings and from simplicity of $\mathcal{F}_1$, one finds $h^1(\mathcal{F}\otimes\mathcal{F}^\vee) = 1 - \chi(\mathcal{F}\otimes\mathcal{F}^\vee) = 8t-3$.

Simplicity of $\mathcal{F}$ and $h^2(\mathcal{F}\otimes\mathcal{F}^\vee) = 0$ give, by Proposition 1.5 (cf. also [12, Proposition 2.10]), that there exists a smooth modular family for $\mathcal{F}$. Furthermore, since $\mathcal{F}$ is Ulrich w.r.t. ξ, with Chern classes as in (11), i.e. $c_1(\mathcal{F}) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2)$ and $c_2(\mathcal{F}) = \xi\cdot\varphi^*\mathcal{O}_{\mathbb{F}_0}(4,6t-2) - (5t+1)F$, the general element $\mathcal{U}$ of the smooth modular family to which $\mathcal{F}$ belongs corresponds to a rank-2 vector bundle, with same Chern classes (namely as those of $\mathcal{F}$), which is Ulrich w.r.t. ξ, as it follows from the facts that, in irreducible flat families, Ulrichness is an open condition (by semi-continuity) and Chern classes are invariants.

Finally, one shows that $\mathcal{U}$ is also slope–stable w.r.t. ξ. Indeed, by Theorem 1.4–(b) (cf. also [9, § 3, (3.2)]), if $\mathcal{U}$ were not a stable bundle, being Ulrich it would be presented as an extension of Ulrich line bundles on X. In such a case, by the classification of Ulrich line bundles on X given in Theorem A-(a), by Chern classes reasons we see that the only possibilities for $\mathcal{U}$ to arise as an extension of Ulrich line bundles should be either extensions (9) or extensions (10). In both cases the dimension of (the projectivization) of the corresponding extension space is either $6t-4$ or $2t$. On the other hand, by semi-continuity on the smooth modular family, one has

$$\begin{aligned} &h^j(\mathcal{U}\otimes\mathcal{U}^\vee) = 0 = h^j(\mathcal{F}\otimes\mathcal{F}^\vee), \quad 2 \leqslant j \leqslant 3, \text{ and} \\ &h^0(\mathcal{U}\otimes\mathcal{U}^\vee) = 1 = h^0(\mathcal{F}\otimes\mathcal{F}^\vee), \end{aligned}$$

thus

$$h^1(\mathcal{U}\otimes\mathcal{U}^\vee) = 1 - \chi(\mathcal{U}\otimes\mathcal{U}^\vee) = 1 - \chi(\mathcal{F}\otimes\mathcal{F}^\vee) = h^1(\mathcal{F}\otimes\mathcal{F}^\vee) = 8t-3,$$

as computed above. In other words, the smooth modular family whose general element is $\mathcal{U}$ is of dimension $8t-3$, which is bigger than both $6t-4$ and $2t$, for any $t \geqslant 1$. This shows that $\mathcal{U}$ general in the smooth modular family corresponds to a stable, and so also slope-stable bundle (cf. Theorem 1.4-(c) above).

By slope-stability of $\mathcal{U}$, we deduce that the moduli space of rank-2 Ulrich bundles with Chern classes as in (11) is not empty and it contains an irreducible component $\mathcal{M} = \mathcal{M}(2)$ where $[\mathcal{U}] \in \mathcal{M}$ is a smooth point, as $h^2(\mathcal{U} \otimes \mathcal{U}^\vee) = 0$. Thus, $\mathcal{M}$ is generically smooth, of dimension $h^1(\mathcal{U} \otimes \mathcal{U}^\vee) = 8t-3$, from which one also deduces that $[\mathcal{U}]$ is a general point in $\mathcal{M}$. Moreover, being Ulrich from (1) one gets $\mu(\mathcal{U}) = 13t-3$.

Note further that $\mathcal{U}^\vee \cong \mathcal{U}(-c_1(\mathcal{U}))$, as $\mathcal{U}$ is of rank 2, and that

$$\begin{aligned} K_X + 4\xi &= -2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2) + 4\xi = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2) \\ &= c_1(\mathcal{F}) = c_1(\mathcal{U}) \end{aligned}$$

and thus

$$\mathcal{U}^\vee \cong \mathcal{U}(-c_1(\mathcal{U})) = \mathcal{U}(-K_X - 4\xi),$$

i.e. that $\mathcal{U}^\vee(K_X + 4\xi) \cong \mathcal{U}$ in other words $\mathcal{U}$ is isomorphic to its Ulrich dual bundle, that is $\mathcal{U}$ is special, as stated.

To prove the last part of the statement, one uses same arguments as in [30, Claim 3.3]. Namely one shows that $M_1 \oplus M_2$ is a smooth point of the Quot-scheme parametrizing simple bundles with given Hilbert polynomal, so there exists a unique irreducible component $\mathcal{R}$ of such a Quot scheme, containing therefore also bundles $\mathcal{F}$ and $\mathcal{F}'$ as in (9) and (10) and all their deformations, in particular containing $\mathcal{U}$. This component $\mathcal{R}$ then projects, via GIT quotient, onto the modular component $\mathcal{M}$ as in the statement. $\square$

2.2. Rank-two Ulrich bundles arising from mixed extensions

Here we consider instead *mixed extensions* as in Cases 2.1 above. One has the following result.

THEOREM 2.3. *Let $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ be a 3-fold scroll over $\mathbb{F}_0$, with $\mathcal{E} = \mathcal{E}_0$ as in Assumptions 1.7. Let $\varphi : X \to \mathbb{F}_0$ be the scroll map and F be the φ-fiber. Then the moduli spaces of rank-2 vector bundles $\mathcal{U}$ on X which are Ulrich w.r.t. ξ and with Chern classes, respectively,*

(1) $c_1(\mathcal{U}) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, -t-2)$ and $c_2(\mathcal{U}) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(9, 3t-3) - (8t+1)F$,

(2) $c_1(\mathcal{U}) = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 3t-2))$ and $c_2(\mathcal{U}) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1) + (6t-4)F$,

(3) $c_1(\mathcal{U}) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-2, t-2)$ and $c_2(\mathcal{U}) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, 7t-3) + (2-7t)F$,

(4) $c_1(\mathcal{U}) = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 5t-2)$ and $c_2(\mathcal{U}) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1) + (t-1)F$

are not empty. Each of them contains a generically smooth irreducible component $\mathcal{M} := \mathcal{M}(2)$ *of dimension, respectively,*

$$\dim(\mathcal{M}) = \begin{cases} 0, & \textit{in case } (1), \\ 10t-6, & \textit{in case } (2), \\ 10t-6, & \textit{in case } (3), \\ 0, & \textit{in case } (4), \end{cases}$$

In cases (1) *and* (4)*,* $\mathcal{M}$ *consists of only one point represented by a polystable bundle (more precisely, in case* (1) *one has* $\mathcal{M} = \{[L_1 \oplus M_1]\}$ *whereas in case* (4) *one has* $\mathcal{M} = \{[L_2 \oplus M_2]\}$*). In cases* (2) *and* (3) *the general point* $[\mathcal{U}] \in \mathcal{M}$ *corresponds to a special and slope-stable vector bundle, of slope* $\mu(\mathcal{U}) = 13t - 3$ *w.r.t.* ξ*.*

Proof. The proof goes similarly as that of Theorem 2.2. The main difference resides in the fact that computations are performed by using extension bundles as in Cases 2.1.

In case (1), we take into account bundles $\mathcal{F}_1$ arising from non-trivial extensions in $\mathrm{Ext}^1(L_1, M_1)$, where $\dim(\mathrm{Ext}^1(L_1, M_1)) = 1$. From [12, Lemma 4.2], $\mathcal{F}_1$ is simple so $h^0(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 1$. The remaining cohomologies $h^i(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee)$, $1 \leqslant i \leqslant 3$, can be easily computed as follows: tensoring with $\mathcal{F}_1^\vee$ the exact sequence defining $\mathcal{F}_1$, one gets

$$0 \to M_1 \otimes \mathcal{F}_1^\vee \to \mathcal{F}_1 \otimes \mathcal{F}_1^\vee \to L_1 \otimes \mathcal{F}_1^\vee \to 0; \tag{20}$$

taking moreover the dual sequence of that defining $\mathcal{F}_1$ and tensoring it with, respectively, M_1 and L_1 gives

$$0 \to M_1 \otimes L_1^\vee (= \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -t)) \to M_1 \otimes \mathcal{F}_1^\vee \to \mathcal{O}_X \to 0 \tag{21}$$

$$0 \to \mathcal{O}_X \to L_1 \otimes \mathcal{F}_1^\vee \to L_1 \otimes M_1^\vee (= -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, t)) \to 0. \tag{22}$$

Clearly $h^i(\mathcal{O}_X) = 0$ if $i \geqslant 1$ and $h^0(\mathcal{O}_X) = 1$; it remains to compute $h^i(X, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -t))$ and $h^i(X, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, t))$, for $i \geqslant 0$. Notice that

$$H^i(X, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, t)) \cong H^i(\mathbb{F}_0, 0) = 0 \text{ for } i \geqslant 0,$$

whereas

$$H^i(X, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -t)) \cong H^i(\mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_0}(-3, -t)).$$

From (4) and (5) we have that $\mathcal{E}$ fits in

$$0 \to 2C_0 + tf \to \mathcal{E} \to C_0 + tf \to 0.$$

If we let $\mathcal{E}(-3,-t) := \mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_0}(-3,-t)$ and if we tensor this exact sequence with $\mathcal{O}_{\mathbb{F}_0}(-3,-t)$, we get

$$0 \to -C_0 \to \mathcal{E}(-3,-t) \to -2C_0 \to 0. \tag{23}$$

From the cohomology sequence associated to (23) it follows therefore that

$$h^i(X, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,-t)) = h^i(\mathbb{F}_0, \mathcal{E}(-3,-t)) = \begin{cases} 0 & \text{if } i = 0,2,3 \\ 1 & \text{if } i = 1 \end{cases}$$

Thus, from (25), (26) and (24), it follows that $h^2(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 0 = h^3(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee)$ and

$$\chi(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = \chi(L_1 \otimes \mathcal{F}_1^\vee) + \chi(M_1 \otimes \mathcal{F}_1^\vee) = 1 + 0 = 1,$$

so, from simplicity of $\mathcal{F}_1$ and the previous vanishings, one has $h^1(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 1 - \chi(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 0$. Since any bundle $\mathcal{F}_1$ arising from non-trivial extensions in $\mathrm{Ext}^1(L_1, M_1)$ is strictly-semistable, from S-equivalence of such bundles, we get the statement.

In case (2), we consider bundles $\mathcal{F}_2$ arising from non-trivial extensions in $\mathrm{Ext}^1(M_2, L_1)$, where $\dim(\mathrm{Ext}^1(M_2, L_1)) = 10t - 5 \geqslant 5$. From [12, Lemma 4.2], one gets that $\mathcal{F}_2$ is simple, i.e. $h^0(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee) = 1$. The remaining cohomologies $h^i(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee)$, $1 \leqslant i \leqslant 3$, can be easily computed as above. We tensor by $\mathcal{F}_2^\vee$ the exact sequence defining $\mathcal{F}_2$ to get

$$0 \to L_1 \otimes \mathcal{F}_2^\vee \to \mathcal{F}_2 \otimes \mathcal{F}_2^\vee \to M_2 \otimes \mathcal{F}_2^\vee \to 0. \tag{24}$$

Moreover, taking the dual sequence of that defining $\mathcal{F}_2$ and tensoring it with, respectively, L_1 and M_2 gives

$$0 \to M_2^\vee \otimes L_1 (= \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,-3t)) \to L_1 \otimes \mathcal{F}_2^\vee \to \mathcal{O}_X \to 0 \tag{25}$$
$$0 \to \mathcal{O}_X \to M_2 \otimes \mathcal{F}_2^\vee \to M_2 \otimes L_1^\vee (= -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,3t)) \to 0 \tag{26}$$

Clearly $h^i(\mathcal{O}_X) = 0$ if $i \geqslant 1$ and $h^0(\mathcal{O}_X) = 1$, so we need to compute $h^i(X, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,-3t))$ and $h^i(X, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,3t))$. Notice that

$$H^i(X, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,3t)) \cong H^i(\mathbb{F}_0, 0) = 0 \text{ for } i \geqslant 0,$$

whereas

$$H^i(X, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,-3t)) \cong H^i(\mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_0}(0,-3t)).$$

From (4) and (5) we have that $\mathcal{E}$ fits in

$$0 \to 2C_0 + tf \to \mathcal{E} \to C_0 + tf \to 0.$$

Set $\mathcal{E}(0,-3t) := \mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_0}(0,-3t) = \mathcal{E}(0,-3t)$ and tensor the previous exact sequence with $\mathcal{O}_{\mathbb{F}_0}(0,-3t)$, so we get

$$0 \to 2C_0 - 2tf \to \mathcal{E}(0,-3t) \to C_0 - 2tf \to 0. \tag{27}$$

From the cohomology sequence associated to (27) one gets

$$\begin{aligned} h^i(X, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,-3t) &= h^i(\mathbb{F}_0, \mathcal{E}(0,-3t)) \\ &= \begin{cases} 0 & \text{if } i = 0,2,3 \\ 10t-5 & \text{if } i = 1 \end{cases} \end{aligned} \tag{28}$$

Thus, from (25), (26) and (24), it follows that $h^2(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee) = 0 = h^3(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee)$ and

$$\chi(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee) = \chi(L_1 \otimes \mathcal{F}_2^\vee) + \chi(M_2 \otimes \mathcal{F}_2^\vee) = -10t + 7,$$

so, from simplicity of $\mathcal{F}_2$ and the previous vanishings, one has $h^1(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 10t - 6$. The conclusion is exactly as in the proof of Theorem 2.2.

Case (3) goes as case (2). We consider bundles $\mathcal{F}_3$ arising from non-trivial extensions in $\mathrm{Ext}^1(L_2, M_1)$ which, from Cases 2.1, is of dimension $10t - 5 \geqslant 5$. From [12, Lemma 4.2] $\mathcal{F}_3$ is simple so $h^0(\mathcal{F}_3 \otimes \mathcal{F}_3^\vee) = 1$. If we tensor by $\mathcal{F}_3^\vee$ the exact sequence defining $\mathcal{F}_3$ we get

$$0 \to M_1 \otimes \mathcal{F}_3^\vee \to \mathcal{F}_3 \otimes \mathcal{F}_3^\vee \to L_2 \otimes \mathcal{F}_3^\vee \to 0;$$

taking also the dual sequence of that defining $\mathcal{F}_3$ and tensoring it with, respectively, M_1 and L_2 gives

$$\begin{gathered} 0 \to L_2^\vee \otimes M_1 (= \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,-3t)) \to M_1 \otimes \mathcal{F}_3^\vee \to \mathcal{O}_X \to 0 \\ 0 \to \mathcal{O}_X \to L_2 \otimes \mathcal{F}_3^\vee \to L_2 \otimes M_1^\vee (= -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,3t)) \to 0. \end{gathered}$$

Observing that the above exact sequences are identical to (25) and (26), one computes the remaining cohomologies $h^i(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee)$, $1 \leqslant i \leqslant 3$, exactly as in case (2) and concludes as in the statement.

Similarly, case (4) goes as case (1). We cosider bundles $\mathcal{F}_4$ arising from non-trivial extensions in $\mathrm{Ext}^1(M_2, L_2)$, which is 1-dimensional. From [12, Lemma 4.2] it follows that $\mathcal{F}_4$ is simple so $h^0(\mathcal{F}_4 \otimes \mathcal{F}_4^\vee) = 1$. To compute the remaining cohomologies $h^i(\mathcal{F}_4 \otimes \mathcal{F}_4^\vee)$, $1 \leqslant i \leqslant 3$, we tensor with $\mathcal{F}_4^\vee$ the exact sequence defining $\mathcal{F}_4$ and get

$$0 \to L_2 \otimes \mathcal{F}_4^\vee \to \mathcal{F}_4 \otimes \mathcal{F}_4^\vee \to M_2 \otimes \mathcal{F}_1^\vee \to 0;$$

taking moreover the dual sequence of that defining $\mathcal{F}_4$ and tensoring it with, respectively, L_2 and M_2 gives

$$\begin{gathered} 0 \to L_2 \otimes M_2^\vee (= \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,-t)) \to L_2 \otimes \mathcal{F}_4^\vee \to \mathcal{O}_X \to 0 \\ 0 \to \mathcal{O}_X \to M_2 \otimes \mathcal{F}_4^\vee \to M_2 \otimes L_2^\vee (= -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3,t)) \to 0. \end{gathered}$$

Observing that the above exact sequences are identical to (21) and (22), one can conclude exactly as in case (1). □

3. Higher-rank sporadic Ulrich bundles on 3-fold scrolls over $\mathbb{F}_0$

In this section we will construct higher-rank, slope-stable, Ulrich vector bundles using both *iterative extensions*, by means of the two *sporadic* Ulrich line bundles

$$\begin{aligned} &M_1 = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, -t-1) \quad \text{and its Ulrich dual} \\ &M_2 = \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1), \end{aligned} \tag{29}$$

as in Theorem A-(a-ii), and *deformations* of such vector-bundle extensions in suitable modular families, generalizing the strategy used in § 2.1 to construct *sporadic* rank-2 Ulrich bundles.

Recall that, from § 2.1, we have

$$\dim(\mathrm{Ext}^1(M_2, M_1)) = h^1(M_1 - M_2) = 6t - 3 \geqslant 3. \tag{30}$$

In order to perform *recursive constructions*, to ease notation we set once and for all $\mathcal{G}_1 := M_1$. From (30), the general element of $\mathrm{Ext}^1(M_2, \mathcal{G}_1) = \mathrm{Ext}^1(M_2, M_1)$ is a non-splitting extension

$$0 \to \mathcal{G}_1 = M_1 \to \mathcal{G}_2 \to M_2 \to 0, \tag{31}$$

where $\mathcal{G}_2 := \mathcal{F}$, as in the proof of Theorem 2.2, is a rank-2 simple vector bundle on X, which is Ulrich w.r.t. ξ and with

$$\begin{aligned} c_1(\mathcal{G}_2) &= c_1(M_1) + c_1(M_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2), \text{ and} \\ c_2(\mathcal{G}_2) &= c_1(M_1) \cdot c_1(M_2) = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2) - (5t+1)F, \end{aligned}$$

as in (13). With a small abuse of notation, we will identify extension (31) with the corresponding rank-2 vector bundle $\mathcal{G}_2$, therefore we will state that $[\mathcal{G}_2] \in \mathrm{Ext}^1(M_2, \mathcal{G}_1)$ is a *general element* of this extension space.

If, in the next step, we considered further extensions $\mathrm{Ext}^1(M_2, \mathcal{G}_2)$, it is easy to see that the dimension of such an extension space drops by one with respect to that of $\mathrm{Ext}^1(M_2, \mathcal{G}_1)$. Therefore, proceeding in this way, after finitely many steps we would have $\mathrm{Ext}^1(M_2, \mathcal{G}_r) = \{0\}$, i.e. $\mathcal{G}_{r+1} = M_2 \oplus \mathcal{G}_r$, for any $r \geqslant r_0$, for some positive integer r_0. To avoid this fact, similarly as in [20, § 4], we proceed by taking *alternating sporadic extensions*, namely

$$0 \to \mathcal{G}_2 \to \mathcal{G}_3 \to M_1 \to 0, \quad 0 \to \mathcal{G}_3 \to \mathcal{G}_4 \to M_2 \to 0, \ \ldots,$$

and so on, that is, defining

$$\epsilon_r := \begin{cases} 1, & \text{if } r \text{ is odd,} \\ 2, & \text{if } r \text{ is even,} \end{cases} \tag{32}$$

we take successive $[\mathcal{G}_r] \in \operatorname{Ext}^1(M_{\epsilon_r}, \mathcal{G}_{r-1})$, for all $r \geqslant 2$, defined by:

$$0 \to \mathcal{G}_{r-1} \to \mathcal{G}_r \to M_{\epsilon_r} \to 0. \tag{33}$$

The fact that we always get *non–trivial* such extensions, for any $r \geqslant 2$, will be proved in Corollary 3.2 below. In any case all vector bundles $\mathcal{G}_r$, recursively defined as in (33), are of rank r and Ulrich w.r.t. ξ, since extensions of Ulrich bundles w.r.t. ξ are again Ulrich w.r.t. ξ. From the fact that any $\mathcal{G}_r$ is recursively defined, Chern classes of $\mathcal{G}_r$, for any $r \geqslant 2$, are obtained as linear combination, with coefficients depending on r, of $c_1(M_i)$ or $c_1(M_i) \cdot c_1(M_j)$, for $1 \leqslant i, j \leqslant 2$. Precisely, Chern classes of $\mathcal{G}_r$ are:

$$c_1(\mathcal{G}_r) := \begin{cases} (r+1)\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,-2t) + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{(r-3)}{2}, r(t-1)\right), \\ \hfill \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{r}{2}, r(t-1)\right), \hfill \text{if } r \text{ is even,} \end{cases} \tag{34}$$

$$c_2(\mathcal{G}_r) = \begin{cases} \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(2r^2-2, (2t-1)r^2-2t+1\right) - \frac{(r-1)(2rt+r+14t-3)}{2}F, \\ \hfill \text{if } r \geqslant 3 \text{ is odd,} \\ \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(2r^2-2r, r(2rt-r-t+1)\right) - \frac{r(2rt+r+t-1)}{2}F, \\ \hfill \text{if } r \text{ is even,} \end{cases}$$

$$c_3(\mathcal{G}_r) = \begin{cases} 4r^3t - 2r^3 - 8r^2t + 4r^2 - 4rt + 2r + 8t - 4, & \text{if } r \geqslant 3 \text{ is odd,} \\ 4r^3t - 2r^3 - 10r^2t + 6r^2 + 4rt - 4r, & \text{if } r \geqslant 4 \text{ is even.} \end{cases}$$

For any $r \geqslant 1$, from (1), the slope of $\mathcal{G}_r$ w.r.t. ξ is $\mu(\mathcal{G}_r) = 13t - 3$. Moreover, from Theorem 1.4-(a), any such bundle $\mathcal{G}_r$ is strictly semistable and slope-semistable, being obtained by extensions of Ulrich bundles.

Remark 3.1. We want to stress *non-sporadic/non-mixed* extensions studied in [30], namely extensions using line bundle pair (L_1, L_2) as in Theorem A-(a-i), existing for any pair (b, k) satisfying (3) and (6) when $e = 0$, have different Chern classes c_2 and c_3. Thus, even when we restrict to the cases $(b, k) = (2t, 3t)$, with $t \geqslant 1$ any integer, moduli spaces determined by (deformations of) bundles $\mathcal{G}_r$ as in (33) are different moduli spaces from those constructed in [30] when $(b, k) = (2t, 3t)$, $t \geqslant 1$.

Among other things, the next result will allow us to prove the aforementioned claim that, from (33), we always get non-trivial extensions (cf. Corollary 3.2). This fact, together with what proved in Lemma 3.3 below, will also imply the existence of simple, so indecomposable, Ulrich vector bundles w.r.t. ξ for any rank $r \geqslant 2$ (cf. Corollary 3.3 below).

LEMMA 3.1. *Let M denote any of the two line bundles M_1 and M_2 as in* (29). *Then, for all integers $r \geqslant 1$, we have*

(i) $h^2(\mathcal{G}_r \otimes M^\vee) = h^3(\mathcal{G}_r \otimes M^\vee) = 0$,

(ii) $h^2(\mathcal{G}_r^\vee \otimes M) = h^3(\mathcal{G}_r^\vee \otimes M) = 0$,

(iii) $h^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+1}}) \geqslant \min\{6t-3,\ 2t+1\} \geqslant 3$.

Proof. For $r = 1$, by definition, we have $\mathcal{G}_1 = M_1$; therefore $\mathcal{G}_1 \otimes M^\vee$ and $\mathcal{G}_1^\vee \otimes M$ are either equal to $\mathcal{O}_X$, if $M = M_1$, or equal to $M_1 - M_2$ and $M_2 - M_1$, respectively, if $M = M_2$. Therefore (i) and (ii) hold true by computations as in § 2.1. As for (iii), by (32) we have that $M_{\epsilon_2} = M_2$ thus $h^1(\mathcal{G}_1 \otimes M_2^\vee) = h^1(M_1 - M_2) = 6t - 3$, as is § 2.1, the latter being always greater than or equal to $\min\{6t-3,\ 2t+1\} \geqslant 3$.

Therefore, we will assume $r \geqslant 2$ and proceed by induction.

Regarding (i), since it holds for $r = 1$, assuming it holds for $r - 1$, then by tensoring (33) with $M^\vee$ we get that

$$h^j(\mathcal{G}_r \otimes M^\vee) = 0, \quad j = 2, 3,$$

because $h^j(\mathcal{G}_{r-1} \otimes M^\vee) = 0$, for $j = 2, 3$, by inductive hypothesis whereas $h^j(M_{\epsilon_r} \otimes M^\vee) = 0$, for $j = 2, 3$, since $M_{\epsilon_r} \otimes M^\vee$ is either $\mathcal{O}_X$, or $M_2 - M_1$, or $M_1 - M_2$.

A similar reasoning, tensoring the dual of (33) by M, proves (ii).

To prove (iii), tensor (33) by $M^\vee_{\epsilon_{r+1}}$ and use that $h^2(\mathcal{G}_{r-1} \otimes M^\vee_{\epsilon_{r+1}}) = 0$ by (i). Thus we have the surjection

$$H^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+1}}) \twoheadrightarrow H^1(M_{\epsilon_r} \otimes M^\vee_{\epsilon_{r+1}}),$$

which implies that $h^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+1}}) \geqslant h^1(M_{\epsilon_r} \otimes M^\vee_{\epsilon_{r+1}})$. According to the parity of r, we have that $M_{\epsilon_r} \otimes M^\vee_{\epsilon_{r+1}}$ equals either $M_1 - M_2$ or $M_2 - M_1$. From computations as in § 2, $h^1(M_1 - M_2) = 6t - 3$ whereas $h^1(M_2 - M_1) = 2t + 1$. Notice that

$$\min\{6t-3,\ 2t+1\} = \begin{cases} 6t - 3 = 2t + 1 = 3, & \text{if } t = 1, \\ 2t + 1 \geqslant 5, & \text{if } t \geqslant 2. \end{cases}$$

Therefore one concludes. □

COROLLARY 3.2. *For any integer $r \geqslant 1$ there exist on X rank-r vector bundles $\mathcal{G}_r$, which are Ulrich w.r.t. ξ, with Chern classes as in* (34), *of slope w.r.t. ξ given by $\mu(\mathcal{G}_r) = 13t - 3$ and which arise as non-trivial extensions as in* (33) *if $r \geqslant 2$.*

Proof. For $r = 1$, we have $\mathcal{G}_1 = M_1$ and the statement holds true from Theorem A-(a-ii) and computations in § 2.

For any $r \geqslant 2$, notice that $\mathrm{Ext}^1(M_{\epsilon_r}, \mathcal{G}_{r-1}) \cong H^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee)$. Therefore, from Lemma 3.1-(iii) there exist non–trivial extensions as in (33), which therefore give rise to bundles $\mathcal{G}_r$ which are Ulrich with respect to ξ, whose Chern classes are exactly as those computed in (34). Since they are Ulrich bundles, the statement about their slope w.r.t. ξ directly follows from (1). $\square$

From Corollary 3.2, at any step we can always pick *non–trivial* extensions of the form (33) and we will henceforth do so. Next result uses similar strategies as in [30, Lemma 4.3].

LEMMA 3.2. *Let $r \geqslant 1$ be an integer. Then we have*

(i) $h^1(\mathcal{G}_{r+1} \otimes M_{\epsilon_{r+1}}^\vee) = h^1(\mathcal{G}_r \otimes M_{\epsilon_{r+1}}^\vee) - 1$,

(ii) $$h^1(\mathcal{G}_r \otimes M_{\epsilon_{r+1}}^\vee) = \begin{cases} \frac{(r+1)}{2} h^1(M_1 - M_2) - \frac{(r-1)}{2} = \frac{r+1}{2}(6t-3) - \frac{(r-1)}{2}, & \textit{if } r \textit{ is odd}, \\ \frac{r}{2} h^1(M_2 - M_1) - \frac{(r-2)}{2} = \frac{r}{2}(2t+1) - \frac{(r-2)}{2}, & \textit{if } r \textit{ is even}. \end{cases}$$

(iii) $h^2(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = h^3(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = 0$,

(iv) $$\chi(\mathcal{G}_r \otimes M_{\epsilon_{r+1}}^\vee) = \begin{cases} \frac{(r+1)}{2}(1 - h^1(M_1 - M_2)) - 1 = \frac{(r+1)}{2}(4 - 6t) - 1, & \textit{if } r \textit{ is odd}, \\ \frac{r}{2}(1 - h^1(M_2 - M_1)) = -rt, & \textit{if } r \textit{ is even}. \end{cases}$$

(v) $$\chi(M_{\epsilon_r} \otimes \mathcal{G}_r^\vee) = \begin{cases} \frac{(r-1)}{2}(1 - h^1(M_1 - M_2)) + 1 = \frac{(r-1)}{2}(4 - 6t) + 1, & \textit{if } r \textit{ is odd}, \\ \frac{r}{2}(1 - h^1(M_2 - M_1)) = -rt, & \textit{if } r \textit{ is even}. \end{cases}$$

(vi) $$\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = \begin{cases} \frac{(r^2-1)}{4}(2 - h^1(M_1-M_2) - h^1(M_2-M_1)) + 1 = \frac{(r^2-1)}{4}(4-8t) + 1, & \textit{if } r \textit{ is odd}, \\ \frac{r^2}{4}(2 - h^1(M_1-M_2) - h^1(M_2-M_1)) = \frac{r^2}{4}(4-8t), & \textit{if } r \textit{ is even}. \end{cases}$$

Proof. (i) Consider the exact sequence in (33), with r replaced by $r+1$. From $\mathrm{Ext}^1(M_{\epsilon_{r+1}}, \mathcal{G}_r) \cong H^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+1}})$ and from the fact that the exact sequence defining $\mathcal{G}_{r+1}$ is constructed by taking a non–zero vector $[\mathcal{G}_{r+1}] \in \mathrm{Ext}^1(M_{\epsilon_{r+1}}, \mathcal{G}_r)$, it follows that the coboundary map

$$H^0(\mathcal{O}_X) \xrightarrow{\partial} H^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+1}})$$

arising from

$$0 \to \mathcal{G}_r \otimes M^\vee_{\epsilon_{r+1}} \to \mathcal{G}_{r+1} \otimes M^\vee_{\epsilon_{r+1}} \to \mathcal{O}_X \to 0, \tag{35}$$

is non–zero, so it is injective; thus, (i) follows from the cohomology of (35).

(ii) Here one uses induction on r. For $r=1$, the right hand side of the formula yields $6t-3$ which is $h^1(\mathcal{G}_1 \otimes M_2^\vee) = h^1(M_1 - M_2)$, see (30). When otherwise $r=2$, the right hand side of the formula is $2t+1$ which is $h^1(\mathcal{G}_2 \otimes M_1^\vee) = h^1(M_2 - M_1) = 2t+1$, as seen in (30), and from the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{G}_2 \otimes M_1^\vee \to M_2 - M_1 \to 0,$$

obtained by (33) with $r=2$ and tensored with $M_1^\vee$, and the fact that $h^j(\mathcal{O}_X) = 0$, for $j=1,2$.

Assuming by induction that formula as in (ii) holds true up to some given integer $r \geqslant 2$, one has to show that it holds also for $r+1$. Considering (33), with r replaced by $r+1$, and tensoring it by $M^\vee_{\epsilon_{r+2}}$ we thus obtain

$$0 \to \mathcal{G}_r \otimes M^\vee_{\epsilon_{r+2}} \to \mathcal{G}_{r+1} \otimes M^\vee_{\epsilon_{r+2}} \to M_{\epsilon_{r+1}} \otimes M^\vee_{\epsilon_{r+2}} \to 0 \tag{36}$$

If r is even, then by definition $M_{\epsilon_{r+2}} = M_2$ whereas $M_{\epsilon_{r+1}} = M_1$. Thus $h^0(M_{\epsilon_{r+1}} \otimes M^\vee_{\epsilon_{r+2}}) = h^0(M_1 - M_2) = 0$ and $h^1(M_{\epsilon_{r+1}} \otimes M^\vee_{\epsilon_{r+2}}) = h^1(M_1 - M_2) = 6t-3$. On the other hand, by Lemma 3.1-(i), $h^2(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+2}}) = 0$. Thus, from (36), we get:

$$h^1(\mathcal{G}_{r+1} \otimes M^\vee_{\epsilon_{r+2}}) = 6t - 3 + h^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_{r+2}}) = 6t - 3 + h^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_r}),$$

as r and $r+2$ have the same parity. Using (i), we have $h^1(\mathcal{G}_r \otimes M^\vee_{\epsilon_r}) = h^1(\mathcal{G}_{r-1} \otimes M^\vee_{\epsilon_r}) - 1$ therefore, by inductive hypothesis with $r-1$ odd, we have $h^1(\mathcal{G}_{r-1} \otimes M^\vee_{\epsilon_r}) = \frac{r}{2}(6t-3) - \frac{(r-2)}{2}$. Summing up, we have

$$h^1(\mathcal{G}_{r+1} \otimes M^\vee_{\epsilon_{r+2}}) = (6t-3) + \frac{r}{2}(6t-3) - \frac{(r-2)}{2} - 1,$$

which is easily seen to be equal to the right hand side expression in (ii), when r is replaced by $r+1$.

If r is odd, the same holds for $r+2$ whereas $r+1$ is even. In this case $M_{\epsilon_{r+2}} = M_1$, $M_{\epsilon_{r+1}} = M_2$ so $h^1(M_{\epsilon_{r+1}} \otimes M^{\vee}_{\epsilon_{r+2}}) = h^1(M_2 - M_1) = 2t+1$ and one applies the same procedure as in the previous case.

(iii) We again use induction on r. For $r = 1$, formula (iii) states that $h^j(M_1 - M_1) = h^j(\mathcal{O}_X) = 0$, for $j = 2, 3$, which is certainly true.

Assume now that (iii) holds up to some integer $r \geqslant 1$; we have to prove that it holds also for $r+1$. Consider the exact sequence (33), where r is replaced by $r+1$, and tensor it by $\mathcal{G}^{\vee}_{r+1}$. From this we get that, for $j = 2, 3$,

$$h^j(\mathcal{G}_{r+1} \otimes \mathcal{G}^{\vee}_{r+1}) \leqslant h^j(\mathcal{G}_r \otimes \mathcal{G}^{\vee}_{r+1}) + h^j(M_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_{r+1}) = h^j(\mathcal{G}_r \otimes \mathcal{G}^{\vee}_{r+1}), \quad (37)$$

the latter equality follows from $h^j(M_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_{r+1}) = 0$, $j = 2, 3$, as in Lemma 3.1-(ii).

Consider the dual exact sequence of (33), where r is replaced by $r+1$, and tensor it by $\mathcal{G}_r$. Thus, Lemma 3.1-(i) yields that, for $j = 2, 3$, one has

$$h^j(\mathcal{G}_r \otimes \mathcal{G}^{\vee}_{r+1}) \leqslant h^j(\mathcal{G}_r \otimes M^{\vee}_{\epsilon_{r+1}}) + h^j(\mathcal{G}_r \otimes \mathcal{G}^{\vee}_r) = h^j(\mathcal{G}_r \otimes \mathcal{G}^{\vee}_r). \quad (38)$$

Now (37)–(38) and the inductive hypothesis yield $h^j(\mathcal{G}_{r+1} \otimes \mathcal{G}^{\vee}_{r+1}) = 0$, for $j = 2, 3$, as desired.

(iv) For $r = 1$, (iv) reads $\chi(M_1 - M_2) = -h^1(M_1 - M_2) = 3 - 6t$, which is true since $h^j(M_1 - M_2) = 0$ for $j = 0, 2, 3$. For $r = 2$, (iv) reads $\chi(\mathcal{G}_2 \otimes M_1^{\vee}) = 1 - h^1(M_2 - M_1) = -2t$ and this holds true because if we take the exact sequence (33), with $r = 2$, tensored by $M_1^{\vee}$ then

$$\chi(\mathcal{G}_2 \otimes M_1^{\vee}) = \chi(\mathcal{O}_X) + \chi(M_2 - M_1) = 1 - h^1(M_2 - M_1) = 1 - (2t+1),$$

as $h^j(M_2 - M_1) = 0$ for $j = 0, 2, 3$.

Assume now that the formula holds up to a certain integer $r \geqslant 2$, we have to prove that it also holds for $r+1$. From (36) we get

$$\chi(\mathcal{G}_{r+1} \otimes M^{\vee}_{\epsilon_{r+2}}) = \chi(\mathcal{G}_r \otimes M^{\vee}_{\epsilon_{r+2}}) + \chi(M_{\epsilon_{r+1}} \otimes M^{\vee}_{\epsilon_{r+2}}).$$

If r is even, the same is true for $r+2$ whereas $r+1$ is odd. Therefore,

$$\begin{aligned} \chi(\mathcal{G}_{r+1} \otimes M^{\vee}_{\epsilon_{r+2}}) &= \chi(\mathcal{G}_r \otimes M_2^{\vee}) + \chi(M_1 - M_2) \\ &= \chi(\mathcal{G}_r \otimes M_2^{\vee}) - h^1(M_1 - M_2). \end{aligned} \quad (39)$$

Then (35), with r replaced by $r-1$, yields

$$\chi(\mathcal{G}_r \otimes M_2^{\vee}) = \chi(\mathcal{G}_{r-1} \otimes M_2^{\vee}) + \chi(\mathcal{O}_X) = \chi(\mathcal{G}_{r-1} \otimes M_2^{\vee}) + 1. \quad (40)$$

Substituting (40) into (39) and using the inductive hypothesis with $r-1$ odd, we get

$$\begin{aligned}\chi(\mathcal{G}_{r+1}\otimes M_2^\vee) &= \chi(\mathcal{G}_{r-1}\otimes M_2^\vee)+1-h^1(M_1-M_2)\\ &= \frac{r}{2}(1-h^1(M_1-M_2))-h^1(M_1-M_2)\\ &= \frac{(r+2)}{2}(1-h^1(M_2-M_1))-1,\end{aligned}$$

proving that the formula holds also for $r+1$ odd.

Similar procedure can be used to treat the case when r is odd. In this case, $M_{\epsilon_{r+1}}=M_2$ whereas $M_{\epsilon_{r+2}}=M_1$. Thus, from the above computations,

$$\chi(\mathcal{G}_{r+1}\otimes M_1^\vee)=\chi(\mathcal{G}_r\otimes M_1^\vee)+\chi(M_2-M_1)=\chi(\mathcal{G}_r\otimes M_1^\vee)-h^1(M_2-M_1).$$

As in the previous case, $\chi(\mathcal{G}_r\otimes M_1^\vee)=1+\chi(\mathcal{G}_{r-1}\otimes M_1^\vee)$ so, applying inductive hypothesis with $r-1$ even, we get $\chi(\mathcal{G}_r\otimes M_1^\vee)=1+\frac{(r-1)}{2}(1-h^1(M_2-M_1))$. Adding up all these quantities, we get

$$\chi(\mathcal{G}_{r+1}\otimes M_{\epsilon_{r+2}}^\vee)=\chi(\mathcal{G}_{r+1}\otimes M_1^\vee)=\frac{r+1}{2}(1-h^1(M_2-M_1)),$$

so formula (iv) holds true also for $r+1$ even.

(v) For $r=1$, (v) reads $\chi(M_1-M_1)=\chi(\mathcal{O}_X)=1$, which is as stated. For $r=2$, (v) reads $\chi(M_2\otimes\mathcal{G}_2^\vee)=1-h^1(M_2-M_1)$, which is once again as stated, as it follows from the dual of sequence (33) tensored by M_2.

Assume now that the formula holds up to a certain integer $r\geqslant 2$ and we need to proving it for $r+1$. Dualizing (33), replacing r by $r+1$ and tensoring it by $M_{\epsilon_{r+1}}$ we find that

$$\begin{aligned}\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_{r+1}^\vee) &= \chi(M_{\epsilon_{r+1}}\otimes M_{\epsilon_{r+1}}^\vee)+\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_r^\vee)\\ &= \chi(\mathcal{O}_{X_n})+\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_r^\vee)=1+\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_r^\vee).\end{aligned}\tag{41}$$

The dual of sequence (33), with r replaced by $r-1$, tensored by $M_{\epsilon_{r+1}}$ yields

$$\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_r^\vee)=\chi(M_{\epsilon_{r+1}}\otimes M_{\epsilon_r}^\vee)+\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_{r-1}^\vee).\tag{42}$$

Substituting (42) into (41) and using the fact that $r+1$ and $r-1$ have the same parity, we get

$$\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_{r+1}^\vee)=1+\chi(M_{\epsilon_{r+1}}\otimes M_{\epsilon_r}^\vee)+\chi(M_{\epsilon_{r-1}}\otimes\mathcal{G}_{r-1}^\vee).$$

If r is even, then $\chi(M_{\epsilon_{r+1}}\otimes M_{\epsilon_r}^\vee)=\chi(M_1-M_2)=-h^1(M_1-M_2)$ whereas, from the inductive hypothesis with $r-1$ odd, $\chi(M_{\epsilon_{r-1}}\otimes\mathcal{G}_{r-1}^\vee)=1+\frac{(r-2)}{2}(1-h^1(M_1-M_2))$. Thus

$$\chi(M_{\epsilon_{r+1}}\otimes\mathcal{G}_{r+1}^\vee)=1-h^1(M_1-M_2)+1+\frac{(r-2)}{2}(1-h^1(M_1-M_2)),$$

the latter equals $1 + \frac{r}{2}(1 - h^1(M_1 - M_2))$, proving that the formula holds also for $r + 1$ odd.

If r is odd, the strategy is similar; in this case one has $\chi(M_{\epsilon_{r+1}} \otimes M_{\epsilon_r}^\vee) = \chi(M_2 - M_1) = -h^1(M_2 - M_1)$ and, by the inductive hypothesis with $r - 1$ even, $\chi(M_{\epsilon_{r-1}} \otimes \mathcal{G}_{r-1}^\vee) = \frac{(r-1)}{2}(1 - h^1(M_2 - M_1))$ so one can conclude.

(vi) We first check the given formula for $r = 1, 2$. We have $\chi(\mathcal{G}_1 \otimes \mathcal{G}_1^\vee) = \chi(M_1 - M_1) = \chi(\mathcal{O}_X) = 1$, which fits with the given formula for $r = 1$. From (33), with $r = 2$, tensored by $\mathcal{G}_2^\vee$ we get

$$\chi(\mathcal{G}_2 \otimes \mathcal{G}_2^\vee) = \chi(M_1 \otimes \mathcal{G}_2^\vee) + \chi(M_2 \otimes \mathcal{G}_2^\vee) \overset{(v)}{=} \chi(M_1 \otimes \mathcal{G}_2^\vee) + 1 - h^1(M_2 - M_1). \quad (43)$$

From the dual of (33), with $r = 2$, tensored by M_1 we get

$$\begin{aligned} \chi(M_1 \otimes \mathcal{G}_2^\vee) &= \chi(M_1 - M_1) + \chi(M_1 - M_2) \\ &= \chi(\mathcal{O}_X) - h^1(M_1 - M_2) = 1 - h^1(M_1 - M_2). \end{aligned} \quad (44)$$

Combining (43) and (44), we get

$$\chi(\mathcal{G}_2 \otimes \mathcal{G}_2^\vee) = 2 - h^1(M_1 - M_2) - h^1(M_2 - M_1),$$

which again fits with the given formula for $r = 2$.

Assume now that the given formula is valid up to a certain integer $r \geqslant 2$; we need to prove it holds for $r + 1$. From (33), in which r is replaced by $r + 1$, tensored by $\mathcal{G}_{r+1}^\vee$ and successively the dual of (33), with r replaced by $r + 1$, tensored by $\mathcal{G}_r$ we get

$$\chi(\mathcal{G}_{r+1} \otimes \mathcal{G}_{r+1}^\vee) = \chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) + \chi(\mathcal{G}_r \otimes M_{\epsilon_{r+1}}^\vee) + \chi(M_{\epsilon_{r+1}} \otimes \mathcal{G}_{r+1}^\vee).$$

If r is even, then $r + 1$ is odd and $M_{\epsilon_{r+1}} = M_1$. From (v) with $(r + 1)$ odd, we get $\chi(M_{\epsilon_{r+1}} \otimes \mathcal{G}_{r+1}^\vee) = 1 + \frac{r}{2}(1 - h^1(M_1 - M_2))$, whereas from (iv) with r even $\chi(\mathcal{G}_r \otimes M_{\epsilon_{r+1}}^\vee) = \frac{r}{2}(1 - h^1(M_2 - M_1))$. Finally, by the inductive hypothesis with r even, $\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = \frac{r^2}{4}(2 - h^1(M_1 - M_2) - h^1(M_2 - M_1))$. Summing–up the three quantities, one gets

$$\chi(\mathcal{G}_{r+1} \otimes \mathcal{G}_{r+1}^\vee) = 1 + \frac{(r+1)^2 - 1}{4}(2 - h^1(M_1 - M_2) - h^1(M_2 - M_1)),$$

proving that the formula holds for $r + 1$ odd.

If r is odd, then $\chi(M_{\epsilon_{r+1}} \otimes \mathcal{G}_{r+1}^\vee) = \frac{r+1}{2}(1 - h^1(M_2 - M_1))$, as it follows from (v) with $(r + 1)$ even, whereas $\chi(\mathcal{G}_r \otimes M_{\epsilon_{r+1}}^\vee) = \frac{(r+1)}{2}(1 - h^1(M_1 - M_2)) - 1$, as predicted by (iv) with r odd. Finally, form the inductive hypothesis with r odd, we have $\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = 1 + \frac{(r^2-1)}{4}(2 - h^1(M_1 - M_2) - h^1(M_2 - M_1))$. If we add up the three quantities, we get

$$\chi(\mathcal{G}_{r+1} \otimes \mathcal{G}_{r+1}^\vee) = \frac{(r+1)^2}{4}(2 - h^1(M_1 - M_2) - h^1(M_2 - M_1)),$$

finishing the proof. □

Notice some fundamental remarks arising from the first step of the previous iterative contruction in (33), which turns out from the proof of Theorem 2.2. We set $\mathcal{G}_1 = M_1$, which is an Ulrich line bundle w.r.t. ξ, of slope $\mu = \mu(M_1) = 13t-3$; by considering non–trivial extensions (31), $\mathcal{G}_2$ turned out to be a simple (so indecomposable) bundle, as it follows from [12, Lemma 4.2] and from the fact that $\mathcal{G}_1 = M_1$ and $M_{\epsilon_2} = M_2$ are both slope–stable, of the same slope $\mu = 13t-3$ w.r.t. ξ and non-isomorphic line bundles. By construction, $\mathcal{G}_2$ turned out to be morover Ulrich, so strictly semistable, of slope $\mu = 13t-3$. On the other hand, in the proof of Theorem 2.2 we showed that $\mathcal{G}_2$ deforms, in an irreducible modular family, to a slope-stable Ulrich bundle $\mathcal{U}_2 := \mathcal{U}$, of the same slope w.r.t. ξ given by $\mu = 13t-3$, same Chern classes $c_i(\mathcal{U}_2) = c_1(\mathcal{G}_2)$, $1 \leqslant i \leqslant 2$. By semi-continuity in the irreducible modular family, cohomological properties as in Lemma 3.1-(i-ii) and Lemma 3.2-(iii-iv-v-vi) hold true when $\mathcal{G}_2$ therein is replaced by $\mathcal{U}_2$. Therefore, from $h^2(\mathcal{U}_2 \otimes \mathcal{U}_2^\vee) = 0$ and simplicity of $\mathcal{U}_2$, by Proposition 1.5 (cf. also [12, Proposition 2.10]) and dimensional computation of $h^1(\mathcal{U}_2 \otimes \mathcal{U}_2^\vee)$, $\mathcal{U}_2$ is a general point of the corresponding modular family and it is also a smooth point, so that the irreducible modular family is generically smooth. Up to shrinking to the open set of smooth points of such an irreducible modular family, we may consider a smooth modular family of simple, slope-stable, Ulrich bundles and the GIT-quotient relation restricted to such a smooth modular family gives rise to an étale cover of an open dense subset of the modular component $\mathcal{M} = \mathcal{M}(2)$ (by the very definition of modular family, cf. [12, pp. 1250083-9/10]), which is therefore generically smooth of the same dimension of the modular family, i.e. $h^1(\mathcal{U}_2 \otimes \mathcal{U}_2^\vee)$, and whose general point is $[\mathcal{U}_2]$, described in Theorem 2.2 (cf. also Theorem B in Introduction).

By induction we can therefore assume that, up to a given integer $r \geqslant 3$, we have already constructed a generically smooth, irreducible modular component $\mathcal{M}(r-1)$ of the moduli space of bundles of rank $(r-1)$, which are Ulrich w.r.t. ξ, with Chern classes $c_i := c_i(\mathcal{G}_{r-1})$ as in (34) (where in the formulas therein r is obviously replaced by $r-1$), for $1 \leqslant i \leqslant 3$, and whose general point $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ is slope-stable, of slope w.r.t. ξ given by $\mu(\mathcal{U}_{r-1}) = 13t-3$ and that satiesfies Lemma 3.1-(i-ii) and Lemma 3.2-(iii-iv-v-vi). Consider therefore extensions

$$0 \to \mathcal{U}_{r-1} \to \mathcal{F}_r \to M_{\epsilon_r} \to 0, \tag{45}$$

with $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ general and with M_{ϵ_r} defined as in (32), (33), according to the parity of r. Notice that

$$\mathrm{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1}) \cong H^1(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee).$$

Lemma 3.3. *In the above set-up, one has*

$$h^1(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) \geqslant \min\{6t-4,\ 2t\} \geqslant 2.$$

In particular, $\mathrm{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1})$ *contains non-trivial extensions as in* (45).

Proof. By assumption $\mathcal{U}_{r-1}$ satisfies Lemma 3.1-(i), so one has

$$h^j(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) = h^j(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) = 0, \ j = 2, 3. \tag{46}$$

Similarly, as by assumptions it also satisfies Lemma 3.2-(iv) (with r replaced by $r-1$), one has

$$\chi(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) = \chi(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee). \tag{47}$$

Thus, equality in (47), together with (46), reads

$$h^0(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) - h^1(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) = h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) - h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee),$$

namely

$$\begin{aligned} &h^1(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) \\ &\qquad = h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) - \big(h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) - h^0(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee)\big) \end{aligned} \tag{48}$$

where $h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) \geqslant \min\{6t-3, \ 2t+1\} \geqslant 3$, as from Lemma 3.1-(iii) where r is replaced by $r-1$. We claim that the following equality

$$h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) = \begin{cases} 0 & \text{if} \quad r \ \text{even} \\ 1 & \text{if} \quad r \ \text{odd} \end{cases} \tag{49}$$

holds true.

Assume for a moment that (49) has been proved; since $\mathcal{U}_{r-1}$ is slope-stable, of the same slope as M_{ϵ_r}, and $\mathcal{U}_{r-1}$ is not isomorphic to M_{ϵ_r}, then $h^0(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) = 0$ as any non-zero homomorphism $M_{\epsilon_r} \to \mathcal{U}_{r-1}$ should be an isomorphism. Thus, using (48), for any $r \geqslant 2$ one gets therefore

$$h^1(\mathcal{U}_{r-1} \otimes L_{\epsilon_r}^\vee) \geqslant h^1(\mathcal{G}_{r-1} \otimes L_{\epsilon_r}^\vee) - 1$$

which, together with Lemma 3.1-(iii), proves the statement.

Thus, we are left with the proof of (49). To prove it, we will use induction on r.

If $r = 2$, then $\mathcal{G}_1 = M_1$, $M_{\epsilon_2} = M_2$, thus $h^0(\mathcal{G}_1 \otimes M_2^\vee) = h^0(M_1 - M_2) = 0$, as it follows from (16) and from (19). If otherwise $r = 3$, then $\mathcal{G}_{r-1} = \mathcal{G}_2$ as in (31) whereas $M_{\epsilon_3} = M_1$, as in (32). Thus, tensoring (31) by $M_1^\vee$, one gets

$$0 \to \mathcal{O}_X \to \mathcal{G}_2 \otimes M_1^\vee \to M_2 - M_1 \to 0;$$

since $h^0(M_2 - M_1) = 0$, from (17) and from (18), then $h^0(\mathcal{G}_2 \otimes M_1^\vee) = h^0(\mathcal{O}_X) = 1$.

Assume therefore that, up to some integer $r - 2 \geqslant 2$, (49) holds true and take $\mathcal{G}_{r-1}$ a non-trivial extension as in (33), with r replaced by $r-1$, namely

$$0 \to \mathcal{G}_{r-2} \to \mathcal{G}_{r-1} \to M_{\epsilon_{r-1}} \to 0. \tag{50}$$

If r is even, then $r-2$ is even and $r-1$ is odd, in particular $M_{\epsilon_{r-1}} = M_1$ and $M_{\epsilon_r} = M_2$. Thus, tensoring (50) with $M_{\epsilon_r}^\vee = M_2^\vee$ gives

$$0 \to \mathcal{G}_{r-2} \otimes M_2^\vee \to \mathcal{G}_{r-1} \otimes M_2^\vee \to M_1 - M_2 \to 0.$$

Since $h^0(M_1 - M_2) = 0$ then

$$h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) = h^0(\mathcal{G}_{r-1} \otimes M_2^\vee) = h^0(\mathcal{G}_{r-2} \otimes M_2^\vee).$$

On the other hand, by (33), with r replaced by $r-2$, namely

$$0 \to \mathcal{G}_{r-3} \to \mathcal{G}_{r-2} \to M_{\epsilon_{r-2}} \to 0, \tag{51}$$

we have $M_{\epsilon_{r-2}} = M_2$, since $r-2$ is even as r is. Thus, tensoring (51) with $M_{\epsilon_r}^\vee$ and taking into account that r is even, one gets

$$0 \to \mathcal{G}_{r-3} \otimes M_2^\vee \to \mathcal{G}_{r-2} \otimes M_2^\vee \to \mathcal{O}_X \to 0.$$

Notice that $\mathcal{G}_{r-3} \otimes M_2^\vee = \mathcal{G}_{r-3} \otimes M_{\epsilon_{r-2}}^\vee$ thus, since $r-3$ is odd, $h^0(\mathcal{G}_{r-3} \otimes M_2^\vee) = 0$ by induction and by (49). On the other hand, the coboundary map

$$H^0(\mathcal{O}_X) \cong \mathbb{C} \xrightarrow{\partial} H^1(\mathcal{G}_{r-3} \otimes M_2^\vee) = H^1(\mathcal{G}_{r-3} \otimes M_{\epsilon_{r-2}}^\vee) \cong \mathrm{Ext}^1(M_{\epsilon_{r-2}}, \mathcal{G}_{r-3})$$

is non-zero since, by iterative construction, $\mathcal{G}_{r-2}$ is taken to be a non-trivial extension; therefore ∂ is injective which implies $h^0(\mathcal{G}_{r-2} \otimes M_2^\vee) = 0$ and so $h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) = 0$, as desired.

Assume now r to be odd thus, $M_{\epsilon_r} = M_1$ whereas $M_{\epsilon_{r-1}} = M_2$. Tensoring (50) with $M_1^\vee$ gives

$$0 \to \mathcal{G}_{r-2} \otimes M_1^\vee \to \mathcal{G}_{r-1} \otimes M_1^\vee \to M_2 - M_1 \to 0.$$

As $h^0(M_2 - M_1) = 0$, then

$$h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) = h^0(\mathcal{G}_{r-1} \otimes M_1^\vee) = h^0(\mathcal{G}_{r-2} \otimes M_1^\vee).$$

Since r is odd, then also $r-2$ is odd and one gets

$$0 \to \mathcal{G}_{r-3} \otimes M_1^\vee \to \mathcal{G}_{r-2} \otimes M_1^\vee \to \mathcal{O}_X \to 0.$$

Notice that $h^0(\mathcal{G}_{r-3} \otimes M_1^\vee) = h^0(\mathcal{G}_{r-3} \otimes M_{\epsilon_{r-2}}^\vee) = 1$, as it follows from (49) with r replaced by $r-2$ which is odd since r is. On the other hand, the fact that $\mathcal{G}_{r-2}$ arises from a non–trivial extension implies as before that the coboundary map

$$H^0(\mathcal{O}_X) \cong \mathbb{C} \xrightarrow{\partial} H^1(\mathcal{G}_{r-3} \otimes M_1^\vee) = H^1(\mathcal{G}_{r-3} \otimes M_{\epsilon_{r-2}}^\vee) \cong \mathrm{Ext}^1(M_{\epsilon_{r-2}}, \mathcal{G}_{r-3})$$

is once again injective. This gives $h^0(\mathcal{G}_{r-2} \otimes M_1^\vee) = h^0(\mathcal{G}_{r-3} \otimes M_1^\vee) = 1$, which implies $h^0(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) = 1$. This concludes the proof of the Lemma. $\square$

Lemma 3.3 ensures that there exist non-trivial extensions arising from (45). Then one has the following consequence.

COROLLARY 3.3. *For a given $r \geqslant 2$, assume that $\mathcal{M}(r-1) \neq \emptyset$ and that $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ general corresponds to a rank-r vector bundle, which is Ulrich w.r.t. ξ and slope-stable, of slope $\mu(\mathcal{U}_{r-1}) = 13t-3$ (where, for $r=2$, $\mathcal{U}_1 = \mathcal{G}_1 = M_1$ and $\mathcal{M}(1) = \{M_1\}$ is a singleton). Consider M_{ϵ_r} as in* (32) *and* (33).

Then $[\mathcal{F}_r] \in \operatorname{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1})$ general is a rank-r vector bundle, which is simple, so indecomposable, Ulrich w.r.t. ξ, with Chern classes as in (34)*, of slope w.r.t. ξ given by $\mu(\mathcal{F}_r) = 13t-3$ and with $h^j(\mathcal{F}_r \otimes \mathcal{F}_r^\vee) = 0$, for $2 \leqslant j \leqslant 3$.*

Proof. Since $\mathcal{U}_{r-1}$ and M_{ϵ_r} are both Ulrich w.r.t. ξ, then it immediately follows that $\mathcal{F}_r$ is of rank r, Ulrich w.r.t. ξ and of slope as stated, by (1).

Now, since $\mathcal{U}_{r-1}$ is slope-stable, with $\mathcal{U}_{r-1}$ not isomorphic to M_{ϵ_r} (if $r > 2$, $\operatorname{rk}(\mathcal{U}_{r-1}) > 1 = \operatorname{rk}(M_{\epsilon_r})$, if otherwise $r = 2$, $\mathcal{U}_1 = M_1$ and $M_{\epsilon_2} = M_2$ are not isomorphic), and since moreover $\mathcal{U}_{r-1}$ and M_{ϵ_r} have the same slope $\mu = 13t-3$ then, by [12, Lemma 4.2], general $[\mathcal{F}_r] \in \operatorname{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1})$ corresponds to a simple, so indecomposable, rank-r vector bundle, with Chern classes as in (34).

To prove the assertions on cohomology groups, consider the dual sequence of (45) and tensor it by $\mathcal{F}_r$, which gives

$$0 \to M_{\epsilon_r}^\vee \otimes \mathcal{F}_r \to \mathcal{F}_r^\vee \otimes \mathcal{F}_r \to \mathcal{U}_{r-1}^\vee \otimes \mathcal{F}_r \to 0.$$

Thus,

$$h^j(\mathcal{F}_r^\vee \otimes \mathcal{F}_r) \leqslant h^j(M_{\epsilon_r}^\vee \otimes \mathcal{F}_r) + h^j(\mathcal{U}_{r-1}^\vee \otimes \mathcal{F}_r), \ 2 \leqslant j \leqslant 3. \tag{52}$$

On the other hand taking (45) tensored, respectively, by $M_{\epsilon_r}^\vee$ and $\mathcal{U}_{r-1}^\vee$ gives

$$0 \to M_{\epsilon_r}^\vee \otimes \mathcal{U}_{r-1} \to M_{\epsilon_r}^\vee \otimes \mathcal{F}_r \to \mathcal{O}_X \to 0$$

and

$$0 \to \mathcal{U}_{r-1}^\vee \otimes \mathcal{U}_{r-1} \to \mathcal{U}_{r-1}^\vee \otimes \mathcal{F}_r \to \mathcal{U}_{r-1}^\vee \otimes M_{\epsilon_r} \to 0,$$

from which one has

$$h^j(M_{\epsilon_r}^\vee \otimes \mathcal{F}_r) \leqslant h^j(M_{\epsilon_r}^\vee \otimes \mathcal{U}_{r-1}) + h^j(\mathcal{O}_X), \ 2 \leqslant j \leqslant 3,$$

and

$$h^j(\mathcal{U}_{r-1}^\vee \otimes \mathcal{F}_r) \leqslant h^j(\mathcal{U}_{r-1}^\vee \otimes \mathcal{U}_{r-1}) + h^j(\mathcal{U}_{r-1}^\vee \otimes M_{\epsilon_r}), \ 2 \leqslant j \leqslant 3.$$

Thus from $h^j(\mathcal{O}_X) = 0$, for $j = 2, 3$, from Lemmas 3.1-(i-ii) and 3.2-(iii) and inductive assumptions on $\mathcal{U}_{r-1}$, one deduces that $h^j(M_{\epsilon_r}^\vee \otimes \mathcal{F}_r) = h^j(\mathcal{U}_{r-1}^\vee \otimes \mathcal{F}_r) = 0$, for $j = 2, 3$, which plugged in (52) gives $h^j(\mathcal{F}_r^\vee \otimes \mathcal{F}_r) = 0$, for $2 \leqslant j \leqslant 3$, as stated. $\square$

Take therefore $[\mathcal{F}_r] \in \operatorname{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1})$ general. From Corollary 3.3 we know that $\mathcal{F}_r$ is simple with $h^2(\mathcal{F}_r^\vee \otimes \mathcal{F}_r) = 0$. Therefore, by [12, Proposition 10.2], $\mathcal{F}_r$ admits a smooth modular family which, with a small abuse of notation, we denote by $\mathcal{M}(r)$ as the modular component of Theorem B in Introduction. Indeed, by definition of smooth modular family as in [12, pp. 1250083-9/10], an open dense subset of it will be an étale cover of the modular component $\mathcal{M}(r)$ we are going to contruct; for this reason and to avoid heavy notation, they will be sometimes identified.

For $r \geqslant 2$, such a smooth modular family $\mathcal{M}(r)$ contains a subscheme, denoted by $\mathcal{M}(r)^{\text{ext}}$, which parametrizes bundles $\mathcal{F}_r$ arising from non–trivial extensions as in (45).

LEMMA 3.4. *Let $r \geqslant 2$ be an integer and let $\mathcal{U}_r$ be a general member of the modular family $\mathcal{M}(r)$. Then $\mathcal{U}_r$ is a vector bundle of rank r, which is Ulrich with respect to ξ, with slope w.r.t. ξ given by $\mu := \mu(\mathcal{U}_r) = 13t - 3$, and with Chern classes as* (34).

Moreover $\mathcal{U}_r$ is simple, in particular indecomposable, with

(i) $$\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = \begin{cases} \frac{(r^2-1)}{4}(4-8t)+1, & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(4-8t), & \text{if } r \text{ is even.} \end{cases}$$

(ii) $h^j(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$, *for* $j = 2, 3$.

Proof. Since $\mathcal{F}_r$ is of rank r and Ulrich w.r.t. ξ, the same holds true for the general member $[\mathcal{U}_r] \in \mathcal{M}(r)$, since Ulrichness is an open property in irreducible families as $\mathcal{M}(r)$. In particular, from (1), one has $\mu(\mathcal{U}_r) = \mu(\mathcal{F}_r) = \mu(\mathcal{U}_{r-1})$. For the same reasons, Chern classes of $\mathcal{U}_r$ coincide with those of $\mathcal{F}_r$ which, in turn, are as in (34).

Since $\mathcal{F}_r$ is simple, as proved in Corollary 3.3, by semi-continuity on $\mathcal{M}(r)$ one has also $h^0(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1$, i.e. $\mathcal{U}_r$ is simple, in particular it is indecomposable.

Property (ii) follows by semi-continuity in the smooth modular family $\mathcal{M}(r)$ when $\mathcal{U}_r$ specializes to $\mathcal{F}_r$ and from Corollary 3.3.

Property (i) follows from Lemma 3.2-(vi), since the given χ depends only on the Chern classes of X, which are fixed, and on the Chern classes of the two factors, which in turn are those as $\mathcal{F}_r$ and so of $\mathcal{G}_r$ as well. □

We will prove that the general member $\mathcal{U}_r$ in the smooth modular family $\mathcal{M}(r)$ is a slope–stable bundle w.r.t. ξ. To prove it, we will make use of the following auxiliary result, whose proof is identical to that of [30, Lemma 4.6], to which the interested reader is referred.

LEMMA 3.5. *Let $r \geqslant 2$ be an integer and assume that the element $\mathcal{F}_r$ of the subscheme $\mathcal{M}(r)^{\text{ext}}$ sits in a non–splitting sequence like* (45), *with* $[\mathcal{U}_{r-1}] \in$

$\mathcal{M}(r-1)$ general. Then, if $\mathcal{D}$ is a destabilizing subsheaf of $\mathcal{F}_r$, then $\mathcal{D}^\vee \cong \mathcal{U}_{r-1}^\vee$ and $(\mathcal{F}_r/\mathcal{D})^\vee \cong M_{\epsilon_r}^\vee$; if furthermore $\mathcal{F}_r/\mathcal{D}$ is torsion–free, then $\mathcal{D} \cong \mathcal{U}_{r-1}$ and $\mathcal{F}_r/\mathcal{D} \cong M_{\epsilon_r}$.

Proof. See the proof of [30, Lemma 4.6]. □

LEMMA 3.6. *Let $r \geqslant 2$ be an integer. Take $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ a general point, where $\mathcal{M}(r-1)$ is the modular component as in Theorem B. Then, the modular family $\mathcal{M}(r)$ as in Lemma 3.4 is generically smooth, of dimension*

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(8t-4), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(8t-4)+1, & \text{if } r \text{ is even.} \end{cases}$$

Furthermore $\mathcal{M}(r)$ properly contains the locally closed subscheme $\mathcal{M}(r)^{\mathrm{ext}}$, namely $\dim(\mathcal{M}(r)^{\mathrm{ext}}) < \dim(\mathcal{M}(r))$.

Proof. Let $\mathcal{U}_r$ be a general member of the smooth modular family $\mathcal{M}(r)$. From Lemma 3.4, one has $h^0(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1$, i.e. it is simple, and $h^j(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$ for $j = 2, 3$.

From the fact that $h^2(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$, it follows that the modular family $\mathcal{M}(r)$ is generically smooth of dimension $\dim(\mathcal{M}(r)) = h^1(\mathcal{U}_r \otimes \mathcal{U}_r^\vee)$ (cf. e.g. [12, Proposition 2.10]). On the other hand, since $h^3(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$ and $h^0(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1$, we have $h^1(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = -\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1$. Therefore, the formula concerning $\dim(\mathcal{M}(r))$ directly follows from Lemma 3.4-(i), since the given χ depends only on the Chern classes of X, which are fixed, and on the Chern classes of the two factors, which in turn are those as $\mathcal{F}_r$ and so of $\mathcal{G}_r$ as well.

Similarly $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ general is by assumptions slope-stable, so in particular simple, thus it satisfies $h^0(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) = 1$. Thus, using Lemma 3.4-(ii), the same reasoning as above shows that

$$\dim(\mathcal{M}(r-1)) = h^1(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) = -\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + 1, \tag{53}$$

where $\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee)$ as in Lemma 3.4-(i) (with r replaced by $r-1$). Morover, by (48), we have

$$\dim(\mathrm{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1})) = h^1(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) \leqslant h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee), \tag{54}$$

where the latter is as in Lemma 3.2-(ii) (with r replaced by $r-1$). Therefore, by the very definition of $\mathcal{M}(r)^{\mathrm{ext}}$ and by (53)-(54), we have

$$\begin{aligned} \dim(\mathcal{M}(r)^{\mathrm{ext}}) &\leqslant \dim(\mathcal{M}(r-1)) + \dim(\mathbb{P}(\mathrm{Ext}^1(M_{\epsilon_r}, \mathcal{U}_{r-1})) \\ &= -\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + 1 + h^1(\mathcal{U}_{r-1} \otimes M_{\epsilon_r}^\vee) - 1 \\ &\leqslant -\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee). \end{aligned}$$

On the other hand, from the above discussion,

$$\dim(\mathcal{M}(r)) = -\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1.$$

Therefore to prove that $\dim(\mathcal{M}(r)^{\mathrm{ext}}) < \dim(\mathcal{M}(r))$ it is enough to show that for any integer $r \geqslant 2$ the following inequality

$$-\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) < -\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1$$

holds true. Notice that the previous inequality reads also

$$-\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1 + \chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) - h^1(\mathcal{G}_{r-1} \otimes M_{\epsilon_r}^\vee) > 0, \tag{55}$$

which is satisfied for any $r \geqslant 2$, as we can easily see.

Indeed use Lemmas 3.4-(i) and 3.2-(ii): if r is even, the left hand side of (55) reads $\frac{(r)}{2}(8t-4) + 2 + \frac{(r-2)}{2}$ which obviously is positive since $r \geqslant 2$ and $t \geq 1$; if r is odd, then $r \geqslant 3$ and the left hand side of (55) reads $\frac{r-1}{2}(6t-5) + \frac{(r-3)}{2}$ which obviously is positive under the assumptions $r \geqslant 3$, $t \geqslant 1$. □

We can now prove slope–stability w.r.t. ξ of the general member of modular family $\mathcal{M}(r)$.

PROPOSITION 3.4. *Let $r \geqslant 1$ be an integer. The general member $\mathcal{U}_r$ in the modular family $\mathcal{M}(r)$ is a bundle which is slope–stable w.r.t. ξ.*

Proof. We use induction on r, the result being obviously true for $r = 1$, where $\mathcal{U}_1 = M_1$, $\mathcal{M}(1) = \{M_1\}$ is a singleton, and $\mathcal{M}(1)^{\mathrm{ext}} = \emptyset$.

Assume therefore $r \geqslant 2$ and, by contradiction, that the general member of $\mathcal{M}(r)$ were not slope–stable, whereas the general point $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ of the modular component corresponds to a bundle which is slope-stable w.r.t. ξ. Then, similarly as in [20, Proposition 4.7], we may find a one-parameter family of bundles $\{\mathcal{U}_r^{(t)}\}$ over the unit disc Δ such that $\mathcal{U}_r^{(t)}$ is a general member of $\mathcal{M}(r)$ for $t \neq 0$ and $\mathcal{U}_r^{(0)}$ lies in $\mathcal{M}(r)^{\mathrm{ext}}$, and such that we have a destabilizing sequence

$$0 \to \mathcal{D}^{(t)} \to \mathcal{U}_r^{(t)} \to \mathcal{Q}^{(t)} \to 0 \tag{56}$$

for $t \neq 0$, which we can take to be saturated, that is, such that $\mathcal{Q}^{(t)}$ is torsion free, whence so that $\mathcal{D}^{(t)}$ and $\mathcal{Q}^{(t)}$ are (Ulrich) vector bundles (see [12, Theorem 2.9] or [9, (3.2)]).

The limit of $\mathbb{P}(\mathcal{Q}^{(t)}) \subset \mathbb{P}(\mathcal{U}_r^{(t)})$ defines a subvariety of $\mathbb{P}(\mathcal{U}_r^{(0)})$ of the same dimension as $\mathbb{P}(\mathcal{Q}^{(t)})$, whence a coherent sheaf $\mathcal{Q}^{(0)}$ of rank $\mathrm{rk}(\mathcal{Q}^{(t)})$ with a surjection $\mathcal{U}_r^{(0)} \to \mathcal{Q}^{(0)}$. Denoting by $\mathcal{D}^{(0)}$ its kernel, we have $\mathrm{rk}(\mathcal{D}^{(0)}) = \mathrm{rk}(\mathcal{D}^{(t)})$ and $c_1(\mathcal{D}^{(0)}) = c_1(\mathcal{D}^{(t)})$. Hence, (56) specializes to a destabilizing sequence for $t = 0$. Lemma 3.5 yields that ${\mathcal{D}^{(0)}}^\vee$ (respectively, ${\mathcal{Q}^{(0)}}^\vee$) is the dual of a member of $\mathcal{M}(r-1)$ (resp., the dual of M_{ϵ_r}). It follows that ${\mathcal{D}^{(t)}}^\vee$ (resp., ${\mathcal{Q}^{(t)}}^\vee$) is a deformation of the dual of a member of $\mathcal{M}(r-1)$ (resp., a deformation of $M_{\epsilon_r}^\vee$), whence that $\mathcal{D}^{(t)}$ is a deformation of a member of $\mathcal{M}(r-1)$, as both are locally free, and $\mathcal{Q}^{(t)} \cong M_{\epsilon_r}$, for the same reason.

In other words, the general member of $\mathcal{M}(r)$ is an extension of M_{ϵ_r} by a member of $\mathcal{M}(r-1)$. Hence $\mathcal{M}(r) = \mathcal{M}(r)^{\text{ext}}$, contradicting Lemma 3.6. □

The collection of the previous results gives the following

THEOREM 3.5. *Let $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ be a 3-fold scroll over $\mathbb{F}_0$, with $\mathcal{E} = \mathcal{E}_0$ satisfying Assumptions 1.7 Let $\varphi : X \to \mathbb{F}_0$ be the scroll map and F be the φ-fiber. Let $r \geqslant 2$ be any integer. Then, for any integer $t \geqslant 1$, the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X which are Ulrich w.r.t. ξ and with Chern classes as in* (34)*, is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension*

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(8t-4), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(8t-4)+1, & \text{if } r \text{ is even.} \end{cases}$$

Moreover the general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 13t-3$.

In particular, there are no slope-stable-Ulrich-rank gaps on X_0 w.r.t. the chosen Chern classes.

Proof. It directly follows from Theorem 2.2, (34), (1) and from Lemmas 3.4, 3.6 and Proposition 3.4 where, as already mentioned, by a small abuse of notation we have used the same symbol $\mathcal{M}(r)$ for the smooth modular family which, via GIT-quotient, gives rise by an étale cover to an open, dense subset of the generically smooth, irreducible component of the corresponding moduli space. □

4. Higher-rank mixed Ulrich bundles on 3-fold scrolls over $\mathbb{F}_0$

In this section we briefly mention how to construct other positive-dimensional *sporadic modular components* of moduli spaces which arise by a similar approach as in § 3, but with the use of *mixed pairs* (L_i, M_j), $1 \leqslant i, j \leqslant 2$, as in Theorem A. Such components will be therefore different from those determined in Theorem B.

Looking back to rank-2 cases, our starting point will be to consider the positive-dimensional component $\mathcal{M} := \mathcal{M}(2)$, where $\mathcal{M}(2)$ as either in Theorem 2.2, or in Theorem 2.3-(2) and (3).

If $[\mathcal{U}_2] \in \mathcal{M}(2)$ is a general point, then one can consider as in (45) extensions $\mathrm{Ext}^1(A, \mathcal{U}_2)$, where A runs among all possible choices $A = L_1, L_2, M_1, M_2$ as in Theorem A-(a). This gives rise to rank-3 vector bundles, which are Ulrich w.r.t. ξ on X and whose Chern classes are given by

$$c_1 := c_1(\mathcal{U}_2) + c_1(A), \quad c_2 := c_2(\mathcal{U}_2) + c_1(\mathcal{U}_2) \cdot c_1(A), \quad c_3 := c_2(\mathcal{U}_2) \cdot c_1(A).$$

Whenever one gets non-trivial extensions, one can compute cohomological properties of the general bundle in such an extension space similarly as done in Lemmas 3.1, 3.2 and 3.3. In such a case, reasoning as in Corollary 3.3, such a general bundle turns out to be simple and whose cohomological properties computed implying that it sits in a smooth modular family as in Proposition 1.5.

Then one deduces properties of the general bundle $\mathcal{U}_3$ in the given modular family as done in Lemma 3.4. Using same strategies as in Lemma 3.6 and in Proposition 3.4, such $\mathcal{U}_3$ gives rise to a general point of a generically smooth modular component $\mathcal{M}(3)$ of computed dimension. Then, one can recursively apply the same procedure to this new $\mathcal{U}_3$ by pairing it in extensions via an Ulrich line bundle A, where A runs among all possible choices $A = L_1, L_2, M_1, M_2$ as in Theorem A-(a).

It is clear from the above description that the number of possible cases to study at each step grows as the rank increases. For this reason, since the procedures to use in any of the cases are exactly as in the previous section, we will limit ourselves to considering one significant case giving rise to *sporadic modular components* which are different from those determined in Theorem B; these will be called *extra sporadic modular components*.

Let us therefore consider our starting step as given by $[\mathcal{U}_2] \in \mathcal{M}(2)$ general as in Theorem 2.2. In particular one has

$$c_1(\mathcal{U}_2) = c_1(M_1) + c_1(M_2) \text{ and } c_2(\mathcal{U}_2) = c_1(M_1) \cdot c_1(M_2),$$

namely $\mathcal{U}_2$ arises as a suitable deformation of vector bundle extensions by means of the *sporadic pair* (M_1, M_2). We then need to take an Ulrich line bundle A, running among all possible choices $A = L_1, L_2, M_1, M_2$ as in Theorem A-(a).

From what proved in § 3, we can avoid to extend $\mathcal{U}_2$ by $A = M_1$, since this case has been already considered therein.

(*i*) if $A = L_1$, one computes that $\dim(\mathrm{Ext}^1(L_1, \mathcal{U}_2)) = h^1(\mathcal{U}_2(-L_1)) = 0$, as it follows by semi-continuity and by (9) tensored by $-L_1$. Therefore, extension via L_1 does not provide an indecomposable bundle and we therefore get rid of this case.

(*ii*) if $A = L_2$, to compute $\dim(\mathrm{Ext}^1(L_2, \mathcal{U}_2)) = h^1(\mathcal{U}_2(-L_2))$ we first observe that, since $\mathcal{U}_2$ and L_2 are both slope-stable and of the same slope, then $h^0(\mathcal{U}_2(-L_2)) = 0$. We consider (9) tensored by $-L_2$ which gives

$$0 \to M_1 - L_2 = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0, -3t) \to \mathcal{F}(-L_2) \to M_2 - L_2 = -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, t) \to 0,$$

from which one gets $h^j(\mathcal{F}(-L_2)) = 0$, for $2 \leqslant j \leqslant 3$, therefore the same holds true for $h^j(\mathcal{U}_2(-L_2)) = 0$, $2 \leqslant j \leqslant 3$, by semi-continuity. Thus, by the invariance of Euler characteristic in irreducible flat families, one gets $h^1(\mathcal{U}_2(-L_2)) = h^1(\mathcal{F}(-L_2))$ where the latter can be computed by the previous exact sequence. Since $h^1(M_2-L_2) = h^1(-\xi+\varphi^*\mathcal{O}_{\mathbb{F}_0}(1,t)) = 0$, then $h^1(\mathcal{F}(-L_2)) = h^1(M_1-L_2)$ which, by Leray, equals $h^1(\mathbb{F}_0, \mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_0}(-3t\, f)) = 10t-5$, as in (28).

Therefore, $\dim(\mathrm{Ext}^1(L_2,\mathcal{U}_2)) = 10t-5$ so we have non-trivial extensions of rank 3 and, as in Corollary 3.3, $[\mathcal{F}_3] \in \mathrm{Ext}^1(L_2,\mathcal{U}_2)$ general turns out to be simple, so indecomposable, Ulrich and with

$$c_1(\mathcal{F}_3) = c_1(\mathcal{U}_2) + c_1(L_2) = c_1(M_1) + c_1(M_2) + c_1(L_2) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, 2t-3)$$

(similar computations for the other Chern classes).

(iii) if otherwise $A = M_2$, as above one computes that $\dim(\mathrm{Ext}^1(M_2,\mathcal{U}_2)) = h^1(\mathcal{U}_2(-M_2)) = 6t-4$, and $[\mathcal{F}_3'] \in \mathrm{Ext}^1(M_2,\mathcal{U}_2)$ general turns out to be simple, so indecomposable, Ulrich and with

$$c_1(\mathcal{F}_3') = c_1(\mathcal{U}_2) + c_1(M_2) = c_1(M_1) + 2c_1(M_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2)$$

(similar computations for the other Chern classes).

To go on, in any of the above cases, we should now consider pairings of the general rank-3 with an Ulrich line bundle A, with A running once again among all possible choices $A = L_1, L_2, M_1, M_2$ as in Theorem A-(a), in order to get non-trivial rank-4 extensions and so on.

We will limit ourselves to perform extensions of the *sporadic* bundle $[\mathcal{F}_3] \in \mathrm{Ext}^1(L_2,\mathcal{U}_2)$ general by means of an Ulrich line bundle A chosen among e.g. *non-sporadic* pairs (L_1, L_2). On the other hand, as observed in § 3, if in the next step we considered further extensions $\mathrm{Ext}^1(L_2,\mathcal{F}_3)$, taking into account the associated coboundary map it is easy to see that the dimension of such an extension space drops by one with respect to that of $\mathrm{Ext}^1(L_2,\mathcal{U}_2)$. Therefore, keeping L_2 fixed on the right side of the extensions, after finitely many steps we would get only splitting bundles. To avoid this fact we proceed once again by taking *alternating extensions*, namely

$$0 \to \mathcal{U}_2 \to \mathcal{F}_3 \to L_2 \to 0, \quad 0 \to \mathcal{F}_3 \to \mathcal{G}_4 \to L_1 \to 0, \ \ldots,$$

and so on, that is, defining

$$\tau_r := \begin{cases} 1, & \text{if } r \geqslant 4 \text{ is even,} \\ 2, & \text{if } r \geqslant 3 \text{ is odd,} \end{cases} \tag{57}$$

we take successive $[\mathcal{G}_r] \in \mathrm{Ext}^1(L_{\tau_r}, \mathcal{G}_{r-1})$, for all $r \geqslant 3$, defined by:

$$0 \to \mathcal{G}_{r-1} \to \mathcal{G}_r \to L_{\tau_r} \to 0, \tag{58}$$

where, for $r = 3$, $\mathcal{G}_{r-1} = \mathcal{G}_2 := \mathcal{U}_2$ and $\mathcal{G}_r = \mathcal{G}_3 = \mathcal{F}_3$. In particular, from the fact that $c_1(\mathcal{U}_2) = c_1(M_1) + c_1(M_2)$ and from (58), one gets $c_1(\mathcal{G}_r) = c_1(M_1) + c_1(M_2) + \lfloor \frac{r-1}{2} \rfloor c_1(L_2) + \lfloor \frac{r-2}{2} \rfloor c_1(L_1)$, namely

$$\begin{aligned} c_1(\mathcal{G}_r) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2) &+ \left(\left\lfloor \tfrac{r-1}{2} \right\rfloor\right)(\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, 2t-1)) \\ &+ \left(\left\lfloor \tfrac{r-2}{2} \right\rfloor\right)(\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, -1)), \end{aligned}$$

which reads

$$c_1(\mathcal{G}_r) := \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{(r-3)}{2}, (r-1)(t-1)+1\right), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{r}{2}, (r-2)(t-1)\right), & \text{if } r \text{ is even,} \end{cases} \tag{59}$$

(similar formulas can be determined for the other Chern classes).

Applying similar computations as in Lemma 3.1, with M_1 and M_2 replaced by L_1 and L_2, we deduce as in Corollary 3.2 that there exist rank-r vector bundles $\mathcal{G}_r$, which are Ulrich w.r.t. ξ, with c_1 as in (59), of slope $\mu(\mathcal{G}_r) = 13t-3$ and which arise as non-trivial extensions as in (58). Applying then similar strategies as in Lemma 3.2, we find recursive formulas:

$$\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = \chi(L_{\tau_r}^\vee \otimes \mathcal{G}_{r-1}) + \chi(\mathcal{G}_{r-1} \otimes \mathcal{G}_{r-1}^\vee) + \chi(\mathcal{O}_X) + \chi(\mathcal{G}_{r-1}^\vee \otimes L_{\tau_r}). \tag{60}$$

By induction, we can argue as in Lemma 3.3 and Corollary 3.3 to get that, starting from $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ general, for any $r \geqslant 2$, corresponding to a rank-$(r-1)$ vector bundle, which is Ulrich w.r.t. ξ and slope-stable, of slope $\mu(\mathcal{U}_{r-1}) = 13t-3$, whose first Chern class is as in (59), then $[\mathcal{F}_r] \in \mathrm{Ext}^1(L_{\tau_r}, \mathcal{U}_{r-1})$ general is a rank-r vector bundle, which is simple, so indecomposable, Ulrich w.r.t. ξ, with first Chern class as in (59), of the same slope and with $h^j(\mathcal{F}_r \otimes \mathcal{F}_r^\vee) = 0$, for $2 \leqslant j \leqslant 3$.

So, from Proposition 1.5, $\mathcal{F}_r$ sits in a smooth modular family $\mathcal{M}(r)$, whose general element $\mathcal{U}_r$ is, reasoning as in Lemma 3.4, Ulrich w.r.t. ξ, of rank r, with slope w.r.t. ξ given by $\mu := \mu(\mathcal{U}_r) = 13t-3$, with first Chern class as in (59), with $h^j(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$, for $j = 2, 3$ (by semi-continuity on $\mathcal{M}(r)$) and with $\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee)$ as in (60) (because $\mathcal{G}_r$ and $\mathcal{U}_r$ have same Chern classes). Using same reasoning as in Lemmas 3.5 and 3.6, one can show that the modular family $\mathcal{M}(r)$ is generically smooth, of dimension

$$h^1(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1 - \chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee),$$

as it follows from the vanishings $h^j(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$, for $j = 2, 3$, and $h^0(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1$, from simplicity of $\mathcal{U}_r$. Thus, one can conclude similarly as in Proposition 3.4.

Therefore, to conclude the proof of Theorem C, one is reduced to compute (60). Applying similar strategies as in Lemma 3.2, from (60) one gets:

$$\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = \begin{cases} \frac{r^2(5-10t)-4r+8t+4}{4}, & \text{if } r \text{ is even,} \\ \frac{2r^2(2-t)-(14t+11)r+(24t+15)}{2}, & \text{if } r \text{ is odd} \end{cases}.$$

Therefore, $\dim(\mathcal{M}(r)) = h^1(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1 - \chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee)$ is as stated in Theorem C.

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References

[1] V. Antonelli, *Characterization of Ulrich bundles on Hirzebruch surfaces*, Rev. Mat. Complut. **34** (2021), no. 1, 43–74.

[2] V. Antonelli, G. Casnati, and O. Genc, *Instanton bundles on* $\mathbb{P}^1 \times \mathbb{F}_1$, Comm. Algebra **40** (2021), no. 8, 3594–3613.

[3] V. Antonelli, G. Casnati, and O. Genc, *Even and odd instanton bundles on Fano threefolds*, Asian J. Math. **26** (2022), no. 1, 81–117.

[4] M. Aprodu and V. Brinzănescu, *Moduli spaces of vector bundles over ruled surfaces*, Nagoya Math. J. **154** (1999), 111–122.

[5] M. Aprodu, L. Costa, and R.M. Miró-Roig, *Ulrich bundles on ruled surfaces*, J. Pure Appl. Algebra **222** (2018), no. 1, 131–138.

[6] M. Aprodu, S. Huh, F. Malaspina, and J. Pons-Llopis, *Ulrich bundles on smooth projective varieties of minimal degree*, Proc. Amer. Math. Soc. **147** (2019), no. 12, 5117–5129.

[7] J. Backelin and J. Herzog, *On Ulrich modules over hypersurface rings*, Commutative Algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, 1989, pp. 63–68.

[8] A. Beauville, *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64.

[9] A. Beauville, *An introduction to Ulrich bundle*, Eur. J. Math. **4** (2018), no. 1, 26–36.

[10] M. C. Beltrametti and A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, De Gruyter Exp. Math., vol. 16, De Gruyter, Berlin, 1995.

[11] J. E. Brosius, *Rank-2 vector bundles on a ruled surface. I*, Math. Ann. **265** (1983), 155–168.

[12] M. Casanellas, R. Hartshorne, F. Geiss, and F.-O. Schreyer, *Stable Ulrich bundles*, Int. J. Math. **23** (2012), no. 8, 1250083, 50 pp.

[13] G. Casnati, *Special Ulrich bundles on non-special surfaces with $p_g = q = 0$*, Int. J. Math. **28** (2017), no. 8, 1750061, 18 pp.

[14] G. Casnati, *Examples of surfaces which are Ulrich-wild*, Proc. Amer. Math. Soc. **148** (2020), no. 12, 5029–5043.

[15] G. Casnati, *Special Ulrich bundles on regular surfaces with non-negative Kodaira dimension*, Manuscripta Math. **167** (2022), no. 1–2, 151–172.

[16] G. Casnati, D. Faenzi, and F. Malaspina, *Rank two aCM bundles on the del Pezzo threefold with Picard number 3*, J. Algebra **429** (2015), 413–446.

[17] G. Casnati, D. Faenzi, and F. Malaspina, *Moduli spaces of rank two aCM bundles on the Segre product of three projective lines*, J. Pure Appl. Algebra **220** (2016), no. 4, 1554–1575.

[18] G. Casnati, D. Faenzi, and F. Malaspina, *Rank two aCM bundles on the del Pezzo fourfold of degree 6 and its general hyperplane section*, J. Pure Appl. Algebra **222** (2018), no. 3, 585–609.

[19] G. Casnati, M. Filip, and F. Malaspina, *Rank two aCMbundles on the del Pezzo threefold of degree 7*, Rev. Mat. Complut. **30** (2017), no. 1, 129–165.

[20] C. Ciliberto, F. Flamini, and A.L. Knutsen, *Ulrich bundles on a general blow-up of the plane*, Ann. Mat. Pura Appl. **202** (2023), no. 4, 1835–1854.

[21] C. Ciliberto, F. Flamini, and A.L. Knutsen, *Ulrich bundles on Del Pezzo threefolds*, J. Algebra **634** (2023), 209–236.

[22] C. Ciliberto, F. Flamini, and A.L. Knutsen, *Elliptic curves, ACM bundles and Ulrich bundles on prime Fano threefolds*, Collect. Math. **75** (2024), no. 3, 795–822.

[23] E. Coskun, *A Survey of Ulrich bundles*, Analytic and algebraic geometry, Hindustan Book Agency, New Delhi, 2017, pp. 85–106.

[24] L. Costa and R. M. Miró-Roig, *Rationality of moduli spaces of vector bundles on rational surfaces*, Nagoya Math. J. **165** (2002), 43–69.

[25] L. Costa and R. M. Miró-Roig, *$GL(V)$-invariant Ulrich bundles on Grassmannians*, Math. Ann. **361** (2015), no. 1–2, 443–457.

[26] L. Costa, R. M. Miró-Roig, and J. Pons-Llopis, *Ulrich bundles – from commutative algebra to algebraic geometry*, De Gruyter Stud. Math., vol. 77, De Gruyter, Berlin, 2021.

[27] Y. Drozd and G. M. Greuel, *Tame and wild projective curves and classification of vector bundles*, J. Algebra **246** (2001), 1–54.

[28] D. Eisenbud, F.-O. Schreyer, and J. Weyman, *Resultants and Chow forms via exterior syzygies*, J. Amer. Math. Soc **16** (2003), no. 3, 537–579.

[29] D. Faenzi and J. Pons-Llopis, *The Cohen-Macaulay representation type of projective arithmetically Cohen-Macaulay varieties*, Epijournal Géom. Algébrique **5** (2021), no. 8, 1–37.

[30] M. L. Fania and F. Flamini, *Ulrich Bundles on some threefold scrolls over $\mathbb{F}_e$*, Adv. Math. **436** (2024), no. 109409, 60 pp.

[31] M. L. Fania, M. Lelli-Chiesa, and J. Pons Llopis, *Ulrich bundles on Three-*

Dimensional Scrolls, Int. Math. Res. Not. IMRN (2021), no. 17, 13478–13507.
[32] M.L. Fania and E. L. Livorni, *Degree nine manifolds of dimension greater than or equal to 3.*, Math. Nachr. **169** (1994), 117–134.
[33] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math., no. 52, Springer, New York, 1977.
[34] A. Hochenegger, *On Ulrich bundles on projective bundles*, Beitr. Algebra Geom. **63** (2022), no. 3, 573–587.
[35] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves. Second edition*, Cambridge Math. Library, Cambridge Univ. Press, Cambridge, 2010.
[36] P. Ionescu, *Embedded Projective Varieties of Small Invariants, III*, Algebraic Geometry L'Aquila, 1988, Lecture Notes in Math., vol. 1417, Springer, 1990, pp. 138–154.
[37] H. Ishihara, *Rank* 2 *Ample Vector Bundles on Some Smooth Rational Surface*, Geom. Dedicata **67** (1997), 309–336.
[38] A. Langer, *On boundedness of semistable sheaves*, Doc. Math. **27** (2022), 1–26.
[39] F. Malaspina, *Ulrich bundles of arbitrary rank on Segre-Veronese varieties*, J. Algebra **625** (2023), 46–65.
[40] B. Ulrich, *Gorenstein rings and modules with high numbers of generators*, Math. Z. **188** (1984), 23–32.

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