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Variable exponents anisotropic elliptic problems with lower order terms

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Abstract. *This paper aims to investigate the existence of distributional solutions in* $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ *(i.e. the anisotropic Sobolev space with variable exponents and zero boundary) for a class of nonlinear anisotropic elliptic equations with variable exponents and a lower-order term that has natural growth with respect to* $|\partial_i u|$, $i = 1, \dots, N$. *The datum* f *on the right-hand side belongs to the space* $L^{(\overline{p}^*)'(\cdot)}(\Omega)$, *where* $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ *is a bounded open Lipschitz domain and* $(\overline{p}^*)'(\cdot)$ *represents the Hölder conjugate of the Sobolev conjugate* $\overline{p}(\cdot)$.

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1. Introduction

We aim to prove that the following problem has at least one distributional solution:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \big(|\partial_i u|^{p_i(x)-2} \partial_i u \big) + u \sum_{i=1}^{N} |u|^{p_i(x)-2} \\ \qquad\qquad + \sum_{i=1}^{N} \partial_i u |\partial_i u|^{p_i(x)-2} = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is an open bounded domain with a Lipschitz boundary $\partial\Omega$, f is in $L^{(\overline{p}^*)'(\cdot)}(\Omega)$, such that $\overline{p}(\cdot)$ the harmonic mean of $\{p_i(\cdot),\, i = 1, \dots, N\}$ and $(\overline{p}^*)'(\cdot) = \frac{N\overline{p}(\cdot)}{1+(N+1)(\overline{p}(\cdot)-1)}$ the Hölder conjugate of the Sobolev conjugate $\overline{p}^*(\cdot) = \frac{N\overline{p}(\cdot)}{N-\overline{p}(\cdot)}$, $\overline{p}(\cdot) < N$.

Problem (1) is classified as a $\overrightarrow{p}(x)-$Laplace type equation since the $\overrightarrow{p}(x)$-anisotropic Laplace differential operator (i.e.,$u \mapsto -\sum_{i=1}^{N} \partial_i \big(|\partial_i u|^{p_i(x)-2} \partial_i u \big)$) is included on its main side. In the variable anisotropic case, you can explore the problems that incorporate this operator and others by referring [15, 16, 17, 18, 23, 24, 25]). Moreover, it contains a non-linear term $u\sum_{i=1}^{N}|u|^{p_i(x)-2}$ that satisfies a sign condition with respect to $|u|$, as well as a non-linear lower-order term $\sum_{i=1}^{N}\partial_i u|\partial_i u|^{p_i(x)-2}$ that has a sign condition with respect to $|\partial u_i|$, $i=1,\dots,N$ and natural growth. We can more easily derive a priori estimations from the problem here and develop approximate solutions with the aid of all these data. The existence results of numerous similar isotropic and anisotropic scalar and variable cases with various data and conditions have been explored; we provide a few of these as examples without limitation in [1, 2, 3, 4, 5, 8, 10, 19, 20, 21, 22, 27].

The Leray-Schauder fixed point theorem was used to demonstrate the existence of a sequence of appropriate approximation solutions (u_n), which served as the foundation for the proof. Then we give prior estimates by proving almost everywhere convergence for the partial derivatives of the solution u_n, which can be turned into strong L^1- convergence. Equipped with this convergence we pass to the limit in the strong L^1 sense for $|\partial_i u_n|^{p_i(x)-2}\partial_i u_n$, and for $u_n|u_n|^{p_i(x)-2}$, and finally we conclude the convergence of u_n to the solution of (1).

The plan of the paper is as follows. The mathematical preliminaries in Section 2 include some embedding theorems and a reminder of the anisotropic Lebesgue-Sobolev spaces with variable exponents. Section 3 contains the main theorem and its proof.

2. Preliminaries

Some fundamental concepts and characteristics of Lebesgue-Sobolev spaces with variable exponents must be given in this section (see [9, 11, 12, 13]).

Let Ω be a bounded open domain of $\mathbb{R}^N$ $(N\geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega})=\{\text{continuous function} \quad p(\cdot):\overline{\Omega}\mapsto\mathbb{R}, \quad p^->1\},$$

such that $p^-=\min_{x\in\overline{\Omega}}p(x)$, and $p^+=\max_{x\in\overline{\Omega}}p(x)$.

Let $p(\cdot)\in\mathcal{C}_+(\overline{\Omega})$. Then the following version of Young's inequality holds for all $a,b\in\mathbb{R}$ and all $\varepsilon>0$,

$$|ab|\leq\varepsilon|a|^{p(x)}+c(\varepsilon)|b|^{p'(x)},$$

where, $p'(\cdot)$ denotes the Hölder conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$ in $\overline{\Omega}$).

In addition, we also have

$$|a+b|^{p(x)}\leq 2^{p^+-1}(|a|^{p(x)}+|b|^{p(x)}).$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R};\ \rho_{p(\cdot)}(u) < \infty\},$$

where the function

$$u \mapsto \rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx$$

is called the convex modular function. It is a reflexive Banach space, under the Luxemburg norm given by

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1\right\}.$$

Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

in this setting holds.

The reflexive Banach space $W^{1,p(\cdot)}(\Omega)$ is defined as

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

and it is endowed with the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$. We define also the reflexive separable Banach space $W_0^{1,p(\cdot)}(\Omega)$ as

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)},$$

endowed with the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$.

If $u \in L^{p(\cdot)}(\Omega)$, then we have (see [9, 11])

$$\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right),$$
$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2}$$

In order to solve our problem (1), we will now introduce the variable exponents anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Let $p_i(\cdot) \in C\left(\overline{\Omega}, [1,+\infty)\right)$, i.e. continuous function $p(\cdot) : \overline{\Omega} \mapsto [1,+\infty)$, $i = 1,\ldots,N$, and we set for every x in $\overline{\Omega}$

$$\overrightarrow{p}(x) = (p_1(x),\ldots,p_N(x)), \quad p_+(x) = \max_{1\leq i\leq N} p_i(x),\ p_-(x) = \min_{1\leq i\leq N} p_i(x),$$
$$p_-^- = \min_{x\in\overline{\Omega}} p_-(x), \quad p_+^+ = \max_{x\in\overline{\Omega}} p_+(x),$$
$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)}}, \quad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N,\\ +\infty, & \text{for } \overline{p}(x) \geq N.\end{cases}$$

We introduce the Banach space

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega)=\Big\{u\in L^{p_+(\cdot)}(\Omega),\partial_i u\in L^{p_i(\cdot)}(\Omega),\ i=1,\dots,N\Big\},$$

under the norm

$$\|u\|_{\overrightarrow{p}(\cdot)}=\|u\|_{p_+(\cdot)}+\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}.$$

The spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are defined as

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)=\overline{C_0^{\infty}(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)},\quad \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)=W^{1,\overrightarrow{p}(\cdot)}(\Omega)\cap W_0^{1,1}(\Omega).$$

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot)\in(\mathcal{C}_+(\overline{\Omega}))^N$. The following embedding results have been proven in [12, 13].

LEMMA 2.1. *If $r\in\mathcal{C}_+(\overline{\Omega})$ and $r(\cdot)<\max(p_+(\cdot),\ \overline{p}^\star(\cdot))$ in $\overline{\Omega}$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)\hookrightarrow L^{r(\cdot)}(\Omega)\ \textit{is compact.}$$

LEMMA 2.2. *If we have*

$$\forall x\in\overline{\Omega},\ p_+(x)<\overline{p}^\star(x). \tag{3}$$

Then the following inequality holds

$$\|u\|_{p_+(\cdot)}\le C\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)},\ \forall u\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{4}$$

where $C>0$ independent of u.Thus,

$$u\mapsto\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}\ \textit{is an equivalent norm on}\ \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

In our work, we need Leray-Schauder's Theorem of existence for approximate solutions.

THEOREM 2.3. *Let X be a Banach space and Ψ a compact operator of $X\times[0,1]$ in X such that*

$$\Psi(x,0)=0,\quad \forall x\in X.$$

Suppose there is a constant M such that

$$\forall(x,\theta)\in X\times[0,1]:(x=\Psi(x,\theta)\Longrightarrow\|x\|_X\le M).$$

Then, the operator Ψ_1 of X in itself given by, for all $x\in X$,

$$\Psi_1(x)=\Psi(x,1),$$

has a fixed point.

3. Statement of results and proofs

DEFINITION 3.1. *We say that u is a distributional solution of the problem* (1) *if $u \in W_0^{1,1}(\Omega)$, and it is such that for all $\varphi \in C_c^\infty(\Omega)$,*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i \varphi \, dx + \int_\Omega u \sum_{i=1}^{N} |u|^{p_i(x)-2} \varphi \, dx + \sum_{i=1}^{N} \int_\Omega \partial_i u |\partial_i u|^{p_i(x)-2} \varphi \, dx = \int_\Omega f(x) \varphi \, dx.$$

Our main result is the following.

THEOREM 3.2. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ be such that $\overline{p} < N$ and* (3) *holds, and assume that $f \in L^{(\overline{p}^*)'(\cdot)}(\Omega)$. Then the problem* (1) *has at least one distributional solution u in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

3.1. Existence of approximate solutions

Let (f_n) be a sequence of bounded functions defined in Ω which converges to f in $L^{(\overline{p}^*)'(\cdot)}(\Omega)$.

LEMMA 3.3. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ be such that $\overline{p} < N$ and* (3) *holds, and assume that $f \in L^{(\overline{p}^*)'(\cdot)}(\Omega)$. Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i \big(|\partial_i u_n|^{p_i(x)-2} \partial_i u_n \big) + u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} & \\ + \sum_{i=1}^{N} \partial_i u_n |\partial_i u_n|^{p_i(x)-2} = f_n, &\quad \text{in } \Omega, \\ u_n = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

in the sense that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n \partial_i \varphi \, dx + \int_\Omega u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} \varphi \, dx + \sum_{i=1}^{N} \int_\Omega \partial_i u_n |\partial_i u_n|^{p_i(x)-2} \varphi \, dx = \int_\Omega f_n \varphi \, dx, \tag{6}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Proof. For fixed $n \in \mathbb{N}^*$ and for all $(v, \delta) \in X \times [0,1]$ where $X = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we consider the operator

$$\begin{aligned} \Psi : X \times [0,1] &\to X \\ (v, \sigma) &\mapsto u = \Psi(v, \sigma), \end{aligned}$$

defined by

$$u = \Psi(v, \sigma) \Leftrightarrow u \text{ is the only weak solution of the problem (7)}$$

where,

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) \\ \qquad = \sigma \left(f_n - \sum_{i=1}^{N} \partial_i v |\partial_i v|^{p_i(x)-2} - v \sum_{i=1}^{N} |v|^{p_i(x)-2} \right) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

and this means that, u verify, $\forall \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the following weak formulation

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i \varphi \, dx \\ &\qquad = \sigma \int_\Omega \left(f_n - \sum_{i=1}^{N} \partial_i v |\partial_i v|^{p_i(x)-2} - v \sum_{i=1}^{N} |v|^{p_i(x)-2} \right) \varphi \, dx. \end{aligned} \tag{8}$$

Now, since v, $\partial_i v \in L^{p_i(\cdot)}(\Omega)$ — due $v \in X$ — we can get for all $(v, \sigma) \in X \times [0,1]$ that

$$\int_\Omega \left| v \sum_{i=1}^{N} |v|^{p_i(x)-2} \right|^{p_i'(x)} dx \le c \sum_{i=1}^{N} \int_\Omega |v|^{p_i(x)} \, dx \le c' \tag{9}$$

and

$$\int_\Omega \left| \sum_{i=1}^{N} \partial_i v |\partial_i v|^{p_i(x)-2} \right|^{p_i'(x)} dx \le C \sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)} \, dx \le C'. \tag{10}$$

Therefore, from (9) and (10), we obtain

$$v \sum_{i=1}^{N} |v|^{p_i(x)-2} \partial_i v, \text{ and } \sum_{i=1}^{N} \partial_i v |\partial_i v|^{p_i(x)-2} \text{ are in } \bigcap_{i=1}^{N} L^{p_i'(x)}(\Omega).$$

Applying the main theorem on monotone operators (see [26, 6, 7, 14]) then yields the existence of the weak solution u of the problem (7) in X. Its uniqueness is a direct consequence of the uniqueness for the homogeneous problem $(= 0)$ when assuming the existence of two weak solutions to (7) and using the independence of f from u.

- It's easy to verify that $\Psi(v, 0) = 0$ for all $v \in X$.

We'll now provide an estimate for the solution of (7). The following should be mentioned first: by using (3), Lemma 2.1, and (2), we obtain

$$\begin{aligned}\int_\Omega |v|^{p_i(x)}\,dx &\leq 1 + \|v\|_{p_i(\cdot)}^{p_i^+} \\ &\leq 2 + \|v\|_{p_i(\cdot)}^{p_+^+} \leq 2 + c\|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}. \qquad (11)\end{aligned}$$

By using (3), (4), and (2), we get

$$\begin{aligned}\sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i(x)}\,dx &\leq N + \sum_{i=1}^N \|\partial_i v\|_{p_i(\cdot)}^{p_i^+} \\ &\leq 2N + \sum_{i=1}^N \|\partial_i v\|_{p_i(\cdot)}^{p_+^+} \\ &\leq 2N + \left(\sum_{i=1}^N \|\partial_i v\|_{p_i(\cdot)}\right)^{p_+^+} = 2N + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}. \qquad (12)\end{aligned}$$

Now, taking $\varphi = u$ as test function in (8), and using (3), Hölder inequality, Young's inequality, the fact that $p_i(\cdot) \leq p_+(\cdot) \leq \overline{p}^*(\cdot)$ in $\overline{\Omega}$ — due (3) —, belonging u to $L^{p_i(\cdot)}(\Omega)$ (i.e. $\rho_{p_i(\cdot)}(u) < \infty$), (11), and (12), we obtain for any fixed choice of each of $\varepsilon > 0$, $\varepsilon' > 0$:

$$\begin{aligned}\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i(x)}\,dx &\leq \int_\Omega \left| f_n - \sum_{i=1}^N \partial_i v |\partial_i v|^{p_i(x)-2} - v\sum_{i=1}^N |v|^{p_i(x)-2}\right| |u|\,dx \\ &\leq \int_\Omega \left(|f_n||u| + \sum_{i=1}^N |u||\partial_i v|^{p_i(x)-1} + \sum_{i=1}^N |u||v|^{p_i(x)-1}\right) dx \\ &\leq 2\big\|f_n\big\|_{p_i'(\cdot)}\big\|u\big\|_{p_i(\cdot)} + \left(C(\varepsilon)\sum_{i=1}^N \int_\Omega |\partial_i v|^{p_i(x)}\,dx + \varepsilon \sum_{i=1}^N \int_\Omega |u|^{p_i(x)}\,dx\right) \\ &+ \left(C'(\varepsilon')\sum_{i=1}^N \int_\Omega |v|^{p_i(x)}\,dx + \varepsilon' \sum_{i=1}^N \int_\Omega |u|^{p_i(x)}\,dx\right) \\ &\leq c\big\|u\big\|_{\overrightarrow{p}(\cdot)} + c'\left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right) + c'' \\ &\leq c\big\|u\big\|_{\overrightarrow{p}(\cdot)} + c'''\left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right). \qquad (13)\end{aligned}$$

On the other hand, we have

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i(x)}\, dx \geq \sum_{i=1}^{N} \min\{\|\partial_i u\|_{p_i(x)}^{p_i^-}, \|\partial_i u\|_{p_i(x)}^{p_i^+}\}.$$

We define for all $i = 1, \ldots, N$; $\quad \xi_i = \begin{cases} p_+^+, & \text{if} \|\partial_i u\|_{p_i(\cdot)} < 1 \\ p_-^-, & \text{if} \|\partial_i u\|_{p_i(\cdot)} \geq 1 \end{cases}$, so that

$$\begin{aligned}
\sum_{i=1}^{N} \min\{\|\partial_i u\|_{p_i(\cdot)}^{p_i^-}, \|\partial_i u\|_{p_i(\cdot)}^{p_i^+}\} &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{\xi_i} \\
&\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{p_-^-} - \sum_{\{i, \xi_i = p_+^+\}} \left(\|\partial_i u\|_{p_i(.)}^{p_-^-} - \|\partial_i u\|_{p_i(\cdot)}^{p_+^+}\right) \\
&\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{p_-^-} - \sum_{\{i, \xi_i = p_+^+\}} \|\partial_i u\|_{p_i(\cdot)}^{p_-^-} \geq \Big(\frac{1}{N} \sum_{i=1}^{N} \|\partial_i u\|_{p_i(.)}\Big)^{p_-^-} - N.
\end{aligned}$$

Then, we get

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i(x)}\, dx \geq \left(\frac{1}{N} \|u\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-} - N. \tag{14}$$

From (13) and (14), we conclude

$$\|u\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \leq C \|u\|_{\overrightarrow{p}(\cdot)} + C' \left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right). \tag{15}$$

If $\|u\|_{\overrightarrow{p}(\cdot)} > 1$, from (15) we have

$$\|u\|_{\overrightarrow{p}}^{p_-^- - 1} \leq C + C' \left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right).$$

Then, there exists $c > 0$ independent of n, such that

$$\|u\|_{\overrightarrow{p}(\cdot)} \leq c \left(1 + \|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)^{\frac{1}{p_-^- - 1}}. \tag{16}$$

If $\|u\|_{\overrightarrow{p}(\cdot)} \leq 1$, we find that (16) is validated in this case only with consideration, for example $c \geq 1$ (The purpose is to combine the two cases $\|u\|_{\overrightarrow{p}(\cdot)} > 1$, and $\|u\|_{\overrightarrow{p}(\cdot)} \leq 1$ into same result (16)).

- We will now prove the continuity of Ψ.

Let's fix $n \in \mathbb{N}^*$, and let (v_m, σ_m) be a sequence of $X \times [0,1]$ converging to (v, σ) in this space. Then, we get

$$v_m \to v, \quad \text{strongly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{17}$$

$$\sigma_m \to \sigma, \quad \text{in } \mathbb{R}. \tag{18}$$

After considering the sequence (u_m) defined by $u_m = \Psi(v_m, \sigma_m)$, $m \in \mathbb{N}^*$, we obtain for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_m|^{p_i(x)-2} \partial_i u_m \partial_i \varphi \, dx = \sigma_m \Bigg(\int_\Omega f_n \varphi \, dx - \sum_{i=1}^{N} \int_\Omega \partial_i v_m |\partial_i v_m|^{p_i(x)-2} \varphi \, dx - \int_\Omega v_m \sum_{i=1}^{N} |v_m|^{p_i(x)-2} \varphi \, dx \Bigg). \tag{19}$$

For v, σ defined in (17), (18), we set $u = \Psi(v, \sigma)$ (i.e. u is the only weak solution of the problem (7) and this is according to the definition of Ψ), then we have for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u \partial_i \varphi \, dx = \sigma \Bigg(\int_\Omega f_n \varphi \, dx - \sum_{i=1}^{N} \int_\Omega \partial_i v |\partial_i v|^{p_i(x)-2} \varphi \, dx - \int_\Omega v \sum_{i=1}^{N} |v|^{p_i(x)-2} \varphi \, dx \Bigg). \tag{20}$$

By (16) and the boundedness of (v_m) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ (due (17)):

$$\left\| u_m \right\|_{\overrightarrow{p}(\cdot)} = \left\| \Psi(v_m, \sigma_m) \right\|_{\overrightarrow{p}(\cdot)} \le c \left(1 + \|v_m\|_{\overrightarrow{p}(x)}^{p_+^+} \right)^{\frac{1}{p_-^- - 1}} \le \varrho, \tag{21}$$

with $\varrho > 0$ independent of m. From (21) we conclude the boundedness of (u_m) in X.

So, there exists $w \in X$ and a subsequence (still denoted by (u_m)) such that

$$u_m \rightharpoonup w \quad \text{weakly in } X. \tag{22}$$

Let us now prove that,

$$\lim_{m \to +\infty} J_{i,m} = 0, \tag{23}$$

where, for $i = 1, \dots, N$,

$$J_{i,m} = \int_\Omega \left(|\partial_i u_m|^{p_i(x)-2} \partial_i u_m - |\partial_i w|^{p_i(x)-2} \partial_i w \right) (\partial_i u_m - \partial_i w) \, dx.$$

After choosing $\varphi = u_m - w$ in (19), we can obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega |\partial_i u_m|^{p_i(x)-2}\partial_i u_m(\partial_i u_m - \partial_i w)\,dx \\
&\qquad = \sigma_m\Big[\int_\Omega f_n(u_m - w) - \sum_{i=1}^{N}\int_\Omega \partial_i v_m|\partial_i v_m|^{p_i(x)-2}(u_m - w)\,dx \\
&\qquad\qquad - \sum_{i=1}^{N}\int_\Omega v_m|v_m|^{p_i(x)-2}(u_m - w)\,dx\Big].
\end{aligned}$$

From this, we can get

$$\begin{aligned}
\sum_{i=1}^{N} J_{i,m} = \sigma_m\Big[\int_\Omega f_n(u_m - w) &- \sum_{i=1}^{N}\int_\Omega \partial_i v_m|\partial_i v_m|^{p_i(x)-2}(u_m - w)\,dx \\
&- \sum_{i=1}^{N}\int_\Omega v_m|v_m|^{p_i(x)-2}(u_m - w)\,dx\Big] \\
&- \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i(x)-2}\partial_i u(\partial_i u_m - \partial_i w)\,dx.
\end{aligned} \tag{24}$$

Since f_n, $\partial_i v_m|\partial_i v_m|^{p_i(x)-2}$, and $v_m|v_m|^{p_i(x)-2}$ are bounded in $L^{p_i'(\cdot)}(\Omega)$, $u_m \to u$ strongly in $L^{r(\cdot)}(\Omega)$ where $r(\cdot)$ defined in Lemma 2.1, the boundedness of $|\partial_i u|^{p_i(x)-2}\partial_i u$ in $L^{p_i'(\cdot)}(\Omega)$, and (22), We find that the right side of the equality (24) go to 0 when $m \to +\infty$, then we obtain (23).

Now we put

$$\Omega_{i,1} = \{x \in \Omega, p_i(x) \geq 2\}, \text{ and } \Omega_{i,2} = \{x \in \Omega, p_i(x) \in (1,2)\}.$$

We recall the following well-known inequalities, that hold for any two real vectors ξ, η ($(\xi,\eta) \neq (0,0)$) and a real $p > 1$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq \begin{cases} 2^{2-p}|\xi - \eta|^p, & \text{if } p \geq 2, \\ (p-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, & \text{if } 1 < p < 2. \end{cases} \tag{25}$$

Therefore, if $p_i(\cdot) \geq 2$ in Ω, we get

$$\begin{aligned}
&2^{2-p_i^+}\int_{\Omega_{i,1}} |\partial_i(u_m - w)|^{p_i(x)}\,dx \\
&\leq \int_{\Omega_{i,1}} \big[|\partial_i u_m|^{p_i(x)-2}\partial_i u_m - |\partial_i w|^{p_i(x)-2}\partial_i u\big]\partial_i(u_m - w)\,dx \leq J_{i,m}.
\end{aligned} \tag{26}$$

If $1 < p_i(\cdot) < 2$ in Ω, we have

$$
\begin{aligned}
&\int_{\Omega_{i,2}} |\partial_i(u_m - w)|^{p_i(x)}\,dx \\
&\le \int_{\Omega_{i,2}} \frac{|\partial_i(u_m - w)|^{p_i(x)}}{(|\partial_i u_m| + |\partial_i w|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\,(|\partial_i u_m| + |\partial_i w|)^{\frac{p_i(x)(2-p_i(x))}{2}}\,dx \\
&\le 2\max\Bigg\{\bigg(\int_{\Omega_{i,2}} \frac{|\partial_i(u_m - w)|^2}{\big(|\partial_i u_m| + |\partial_i w|\big)^{2-p_i(x)}}\,dx\bigg)^{\frac{p_i^-}{2}}, \\
&\qquad\qquad \bigg(\int_{\Omega_{i,2}} \frac{|\partial_i(u_m - w)|^2}{(|\partial_i u_m| + |\partial_i w|)^{2-p_i(x)}}\,dx\bigg)^{\frac{p_i^+}{2}}\Bigg\} \\
&\quad\times\max\Bigg\{\bigg(\int_{\Omega} \big(|\partial_i u_m| + |\partial_i w|\big)^{p_i(x)}\,dx\bigg)^{\frac{2-p_i^+}{2}}, \\
&\qquad\qquad \bigg(\int_{\Omega} \big(|\partial_i u_m| + |\partial_i w|\big)^{p_i(x)}\,dx\bigg)^{\frac{2-p_i^-}{2}}\Bigg\} \\
&\le 2c\max\Big\{\Big(J_{i,m}\Big)^{\frac{p_i^-}{2}}, \Big(J_{i,m}\Big)^{\frac{p_i^+}{2}}\Big\} \times \bigg(1 + (\rho_{p_i}(|\partial_i u_m| + |\partial_i w|))^{\frac{2-p_-^-}{2}}\bigg).
\end{aligned} \tag{27}
$$

Since u_m, $w \in X$, and (23), after letting $m \to +\infty$ in (26) and in (27), we get

$$u_m \to w \ \text{ Strongly in } X. \tag{28}$$

So, we can pass to the limit in (19) as $m \to +\infty$, and using (28), we get for all $\varphi \in X$,

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} |\partial_i w|^{p_i(x)-2}\partial_i w \partial_i \varphi\,dx \\
&= \sigma\left(\int_{\Omega} f_n \varphi\,dx - \sum_{i=1}^{N}\int_{\Omega} \partial_i v|\partial_i v|^{p_i(x)-2}\varphi\,dx - \sum_{i=1}^{N}\int_{\Omega} v|v|^{p_i(x)-2}\varphi\,dx\right),
\end{aligned}
$$

and this implies that $w = \Psi(v, \sigma)$.

From the uniqueness of the weak solution of problem (7) then conclude that $w = u = \Psi(v, \sigma)$ where u defined in (20). So,

$$\Psi(v_m, \sigma_m) = u_m \to u = \Psi(v, \sigma) \quad \text{strongly in } X. \tag{29}$$

which shows the continuity of Ψ.

Compactness of Ψ: Let $\tilde{B}$ be a bounded of $X \times [0,1]$. Thus $\tilde{B} \subset B \times [0,1]$, with B a bounded of X, which can be assumed to be a ball of center O and of radius $r > 0$. For $u \in \Psi(\tilde{B})$, there exists $(v, \sigma) \in B \times [0,1]$ ($\|v\|_{\overrightarrow{p}(\cdot)} \leq r$), such that

$$u = \Psi(v, \sigma).$$

Thanks to (16), we can obtain

$$\|u\|_{\overrightarrow{p}(\cdot)} \leq c\left(1 + r^{p_+^+}\right)^{\frac{1}{p_-^- - 1}} = \rho.$$

This proves that Ψ applies $\tilde{B}$ in the closed ball of center O and radius ρ — ρ depend on r — in X.

Let then (u_m) be a sequence of elements of $\Psi(\tilde{B})$, therefore $u_m = \Psi(v_m, \delta_m)$ with $(v_m, \delta_m) \in \tilde{B}$. Since u_m remains in a bounded of X, it is possible to extract a subsequence (still denoted (u_m)) which converges weakly to an element $u = (v, \delta) \in \Psi(\tilde{B})$, such that $(v, \sigma) \in B \times [0,1]$ and B were previously defined. So, we can write

$$\Psi(v_m, \delta_m) = u_m \rightharpoonup u = \Psi(v, \delta) \quad \text{Weakly in } X.$$

Then, like getting (29), thanks to the continuity of Ψ, we can conclude that

$$\Psi(v_m, \delta_m) = u_m \to u = \Psi(v, \delta) \quad \text{Strongly in } X.$$

This proves that $\overline{\Psi(\tilde{B})}^X$ is compact. So Ψ is compact.

Now, let's prove that; $\exists M > 0$,

$$\forall (v, \sigma) \in X \times [0,1] : v = \Psi(v, \sigma) \Rightarrow \|v\|_X \leq M.$$

For that, we give the estimate of elements of X such that $v = \Psi(v, \sigma)$, then we have for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$,

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)-2} \partial_i v \partial_i \varphi \, dx \\ = \sigma \int_\Omega \left(f_n - \sum_{i=1}^{N} \partial_i v |\partial_i v|^{p_i(x)-2} - v \sum_{i=1}^{N} |v|^{p_i(x)-2} \right) \varphi \, dx. \quad (30)\end{aligned}$$

After choosing $\varphi = v$ as a test function in the weak formulation (30) and dropping the nonnegative term , and using Hölder inequality, Young's inequality, Lemma 2.1, the fact that $p_i(\cdot) \leq p_+(\cdot) \leq \overline{p}^*(\cdot)$ in $\overline{\Omega}$ — due (3) —, and the belonging of v, $\partial_i v$ to $L^{p_i(\cdot)}(\Omega)$ — due $v \in X$ —, we obtain for all $\varepsilon > 0$:

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)}\,dx \\
&\le 2\|f_n\|_{p_i'(\cdot)}\|v\|_{p_i(\cdot)} + C(\varepsilon)\sum_{i=1}^{N}\int_\Omega |v|^{p_i(x)}\,dx + \varepsilon \sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i(x)}\,dx \\
&\le c\|f_n\|_{p_i'(\cdot)}\|v\|_{\overrightarrow{p}(\cdot)} + C'(\varepsilon) + \varepsilon \sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i(x)}\,dx.
\end{aligned} \tag{31}$$

After choosing $\varepsilon = \frac{1}{2}$ in (31), we get

$$\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i(x)}\,dx \le c\|v\|_{\overrightarrow{p}(\cdot)} + c',$$

where, c and c' are two positive constants independent to n.

Then, with a method of proof similar to (16) we can get that, there exists $C > 0$ independent to n such that

$$\|v\|_{\overrightarrow{p}(\cdot)} \le C. \tag{32}$$

It then follows from Theorem 2.3 that the operator $\Psi_1 : X \to X$ defined by $\Psi_1(u) = \Psi(u,1)$ has a fixed point, which shows the existence of a solution of the approximated problems (5) in the sense of (6). $\square$

3.1.1. A priori estimates

Lemma 3.4. *Let f, p_i, $i = 1,\dots,N$ be restricted as in Theorem 3.2. Then*

$$(u_n) \text{ is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{33}$$

Proof. After choosing $\varphi = u_n$ as a test function in the weak formulation (6), then like the proof of (32) we can easily get (33). $\square$

Lemma 3.5. *There exists a subsequence (still denoted (u_n)) such that, for all $i = 1,\dots,N$*

$$\partial_i u_n \to \partial_i u \ \text{ a.e. in } \Omega. \tag{34}$$

Proof. From (33) the sequence (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. So, there exists a function $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega. \tag{35}$$

We consider the function

$$\Theta_n = \sum_{i=1}^{N} \int_\Omega \left(|\partial_i u_n|^{p_i(x)-2} \partial_i u_n - |\partial_i u|^{p_i(x)-2} \partial_i u \right) (\partial_i u_n - \partial_i u)\, dx,$$

and let's prove that,

$$\lim_{n\to+\infty} \Theta_n = 0. \tag{36}$$

We can write Θ_n in the following form

$$\begin{aligned}\Theta_n = &\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx \\ &- \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx = I_n - J_n,\end{aligned}$$

where

$$\begin{aligned} I_n &= \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx, \\ J_n &= \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx. \end{aligned}$$

After choosing $\varphi = u_n - u$ in (6), with the use of (35), and boundedness of both $(u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2})$ and $(\sum_{i=1}^{N} \partial_i u_n |\partial_i u_n|^{p_i(x)-2})$ in $L^{p_i'(\cdot)}$ — $p_i'(\cdot)$ is the Sobolev conjugate of $p_i(\cdot)$ —, we can obtain

$$\lim_{n\to+\infty} I_n = 0. \tag{37}$$

Since $(\partial_i u_n)$ is bounded in $L^{p_i(\cdot)}$ (due (33)), then there exists a function $w \in L^{p_i(\cdot)}$ and a subsequence (still denoted by $(\partial_i u_n)$) such that

$$\partial_i u_n \rightharpoonup w \quad \text{weakly in } L^{p_i(x)}. \tag{38}$$

Through (38) and the boundedness of $|\partial_i u_n|^{p_i(x)-2} \partial_i u_n$ in $L^{p_i'(x)}$ we conclude that

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - w)\, dx = 0. \tag{39}$$

by combining (37) and (39), we get

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u - w)\, dx = 0. \tag{40}$$

Then, (40) implies that $w = \partial_i u$. Therefore,

$$\partial_i u_n \rightharpoonup \partial_i u \quad \text{weakly in } L^{p_i(x)}. \tag{41}$$

From (41) and the boundedness of $|\partial_i u|^{p_i(x)-2}\partial_i u$ in $L^{p_i'(x)}$ we conclude that

$$\lim_{n\to+\infty} J_n = 0. \tag{42}$$

From (37) and (42) we obtain (36). Through (25) we conclude that, for all $i = 1, \dots, N$

$$T_{i,n} > 0, \tag{43}$$

where

$$T_{i,n} = \left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n - |\partial_i u|^{p_i(x)-2}\partial_i u\right)(\partial_i u_n - \partial_i u).$$

Then, (43) and (36) gives us, for all $i = 1, \dots, N$

$$T_{i,n} \to 0, \quad \text{strongly in } L^1(\Omega).$$

Extracting a subsequence, still denoted by (u_n), we have for all $i = 1, \dots, N$

$$T_{i,n} \to 0 \quad \text{a.e. in } \Omega. \tag{44}$$

Then there exists a subset $\Omega_0 \subset \Omega$ with $|\Omega_0| = 0$, such that, for $x \in \Omega\backslash\Omega_0$,

$$|\partial_i u(x)| < \infty \text{and, } T_{i,n} \to 0$$

From (44), we have $T_{i,n} \leq h(x)$ for some function h. Let us prove that there exists a function g such that

$$\left|\partial_i u_n(x)\right| \leq g(x). \tag{45}$$

By inequality (25), we obtain

$$h(x) \geq \begin{cases} c\left((|\partial_i u_n| - |\partial_i u|)^{p_-^-} - 1\right), & \text{if } p_i(x) \geq 2 \\ c'\left(\frac{|\partial_i u_n| - |\partial_i u|}{1+|\partial_i u_n|+|\partial_i u|}\right)^2, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

and this implies (45). We proceed by contradiction to prove that

$$\partial_i u_n(x) \to \partial_i u(x) \text{ in } \Omega\backslash\Omega_0.$$

Assume that there exists $x_0 \in \Omega\backslash\Omega_0$ such that $\partial_i u_n(x_0)$ does not converge to $\partial_i u(x_0)$. By using Theorem of Bolzano Weierstrass we obtain that, $\partial_i u_n(x_0) \to b$, $b \in \mathbb{R}$, up to a subsequence. Passing to the limit in

$$\left((|\partial_i u_n(x_0)|^{p_i(x_0)-2}\partial_i u_n(x_0) - |\partial_i u x_0)|^{p_i(x_0)-2}\partial_i u(x_0)\right)(\partial_i u_n(x_0) - \partial_i u(x_0)),$$

we get

$$\left(|b|^{p_i(x_0)-2}b - |\partial_i u(x_0)|^{p_i(x_0)-2}\partial_i u(x_0)\right)(b - \partial_i u(x_0)) = 0,$$

and by using (25), we conclude that $b = \partial_i u(x_0)$. This leads to (34). $\square$

3.2. Proof of the Theorem 3.2 :

From (34), we obtain , for all $i = 1, \dots, N$

$$|\partial_i u_n|^{p_i(x)-2}\partial_i u_n \to |\partial_i u|^{p_i(x)-2}\partial_i u \quad \text{a.e. in } \Omega. \tag{46}$$

By (33) we can get, for all $i = 1, \dots, N$

$$\int_\Omega ||\partial_i u_n|^{p_i(x)-2}\partial_i u_n|^{p_i'(x)}\, dx = \int_\Omega |\partial_i u_n|^{p_i(x)}\, dx \le c, \quad p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot)-1}. \tag{47}$$

Then, (47) implies that, for all $i = 1, \dots, N$

$$\Big(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\Big) \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega). \tag{48}$$

By Young's inequality and since $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we get for all $\varepsilon > 0$

$$\begin{aligned}\int_\Omega \big||\partial_i u_n|^{p_i(x)-2}\partial_i u_n\big|\, dx &= \int_\Omega |\partial_i u_n|^{p_i(x)-1}\, dx \\ &\le C(\varepsilon) + \varepsilon \int_\Omega |\partial_i u_n|^{p_i(x)}\, dx \\ &\le C(\varepsilon) + \varepsilon c = C'(\varepsilon).\end{aligned}$$

Then, for any fixed choice for ε, we conclude that, for all $i = 1, \dots, N$

$$\Big(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\Big) \in L^1(\Omega). \tag{49}$$

So, by (49), (46), (48), and Vitali's theorem, we derive, for all $i = 1, \dots, N$

$$|\partial_i u_n|^{p_i(x)-2}\partial_i u_n \to |\partial_i u|^{p_i(x)-2}\partial_i u \text{ strongly in } L^1(\Omega).$$

Now, from (35), we conclude that

$$u_n|u_n|^{p_i(x)-2} \to u|u|^{p_i(x)-2} \quad \text{a.e. in } \Omega. \tag{50}$$

On the other hand, since $u_n \in L^{p_i(\cdot)}(\Omega)$, we obtain for all $i = 1, \dots, N$

$$\int_\Omega \big|u_n|u_n|^{p_i(x)-2}\big|^{p_i'(x)}\, dx = \int_\Omega |u_n|^{p_i(x)} \le C. \tag{51}$$

Then, (51) implies that, for all $i = 1, \dots, N$

$$u_n|u_n|^{p_i(x)-2} \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega). \tag{52}$$

Like the proof of (49) with using that $u_n \in L^{p_i(\cdot)}(\Omega)$, we can obtain for all $i = 1, \dots, N$

$$\Big(u_n|u_n|^{p_i(x)-2}\Big) \in L^1(\Omega). \tag{53}$$

So, by (53), (50), (52), and Vitali's theorem, we derive, for all $i = 1, \dots, N$

$$u_n|u_n|^{p_i(x)-2} \to u|u|^{p_i(x)-2} \text{ strongly in } L^1(\Omega).$$

So, we can easily pass to the limit in (6). Thus, Theorem 3.2 was proven.

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