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A priori estimates for convective quasilinear equations and systems

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Dedicated to Enzo Mitidieri on the occasion of his 70th birthday

Abstract. *The paper concerns universal a priori estimates for positive solutions to a large class of elliptic quasilinear equations and systems involving the p-Laplacian operator on arbitrary domains of* $\mathbb{R}^N$ *and a convective term in the reaction. Our main theorems, new even for the Laplacian operator, extend previous estimates by Poláčik, Quitter and Souplet in [38] to very general nonlinearities admitting solely a lower bound, yielding a curious dichotomy. The main ingredients are a key doubling property, a rescaling argument, different from the classical blow-up technique of Gidas and Spruck, and Liouville-type theorems for inequalities. A discussion on the sharpness of the exponent in the power type term is also included.*

Keywords: elliptic problems, quasilinear equations and systems, a priori estimates.
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1. Introduction

In this paper, we focus on deriving universal pointwise a priori estimates for C^1 classical nonnegative solutions to elliptic quasilinear problems of the form

$$-\Delta_p u = f(x, u, Du) \quad \text{in } \Omega, \tag{1}$$

and

$$\begin{cases} -\Delta_{p_1} u = f_1(x, u, v, Du, Dv) & \text{in } \Omega, \\ -\Delta_{p_2} v = f_2(x, u, v, Du, Dv) & \text{in } \Omega, \end{cases} \tag{2}$$

where $\Omega \subseteq \mathbb{R}^N$ is an arbitrary domain, $1 < p, p_1, p_2 < N$ and f, f_1, f_2 are nonnegative continuous functions defined in $\Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N$, with f satisfying

$$f(x, t, \xi) \geq \ell(x) t^q |\xi|^\theta \tag{3}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^N$ and t sufficiently large, where ℓ is a nontrivial nonnegative continuous function and $q, \theta > 0$, while the assumptions of f_1, f_2 will become clear during the paper.

The word *universal* is taken from [41], where Serrin and Zou mean that the bounds obtained are not only independent of the solutions but no boundary conditions of any type are assumed on the solutions.

Our model problem can be seen as a generalization of the viscous stationary case of the Hamilton-Jacobi equation, whose model is given by

$$\begin{cases} u_t - \Delta u = |Du|^\theta, & t > 0, \quad x \in \Omega, \\ u(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $\theta > 1$. Problem (4) is related to the Kardar-Parisi-Zhang equation in the theory of growth and roughening of surfaces (see [5] for details and references). The equation in (4) is one of the simplest examples of parabolic PDEs with nonlinearity depending on the first-order spatial derivates, and in some sense, its stationary case can be seen as the analogous of the celebrated Lane-Emden equation $-\Delta u = u^q$, $q > 1$, $u > 0$. For results in this direction, we mention [1, 29, 43] and the references therein.

A priori estimates can generally be categorized as uniform estimates, integral estimates, or pointwise estimates.

Uniform a priori estimates for the Lane-Emden equation in $\mathbb{R}^N$ were first established by Gidas and Spruck in [27, 28], using the well-know blow-up technique. This method is essential for proving existence results for associated Dirichlet problems within bounded domains. For systems, relevant results can be found in the work of Clément, de Figueiredo, and Mitidieri in [12]. The approach involves showing that the failure of a uniform estimate leads to a non-trivial solution of a limit problem in $\mathbb{R}^N$ or in the half-space $\mathbb{R}^N_+$. From there, an appropriate Liouville-type theorem justifies the claim. The challenge with this technique in quasilinear cases arises from the limited availability of Liouville-type theorems in half-spaces. Consequently, some papers, such as [2, 13, 40], impose geometric constraints on the domain to avoid the half-space scenario.

Integral a priori estimates were developed by Mitidieri and Pohozaev for inequalities to prove nonexistence results through their nonlinear capacity method, detailed in [35] and [36]. These results apply to both coercive and noncoercive systems of inequalities. Solutions in this context belong solely to local Sobolev spaces, without reliance on a maximum principle or assumptions regarding behavior at infinity. The derivation of such estimates hinges on the application of appropriate test functions. By optimally selecting a test function, one is led to a nonlinear minimax problem, which in turn produces a nonlinear capacity linked to the associated nonlinear problem. The Liouville property (nonexistence) is proven by analyzing the asymptotic behavior of this capacity concerning a particular parameter. As noted in [37], there is no established regularity theory for differential inequalities, marking a significant distinction from differential equations. The determination of an appropriate solution class for differential inequalities becomes critical, as there are instances where solutions exist within one class of functions but not in others. More general a priori bounds for solutions of a broad class of quasilinear degenerate elliptic inequalities have been proved in [15]. As an outcome, the authors deduce sharp Liouville theorems for inequalities associated with elliptic operators, such as p-Laplacian, the mean curvature and the generalized mean curvature operator, even in general settings as Carnot groups as well as results on the sign of solutions for quasilinear coercive/noncoercive inequalities. In this direction, we refer also to [14, 16, 17, 18, 22].

Finally, pointwise a priori estimates of the form

$$u(x) + |Du(x)|^{\gamma_1} \leq C(1 + \mathrm{dist}^{-\gamma_2}(x, \partial\Omega)), \qquad x \in \Omega, \tag{5}$$

with γ_1, $\gamma_2 > 0$ depending on the parameters of the differential equation under consideration, are obtained in [9] by the direct Bernstein method, in [41] by a Harnack-type theorem and in [38] with the use of *doubling lemma*. This latter key lemma is used in order roughly to avoid again the half-space case for the limit problem to which a Liouville-type theorem is applied to reach the conclusion. On the other hand, in [41] it is emphasized that roughly Liouville theorems can be seen as a consequence and a limiting case of pointwise a priori estimates, cfr. [19], since $\mathrm{dist}(x, \partial\Omega)$ can be chosen arbitrarily large when solutions are considered in the entire $\mathbb{R}^N$. As observed in [38], even if blow-up technique and rescaling procedure via doubling lemma are in some sense similar, there is a key difference: in the first case u is smooth up to the boundary, while in the second case, it is possible to rescale directly about a sequence of points of global maxima, the size of the solution is automatically dominated around.

In [3], by using doubling lemma, local estimates proved in [38] are extended to the following prototype

$$-\Delta_p u = a(x)\, u^q - b(x) u^s |Du|^\theta \quad \text{in } \Omega,$$

with $1 < p < N$, $0 \leq s < q$, $0 < \theta < p$, q subcritical with respect to Sobolev's critical exponent and a, b continuous functions, which describes, for instance, the evolution of the population density of a biological species, under the effect of predation. Then, in [4], existence results for a quasilinear convective Dirichlet problem in a bounded domain are obtained from estimates of the type (5) combined with a blow-up technique and a fixed point theorem.

In summary, a priori estimates are strictly connected to Liouville-type results.

The purpose of the present paper is to prove a priori estimates of the form (5), first to positive solutions to problem (1). To this aim, we recall some well-known results on Liouville-type theorems for the p-Laplacian. We start with the result by Mitidieri and Pohozaev in [35] where it is proved that the elliptic inequality $-\Delta_p u \geq u^q$ in $\mathbb{R}^N$ admits no positive solutions if $1 < q \leq p_* - 1$, where $p_* = p(N-1)/(N-p)$ is the so-called Serrin's critical exponent. Later, Serrin and Zou in [41] generalize the result by Gidas and Spruck to the equality $-\Delta_p u = u^q$ in $\mathbb{R}^N$, obtaining that no positive solutions exist if $1 < q < p^* - 1$, with $p^* = Np/(N-p)(> p_*)$. These results are sharp. For a detailed description of the role of the two critical exponents, also in connection with the behavior of singular solutions near an isolated singularity of $-\Delta u = |u|^{q-1}u$ in the punctured ball, see [26].

For the Lane-Emden system of equations:

$$\begin{cases} -\Delta u = v^p & \text{in } \mathbb{R}^N, \\ -\Delta v = u^q & \text{in } \mathbb{R}^N, \end{cases} \tag{6}$$

the situation is more nuanced. The Lane-Emden conjecture asserts that there are no positive classical solutions to (6) if and only if the inequality holds

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2.$$

However, this conjecture has not been completely resolved. Mitidieri settled the radial case in [33]. While Souplet in [38] and [42], respectively, solved the Lane-Emden conjecture in dimensions 3 and 4 in the non radial case. For the inequality version of system (6), Mitidieri determined the dividing curve for existence and nonexistence in [34]. More precisely, in [34] it is proved that nonexistence occurs if and only if

$$\max\left\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right\} \geq \frac{N-2}{2},$$

with $pq > 1$. For quasilinear versions of systems of inequalities of type (6), we refer to Mitidieri and Pohozaev in [37] as well as [22].

For nonlinearities dependent on the gradient, Lions, using the Bernstein technique, proved that any C^2 solution to the elliptic Hamilton-Jacobi equation in $\mathbb{R}^N$

$$\Delta u = |Du|^\theta$$

must be constant for $\theta > 1$, establishing a Liouville-type result [31]. Later Bidaut-Véron, Garcia-Huidobro and Véron, showed that for any C^1 solution to the quasilinear version of the Hamilton-Jacobi equation in an arbitrary domain $\Omega \subset \mathbb{R}^N$

$$\Delta_p u = |Du|^\theta \quad \text{in} \quad \Omega,$$

with $1 < p \le N$ and $\theta > p-1$, the following estimate holds

$$|Du(x)| \le C \text{dist}^{1/(\theta-p+1)}(x, \partial\Omega).$$

As a consequence, a Liouville-type result holds when $\Omega = \mathbb{R}^N$, see [7]. This result is in the same spirit as the work of Dancer [19] and also relates to the findings in [41]. A further generalization involving a reaction that depends not only on the gradient, but also on u, specifically:

$$-\Delta_p u = u^q |Du|^\theta \quad \text{in} \quad \mathbb{R}^N, \tag{7}$$

was introduced in its radial form for $p = 2$ in [10]. In Theorem 7.4 of that work, a sharp non existence result was established, and an explicit solution was constructed in the critical case. On the other hand, Bidaut-Véron considered the quasilinear case in [6] by showing that when $1 < p < N$, $q \ge 0$ and $\theta \ge p$, any positive C^1 solution to (7) must be constant, generalizing a previous work by Filippucci, Pucci and Souplet in [25], for the case $p = 2$, which assumed boundedness of the solution. For the case where $\theta < p$, Liouville-type results are only known for certain subregions. Mitidieri and Pohozaev proved in [37] that nonconstant, nonnegative supersolutions of equation (7) can only exist if $q > 0$ and

$$q > p_* - 1 - \frac{\theta(N-1)}{N-p}. \tag{8}$$

For the sharpness to this lower bound, we refer to Section 4. We quote [9] for pointwise a priori estimates of the gradient of solutions to (7) with $p = 2$ in arbitrary domains Ω of $\mathbb{R}^N$ by Bernstein method under further restrictions on q, θ, but in the same *supercritical range* (8), given in this case by $(N-2)q + (N-1)\theta > N$, see Theorem B in [9]. Passing to (7), pointwise gradient estimates for signed solutions are proved in [8], while in [6] the author improves the results in [9] for $q \le 0$.

The main results of the present paper deal with pointwise a priori estimates in the scalar case (1) and in the vectorial case (2). As a corollary of our first main result, we obtain the following.

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain and $1 < p < N$. Let $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ be a continuous function, and there exists $C_1 > 0$ such that*

$$0 \le f(x,t,\xi) \le C_1\big(1 + t^\sigma + t^q |\xi|^\theta\big) \tag{9}$$

for all $x \in \Omega$ and for $t \geq 0$ and $\xi \in \mathbb{R}^N$, with

$$0 < \sigma < \frac{pq + \theta}{p - \theta}, \tag{10}$$

$$q + \theta > p - 1, \tag{11}$$

$$0 \leq \theta < p - \frac{N - p}{N - 1}, \qquad 0 < q \leq p_* - 1 - \frac{\theta(N-1)}{N - p}. \tag{12}$$

Moreover, assume (3) *with ℓ a positive constant.*

Then, only one of the two possibilities is in force: either

(I) there exists $C = C(N, p, q, \theta) > 0$ (independent of Ω and u) such that for any nonnegative nonconstant solution u of (1) *there holds*

$$u + |Du|^{\gamma/(\gamma+1)} \leq C(1 + dist^{-\gamma}(x, \partial\Omega)), \quad x \in \Omega, \tag{13}$$

with

$$\gamma = \frac{p - \theta}{q + \theta - p + 1}. \tag{14}$$

In particular, if $\Omega = B_R \backslash \{0\}$ for some $R > 0$, then

$$u + |Du|^{\gamma/(\gamma+1)} \leq C(1 + |x|^{-\gamma}), \quad 0 < x \leq \frac{R}{2}. \tag{15}$$

or

(II) there exist a sequence of domains $(\Omega_k)_k \subset \Omega$, of points $(x_k)_k \in \Omega_k$ and of nonnegative solutions $(u_k)_k$ in Ω_k of (1) *such that*

$$u_k(x_k) \to \infty, \qquad \frac{|Du_k(x_k)|^{\gamma/(\gamma+1)}}{u_k(x_k)}, \ \frac{d^{-\gamma}(x_k, \partial\Omega_k)}{u_k(x_k)} \to 0 \tag{16}$$

as $k \to \infty$.

We point out that condition $(12)_2$ with $p = 2$ and with the strict sign in the inequality is equivalent to

$$(N-2)q + (N-1)\theta < N$$

which is the so-called *subcritical range* in [9], giving that $\theta < N/(N-1)(< 2)$, namely $(12)_1$ with $p = 2$.

Differently from [3], Corollary 1.1 exhibits a curious dichotomy, indeed in addition to the pointwise a priori estimate (13), it also gives the alternative (II), because of the very general assumptions on the nonlinearity, which does not fall in the setting of [38] or [3], where the equation treated admits only $u = 0$ as constant solution.

It would be interesting to understand if condition (II) could be removed by using a different technique. Here the proof technique used brings out many difficulties due to the presence of the gradient term, and even the management of the high number of parameters involved is quite demanding. In particular, the proof proceeds by contradiction, the failure of (13) gives a sequence of solutions $(u_k)_k$ increasingly large whose growth can be controlled in a suitable neighbourhood. Now an appropriate rescaling produces a bounded positive solution of a limit problem in $\mathbb{R}^N$. A Liouville-type theorem for inequalities gives the required contradiction. One of the main difficulties lies on the fact that any possible solution to the limit problem must be constant, but not necessarily trivial, see [21, 22]. For general results for elliptic equations with gradient terms, we mention [20]. This is the reason why we have an alternative to the a priori estimates depicted by a particular sequence of solutions blowing up accordingly to (16).

Section 3 is devoted to two different generalizations of Corollary 1.1 by requiring either a possible dependence on x on the nonlinearity f, cfr. Theorem 3.1, or replacing the lower bound of f with an asymptotical behaviour of f, cfr. Theorem 3.2.

The last part of the paper is dedicated to the vectorial case, namely, we give some a priori estimates for nonnegative solutions (u, v), i.e. $u \geq 0$ and $v \geq 0$, of elliptic systems of the form (2) with f_1, f_2 satisfying suitable assumptions which guarantee a lower bound only for either u or v large, as it will be clear in the next statement. Our results, which are new even in the Laplacian case, extend previous ones in [38], where Lane-Emden type systems are considered. For simplicity, we now state only a corollary relative to the problem involving the pure Laplacian operator, namely

$$\begin{cases} -\Delta u = f_1(x, u, v, Du, Dv) & \text{in } \Omega, \\ -\Delta v = f_2(x, u, v, Du, Dv) & \text{in } \Omega. \end{cases} \tag{17}$$

Corollary 1.2. *Let Ω be an arbitrary domain of $\mathbb{R}^N$, $N \geq 3$, $f_1, f_2 : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ be continuous functions and there exist $C_1', C_2' > 0$ such that*

$$0 \leq f_1(x, s, t, \eta, \xi) \leq C_1'(1 + t^{\sigma_1} + s^{\tau_1} + t^{q_1}|\xi|^{\theta_1}), \tag{18}$$

and

$$0 \leq f_2(x, s, t, \eta, \xi) \leq C_2'(1 + s^{\sigma_2} + t^{\tau_2} + s^{q_2}|\eta|^{\theta_2}), \tag{19}$$

for all $x \in \Omega$ and for u, $v \geq 0$ and η, $\xi \in \mathbb{R}^N$, with

$$\theta_1, \theta_2 < 2, \qquad 1 < q_2 + \theta_2, \qquad 1 < q_1 + \theta_1,$$

and

$$\sigma_1 < \frac{\alpha'+2}{\beta'}, \qquad \sigma_2 < \frac{\beta'+2}{\alpha'}, \qquad \tau_1 < \frac{\alpha'+2}{\alpha'}, \qquad \tau_2 < \frac{\beta'+2}{\beta'},$$

where

$$\alpha' = \frac{(2-\theta_1)+(q_1+\theta_1)(2-\theta_2)}{(q_1+\theta_1)(q_2+\theta_2)-1}, \quad \beta' = \frac{(2-\theta_2)+(q_2+\theta_2)(2-\theta_1)}{(q_1+\theta_1)(q_2+\theta_2)-1}.$$

Moreover, there exist $\overline{C}_1', \overline{C}_2' > 0$ *such that*

$$f_1(x,s,t,\eta,\xi) \geq \overline{C}_1' t^{q_1}|\xi|^{\theta_1}, \quad x \in \Omega,\ t \text{ large},\ s \geq 0,\ \xi,\eta \in \mathbb{R}^N$$

$$f_2(x,s,t,\eta,\xi) \geq \overline{C}_2' s^{q_2}|\eta|^{\theta_2}, \quad x \in \Omega,\ s \text{ large},\ t \geq 0,\ \xi,\eta \in \mathbb{R}^N.$$

Then, if

$$\max\left\{\frac{(2-\theta_1)+(q_1+\theta_1)(2-\theta_2)}{(q_1+\theta_1)(q_2+\theta_2)-1}, \frac{(2-\theta_2)+(q_2+\theta_2)(2-\theta_1)}{(q_1+\theta_1)(q_2+\theta_2)-1}\right\} \geq N-2,$$

only one of the two following possibilities is in force: either

(I)′ there exist $C = C(N, q_1, q_2, \theta_1, \theta_2) > 0$ *(independent of* Ω *and* u*) such that for any nonnegative solution* (u,v) *of* (17) *the following estimates holds*

$$u + |Du|^{\alpha'/(\alpha'+1)} \leq C(1 + dist^{-\alpha'}(x,\partial\Omega)), \quad x \in \Omega, \tag{20}$$

$$v + |Dv|^{\beta'/(\beta'+1)} \leq C(1 + dist^{-\beta'}(x,\partial\Omega)), \quad x \in \Omega. \tag{21}$$

In particular, if $\Omega = B_R\backslash\{0\}$ *for some* $R > 0$*, then for* $x \in (0, R/2]$

$$u + |Du|^{\alpha'/(\alpha'+1)} \leq C(1 + |x|^{-\alpha'}), \quad v + |Dv|^{\beta'/(\beta'+1)} \leq C(1 + |x|^{-\beta'}),$$

or

(II)′ there exist a sequence of domains $(\Omega_k)_k \subset \Omega$*, of points* $(x_k)_k \in \Omega_k$ *and of nonnegative solutions* $(u_k, v_k)_k$ *in* Ω_k *of* (17) *such that at least one of the following conditions holds*

$$u_k(x_k) \to \infty, \qquad \frac{|Du_k(x_k)|^{\alpha'/(\alpha'+1)}}{u_k(x_k)}, \frac{d^{-\alpha'}(x_k,\partial\Omega_k)}{u_k(x_k)} \to 0 \tag{22}$$

$$v_k(x_k) \to \infty, \qquad \frac{|Dv_k(x_k)|^{\beta'/(\beta'+1)}}{v_k(x_k)}, \frac{d^{-\beta'}(x_k,\partial\Omega_k)}{v_k(x_k)} \to 0 \tag{23}$$

as $k \to \infty$*.*

The technique used to prove Corollary 1.2 is similar to that of Corollary 1.1, with the use of a Liouville-type result for a system of elliptic inequalities. Here additional difficulties appear due to the vectorial case and the fact that in general maximum principle does not hold for systems. Furthermore, we emphasize that the lower bounds for the nonlinearities given in Corollary 1.2 actually are allowed to depend only on two components. Indeed, we do not manage to deal with nonlinearities satisfying $f_1(x,s,t,\eta,\xi) \geq s^{\varrho_1}t^{q_1}|\eta|^{\sigma_1}|\xi|^{\theta_1}$, $f_2(x,s,t,\eta,\xi) \geq s^{q_2}t^{\varrho_2}|\eta|^{\theta_2}|\xi|^{\sigma_2}$ since, as far as we know, Liouville-type results are not available in a such general setting, even we are not able to handle the multipower case $f_1 \geq g_1 := s^{\varrho_1}t^{q_1}|\xi|^{\theta_1}$ and $f_2 \geq g_2 := s^{q_1}t^{\varrho_2}|\eta|^{\theta_2}$, for which a Liouville-type result is proved in [23], but technical difficulties arise due essentially to the fact that $g_1(0,t,\xi) = 0$, as well as $g_2(s,0,\eta) = 0$.

The present paper is organized as follows. Section 2, contains some preliminary results, such as doubling lemma and Liouville-type results for elliptic inequalities with gradient terms, both in the scalar and in the vectorial case. The main theorem of the paper in the scalar case is given and proved in Section 3, which also contains the proof of Corollary 1.1. Moreover, the sharpness of the assumption (12) with respect to q is contained in Section 4. Finally, a priori estimates in the vectorial case are obtained in Section 5 together with the proof of Corollary 1.2.

2. Preliminaries

We start this section by recalling the doubling lemma, due to Poláčik, Quitter and Souplet in [38], a key tool in the proof of the main results of the paper.

LEMMA 2.1. (*Theorem* 5.1, *[38]*) *Let (X, d) be a complete metric space, and let $\emptyset \neq D \subset \Sigma \subset X$ with Σ closed. Set $\Gamma = \Sigma \backslash D$. Finally, let $M : D \to (0,\infty)$ be bounded on compact subsets of D, and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y) > 2k\, dist^{-1}(y,\Gamma),$$

then there exist $x \in D$ such that

$$M(x) > 2k\, dist^{-1}(x,\Gamma), \qquad M(x) \geq M(y), \tag{24}$$

and

$$M(z) \leq 2M(x) \;\; for \;\; all \;\; z \in D \cup \overline{B}_X(x, kM^{-1}(x)).$$

REMARK 2.2. In the Euclidean subcase $X = \mathbb{R}^N$ with Ω an open subset of $\mathbb{R}^N$, put $D = \Omega$, $\Sigma = \overline{D}$, $\Gamma = \partial\Omega$. Then we have $\overline{B}(x, kM^{-1}(x)) \subset D$. Indeed, since D is open, (24) implies that

$$\mathrm{dist}(x, D^c) = \mathrm{dist}(x,\Gamma) > 2kM^{-1}(x).$$

A second key ingredient for the proof of the principal theorems in our paper is given by Liouville-type results for elliptic inequalities involving gradient terms.

THEOREM 2.3. *Let $1 < p < N$, $\theta < p$, $c_0, R_0 \in \mathbb{R}^+$ and $\zeta \in \mathbb{R}$ such that*

$$a(x) \geq c_0|x|^{-\zeta} \quad \text{for all } x \text{ with } |x| \geq R_0,$$

with a a nonnegative misurable function. Then, any nonnegative solution u of the inequality

$$-\Delta_p u \geq a(x)u^q|Du|^\theta \quad \text{in } \mathbb{R}^N, \tag{25}$$

with $a(x)|Du|^p, a(x)u^q|Dv|^\theta \in L^1_{loc}(\mathbb{R}^N)$, is necessarily constant in $\mathbb{R}^N$, if

$$q + \theta > p - 1, \quad \zeta < p - \theta,$$

and (12) *is in force.*

Actually, Theorem 2.3 was first proved by Mitidieri and Pohozaev in [37] for solutions belonging to a local space depending on a parameter, cfr. Theorem 15.1, where $a(x) = 1$ and the strict inequality in (12) holds. Later, in [21] it was included the weight $a(x)$ and the equality sign in (12), but with the stronger condition $\theta < p - 1$, removed in [22]. As emphasized in the Introduction of Part I in [37], the nonlinear capacity procedure, developed by Mitidieri and Pohozaev, for problems with linear principal part can be applied with no assumption on the sign of solutions; while the same procedure cannot be applied to quasilinear problems without assuming that solutions are nonnegative.

Liouville theorems for p-Laplacian equations with gradient terms for coercive problems, involving Keller-Osserman condition, have been studied first by Martio and Porru in [32], then also by Filippucci, Pucci and Rigoli in [24] and by Farina and Serrin in [20], where solutions of any sign are considered.

Finally, a priori estimates and Liouville-type results in the superlinear case $\theta \geq p$ are discussed in the Introduction.

We conclude this section with a Liouville-type result for systems of inequalities, see [22], which will be crucial in Section 5.

THEOREM 2.4. *Let $p_1, p_2 > 1$ then any nonnegative entire solution (u, v) of system*

$$\begin{cases} -\Delta_{p_1} u \geq v^{q_1}|Dv|^{\theta_1} & \text{in } \mathbb{R}^N, \\ -\Delta_{p_2} v \geq u^{q_2}|Du|^{\theta_2} & \text{in } \mathbb{R}^N, \end{cases} \tag{26}$$

belonging to

$$\{u : \mathbb{R}^N \to \mathbb{R}^+_0 : |Du|^{p_1},\ u^{q_2}|Du|^{\theta_2} \in L^1_{loc}(\mathbb{R}^N)\} \\ \times \{v : \mathbb{R}^N \to \mathbb{R}^+_0 : |Dv|^{p_2},\ v^{q_1}|Dv|^{\theta_1} \in L^1_{loc}(\mathbb{R}^N)\},$$

is necessarily constant in $\mathbb{R}^N$*, provided that*

$$N > \max\{p_1, p_2\},$$

$$p_1 - 1 < q_2 + \theta_2, \qquad p_2 - 1 < q_1 + \theta_1, \tag{27}$$

and

$$\max\left\{ \frac{(p_2-1)(p_1-\theta_1) + (q_1+\theta_1)(p_2-\theta_2)}{(q_1+\theta_1)(q_2+\theta_2) - (p_1-1)(p_2-1)} - \frac{N-p_1}{p_1-1}, \right.$$
$$\left. \frac{(p_1-1)(p_2-\theta_2) + (q_2+\theta_2)(p_1-\theta_1)}{(q_1+\theta_1)(q_2+\theta_2) - (p_1-1)(p_2-1)} - \frac{N-p_2}{p_2-1} \right\} \geq 0. \tag{28}$$

If, furthermore, $\theta_1 = \theta_2 = 0$*, then* $(u, v) \equiv (0, 0)$ *in* $\mathbb{R}^N$.

For the case $\theta_1 = \theta_2 = 0$ in (26) we refer to Theorem 22.1 by Mitideri and Pohozaev in [37] and its generalization.

For Liouville-type results for the equation (7) we refer to [11], where Corollary B-1 in [9], devoted to the case $p = 2$ in (7), is generalized.

Theorem 2.5 (Theorem 1.1, [11]). *Let* $N \geq 2$*,* $p > 1$*,* $q > 0$ *and* $0 < \theta < p$*.*

Then, any nonnegative solution of (7) *is constant if* $q + \theta > p - 1$ *and*

(*i*) $q \geq 1$, $q + \theta < \frac{4+N}{N}(p-1) =: Q_1$

(*ii*) $0 < q < 1$, $q + \theta < \left(1 + \frac{(q+1)^2}{qN}\right)(p-1) =: Q_2$,

where $Q_1 \leq Q_2$.

Remark 2.6. If we compare (12), which can be written as

$$q + \theta \leq \frac{(N+q)(p-1)}{N-1},$$

with the above conditions (*i*) and (*ii*) we deduce that if $0 < q < (3N-4)/N \in (1, 3)$, then Theorem 2.5, gives a wider range for $q + \theta$ with respect to the one given in Theorem 2.3, as it happens in the case without a gradient term if we consider Liouville-type results for equations and inequalities, where the upper bounds are given respectively by the Sobolev exponent p^* and the Serrin exponent p_*. Unexpectedly, if $q \geq (3N-4)/N$, then the range of q in Theorem 2.3 is larger than the one of Theorem 2.5.

3. Main results in the scalar case

The present section is devoted to the main result of the paper in the scalar case.

THEOREM 3.1. *Let Ω be an arbitrary domain of $\mathbb{R}^N$, $1 < p < N$, consider $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ a continuous function satisfying* (9) *with q, θ, σ as in* (10), (11) *and* (12). *Moreover, suppose that there exists a continuous function $\ell \geq 0$, with*

$$\lim_{z \to x \in \overline{\Omega}} \ell(z) \in (0, \infty)$$

both for Ω bounded or not, such that (3) *holds.*

Then, either (I) *or* (II) *in Corollary 1.1 holds for nonnegative solutions of* (1).

Proof. Let γ as in (14). Assume that the estimate (13) fails. Then, there exist sequences $(\Omega_k)_k \subseteq \Omega$, $y_k \in \Omega_k$ and u_k positive non constant solution of (1) on Ω_k, such that

$$\frac{(u_k(y_k) + |Du_k(y_k)|^{\gamma/(\gamma+1)})^{1/\gamma}}{(1 + \operatorname{dist}^{-\gamma}(y_k, \partial\Omega_k))^{1/\gamma}} \to \infty,$$

as $k \to \infty$. Now, by setting

$$M_k(y_k) := u_k(y_k)^{1/\gamma} + |Du_k(y_k)|^{1/(\gamma+1)}, \qquad k \geq 1, \tag{29}$$

we can suppose that

$$\begin{aligned} M_k(y_k) &\geq C(u_k(y_k) + |Du_k(y_k)|^{\gamma/(\gamma+1)})^{1/\gamma} \\ &\geq 2k(1 + \operatorname{dist}^{-\gamma}(y_k, \partial\Omega_k))^{1/\gamma} \geq 2k \operatorname{dist}^{-1}(y_k, \partial\Omega_k), \end{aligned}$$

where C is a positive constant. By Lemma 2.1 and Remark 2.2, it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial\Omega_k), \quad M_k(x_k) \geq M_k(y_k), \tag{30}$$

and for all z such that $|z - x_k| \leq k M_k^{-1}(x_k)$, then

$$M_k(z) \leq 2M_k(x_k). \tag{31}$$

Now we rescale u_k by setting

$$\tilde{u}_k(y) = \lambda_k^{\gamma} u_k(z), \quad z = x_k + \lambda_k y, \quad |y| \leq k, \text{ with } \lambda_k = M_k^{-1}(x_k). \tag{32}$$

Since $M_k(x_k) \geq M_k(y_k) \geq 2k(1 + \operatorname{dist}^{-\gamma}(y_k, \partial\Omega_k))^{1/\gamma} > 2k$, by (32), we have

$$M_k(x_k) \to \infty, \quad \lambda_k \to 0, \qquad k \to \infty.$$

In particular, since $(\partial/\partial y_i)\tilde{u}_k(y) = \lambda_k^{\gamma+1}(\partial/\partial z_i)u_k(z)$, we immediately have

$$\Delta_p\tilde{u}_k = div_y(|D\tilde{u}_k|^{p-2}D\tilde{u}_k) = \lambda_k^{\gamma(p-1)+p}\Delta_p u_k,$$

so that, since u_k is a non constant solution of (1), then $\tilde{u}_k$ is a non constant solution of

$$-\Delta_p\tilde{u}_k = f_k(x_k + \lambda_k y, \tilde{u}_k, D\tilde{u}_k), \quad |y| \le k, \tag{33}$$

with

$$f_k(z, \tilde{u}_k, D\tilde{u}_k) = \lambda_k^{\gamma(p-1)+p} f(x_k + \lambda_k y, \lambda_k^{-\gamma}\tilde{u}_k, \lambda_k^{-\gamma-1}D\tilde{u}_k) \tag{34}$$

Moreover, from (29), (31) and the definition of λ_k, we get

$$\begin{aligned}[\tilde{u}_k^{1/\gamma} + |D\tilde{u}_k|^{1/(\gamma+1)}](0) &= \lambda_k[u_k^{1/\gamma} + |Du_k|^{1/(\gamma+1)}](x_k) \\ &= \lambda_k M_k(x_k) = 1,\end{aligned} \tag{35}$$

and, when $|y| \le k$, it holds

$$\begin{aligned}[\tilde{u}_k^{1/\gamma} + |D\tilde{u}_k|^{1/(\gamma+1)}](y) &= \lambda_k[u_k^{1/\gamma} + |Du_k|^{1/(\gamma+1)}](z) \\ &= \lambda_k M_k(z) \le 2\lambda_k M_k(x_k) = 2.\end{aligned} \tag{36}$$

In particular, by virtue of assumption (9) on f, for all $|y| \le k$, we have

$$\begin{aligned}f_k(z, \tilde{u}_k, D\tilde{u}_k) &\le C_1\left(\lambda_k^{\gamma(p-1)+p} + \lambda_k^{\gamma(p-1)+p-\sigma\gamma}\tilde{u}_k^{\sigma} + \tilde{u}_k^{q}|D\tilde{u}_k|^{\theta}\right) \\ &\le C_1\left(\lambda_k^{\gamma(p-1)+p} + \lambda_k^{\gamma(p-1)+p-\sigma\gamma}2^{\gamma\sigma} + 2^{\gamma(q+\theta)+\theta}\right),\end{aligned}$$

thanks to (14) and (36). Now, note that the following holds

$$\gamma(p-1) + p - \sigma\gamma > 0,$$

by (10) since $\gamma > 0$. Consequently, we immediately get

$$0 \le f_k(z, \tilde{u}_k, D\tilde{u}_k) \le C, \qquad C > 0, \tag{37}$$

since $\lambda_k \to 0$ as $k \to \infty$.

By using standard regularity results (cfr. [30, 44]) since $\tilde{u}_k$ and $D\tilde{u}_k$ are bounded on compact subsets of Ω by (36), we deduce that there exists $\rho \in (0,1)$ such that $\tilde{u}_k$ is bounded in $C^{1,\rho}(\overline{\Omega}_k)$, namely $\tilde{u}_k$ is bounded in $C^{1,\rho}_{loc}(\mathbb{R}^N)$. Therefore $\tilde{u}_k$ converges in $C^1_{loc}(\mathbb{R}^N)$ to a certain function $\tilde{u} \ge 0$. Moreover, by letting $k \to \infty$ in (35) we have that $\tilde{u}(0) + |D\tilde{u}(0)|^{1/\gamma} = 1$ thus $\tilde{u}$ is nontrivial and, by letting $k \to \infty$ in (33) by virtue of (37), we have that $\tilde{u}$ satisfies $-\Delta_p\tilde{u} \ge 0$ in $\mathbb{R}^N$, and by the strong maximum principle for the p-Laplacian (see [39] and [45]) it results $\tilde{u}(y) > 0$ for all $y \in \mathbb{R}^N$.

Thus, for all $y \in \mathbb{R}^N$, being $\tilde{u} > 0$, it follows that $\lambda_k^{-\gamma}\tilde{u}_k(y) \to \infty$ as $k \to \infty$. So that assumption (3) can be applied to f in (34), yielding for k large

$$\begin{aligned} f_k(z, \tilde{u}_k, D\tilde{u}_k) &\geq \lambda_k^{-\gamma(q+\theta-p+1)+p-\theta}\ell(x_k + \lambda_k y)\tilde{u}_k^q|D\tilde{u}_k|^\theta \\ &= \ell(x_k + \lambda_k y)\tilde{u}_k^q|D\tilde{u}_k|^\theta \end{aligned} \tag{38}$$

thanks to the choice of γ.

Consequently, by letting $k \to \infty$ in (33), if Ω is bounded, then we have $x_k \to \overline{x} \in \overline{\Omega}$, up to subsequences, while if Ω is unbounded and if $x_k \to \infty$, denoting again $\ell(\overline{x}) := \lim_{k\to\infty} \ell(x_k) \in (0, \infty)$, yielding from (38) and (33), we have

$$-\Delta_p \tilde{u} \geq \ell(\overline{x})\tilde{u}^q|D\tilde{u}|^\theta \quad \text{in} \quad \mathbb{R}^N.$$

Since condition (12) and (11) are in force, then Theorem 2.3, applied with $\zeta = 0$ and $a(x) = c_0 = \ell(\overline{x})$, gives that $\tilde{u}$ necessarily is constant.

Moreover, because of $\tilde{u}(0)+|D\tilde{u}(0)|^{1/\gamma} = 1$, then $\tilde{u} \equiv 1$. This means by (32) that, for all $y \in \mathbb{R}^N$,

$$u_k(x_k + \lambda_k y)M_k(x_k)^{-\gamma} \to 1, \qquad k \to \infty \tag{39}$$

namely, since $M_k(x_k) \to \infty$ as $k \to \infty$, then $(u_k(x_k))_k$ is an unbounded sequence with

$$u_k(x_k) \sim M_k(x_k)^\gamma, \quad k \to \infty$$

and

$$\frac{|Du_k(x_k)|^{\gamma/(\gamma+1)}}{u_k(x_k)} \to 0, \quad k \to \infty, \tag{40}$$

where here we have used the definition of $M_k(x_k)$.

Now, we claim that

$$\frac{d^{-\gamma}(x_k, \partial\Omega_k)}{u_k(x_k)} \to 0, \quad k \to \infty. \tag{41}$$

First, from (30), we have

$$\frac{d^{-\gamma}(x_k, \partial\Omega_k)}{M_k(x_k)^\gamma} < \frac{1}{(2k)^\gamma}. \tag{42}$$

From (36), for k sufficiently large, then

$$M_k(x_k)^\gamma \leq 2u_k(x_k). \tag{43}$$

So that, taking into account (42), (43), for k sufficiently large, we get

$$\frac{1}{(2k)^\gamma} > \frac{d^{-\gamma}(x_k, \partial\Omega_k)}{M_k(x_k)^\gamma} \geq \frac{d^{-\gamma}(x_k, \partial\Omega_k)}{2u_k(x_k)}$$

that is the claim (41).

We have proved that if (I) does not hold, then (II) is in force. On the other hand, if (II) holds, then by (40) and (41), we have

$$\frac{u_k(x_k) + |Du_k(x_k)|^{\gamma/(\gamma+1)}}{1 + d^{-\gamma}(x_k, \partial\Omega_k)} = \frac{1 + \frac{|Du_k(x_k)|^{\gamma/(\gamma+1)}}{u_k(x_k)}}{\frac{1}{u_k(x_k)} + \frac{d^{-\gamma}(x_k,\partial\Omega_k)}{u_k(x_k)}} \to \infty,$$

namely, (I) fails. The proof is so concluded.

Thus, Theorem 3.1 is proved since either (I) or (II) should hold. □

In the next result, we replace (3) with another type of assumption which takes inspiration from [38].

Theorem 3.2. *Let Ω be an arbitrary domain of $\mathbb{R}^N$, $1 < p < N$, consider $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ a continuous function satisfying* (9) *with q, θ, σ as in* (10)*,* (11) *and* (12)*. Moreover, assume that for all $x \in \overline{\Omega}$,*

$$\lim_{t\to\infty, \Omega\ni z\to x} t^{-q-(\gamma+1)\theta/\gamma}\xi^{-\theta} f(z, t, t^{(\gamma+1)/\gamma}\xi) = m(x) \in (0, \infty) \tag{44}$$

uniformly for $\xi \neq 0$ bounded. Moreover, if Ω is unbounded, then we assume it also holds for $x = \infty$.

Then, either (I) *or* (II) *in Corollary 1.1 holds for nonnegative solutions of* (1)*.*

Proof. We proceed word by word as in the proof of Theorem 3.1 up to the application of the strong maximum principle which gives $\tilde{u} > 0$ in $\mathbb{R}^N$. Fixing $y \in \mathbb{R}^N$ and denoting

$$\mu_k := \lambda_k^{-\gamma}\tilde{u}_k, \qquad \xi_k := \tilde{u}_k^{-(\gamma+1)/\gamma} D\tilde{u}_k,$$

we may write

$$\tilde{u}_k^q |D\tilde{u}_k|^\theta = \mu_k^{q+(\gamma+1)\theta/\gamma} \lambda_k^{\gamma q+(\gamma+1)\theta} \xi_k^\theta.$$

By the definition of γ, we have

$$\gamma(p-1) + p - \gamma q - (\gamma+1)\theta = 0$$

so that (34) gives

$$f_k(z, \tilde{u}_k, D\tilde{u}_k) = \mu_k^{-q-(\gamma+1)\theta/\gamma} \xi_k^{-\theta} \tilde{u}_k^q |D\tilde{u}_k|^\theta f(z, \mu_k, \mu_k^{(\gamma+1)/\gamma}\xi_k)$$

Note that, $\mu_k \to \infty$ as $k \to \infty$, furthermore, $\xi_k \neq 0$ since $\tilde{u}_k$ are non constant positive functions for all k and ξ_k are bounded by (36) and $\tilde{u} > 0$. If $(x_k)_k$

is bounded, then we may assume that $x_k \to \overline{x} \in \overline{\Omega}$ by extracting a further subsequence, and assumption (44) implies that

$$f_k(z, \tilde{u}_k, D\tilde{u}_k) \to m(\overline{x})\tilde{u}^{q_1}|D\tilde{u}|^{\theta_1} \tag{45}$$

Otherwise, if Ω is unbounded and $x_k \to \infty$ (along some subsequence), then the additional assumption on f implies that (45) still holds with $\overline{x} = \infty$. Consequently, $\tilde{u}$ verifies

$$-\Delta_p \tilde{u} = m(\overline{x})\tilde{u}^q|D\tilde{u}|^\theta \quad \text{in} \quad \mathbb{R}^N.$$

By (12), Theorem 2.3 gives that $\tilde{u}$ is necessarily constant and, by (35), then $\tilde{u} \equiv 1$. Now we can repeat the proof of Theorem 3.1 from (39). □

Remark 3.3. We point out that the conclusion of Theorem 3.2 can be improved since (12) can be replaced by (i) and (ii) in Theorem 2.5, which in some cases give a wider range for q, as discussed in Remark 2.6. This depends on the fact that, as it is evident in the proof of Theorem 3.2, the limit problem is an equation of the form (7), to which we have applied a Liouville-type result for inequalities, Theorem 2.3. On the other hand, in Theorem 3.1 the range of q cannot be enlarged since the limit problem is an inequality of the form (25), so that Theorem 2.3 gives the conclusion.

As a trivial consequence of Theorem 3.1 (or Theorem 3.2) when $\ell(x) \equiv 1$, we get Corollary 1.1.

4. On the sharpness of condition (12)

We end the scalar part of the paper by reasoning on the sharpness of condition $(12)_2$. Indeed, not only it is optimal for the nonexistence of supersolutions to the equation (7), as observed in the Introduction, but it reveals also optimal for pointwise a priori estimates, as we will see later.

In the first case, it is a straightforward calculation to exhibit supersolutions of (7) when (12) fails, that is (8) holds, or equivalently

$$q(N-p) > N(p-1) - \theta(N-1). \tag{46}$$

To this aim, let $u(r) = u(|x|) = C(1+|x|^2)^{-\mathfrak{m}/2} = C(1+r^2)^{-\mathfrak{m}/2}$, $r = |x| > 0$, $C > 0$ and $\mathfrak{m} > 0$, then $u \in C^1(\mathbb{R}^N)$ and satisfies (for simplicity we consider $C = 1$)

$$\begin{aligned} -\Delta_p u = {}& \mathfrak{m}^{p-1}(1+r^2)^{-(\mathfrak{m}+2)(p-1)/2} r^{p-2} \\ & \cdot \big[(N+p-2) - (\mathfrak{m}+2)(p-1)(1+r^2)^{-1}r^2\big]. \end{aligned}$$

On the other hand, we have

$$u^q|Du|^\theta = \mathfrak{m}^\theta r^\theta (1+r^2)^{-\mathfrak{m}(\theta+q)/2-\theta}.$$

Thus, u is a supersolution of (7) if

$$\begin{aligned}\big[(N+p-2)-(\mathfrak{m}+2)(p-1)(1+r^2)^{-1}r^2\big]\\ \geq \big[N+p-2-(\mathfrak{m}+2)(p-1)\big]\\ \geq \mathfrak{m}^{\theta-p+1}r^{\theta-p+2}(1+r^2)^{-\mathfrak{m}(\theta+q)/2-\theta+(\mathfrak{m}+2)(p-1)/2} =: \varphi(r).\end{aligned}$$

Note that

$$\varphi(r)\sim r^{\theta-p+2} \quad \text{if } r\sim 0^+,$$
$$\varphi(r)\sim r^{-\mathfrak{m}(\theta+q)-\theta+(\mathfrak{m}+2)(p-1)-p+2} \quad \text{if } r\sim\infty,$$

so that φ is bounded in $\mathbb{R}_0^+$ by requiring $\theta\geq p-2$ and $-\mathfrak{m}(\theta+q)-\theta+\mathfrak{m}(p-1)+p\leq 0$ and $(N+p-2)-(\mathfrak{m}+2)(p-1)>0$, yielding $\theta\geq p-2$ and

$$\frac{p-\theta}{q+\theta-p+1}\leq\mathfrak{m}<\frac{N-p}{p-1}. \tag{47}$$

In turn, it is possible to choose $\mathfrak{m}$ suitable if

$$\frac{p-\theta}{q+\theta-p+1}<\frac{N-p}{p-1},$$

namely

$$(q+\theta)(N-p)-(p-1)(N-p)>(p-\theta)(p-1),$$

which is in force by (46). Hence, (7) admits supersolutions of the form $C(1+|x|^2)^{-\mathfrak{m}/2}$, with $C>0$, provided that $\theta>0$, $\theta\geq p-2$ and (46) holds.

Most of all, condition (12) is also optimal on q to reach pointwise a priori estimates of the form (13) for solutions of the equation

$$-\Delta_p u = a(x)\,u^q|Du|^\theta \quad \text{in } B_R\setminus\{0\}, \tag{48}$$

as shown by the following counterexample. Indeed, if condition (12) does not hold, namely (46) is in force, then there are solutions of (48) for which the estimate (15) fails.

Precisely, let $u(r)=u(|x|)=|x|^{-\mathfrak{m}}=r^{-\mathfrak{m}}$, $r>0$ and $\mathfrak{m}>0$ defined in $B_R\setminus\{0\}$, with $R>0$ to be chosen. Obviously, u satisfies

$$-\Delta_p u = \mathfrak{m}^{p-1}[N-1-(\mathfrak{m}+1)(p-1)]r^{-\mathfrak{m}(p-1)-p}$$

and it holds

$$a(x)\,u^q|Du|^\theta = a(x)\mathfrak{m}^\theta r^{-\mathfrak{m}q-\theta(\mathfrak{m}+1)}.$$

Consequently, u is a radial solution of (48) if the following holds

$$\mathfrak{m}^{p-1}[N-p-\mathfrak{m}(p-1)]r^{-\mathfrak{m}(p-1)-p} = a(x)\mathfrak{m}^\theta r^{-\mathfrak{m}q-\theta(\mathfrak{m}+1)},$$

which is in force if and only if

$$a(x) = \mathfrak{m}^{p-1-\theta}[N - p - \mathfrak{m}(p-1)]r^{\mathfrak{m}(\theta+q-p+1)-p+\theta}.$$

Consequently, $a \in C(B_R)$, $R > 0$, and $a(x) > 0$ if

$$\begin{cases} N - p - \mathfrak{m}(p-1) > 0, \\ \mathfrak{m}(\theta + q - p + 1) - p + \theta \geq 0. \end{cases} \tag{49}$$

By (11), we have $\theta + q - p + 1 > 0$ so that, from (49), the interval in which $\mathfrak{m}$ can be chosen is (47), which is already been observed that is nonempty thanks to (46). With this choice for $\mathfrak{m}$ and with p, q, θ satisfying (11), all the assumptions of Corollary 1.1 hold *except condition* (12). We claim that estimate (15) does not occur. First, we compute

$$u + |Du|^{(p-\theta)/(q+1)} = r^{-\mathfrak{m}} + \mathfrak{m}^{(p-\theta)/(q+1)}r^{-(\mathfrak{m}+1)(p-\theta)/(q+1)}.$$

Thus, condition (15) reduces to prove the boundness of the following function for any $0 < r \leq R/2$

$$r^{-\mathfrak{m}} + \mathfrak{m}^{(p-\theta)/(q+1)}r^{-(\mathfrak{m}+1)(p-\theta)/(q+1)} - c - cr^{-(p-\theta)/(\theta+q-p+1)}, \quad c > 0.$$

Namely, we have to prove the boundedness of $r^{-\mathfrak{m}}\psi(r)$ close to 0, where

$$\psi(r) := 1 + \mathfrak{m}^{(p-\theta)/(q+1)}r^{[\mathfrak{m}(q+\theta-p+1)-(p-\theta)]/(q+1)} - cr^{\mathfrak{m}-(p-\theta)/(\theta+q-p+1)}.$$

By the choice of $\mathfrak{m}$ in (47), we get $\psi(r) \to 1$ as $r \to 0^+$, so that $r^{-\mathfrak{m}}\psi(r) \to \infty$ as $r \to 0^+$. Consequently, the estimate (15) fails.

5. Main results in the vectorial case

As observed by Poláčik, Quittner and Souplet in Remark 7.4 in [38], universal estimates can be similarly obtained also for systems. To this aim, define

$$\alpha = \frac{(p_2-1)(p_1-\theta_1) + (q_1+\theta_1)(p_2-\theta_2)}{(q_1+\theta_1)(q_2+\theta_2) - (p_1-1)(p_2-1)}, \tag{50}$$

$$\beta = \frac{(p_1-1)(p_2-\theta_2) + (q_2+\theta_2)(p_1-\theta_1)}{(q_1+\theta_1)(q_2+\theta_2) - (p_1-1)(p_2-1)}. \tag{51}$$

The assumptions (27) and (28) guarantee the positivity of α, β. Of course, in the scalar case $\alpha = \beta = \gamma$, with γ defined in (14).

Now we are ready to present the first main result concerning systems.

THEOREM 5.1. *Let Ω be an arbitrary domain of $\mathbb{R}^N$, $1 < p_1, p_2 < N$ and* (27). *Let $f_1, f_2 : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ be continuous functions satisfying* (18) *and* (19) *with*

$$\sigma_1 < \frac{\alpha(p_1 - 1) + p_1}{\beta}, \qquad \sigma_2 < \frac{\beta(p_2 - 1) + p_2}{\alpha}, \tag{52}$$

and

$$\tau_1 < \frac{\alpha(p_1 - 1) + p_1}{\alpha}, \qquad \tau_2 < \frac{\beta(p_2 - 1) + p_2}{\beta}, \tag{53}$$

where α, β are given in (50), (51).

Suppose that there exist two functions $\ell_1, \ell_2 \geq 0$, with

$$\lim_{z \to x \in \overline{\Omega}} \ell_i(z) \in (0, \infty)$$

for $i = 1, 2$, both for Ω bounded or not, such that

$$f_1(x, s, t, \eta, \xi) \geq \ell_1(x) t^{q_1} |\xi|^{\theta_1}, \qquad x \in \Omega,\ t \text{ large},\ s \geq 0,\ \xi, \eta \in \mathbb{R}^N \tag{54}$$

and

$$f_2(x, s, t, \eta, \xi) \geq \ell_2(x) s^{q_2} |\eta|^{\theta_2}, \qquad x \in \Omega,\ s \text{ large},\ t \geq 0,\ \xi, \eta \in \mathbb{R}^N. \tag{55}$$

If (28) *holds, namely*

$$\max\left\{\alpha - \frac{N - p_1}{p_1 - 1}, \beta - \frac{N - p_2}{p_2 - 1}\right\} \geq 0,$$

then, either (I)$'$ *or* (II)$'$ *in Corollary 1.2 holds for nonnegative solutions of* (2).

REMARK 5.2. The upper bounds for θ_1, θ_2 implied by (28), differently from $(12)_1$ for the scalar case, appear difficult to be explicitly evaluated. By the way, the positivity of α, β implies that at least one of the two conditions $\theta_1 < p_1$ or $\theta_2 < p_2$ hold.

Proof. Assume that either the estimate (20) or (21) fails. Then, there exist sequences $(\Omega_k)_k \subseteq \Omega$, $y_k \in \Omega_k$ and (u_k, v_k) positive non constant solutions of (2) on Ω_k, such that

$$M_k(y_k) \geq 2k \operatorname{dist}^{-1}(y_k, \partial\Omega_k)$$

where

$$M_k := u_k^{1/\alpha} + v_k^{1/\beta} + |Du_k|^{1/(\alpha+1)} + |Dv_k|^{1/(\beta+1)}, \qquad k \geq 1. \tag{56}$$

By Lemma 2.1 and Remark 2.2, it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial\Omega_k), \quad M_k(x_k) \geq M_k(y_k),$$

and for all z such that $|z - x_k| \le kM_k^{-1}(x_k)$, then

$$M_k(z) \le 2M_k(x_k). \tag{57}$$

Now we rescale (u_k, v_k) by setting $\lambda_k = M_k^{-1}(x_k)$ and

$$\tilde{u}_k(y) = \lambda_k^{\alpha} u_k(z), \quad \tilde{v}_k(y) = \lambda_k^{\beta} v_k(z), \quad z = x_k + \lambda_k y, \quad |y| \le k. \tag{58}$$

Since $M_k(x_k) \ge M_k(y_k) > 2k$, by (58), we also have

$$M_k(x_k) \to \infty, \quad \lambda_k \to 0, \qquad k \to \infty.$$

In particular, since $(\partial/\partial y_i)\tilde{u}_k(y) = \lambda_k^{\alpha+1}(\partial/\partial z_i)u_k(z)$, we immediately have

$$\Delta_{p_1}\tilde{u}_k = div_y(|D\tilde{u}_k|^{p_1-2}D\tilde{u}_k) = \lambda_k^{\alpha(p_1-1)+p_1}\Delta_{p_1}u_k,$$

similarly,

$$\Delta_{p_2}\tilde{v}_k = div_y(|D\tilde{v}_k|^{p_2-2}D\tilde{v}_k) = \lambda_k^{\beta(p_2-1)+p_2}\Delta_{p_2}v_k,$$

so that, since (u_k, v_k) is a solution of (2), then $(\tilde{u}_k, \tilde{v}_k)$ is a solution in $|y| \le k$ of

$$\begin{cases} -\Delta_{p_1}\tilde{u}_k = f_{1,k}(z, \tilde{u}_k, \tilde{v}_k, D\tilde{u}_k, D\tilde{v}_k), \\ -\Delta_{p_2}\tilde{v}_k = f_{2,k}(z, \tilde{u}_k, \tilde{v}_k, D\tilde{u}_k, D\tilde{v}_k), \end{cases} \tag{59}$$

with

$$\begin{aligned} &f_{1,k}(z, \tilde{u}_k, \tilde{v}_k, D\tilde{u}_k, D\tilde{v}_k) \\ &\qquad = \lambda_k^{\alpha(p_1-1)+p_1} f_1(z, \lambda_k^{-\alpha}\tilde{u}_k, \lambda_k^{-\beta}\tilde{v}_k, \lambda_k^{-\alpha-1}D\tilde{u}_k, \lambda_k^{-\beta-1}D\tilde{v}_k), \end{aligned} \tag{60}$$

$$\begin{aligned} &f_{2,k}(z, \tilde{u}_k, \tilde{v}_k, D\tilde{u}_k, D\tilde{v}_k) \\ &\qquad = \lambda_k^{\beta(p_2-1)+p_2} f_2(z, \lambda_k^{-\alpha}\tilde{u}_k, \lambda_k^{-\beta}\tilde{v}_k, \lambda_k^{-\alpha-1}D\tilde{u}_k, \lambda_k^{-\beta-1}D\tilde{v}_k). \end{aligned} \tag{61}$$

Moreover, from (56), (57) and the definition of λ_k, we get

$$[\tilde{u}_k^{1/\alpha} + \tilde{v}_k^{1/\beta} + |D\tilde{u}_k|^{1/(\alpha+1)} + |D\tilde{v}_k|^{1/(\beta+1)}](0) = \lambda_k M_k(x_k) = 1, \tag{62}$$

and, when $|y| \le k$, it holds

$$\begin{aligned} &[\tilde{u}_k^{1/\alpha} + \tilde{v}_k^{1/\beta} + |D\tilde{u}_k|^{1/(\alpha+1)} + |D\tilde{v}_k|^{1/(\beta+1)}](y) \\ &\qquad = \lambda_k M_k(z) \le 2\lambda_k M_k(x_k) = 2. \end{aligned} \tag{63}$$

In particular, by using (18), (19), for all $|y| \le k$, we have

$$\begin{aligned} f_{1,k}(z, &\tilde{u}_k, \tilde{v}_k, D\tilde{u}_k, D\tilde{v}_k) \\ &\le C_1\left(\lambda_k^{\alpha(p_1-1)+p_1} + \lambda_k^{\omega_1}\tilde{v}_k^{\sigma_1} + \lambda_k^{\kappa_1}\tilde{u}_k^{\tau_1} + \lambda_k^{\delta_1}\tilde{v}_k^{q_1}|D\tilde{v}_k|^{\theta_1}\right) \\ &\le C_1\left(\lambda_k^{\alpha(p_1-1)+p_1} + \lambda_k^{\omega_1}2^{\beta\sigma_1} + \lambda_k^{\kappa_1}2^{\alpha\tau_1} + \lambda_k^{\delta_1}2^{\beta(q_1+\theta_1)+\theta_1}\right), \end{aligned}$$

and analogously

$$f_{2,k}(z,\tilde u_k,\tilde v_k,D\tilde u_k,D\tilde v_k) \le C_2\left(\lambda_k^{\beta(p_2-1)+p_2}+\lambda_k^{\omega_2}2^{\alpha\sigma_2}+\lambda_k^{\kappa_2}2^{\beta\tau_2}+\lambda_k^{\delta_2}2^{\alpha(q_2+\theta_2)+\theta_2}\right),$$

thanks to (63) and the choice of α and β in (50), (51) where

$$\omega_1=\alpha(p_1-1)+p_1-\beta\sigma_1,\qquad \omega_2=\beta(p_2-1)+p_2-\alpha\sigma_2$$

$$\kappa_1=\alpha(p_1-1)+p_1-\alpha\tau_1,\qquad \kappa_2=\beta(p_2-1)+p_2-\beta\tau_2,$$

$$\delta_1=\alpha(p_1-1)+p_1-\beta q_1-(\beta+1)\theta_1,\quad \delta_2=\beta(p_2-1)+p_2-\alpha q_2-(\alpha+1)\theta_2.$$

Actually, by the definition of α,β, then $\delta_1=\delta_2=0$. While, since $p_1,p_2>1$ and $\alpha,\beta>0$, then

$$\alpha(p_1-1)+p_1>0\,,\qquad \beta(p_2-1)+p_2>0,$$

so $\omega_1,\omega_2>0$ and $\kappa_1,\kappa_2>0$ are in force by (52) and (53).

Consequently, we immediately get

$$0\le f_{i,k}(z,\tilde u_k,\tilde v_k,D\tilde u_k,D\tilde v_k)\le C,\qquad C>0,\qquad i=1,2, \tag{64}$$

since $\lambda_k\to 0$ as $k\to\infty$.

By using standard regularity results (cfr. [30], [44]) since $\tilde u_k,\tilde v_k$ and $D\tilde v_k$, $D\tilde u_k$ are bounded on compact subsets of Ω by (63), we deduce that there exists $\rho\in(0,1)$ such that $\tilde u_k,\tilde v_k$ are bounded in $C^{1,\rho}(\overline{\Omega}_k)$, namely $\tilde u_k,\tilde v_k$ are bounded in $C^{1,\rho}_{loc}(\mathbb{R}^N)$. Therefore $\tilde u_k,\tilde v_k$ converges in $C^1_{loc}(\mathbb{R}^N)$ to certain functions $\tilde u,\tilde v\ge 0$, respectively. Letting $k\to\infty$ in (63), then $\tilde u,\tilde v$ are bounded, while by (62) we have that $(\tilde u,\tilde v)\ne(0,0)$, so at least one function between $\tilde u,\tilde v$ is nontrivial. Moreover, if $k\to\infty$ in (59) by virtue of (64), we have that $\tilde u,\tilde v$ satisfy $-\Delta_{p_1}\tilde u\ge 0$ and $-\Delta_{p_1}\tilde v\ge 0$ in $\mathbb{R}^N$.

Assume for instance $\tilde u\ne 0$. Thus, by the strong maximum principle applied to $\tilde u$ (see [39] and [45]) it results $\tilde u(y)>0$ for all $y\in\mathbb{R}^N$. Observe that, for all $y\in\mathbb{R}^N$, being $\tilde u>0$, it follows that

$$\lambda_k^{-\alpha}\tilde u_k(y)\to\infty,\quad k\to\infty \tag{65}$$

By applying (55), which holds for s large, in (61), we get

$$\begin{aligned} f_{2,k}(z,&\tilde u_k,\tilde v_k,D\tilde u_k,D\tilde v_k)\\ &\ge \lambda_k^{\beta(p_2-1)+p_2-\alpha q_2-(\alpha+1)\theta_2}\ell_2(x_k+\lambda_k y)\tilde u_k^{q_2}|D\tilde u_k|^{\theta_2}\\ &=\ell_2(x_k+\lambda_k y)\tilde u_k^{q_2}|D\tilde u_k|^{\theta_2}, \end{aligned} \tag{66}$$

where the last equality derives by the definition of α and β in (50), (51). By letting $k \to \infty$, if Ω is bounded, then we have $x_k \to \overline{x} \in \overline{\Omega}$, up to subsequences, while if Ω is unbounded and if $x_k \to \infty$, denoting $\ell_2(\overline{x}) := \lim_{k\to\infty} \ell_2(x_k) \in (0,\infty)$, then from (59), and (66), we obtain that $(\tilde{u},\tilde{v})$ verifies

$$-\Delta_{p_2}\tilde{v} \geq \ell_2(\overline{x})\tilde{u}^{q_2}|D\tilde{u}|^{\theta_2} \quad \text{in} \quad \mathbb{R}^N. \tag{67}$$

It is evident that, from (67), if $\tilde{v} \equiv 0$, then $\tilde{u}$ must be a positive constant, precisely, by (62), $\tilde{u} \equiv 1$, in other words, we are in the situation where

$$\tilde{v} \equiv 0, \qquad \tilde{u} \equiv 1, \tag{68}$$

yielding (22) in (II)$'$ as we will show later.

Otherwise, if $\tilde{v}$ is nontrivial, then $\tilde{v}(y) > 0$ by the strong maximum principle and we have

$$\lambda_k^{-\beta}\tilde{v}_k(y) \to \infty, \quad k \to \infty.$$

In turn, we can apply (54), which holds for t large, in (60), yielding

$$\begin{aligned} f_{1,k}(z,&\tilde{u}_k, \tilde{v}_k, D\tilde{u}_k, D\tilde{v}_k) \\ &\geq \lambda_k^{\alpha(p_1-1)+p_1-\beta q_1-(\beta+1)\theta_1}\ell_1(x_k+\lambda_k y)\tilde{v}_k^{q_1}|D\tilde{v}_k|^{\theta_1} \\ &= \ell_1(x_k+\lambda_k y)\tilde{v}_k^{q_1}|D\tilde{v}_k|^{\theta_1}, \end{aligned} \tag{69}$$

again the last equality holds thanks to the choice of α and β in (50), (51), respectively. Hence, as above, by letting $k \to \infty$, and denoting $\ell_1(\overline{x}) := \lim_{k\to\infty} \ell_1(x_k) \in (0,\infty)$, then from (59), (69) and (67), we obtain that $(\tilde{u},\tilde{v})$ verifies

$$\begin{cases} -\Delta_{p_1}\tilde{u} \geq \ell_1(\overline{x})\tilde{v}^{q_1}|D\tilde{v}|^{\theta_1} & \text{in} \quad \mathbb{R}^N \\ -\Delta_{p_2}\tilde{v} \geq \ell_2(\overline{x})\tilde{u}^{q_2}|D\tilde{u}|^{\theta_2} & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{70}$$

Consequently, Theorem 2.4 can be applied thanks to (28), giving that $\tilde{u},\tilde{v}$ are necessarily positive constants and, because of (62), then $\tilde{u}+\tilde{v} \equiv 1$ in $\mathbb{R}^N$. Actually, this latter equality holds also in the case (68). This means, by (58), that, for all $y \in \mathbb{R}^N$,

$$u_k(x_k+\lambda_k y)M_k(x_k)^{-\alpha} + v_k(x_k+\lambda_k y)M_k(x_k)^{-\beta} \to 1, \qquad k \to \infty$$

namely, since $M_k(x_k) \to \infty$ as $k \to \infty$, at least one of the two sequences $(u_k(x_k))_k$ and $(v_k(x_k))_k$ is an unbounded sequence.

If only one sequence is unbounded, let us assume $(u_k(x_k))_k$ for simplicity, that is exactly the case (68), then

$$u_k(x_k) \sim M_k(x_k)^{\alpha}, \quad k \to \infty$$

and

$$\frac{|Du_k(x_k)|^{\alpha/(\alpha+1)} + |Dv_k(x_k)|^{\beta/(\beta+1)}}{u_k(x_k)} \to 0, \quad k \to \infty, \tag{71}$$

where we have used the definition of $M_k(x_k)$. In particular,

$$\frac{|Du_k(x_k)|^{\alpha/(\alpha+1)}}{u_k(x_k)} \to 0, \quad k \to \infty.$$

As in the proof of Theorem 3.1, it holds

$$\frac{d^{-\alpha}(x_k, \partial\Omega_k)}{u_k(x_k)} \to 0, \quad k \to \infty.$$

Thus, (22) is proved. On the other hand, if (68) is replaced by $\tilde{u} \equiv 0$ and $\tilde{v} \equiv 1$, which is obtained by arguing conversely in the proof, we immediately reach (23) in (II)$'$.

Finally, if $(u_k(x_k))$ and $(v_k(x_k))$ are both unbounded, then

$$u_k(x_k)M_k(x_k)^{-\alpha} + v_k(x_k)M_k(x_k)^{-\beta} \to 1, \qquad k \to \infty$$

that is, there exist $\Theta_1, \Theta_2 > 0$ (being $0 < \tilde{u}, \tilde{v} < 1$) such that $\Theta_1 + \Theta_2 = 1$ and

$$u_k(x_k) \sim \Theta_1 M_k(x_k)^{\alpha} \quad v_k(x_k) \sim \Theta_2 M_k(x_k)^{\beta}, \quad k \to \infty.$$

Moreover, in addition to (71), it holds

$$\frac{|Du_k(x_k)|^{\alpha/(\alpha+1)} + |Dv_k(x_k)|^{\beta/(\beta+1)}}{v_k(x_k)} \to 0, \quad k \to \infty, \tag{72}$$

where we have used the definition of $M_k(x_k)$. In particular, combining (71) and (72), we have

$$\frac{|Du_k(x_k)|^{\alpha/(\alpha+1)}}{u_k(x_k)}, \frac{|Dv_k(x_k)|^{\beta/(\beta+1)}}{v_k(x_k)} \to 0, \quad k \to \infty.$$

From which we can deduce (22) and (23) in (II)$'$. As in the proof of Theorem 3.1, it is not difficult to prove that if (II)$'$ holds, then (I)$'$ fails. The proof of Theorem 5.1 is concluded. □

Now, as in the scalar case, we state a similar theorem where, instead of assumptions (54) and (55), we assume a limit condition as (44), inspired by [38] where, however, systems involving convection terms are not treated.

THEOREM 5.3. *Let Ω be an arbitrary domain of $\mathbb{R}^N$, $1 < p_1, p_2 < N$. Let $f_1, f_2 : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ be continuous functions satisfying* (18), (19) *with* (27), (52), (53) *where α, β given in* (50), (51).

Assume that for all $x \in \overline{\Omega}$,

$$\lim_{t\to\infty,\Omega\ni z\to x} t^{-q_1-(\beta+1)\theta_1/\beta}\eta^{-\theta_1} f_1(z,s,t,s^{(\alpha+1)/\alpha}\xi, t^{(\beta+1)/\beta}\eta) = m_1(x) \in (0,\infty), \tag{73}$$

uniformly for $\eta \neq 0$ *bounded,* $s \in \mathbb{R}_0^+$*, and*

$$\lim_{s\to\infty,\Omega\ni z\to x} s^{-q_2-(\alpha+1)\theta_2/\alpha}\xi^{-\theta_2} f_2(z,s,t,s^{(\alpha+1)/\alpha}\xi, t^{(\beta+1)/\beta}\eta) = m_2(x) \in (0,\infty), \tag{74}$$

uniformly for $\xi \neq 0$ *bounded,* $t \in \mathbb{R}_0^+$*. Moreover, if* Ω *is unbounded, then we assume that* (73), (74) *also hold for* $x = \infty$.

If (28) *holds, then, either* (I)$'$ *or* (II)$'$ *in Corollary 1.2 holds for nonnegative solutions of* (2).

Proof. We proceed word by word as in the proof of Theorem 5.1 up to the application of the maximum principle to $\tilde{u} = \lim_{k\to\infty}\tilde{u}_k$ which gives that $\tilde{u}(y) > 0$ for all $y \in \mathbb{R}^N$, so that (65) is in force.

Fixing $y \in \mathbb{R}^N$ and denoting

$$\mu_k := \lambda_k^{-\alpha}\tilde{u}_k, \qquad \nu_k := \lambda_k^{-\beta}\tilde{v}_k$$

$$\xi_k := \tilde{u}_k^{-(\alpha+1)/\alpha} D\tilde{u}_k, \qquad \eta_k := \tilde{v}_k^{-(\beta+1)/\beta} D\tilde{v}_k$$

we may write

$$\begin{aligned}\tilde{v}_k^{q_1}|D\tilde{v}_k|^{\theta_1} &= (\nu_k(\tilde{u}_k\mu_k^{-1})^{\beta/\alpha})^{q_1+(\beta+1)\theta_1/\beta}\eta_k^{\theta_1}\\ &= (\nu_k\lambda_k^{\beta})^{q_1+(\beta+1)\theta_1/\beta}\eta_k^{\theta_1} = \nu_k^{q_1+(\beta+1)\theta_1/\beta}\lambda_k^{\beta q_1+(\beta+1)\theta_1}\eta_k^{\theta_1},\end{aligned}$$

and similarly

$$\tilde{u}_k^{q_2}|D\tilde{u}_k|^{\theta_2} = \mu_k^{q_2+(\alpha+1)\theta_2/\alpha}\lambda_k^{\alpha q_2+(\alpha+1)\theta_2}\xi_k^{\theta_2}.$$

By the definition of α and β, respectively in (50), (51), we have

$$\alpha(p_1-1)+p_1-\beta q_1-(\beta+1)\theta_1 = 0$$

$$\beta(p_2-1)+p_2-\alpha q_2-(\alpha+1)\theta_2 = 0$$

so that

$$\begin{aligned}&f_{1,k}(z,\tilde{u}_k,\tilde{v}_k,D\tilde{u}_k,D\tilde{v}_k)\\ &\quad = \nu_k^{-q_1-(\beta+1)\theta_1/\beta}\eta_k^{-\theta_1}\tilde{v}_k^{q_1}|D\tilde{v}_k|^{\theta_1} f_1(z,\mu_k,\nu_k,\mu_k^{(\alpha+1)/\alpha}\xi_k,\nu_k^{(\beta+1)/\beta}\eta_k)\end{aligned}$$

and

$$\begin{aligned}&f_{2,k}(z,\tilde u_k,\tilde v_k,D\tilde u_k,D\tilde v_k)\\&\quad=\mu_k^{-q_2-(\alpha+1)\theta_2/\alpha}\xi_k^{-\theta_2}\tilde u_k^{q_2}|D\tilde u_k|^{\theta_2}f_1(z,\mu_k,\nu_k,\mu_k^{(\alpha+1)/\alpha}\xi_k,\nu_k^{(\beta+1)/\beta}\eta_k).\end{aligned}$$

Note that, $\mu_k\to\infty$ as $k\to\infty$, by (65), while $\xi_k\neq 0$ since $\tilde u_k$ are non constant positive functions for all k and ξ_k are bounded by (63) and $\tilde u>0$. If $(x_k)_k$ is bounded, we may assume that $x_k\to\overline{x}\in\overline{\Omega}$ by extracting a further subsequence, Analogously, if Ω is unbounded and $x_k\to\infty$ (along some subsequence). Thus, assumption (74) imply that

$$f_{2,k}(z,\tilde u_k,\tilde v_k,D\tilde u_k,D\tilde v_k)\to m_2(\overline{x})\tilde u^{q_2}|D\tilde u|^{\theta_2}.$$

In particular, $(\tilde u,\tilde v)$ satisfies

$$-\Delta_{p_2}\tilde v=m_2(\overline{x})\tilde u^{q_2}|D\tilde u|^{\theta_2}\quad\text{in}\quad\mathbb{R}^N,$$

yielding $\tilde u\equiv 1$ if $\tilde v\equiv 0$, by virtue of (62), namely we fall in (68). Differently, if $\tilde v\not\equiv 0$, then $\tilde v>0$ in $\mathbb{R}^N$ by the strong maximum principle, so that $\nu_k\to\infty$ as $k\to\infty$ and we can apply (73) which gives

$$f_{1,k}(z,\tilde u_k,\tilde v_k,D\tilde u_k,D\tilde v_k)\to m_1(\overline{x})\tilde v^{q_1}|D\tilde v|^{\theta_1},$$

being $\eta_k\neq 0$ and bounded, arguing as before.

Consequently, $(\tilde u,\tilde v)$ verifies

$$\begin{cases}-\Delta_{p_1}\tilde u=m_1(\overline{x})\tilde v^{q_1}|D\tilde v|^{\theta_1} & \text{in}\quad\mathbb{R}^N\\-\Delta_{p_2}\tilde v=m_2(\overline{x})\tilde u^{q_2}|D\tilde u|^{\theta_2} & \text{in}\quad\mathbb{R}^N,\end{cases}$$

yielding $\tilde u,\tilde v$ necessarily constant functions by virtue of Theorem 2.4 thanks to (28), which is a contradiction. Now we can proceed as below (70). □

Finally, Corollary 1.2, whose statement is given in the Introduction, is merely the application of Theorem 5.1 (or 5.3) with $p_1=p_2=2$.

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