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Fast uniform stabilization of the linearized magnetohydrodynamics system by finite-dimensional localized feedback controllers

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Dedicated to ENZO MITIDIERI on the occasion of his retirement

Abstract. *This research project considers the d-dimensional MagnetoHydroDynamics (MHD) system defined on a sufficiently smooth bounded domain, $d = 2, 3$ with homogeneous boundary conditions, and subject to external sources assumed to cause instability. The initial conditions for both fluid and magnetic equations are taken of low regularity. We then seek to uniformly stabilize such MHD system in the vicinity of an unstable equilibrium pair, in the critical setting of correspondingly low regularity spaces, by means of explicitly constructed, static, feedback controls, which are localized on an arbitrarily small interior subdomain. In addition, the actuators will be minimal in number. The resulting space of well-posedness and stabilization is a suitable product space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$, $1 < p < \frac{2q}{2q-1}$, $q > d$, of tight Besov spaces for the fluid velocity component and the magnetic field component (each "close" to $\mathbf{L}^3(\Omega)$ for $d = 3$). It is known that such Besov space does not recognize compatibility conditions at the boundary, yet it provides a "minimal" level of regularity necessary to handle the nonlinear terms. In this paper we provide a solution of the first step: uniform stabilization of the linearized MHD. Showing maximal L^p-regularity up to $T = \infty$ for the feedback stabilized linearized system is critical for the analysis of well-posedness and stabilization of the feedback nonlinear problem. The solution of the nonlinear stabilization problem is to be given in a successive paper [29].*

Keywords: Magnetohydrodynamics, Unform stabiliaztion, Feedback controls.
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1. Introduction. Statement of Main Results.

The Magnetohydrodynamics (henceforth referred to as MHD) equation refers to phenomena arising in electrically conducting magnetic fluids. It is caused by the induction of current in a conductive fluid flow due to a magnetic field and moreover by polarization of the fluid and reciprocal changes in the magnetic field. There is a massive literature on this subject to which we refer. MHD has been used extensively in Plasma Confinement, Liquid Metal cooling of nuclear reactors and Electro Magnetic Casting (EMC). The system of MHD equations consists of the Navier-Stokes equations of a viscous incompressible fluid flow suitably coupled by high-order coupling with Maxwell-Ohm equations (of parabolic character) of an electromagnetic field [41, 47, 59, 60].

"*Turbulence is the most important unsolved problem of classical Physics*"– Richard Feynman.

Regardless of the presence of electromagnetism, turbulence is a generic phenomenon of large scale fluid flows. This is known as Hydrodynamic (HD) turbulence. In the present project, we continue our analysis of the localized feedback stabilization of fluids such as in [24]-[28] in general bounded 2d/3d domains by means of finitely many controllers (with minimal number of actuators), by considering at first the case of spatially localized interior feedback controllers. We do so in a functional setting of low regularity, namely with initial conditions for both the fluid equation and the magnetic equation taken in Besov space of tight indices ("close" to $L^3(\Omega)$ for $d = 3$). The reason for seeking such generality (over the traditional L^2-based Sobolev setting in much of the literature of feedback stabilization of parabolic PDE-dynamics) is that this is the "right setting" for our next step: the study of the more challenging stabilization problem of MHD systems on general 3d domains by means this time of finitely many, localized, boundary-based static feedback controls. That in the boundary-based case the aforementioned Besov setting is the critical one was demonstrated in the case of the 3d Navier-Stokes equations. It was in this Besov setting ("close" to $L^3(\Omega)$ for $d = 3$), that a 20-year old problem was resolved in the affirmative [27]: That is, that the 3d Navier-Stokes equations can be stabilized uniformly in the vicinity of an original unstable equilibrium solution by a boundary-based, localized, static feedback controller, that moreover is finite-dimensional (and explicitly constructed). Finite dimensionality of such stabilizing, static, localized, boundary controller in the $L^2(\Omega)$-based Sobolev setting was shown previously in [31] only in the $d = 2$ case. In the solution [31] of the $d = 3$ case, in the $L^2(\Omega)$-based Sobolev setting, finite dimensionality of the feedback boundary localized controller requires the additional assumption that the initial condition be compactly supported on Ω. It is the aforementioned Besov setting that allows the analysis to eliminate such assumption of compactly-supported initial data. The technical reasons are as follows.

The chosen Besov space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ introduced in [24]-[28] with tight indices ($1 < p < \frac{6}{5}$ for $d = 3$ and $1 < p < \frac{4}{3}$ for $d = 2$) needs to satisfy two competing requirements. On the one hand, be of sufficiently low topological level as to not recognize the boundary conditions in $\mathbf{L}^q_\sigma(\Omega)$ in (1.15) below, to be compatible with the initial data. On the other hand, of sufficiently high topological level as to be able to handle the nonlinear analysis of well-posedness and stabilization of the resultant closed-loop feedback Navier-Stokes problem. Thus, the generality of the Besov setting in the present paper is also intended to be a testing ground to attack in the future the more challenging boundary-based stabilization problem for the MHD systems.

1.1. Controlled Dynamic Magnetohydrodynamic Equations

Let, at first, Ω be an open connected bounded domain in $\mathbb{R}^d, d = 2, 3$ with sufficiently smooth boundary $\Gamma = \partial\Omega$. More specific requirements will be given below. Let ω be an arbitrarily small open smooth subset of the interior Ω, $\omega \subset \Omega$, of positive measure. Let m denote the characteristic function of ω: $m(\omega) \equiv 1$, $m(\Omega\backslash\omega) \equiv 0$. We consider the following Magnetohydrodynamic equations in the d-velocity field $y = \{y_1, \dots, y_d\}$, the scalar pressure π and the magnetic field $B = \{B_1, \dots, B_d\}$. They are perturbed by exterior forces f, g and subject to the action of a pair u, v of interior localized controls supported on an arbitrary small subset ω of Ω, to be described below, where $Q = (0,\infty) \times \Omega$, $\Sigma = (0, \infty) \times \Gamma$:

$$y_t - \nu_f \Delta y + (y\cdot\nabla)y + \nabla\pi + \frac{1}{2}\nabla(B\cdot B) - (B\cdot\nabla)B = m(x)u(t,x) + f(x) \text{ in } Q, \quad (1.1a)$$

$$B_t - \nu_m \Delta B + (y\cdot\nabla)B - (B\cdot\nabla)y = m(x)v(t,x) + g(x) \text{ in } Q, \quad (1.1b)$$

$$\operatorname{div} y = 0, \quad \operatorname{div} B = 0 \text{ in } Q, \quad (1.1c)$$

$$y = 0,\ B\cdot n = 0,\ (\text{curl } B)\times n = 0 \text{ on } \Sigma, \quad (1.1d)$$

$$y(0,x) = y_0, \quad B(0,x) = B_0 \text{ on } \Omega, \quad (1.1e)$$

while n is the unit outward normal on $\partial\Omega$. The coefficients ν_f, ν_m are the positive kinematic viscosity and the magnetic viscosity, respectively. The B-equation (1.1b) is usually written with the term ν_m curl curl B. We have invoked the formula $\text{curl curl}\, B = -\Delta B + \nabla \text{div} B$ as well as div $B \equiv 0$ in (1.1c), to rewrite it in a more convenient form. Furthermore, we denote the total pressure ϱ in the dynamic equation (1.1a) as $\varrho := \pi + \frac{1}{2}(B\cdot B)$ and in the static case $\varrho_e := \pi_e + \frac{1}{2}(B_e \cdot B_e)$.

Remark 1.1. In the preceding model, we imposed the following boundary conditions on the magnetic equation:

$$B \cdot n = 0, \ (\text{curl } B) \times n = 0 \text{ on } \Sigma. \tag{1.2}$$

Alternatively, one could consider Dirichlet boundary conditions for the magnetic field:

$$B = 0 \text{ on } \Sigma, \tag{1.3}$$

as outlined in equations (E.S) or (S.S) of [61].

1.2. Stationary Magnetohydrodynamics equations

The following result represents our basic starting point. See [3]

Theorem 1.1. *Consider the following steady-state Magnetohydrodynamics equations in* Ω

$$-\nu_f \Delta y_e + (y_e \cdot \nabla) y_e + \nabla \varrho_e - (B_e \cdot \nabla) B_e = f(x) \qquad \text{in } \Omega, \tag{1.4a}$$

$$-\nu_m \Delta B_e + (y_e \cdot \nabla) B_e - (B_e \cdot \nabla) y_e = g(x) \qquad \text{in } \Omega, \tag{1.4b}$$

$$div\, y_e = 0, \quad div\, B_e = 0 \qquad \text{in } \Omega, \tag{1.4c}$$

$$y_e = 0, \ B_e \cdot n = 0, \ (curl\ B_e) \times n = 0 \qquad \text{on } \Gamma \tag{1.4d}$$

where $\varrho_e = \pi_e + \frac{1}{2}(B_e \cdot B_e)$*. Let* $1 < q < \infty$*. For any* $f, g \in \mathbf{L}^q(\Omega)$*, there exits a solution (not necessarily unique)* $(y_e, B_e, \pi_e) \in \mathbf{W}^{2,q}(\Omega) \times \mathbf{W}^{2,q}(\Omega) \times W^{1,q}(\Omega) \equiv (W^{2,q}(\Omega))^d \times (W^{2,q}(\Omega))^d \times W^{1,q}(\Omega), \ q > d.$

1.3. Translated MHD system

We return to Theorem 1.1 which provides an equilibrium triplet $\{y_e, B_e, \pi_e\}$. Then, we translate by $\{y_e, B_e, \pi_e\}$ the original MHD problem (1.1). Thus we introduce new variables

$$z = y - y_e, \quad \mathbb{B} = B - B_e \quad p = \varrho - \varrho_e \tag{1.5a}$$

and obtain the translated problem

$$z_t - \nu_f \Delta z + (y_e \cdot \nabla) z + (z \cdot \nabla) y_e - (B_e \cdot \nabla) \mathbb{B}$$
$$- (\mathbb{B} \cdot \nabla) B_e + (z \cdot \nabla) z - (\mathbb{B} \cdot \nabla) \mathbb{B} + \nabla p = mu \qquad \text{in } Q, \tag{1.5b}$$

$$\mathbb{B}_t - \nu_m \Delta \mathbb{B} + (y_e \cdot \nabla) \mathbb{B} - (\mathbb{B} \cdot \nabla) y_e - (B_e \cdot \nabla) z$$
$$+ (z \cdot \nabla) B_e + (z \cdot \nabla) \mathbb{B} - (\mathbb{B} \cdot \nabla) z = mv \qquad \text{in } Q, \tag{1.5c}$$

$$\text{div } z = 0, \quad \text{div } \mathbb{B} = 0 \qquad \text{in } Q, \tag{1.5d}$$

$$z = 0, \ \mathbb{B} \cdot n = 0, \ (\text{curl } \mathbb{B}) \times n = 0 \qquad \text{on } \Sigma, \tag{1.5e}$$

$$z(0, x) = y_0(x) - y_e(x), \quad \mathbb{B}(0, x) = B_0(x) - B_e(x) \qquad \text{on } \Omega. \tag{1.5f}$$

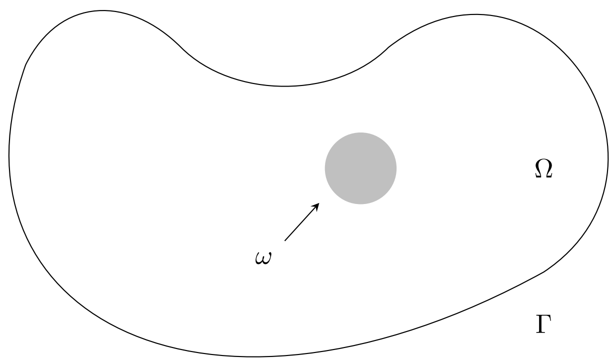

Figure 1: The localized interior set ω.

1.4. Translated Linearized MHD system

The translated linearized problem in the variables $\{w,\mathbb{W}\}$ corresponding to (1.5) is

$$\begin{aligned}
w_t - \nu_f \Delta w + (y_e\cdot\nabla)w + (w\cdot\nabla)y_e - (B_e\cdot\nabla)\mathbb{W} & \\
-(\mathbb{W}\cdot\nabla)B_e + \nabla p = mu &\qquad \text{in } Q, \qquad (1.6a)\\
\mathbb{W}_t - \nu_m \Delta \mathbb{W} + (y_e\cdot\nabla)\mathbb{W} - (\mathbb{W}\cdot\nabla)y_e - (B_e\cdot\nabla)w & \\
+(w\cdot\nabla)B_e = mv &\qquad \text{in } Q, \qquad (1.6b)\\
\operatorname{div} w = 0, \quad \operatorname{div}\mathbb{W} = 0 &\qquad \text{in } Q, \qquad (1.6c)\\
w = 0,\ \mathbb{W}\cdot n = 0,\ (\text{curl } \mathbb{W})\times n = 0 &\qquad \text{on } \Sigma, \qquad (1.6d)\\
w(0,x) = y_0 - y_e, \quad \mathbb{W}(0,x) = B_0 - B_e &\qquad \text{on } \Omega. \qquad (1.6e)
\end{aligned}$$

In line with the literature of Navier-Stokes equations, it will be convenient to introduce the following first order operators

$$\mathcal{L}^{+}_{y_e} w = (y_e\cdot\nabla)w + (w\cdot\nabla)y_e, \tag{1.7}$$

$$\mathcal{L}^{+}_{B_e} \mathbb{W} = (B_e\cdot\nabla)\mathbb{W} + (\mathbb{W}\cdot\nabla)B_e, \tag{1.8}$$

$$\mathcal{L}^{-}_{y_e} \mathbb{W} = (y_e\cdot\nabla)\mathbb{W} - (\mathbb{W}\cdot\nabla)y_e, \tag{1.9}$$

$$\mathcal{L}^{-}_{B_e} w = \left[(B_e\cdot\nabla)w - (w\cdot\nabla)B_e\right], \tag{1.10}$$

$\mathcal{L}^+_{y_e}$ and $\mathcal{L}^+_{B_e}$ being the Oseen operators for y_e and B_e respectively. With this notation we return to the translated linearized system $\{w, \mathbb{W}\}$ in (1.6), and rewrite it as

$$\begin{cases} w_t - \nu_f \Delta w + \mathcal{L}^+_{y_e}(w) - \mathcal{L}^+_{B_e}(\mathbb{W}) + \nabla p = mu & \text{in } Q, \quad (1.11a) \\ \mathbb{W}_t - \nu_m \Delta \mathbb{W} + \mathcal{L}^-_{y_e}(\mathbb{W}) - \mathcal{L}^-_{B_e}(w) = mv & \text{in } Q, \quad (1.11b) \\ \operatorname{div} w \equiv 0, \quad \operatorname{div} \mathbb{W} \equiv 0 & \text{in } Q, \quad (1.11c) \\ w \equiv 0, \quad \mathbb{W} \cdot n \equiv 0, \quad (\operatorname{curl} \mathbb{W}) \times n \equiv 0 & \text{in } \Sigma, \quad (1.11d) \\ w(0,x) = y_0 - y_e, \quad \mathbb{W}(0,x) = B_0 - B_e & \text{on } \Omega. \quad (1.11e) \end{cases}$$

1.5. A preliminary, qualitative statement of the main result of the present paper

While we refer to the subsequent Theorem 2.2 (to be proved in Section 5) for a complete, quantitative statement of the main result, here we wish to provide a preliminary, orientative, qualitative version, to enlighten and guide further reading.

Let $1 < q < \infty$. Let the linearized problem (1.11) in $\{w, \mathbb{W}\}$ be "unstable" (Section 2.2) with N unstable eigenvalues $\{\lambda_j\}_{j=1}^N$, M of which are distinct:

$$\ldots \leq Re\ \lambda_{N+2} \leq Re\ \lambda_{N+1} < 0 \leq Re\ \lambda_N \leq \ldots \leq Re\ \lambda_2 \leq Re\ \lambda_1. \tag{1.12}$$

Let $\varepsilon > 0$ and set $\gamma_0 = |Re\ \lambda_{N+1}| - \varepsilon$. We shall then construct (in fact, in many ways) vectors $\mathbf{p}_1, \ldots, \mathbf{p}_K$ and vectors $\mathbf{u}_1, \ldots, \mathbf{u}_K$ in the appropriate functional setting, $K =$ maximal geometric multiplicity of the unstable eigenvalues, such that the linearized MHD system (1.11) with feedback control $\mathbf{u}_N = \{u_N, v_N\}$ acting on ω, of the form given by

$$\mathbf{u}_N = \begin{bmatrix} u_N \\ v_N \end{bmatrix} = \sum_{k=1}^{K} \left(\begin{bmatrix} w(t) \\ \mathbb{W}(t) \end{bmatrix}_N, \mathbf{p}_k \right) \mathbf{u}_k \tag{1.13}$$

generates a feedback semigroup $S_F(t)$ (to be called $e^{\mathbb{A}_{F,q} t}$ in Theorem 2.2), which in the appropriate L^q- or Besov functional setting possesses the following properties: it is analytic; even more, it has L^q-maximal regularity up to $T = \infty$ (Section 7 of [29]); and is uniformly stable with decay $\gamma_0 > 0$:

$$\|S_F(t)\| \leq M e^{-\gamma_0 t}, \quad t \geq 0,\ M \geq 1. \tag{1.14}$$

Indeed, a known, minor modification of the proof taking λ_N an arbitrarily preselected eigenvalue, produces an arbitrarily prescribed decay rate. Moreover, using critically the property that $S_F(t)$ has L^q-maximal regularity up to $T = \infty$, the technical proof of [29] produces uniform stabilization of the original nonlinear MHD system (1.4), in the vicinity of the (preselected) equilibrium pair $\{y_e, B_e\}$. For a comprehensive understanding of maximal regularity theory, the interested reader is referred to [11].

1.6. Comparison with the literature

A detailed technical comparison with the literature is provided in [29]. Here we quote two references. In paper [5] (where $B \cdot n = 0$ on $\partial\Omega$ is used rather than $B \times n = 0$ on $\partial\Omega$), the authors select localized, interior, proportional, infinite dimensional feedback controls of the form (used in past literature) $u = -m_1 k_k (y - y_e)$; $\quad v = -m_2 k_2 (B - B_e)$ under several additional assumptions on the constants k_i and the characteristic functions m_i. In contrast, the $L^2(\Omega)$-Sobolev treatment of [33] is based on the same decomposition technique (described in Section 2.2 of the present paper) that was introduced in [50]. The stabilizing feedback operator is finite dimensional of an unknown dimension, and moreover not explicit. Additional technical and conceptual differences are provided in [29]. Among the several critical differences, we mention in particular two. First, in the Hilbert setting of [33], maximal regularity and analyticity of the strongly continuous semigroup are equivalent properties [46]. In the Banach setting, in particular the Besov setting of the research of the present authors, one needs to establish maximal regularity, as this property implies, but it is generally not implied by, analyticity of the strongly continuous (s.c.) semigroup. Establishing such maximal regularity property, in the present Besov setting of the s.c. analytic semigroup that stabilizes the linearized problem is a challenging task. This is carried out in [29, Section 7], following the strategy in [24, 28]. Second, the finite dimensional approach introduced in [50], and followed in both [33] and the present paper, critically requires at the very outset a unique continuation property (UCP) result to assert the Kalman algebraic, finite rank condition of the finite dimensional unstable component of the overall system. This is the "ignition key" of paper [55], the basic preliminary task needed at the very beginning of the analysis of the stabilization of a parabolic system. See the several illustrations in [55]. In our work, it amounts to establishing a UCP for a static, over-determined eigenvalue problem [(3.26) in Theorem 3.3 of the present paper]. This is a delicate property which is established by Carleman-type estimates [30], following the strategy of [53, 54, 56]. In contrast, [33] establishes a UCP for the dynamic coupled problem (3.3) in [33], by virtue of Carleman-type inequalities for parabolic equations. To this end, [33] "makes use of refined estimates for elliptic equations obtained in Imanuvilov and Puel [20] and couple them with estimates for the parabolic part of the system" [33, p. 973]. Our approach is much more direct.

1.7. Helmholtz decomposition

To eliminate the pressure term in the fluid equation (∇p in the nonlinear (1.5b), or ∇p in the linear (1.6a)) one needs, as usual, to introduce the Helmholtz (Leray) decomposition. A first difficulty one faces in extending the local exponential stabilization result for the interior localized problem (1.1) from the Hilbert-space setting in [4, 6, 31], and references therein to the $\mathbf{L}^q$ setting is the question of the existence of a Helmholtz (Leray) projection for the domain Ω in $\mathbb{R}^d$. More precisely: Given an open set $\Omega \subset \mathbb{R}^d$, the Helmholtz decomposition answers the question as to whether $\mathbf{L}^q(\Omega)$ can be decomposed into a direct sum of the solenoidal vector space $\mathbf{L}^q_\sigma(\Omega)$ and the space $\mathbf{G}^q(\Omega)$ of gradient fields. Here,

$$\begin{aligned} \mathbf{L}^q_\sigma(\Omega) &= \overline{\{\mathbf{y} \in \mathbf{C}^\infty_c(\Omega) : \text{div } \mathbf{y} = 0 \text{ in } \Omega\}}^{\|\cdot\|_q} \\ &= \{\mathbf{g} \in \mathbf{L}^q(\Omega) : \text{div } \mathbf{g} = 0;\ \mathbf{g}\cdot n = 0 \text{ on } \partial\Omega\}, \\ &\qquad\qquad \text{for any locally Lipschitz domain } \Omega \subset \mathbb{R}^d, d \geq 2 \\ \mathbf{G}^q(\Omega) &= \{\mathbf{y} \in \mathbf{L}^q(\Omega) : \mathbf{y} = \nabla p,\ p \in W^{1,q}_{loc}(\Omega)\} \text{ where } 1 \leq q < \infty. \end{aligned} \tag{1.15}$$

Both of these are closed subspaces of $\mathbf{L}^q$. Henceforth in this paper, we assume that the bounded domain $\Omega \subset \mathbb{R}^d$ under consideration admits a Helmholtz decomposition $L^q(\Omega)$; i.e. that it can be decomposed into the direct sum (non-orthogonal except for $q = 2$)

$$\mathbf{L}^q(\Omega) = \mathbf{L}^q_\sigma(\Omega) \oplus \mathbf{G}^q(\Omega). \tag{1.16}$$

The unique linear, bounded and idempotent (i.e. $P_q^2 = P_q$) projection operator $P_q : \mathbf{L}^q(\Omega) \longrightarrow \mathbf{L}^q_\sigma(\Omega)$ having $\mathbf{L}^q_\sigma(\Omega)$ as its range and $\mathbf{G}^q(\Omega)$ as its null space is called the Helmholtz projection. Paper [27] collects results of the literature where the Helmholtz decomposition holds true: e.g. a bounded C^1-domain in $\mathbb{R}^d$, $1 < q < \infty$ [15, Theorem 1.1, p. 107; Theorem 1.2, p. 114]; a bounded convex domain, $d \geq 2$, $1 < q < \infty$ [13], any open set in $\mathbb{R}^d$, for $q = 2$ [10]; and where fails (for some $q \neq 2$) [34].

1.8. Functional framework: definition of Besov spaces $\mathbf{B}^s_{q,p}(\Omega)$ on domains of class C^1 as real interpolation of Sobolev spaces:

Let m be a positive integer, $m \in \mathbb{N}, 0 < s < m, 1 \leq q < \infty, 1 \leq p \leq \infty$, then we define [16, p. 1398] the real interpolation space

$$\mathbf{B}^s_{q,p}(\Omega) = (\mathbf{L}^q(\Omega), \mathbf{W}^{m,q}(\Omega))_{\frac{s}{m},p} \tag{1.17a}$$

This definition does not depend on $m \in \mathbb{N}$ [57, p. xx]. This clearly gives

$$\mathbf{W}^{m,q}(\Omega) \subset \mathbf{B}^s_{q,p}(\Omega) \subset \mathbf{L}^q(\Omega) \quad \text{and} \quad \|y\|_{\mathbf{L}^q(\Omega)} \leq C \|y\|_{\mathbf{B}^s_{q,p}(\Omega)}. \tag{1.17b}$$

We shall be particularly interested in the following special real interpolation space of the $\mathbf{L}^q$ and $\mathbf{W}^{2,q}$ spaces $\left(m = 2, s = 2 - \frac{2}{p}\right)$:

$$\mathbf{B}^{2-\frac{2}{p}}_{q,p}(\Omega) = \left(\mathbf{L}^q(\Omega), \mathbf{W}^{2,q}(\Omega)\right)_{1-\frac{1}{p},p}. \tag{1.18}$$

Our interest in (1.18) is due to the following characterization [2, Theorem 3.4], [16, p. 1399]: if $A_{1,q}$ denotes the Stokes operator to be introduced below in (1.20), then

$$\left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p} = \left\{\mathbf{g} \in \mathbf{B}^{2-2/p}_{q,p}(\Omega) : \ \text{div } \mathbf{g} = 0, \ \mathbf{g}|_\Gamma = 0\right\}$$
$$\text{if } \frac{1}{q} < 2 - \frac{2}{p} < 2 \tag{1.19a}$$

$$\left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p} = \left\{\mathbf{g} \in \mathbf{B}^{2-2/p}_{q,p}(\Omega) : \ \text{div } \mathbf{g} = 0, \ \mathbf{g} \cdot n|_\Gamma = 0\right\}$$
$$\equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \text{ if } 0 < 2 - \frac{2}{p} < \frac{1}{q}; \text{ or } 1 < p < \frac{2q}{2q-1}. \tag{1.19b}$$

Notice that, in (1.19b), the condition $g \cdot n|_\Gamma = 0$ is an intrinsic condition of the space $\mathbf{L}^q_\sigma(\Omega)$ in (1.15), not an extra boundary condition as $g|_\Gamma = 0$ in (1.19a). This way in case (1.19b), we define the subspace $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ of $\mathbf{B}^{2-2/p}_{q,p}(\Omega)$, which is a critical state space in the present study.

Remark 1.2. In the analysis of well-posedness and stabilization of the non-linear MHD problem (1.1), with interior localized controls $\{u, v\}$ in feedback form to be carried out in the successive paper [29], we shall need to impose the constraint $q > 3$, to obtain the embedding $\mathbf{W}^{1,q}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$ in our case of interest $d = 3$, as already noted at the end of Section 1.7. What is then the allowable range of the parameter p in such case $q > 3$? The intended goal of the present paper is to obtain the sought-after stabilization result in a function space, such as a $\mathbf{B}^{2-2/p}_{q,p}(\Omega)$-space, that does not recognize boundary conditions of the initial condition (I.C.), for otherwise it will force compatibility condition on the boundary, and hence reduce the class of control problems under consideration. Thus, we need to avoid the case in (1.19a), as this implies a Dirichlet homogeneous B.C. Instead, we need to fit into the case (1.19b), where the condition $g \cdot n = 0$ on Γ is an intrinsic condition of the space $\mathbf{L}^q_\sigma(\Omega)$, as already noted below (1.19b). For $d = 3$, we shall then impose the condition $2 - \dfrac{2}{p} < \dfrac{1}{q} < \dfrac{1}{3}$ and then obtain that p must satisfy $p < \dfrac{6}{5}$. Moreover, analyticity and maximal regularity of the Stokes problem will require $p > 1$. Thus, in conclusion, the allowed range of the parameters p, q under which we shall solve the well-posedness and stabilization problem of the nonlinear MHD feedback system (9.1) in [29] for $d = 3$, following [24]-[28] in the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ defined in (1.19b), which - as intended - does not recognize boundary conditions is: $q > 3,\ 1 < p < \dfrac{6}{5}$.

1.9. Abstract translated nonlinear model

Premise. The fluid component (z in the nonlinear model (1.5b), w in the linear model (1.6a)) is as usual subject to the application of the Helmholtz projector to eliminate the pressure term ∇p and thus fall into the solenoidal space $\mathbf{L}^q_\sigma(\Omega)$, taking advantage of the two conditions: $z \equiv 0$ on Σ, $\operatorname{div} z \equiv 0$ in Q, property of this space, and similarly for w. The equation of the magnetic component ($\mathbb{B}$ in the nonlinear model (1.5c), $\mathbb{W}$ in the linear model (1.6b)) enjoys similar properties: $\mathbb{B} \cdot n \equiv 0$ on Σ, $\operatorname{div} \mathbb{B} \equiv 0$ on Q; $\mathbb{W} \cdot n \equiv 0$ on Σ, $\operatorname{div} \mathbb{W} \equiv 0$ in Q, intrinsic to the solenoidal space $\mathbf{L}^q_\sigma(\Omega)$. Accordingly, we shall apply the Helmholtz projection also to the magnetic equations which therefore will be studied in $\mathbf{L}^q_\sigma(\Omega)$ as well. This will justify the definition of the operator $A_{2,q}$ and $\mathcal{N}_q$ below in (1.21), (1.26).

Let $1 < q < \infty$ be fixed. We set recalling (1.7)-(1.10)

$$
\begin{aligned}
A_{1,q} f &= -P_q \Delta f, \\
\mathcal{D}(A_{1,q}) &= \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega), && (1.20) \\
A_{2,q} F &= -P_q \Delta F, \\
\mathcal{D}(A_{2,q}) &= \left\{ F \in \mathbf{W}^{2,q}(\Omega) \cap \mathbf{L}^q_\sigma(\Omega),\ (\text{curl } F) \times n \equiv 0 \text{ on } \Gamma \right\}, && (1.21) \\
A_{o,y_e,q} f &= P_q \mathcal{L}^+_{y_e} f = P_q \left[(y_e \cdot \nabla) f + (f \cdot \nabla) y_e \right], \\
\mathcal{D}(A_{o,y_e,q}) &= \mathcal{D}(A^{1/2}_{1,q}) \subset \mathbf{L}^q_\sigma(\Omega), && (1.22) \\
A_{o,B_e,q} F &= P_q \mathcal{L}^+_{B_e} F = P_q \left[(B_e \cdot \nabla) F + (F \cdot \nabla) B_e \right], \\
\mathcal{D}(A_{o,B_e,q}) &= \mathcal{D}(A^{1/2}_{2,q}) \subset \mathbf{L}^q_\sigma(\Omega), && (1.23) \\
L^-_{B_e} f &= P_q \mathcal{L}^-_{B_e} f = P_q \left[(B_e \cdot \nabla) f - (f \cdot \nabla) B_e \right], \\
\mathcal{D}(L^-_{B_e}) &= \mathcal{D}(A^{1/2}_{2,q}) \subset \mathbf{L}^q_\sigma(\Omega), && (1.24) \\
L^-_{y_e} F &= P_q \mathcal{L}^-_{y_e} F = P_q [(y_e \cdot \nabla) F - (F \cdot \nabla) y_e], \\
\mathcal{D}(L^-_{y_e}) &= \mathcal{D}(A^{1/2}_{2,q}) \subset \mathbf{L}^q_\sigma(\Omega), && (1.25) \\
\mathcal{N}_q \left(\begin{bmatrix} f \\ F \end{bmatrix} \right) &= \begin{bmatrix} P_q[(f \cdot \nabla) f - (F \cdot \nabla) F] \\ P_q[(f \cdot \nabla) F - (F \cdot \nabla) f] \end{bmatrix}. && (1.26)
\end{aligned}
$$

For $\mathcal{D}(A^{1/2}_{1,q})$ see (1.41b) below. $A_{1,q}$ is of course the Stokes operator of the fluid component and $A_{o,y_e,q}$ the corresponding Oseen perturbation operator, while $\mathcal{A}_{y_e,q} = -(\nu_f A_{1,q} + A_{o,y_e,q})$ is the corresponding Oseen operator. Similarly,

$A_{2,q}$ is the magnetic operator with $A_{o,B_e,q}$ the Oseen perturbation operator of the magnetic component. Next we apply the Helmholtz projector P_q to the z-equation (1.5b), recall (1.7), (1.8) and thus eliminate the pressure term, $P_q \nabla p = 0$:

$$z_t - \nu_f (P_q \Delta) z + (P_q \mathcal{L}^+_{y_e}) z - (P_q \mathcal{L}^+_{B_e} \mathbb{B}) + P_q[(z \cdot \nabla) z - (\mathbb{B} \cdot \nabla)\mathbb{B}] = P_q m u. \quad (1.27a)$$

Similarly, we apply P_q to the $\mathbb{B}$-equation in (1.5c) and obtain via (1.9), (1.10):

$$\mathbb{B}_t - \nu_m (P_q \Delta)\mathbb{B} + P_q \mathcal{L}^-_{y_e} \mathbb{B} - P_q \mathcal{L}^-_{B_e} z + P_q[(z \cdot \nabla)\mathbb{B} - (\mathbb{B} \cdot \nabla) z] = P_q m v. \quad (1.27b)$$

We combine (1.27a) and (1.27b) to obtain the following PDE problem

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} = \begin{bmatrix} \nu_f P_q \Delta & 0 \\ 0 & \nu_m P_q \Delta \end{bmatrix} \begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} + \begin{bmatrix} -P_q \mathcal{L}^+_{y_e} & P_q \mathcal{L}^+_{B_e} \\ P_q \mathcal{L}^-_{B_e} & -P_q \mathcal{L}^-_{y_e} \end{bmatrix} \begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} \\ \qquad\qquad - \mathcal{N}_q \begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} + \begin{bmatrix} P_q m u \\ P_q m v \end{bmatrix} \text{ in } Q, & (1.28a) \\ \text{div } z \equiv 0, \quad \text{div } \mathbb{B} \equiv 0 \text{ in } Q, & (1.28b) \\ z \equiv 0, \quad \mathbb{B} \cdot n \equiv 0, \quad \text{curl } \mathbb{B} \times n \equiv 0 \text{ in } \Sigma, & (1.28c) \\ z(0,x) = y_0 - y_e, \quad \mathbb{B}(0,x) = B_0 - B_e \text{ on } \Omega. & (1.28d) \end{cases}$$

Taking advantage of the divergence free conditions div $z = 0$, div $\mathbb{B} = 0$ in Q along with $z \equiv 0$ on Σ, and recalling $\mathbf{L}^q_\sigma(\Omega)$ in (1.15), we see that the corresponding abstract equation of the PDE-coupled system (1.28) is

$$\frac{d}{dt}\begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} = \begin{bmatrix} -\nu_f A_{1,q} & 0 \\ 0 & -\nu_m A_{2,q} \end{bmatrix} \begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} + \begin{bmatrix} -A_{o,y_e,q} & A_{o,B_e,q} \\ L^-_{B_e} & -L^-_{y_e} \end{bmatrix} \begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} - \mathcal{N}_q \begin{bmatrix} z \\ \mathbb{B} \end{bmatrix} + \begin{bmatrix} P_q m u \\ P_q m v \end{bmatrix} \text{ in } \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega). \quad (1.29)$$

1.10. Abstract translated linearized model

The linearized versions of the PDE-problem (1.28) and its corresponding abstract equation (1.29) are, respectively

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} = \begin{bmatrix} \nu_f P_q \Delta & 0 \\ 0 & \nu_m P_q \Delta \end{bmatrix} \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} -P_q \mathcal{L}^+_{y_e} & P_q \mathcal{L}^+_{B_e} \\ P_q \mathcal{L}^-_{B_e} & -P_q \mathcal{L}^-_{y_e} \end{bmatrix} \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} P_q m u \\ P_q m v \end{bmatrix} \text{ in } Q, & (1.30a) \\ \text{div } w \equiv 0, \quad \text{div } \mathbb{W} \equiv 0 \text{ in } Q, & (1.30b) \\ w \equiv 0, \quad \mathbb{W} \cdot n \equiv 0, \quad \text{curl } \mathbb{W} \times n \equiv 0 \text{ in } \Sigma & (1.30c) \end{cases}$$

along with the initial conditions $[w(0), \mathbb{W}(0)] = [w_0, \mathbb{W}_0]$, and

$$\frac{d}{dt}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} = \begin{bmatrix} -\nu_f A_{1,q} & 0 \\ 0 & -\nu_m A_{2,q} \end{bmatrix}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} -A_{o,y_e,q} & A_{o,B_e,q} \\ L^-_{B_e} & -L^-_{y_e} \end{bmatrix}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} P_q mu \\ P_q mv \end{bmatrix} \text{ in } \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega). \tag{1.31}$$

1.11. The Stokes operator $(-A_{1,q})$ and the magnetic operator $(-A_{2,q})$ generate strongly continuous, analytic, uniformly stable semigroups $e^{-A_{1,q}t}$ and $e^{-A_{2,q}t}$ on $\mathbf{L}^q_\sigma(\Omega)$, $1 < q < \infty$. [17, 35, 42, 47]

THEOREM 1.2. *Let $d \geq 2$, $1 < q < \infty$ and let Ω be a bounded domain in $\mathbb{R}^d$ of class C^3. Then*

Part A: On the Stokes operator $(-A_{1,q})$ and the corresponding Oseen operator $\mathcal{A}_{y_e,q} = -(\nu_f A_{1,q} + A_{o,y_e,q})$

(i) the Stokes operator $-A_{1,q} = P_q\Delta$ in (1.20), repeated here as

$$-A_{1,q}\psi = P_q\Delta\psi, \quad \psi \in \mathcal{D}(A_{1,q}) = \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega) \tag{1.32}$$

generates a s.c analytic semigroup $e^{-A_{1,q}t}$ on $\mathbf{L}^q_\sigma(\Omega)$. See [17, 35, 47] and the review paper [19, Theorem 2.8.5, p. 17]. For $q = 2$, $A_{1,q=2}$ is positive self-adjoint on $\mathbf{L}^2_\sigma(\Omega)$.

(ii) With reference to (1.31), the Oseen operator $\mathcal{A}_{y_e,q}$

$$\mathcal{A}_{y_e,q} = -(\nu_f A_{1,q} + A_{o,y_e,q}), \quad \mathcal{D}(\mathcal{A}_{y_e,q}) = \mathcal{D}(A_{1,q}) \subset \mathbf{L}^q_\sigma(\Omega) \tag{1.33}$$

generates a s.c analytic semigroup $e^{\mathcal{A}_{y_e,q}t}$ on $\mathbf{L}^q_\sigma(\Omega)$. This follows as $A_{o,y_e,q}$ is relatively bounded with respect to $A^{1/2}_{1,q}$, defined in (1.41b) see (1.22): thus a standard theorem on perturbation of an analytic semigroup generator applies [36, Corollary 2.4, p. 81].

(iii) One has

$$\begin{cases} 0 \in \rho(A_{1,q}) = \text{ the resolvent set of the Stokes operator } A_{1,q} & (1.34\text{a}) \\ A^{-1}_{1,q} : \mathbf{L}^q_\sigma(\Omega) \longrightarrow \mathbf{L}^q_\sigma(\Omega) \text{ is compact.} & (1.34\text{b}) \end{cases}$$

Similarly, the operator $\mathcal{A}_{y_e,q}$ has compact resolvent on $\mathbf{L}^q_\sigma(\Omega)$.

(iv) *The s.c. analytic Stokes semigroup $e^{-A_{1,q}t}$ is uniformly stable on $\mathbf{L}^q_\sigma(\Omega)$: there exist constants $M \geq 1, \delta > 0$ (possibly depending on q) such that*

$$\left\| e^{-A_{1,q}t} \right\|_{\mathcal{L}(\mathbf{L}^q_\sigma(\Omega))} \leq Me^{-\delta t}, \ t > 0. \tag{1.35}$$

Part B: On the magnetic operator $(-A_{2,q})$ and the corresponding perturbation operator $\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^-_{y_e})$

(v) *Similarly, the operator $-A_{2,q}$ in (1.22) generates a s.c. analytic semigroup $e^{-A_{2,q}t}$ on $\mathbf{L}^q_\sigma(\Omega)$, which moreover is uniformly stable here: there exist constants $M \geq 1$, $\delta > 0$ (possibly depending on q), such that*

$$\left\| e^{-A_{2,q}t} \right\|_{\mathcal{L}(\mathbf{L}^q_\sigma(\Omega))} \leq Me^{-\delta t}, \ t > 0 \tag{1.36a}$$

[34], repeated also in [37], [33, p. 968]. Moreover, $A_{2,q}$ has compact resolvent on $\mathbf{L}^q_\sigma(\Omega)$. For $q = 2$, $A_{2,q=2}$ is positive self adjoint on $\mathbf{L}^2_\sigma(\Omega)$ [34, p. 3382], [60], [14, Lemma 1]

(vi) *Likewise, with reference to (1.31), the operator*

$$\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^-_{y_e}), \quad \mathcal{D}(\mathcal{A}_{2,q}) = \mathcal{D}(A_{2,q}) \tag{1.36b}$$

generates a s.c. analytic semigroup $e^{\mathcal{A}_{2,q}t}$ on $\mathbf{L}^q_\sigma(\Omega)$ and it has a compact resolvent on $\mathbf{L}^q_\sigma(\Omega)$, as $\mathcal{D}(L^-_{y_e}) = \mathcal{D}(A^{\frac{1}{2}}_{2,q})$ on $\mathbf{L}^q_\sigma(\Omega)$ by (1.25).

Part C: On the linear diagonal principal part operator $\mathbb{A}_{o,q}$ of the linear $[w, \mathbb{W}]$-problem in (1.31)

(vii) *With reference to (1.31), the operator*

$$\mathbb{A}_{o,q} = \begin{bmatrix} -\nu_f A_{1,q} & 0 \\ 0 & -\nu_m A_{2,q} \end{bmatrix} :$$
$$\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \supset \mathcal{D}(\mathbb{A}_{o,q}) = \mathcal{D}(A_{1,q}) \times \mathcal{D}(A_{2,q}) \to \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega), \tag{1.37}$$

generates a s.c. analytic semigroup $e^{\mathbb{A}_{o,q}t}$ on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$, which is uniformly stable here: there exist constants $M \geq 1$, $\delta > 0$, such that

$$\left\| e^{\mathbb{A}_{o,q}t} \right\|_{\mathcal{L}(\mathbf{Y}^q_\sigma(\Omega))} \leq Me^{-\delta t}, \ t > 0, \ \mathbf{Y}^q_\sigma(\Omega) \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega). \tag{1.38}$$

Moreover, $\mathbb{A}_{o,q}$ has a compact resolvent on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \equiv \mathbf{Y}^q_\sigma(\Omega)$.

(viii) The operator

$$\widetilde{\mathbb{A}}_q = \mathbb{A}_{o,q} + \Pi = \begin{bmatrix} -\nu_f A_{1,q} & 0 \\ 0 & -\nu_m A_{2,q} \end{bmatrix} + \begin{bmatrix} -A_{o,y_e,q} & A_{o,B_e,q} \\ L^-_{B_e} & -L^-_{y_e} \end{bmatrix}$$
$$\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \supset \mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(A_{1,q}) \times \mathcal{D}(A_{2,q}) \to \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \equiv \mathbf{Y}^q_\sigma(\Omega)$$
$$\mathcal{D}(\Pi) \equiv \mathcal{D}((-\mathbb{A}_{o,q})^{1/2}), \quad by\ (1.22)\text{-}(1.25) \tag{1.39}$$

is the generator of the s.c. analytic semigroup $e^{\widetilde{\mathbb{A}}_q t}$ *on* $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \equiv \mathbf{Y}^q_\sigma(\Omega)$*, has compact resolvent here, so that the problem* (1.31) *is rewritten as*

$$\frac{d}{dt}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} = \widetilde{\mathbb{A}}_q \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix}, \quad \begin{bmatrix} w(0) \\ \mathbb{W}(0) \end{bmatrix} = \begin{bmatrix} w_0 \\ \mathbb{W}_0 \end{bmatrix} \ \ on\ \mathbf{Y}^q_\sigma(\Omega). \tag{1.40}$$

1.12. Domains of fractional powers, $\mathcal{D}(A^\alpha_{1,q})$, $0 < \alpha < 1$ of the Stokes operator $A_{1,q}$ and $\mathcal{D}(A^\alpha_{2,q})$, $0 < \alpha < 1$, of the magnetic operator $A_{2,q}$ on $\mathbf{L}^q_\sigma(\Omega)$, $1 < q < \infty$.

THEOREM 1.3. *For the domains of fractional powers* $\mathcal{D}(A^\alpha_{1,q}), 0 < \alpha < 1$*, of the Stokes operator* $A_{1,q}$ *in* (1.20) = (1.32)*, the following complex interpolation relation holds true [18] and [19, Theorem 2.8.5, p. 18]*

$$[\mathcal{D}(A_{1,q}), \mathbf{L}^q_\sigma(\Omega)]_{1-\alpha} = \mathcal{D}(A^\alpha_{1,q}),\ 0 < \alpha < 1,\ 1 < q < \infty; \tag{1.41a}$$

in particular

$$[\mathcal{D}(A_{1,q}), \mathbf{L}^q_\sigma(\Omega)]_{\frac{1}{2}} = \mathcal{D}(A^{1/2}_{1,q}) \equiv \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega). \tag{1.41b}$$

Thus, on the space $\mathcal{D}(A^{1/2}_{1,q})$*, the norms*

$$\|\nabla \cdot\|_{\mathbf{L}^q(\Omega)} \ \ and\ \ \|\ \|_{\mathbf{W}^{1,q}(\Omega)} \tag{1.41c}$$

are equivalent via Poincaré inequality.
Similarly, for the domains of fractional powers $\mathcal{D}(A^\alpha_{2,q})$ *of the magnetic operator* $A^\alpha_{2,q}$ *in* (1.21)

$$[\mathcal{D}(A_{2,q}), \mathbf{L}^q_\sigma(\Omega)]_{1-\alpha} = \mathcal{D}(A^\alpha_{2,q}),\ 0 < \alpha < 1,\ 1 < q < \infty;. \tag{1.42}$$

1.13. Linearized fluid and magnetic operators on the Besov space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$

Part A: The Stokes operator $-A_{1,q}$ and the Oseen operator $\mathcal{A}_{y_e,q}$, $1 < q < \infty$ generate s.c. analytic semigroups on the Besov spaces

$$(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} = \left\{ \mathbf{g} \in \mathbf{B}^{2-2/p}_{q,p}(\Omega) : \ \text{div}\ \mathbf{g} = 0,\ \mathbf{g}|_\Gamma = 0 \right\} \quad \text{if } \frac{1}{q} < 2 - \frac{2}{p} < 2; \tag{1.43a}$$

$$(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} = \left\{ \mathbf{g} \in \mathbf{B}^{2-2/p}_{q,p}(\Omega) : \ \text{div}\ \mathbf{g} = 0,\ \mathbf{g} \cdot n|_\Gamma = 0 \right\} \equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \quad \text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}. \tag{1.43b}$$

Theorem 1.2(i) states that the Stokes operator $-A_{1,q}$ generates a s.c analytic semigroup on the space $\mathbf{L}^q_\sigma(\Omega)$, $1 < q < \infty$, hence on the space $\mathcal{D}(A_{1,q})$ in (1.32), with norm $\| \cdot \|_{\mathcal{D}(A_{1,q})} = \|A_{1,q} \cdot \|_{\mathbf{L}^q_\sigma(\Omega)}$ as $0 \in \rho(A_{1,q})$. Then, one obtains that the Stokes operator $-A_{1,q}$ generates a s.c. analytic semigroup on the real interpolation spaces in (1.43). Next, the Oseen operator $\mathcal{A}_{y_e,q} = -(\nu_f A_{1,q} + A_{o,y_e,q})$ in (1.33) likewise generates a s.c. analytic semigroup $e^{\mathcal{A}_{y_e,q}t}$ on $\mathbf{L}^q_\sigma(\Omega)$ since $A_{o,y_e,q}$ is relatively bounded w.r.t. $A^{1/2}_{1,q}$, as $A_{o,y_e,q}A^{-1/2}_{1,q}$ is bounded on $\mathbf{L}^q_\sigma(\Omega)$. Moreover $\mathcal{A}_{y_e q}$ generates a s.c. analytic semigroup on $\mathcal{D}(\mathcal{A}_{y_e,q}) = \mathcal{D}(A_{1,q})$ (equivalent norms). Hence $\mathcal{A}_{y_e,q}$ generates a s.c. analytic semigroup on the real interpolation space of (1.43). Here below, however, we shall formally state the result only in the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ for the case $2 - 2/p < 1/q$. i.e. $1 < p < 2q/_{2q-1}$, as this does not contain B.C. The objective of the present paper is precisely to obtain stabilization results on spaces that do not recognize B.C.

THEOREM 1.4. *Let $1 < q < \infty, 1 < p < 2q/_{2q-1}$.*

(i) The Stokes operator $-A_{1,q}$ in (1.32) *generates a s.c. analytic semigroup $e^{-A_{1,q}t}$ on the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ defined in* (1.19b) = (1.43b) *which moreover is uniformly stable, as in* (1.35),

$$\left\| e^{-A_{1,q}t} \right\|_{\mathcal{L}\left(\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\right)} \le Me^{-\delta t}, \quad t > 0. \tag{1.44}$$

(ii) The Oseen operator $\mathcal{A}_{y_e,q}$ in (1.33) *generates a s.c. analytic semigroup $e^{\mathcal{A}_{y_e,q}t}$ on the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ in* (1.19b) = (1.43b).

Part B: The magnetic operator $-A_{2,q}$ and its corresponding perturbation $\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^-_{y_e})$ generate s.c. analytic semigroups on $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ for $1 < p < \dfrac{2q}{2q-1}$.

For $0 < 2 - \dfrac{2}{p} < \dfrac{1}{q}$, or $1 < p < \frac{2q}{2q-1}$, the homogeneous Dirichlet BC $\psi|_\Gamma = 0$ of the Stokes operator $(-A_{1,q})$ in (1.32) is not recognized in the interpolation characterization (1.43b). For the same range of p, then the higher level BC $(\text{curl } B) \times n = 0$ on Γ is likewise not recognized, and we have as in [1, (1.15a-b), Theorem 3.4], [16, p. 1399].

Since the operator $A_{2,q}$ has different boundary conditions from the Stokes operator $A_{1,q}$, we use the maximal L^p-regularity for the Stokes operator with Neumann, Robin or Navier boundary conditions to conclude that the magnetic operator $A_{2,q}$ has maximal L^p-regularity. The magnetic boundary conditions that we have to consider are first order boundary conditions which are comparable with such first order Neumann, Robin or Navier boundary conditions. More specifically

$$B \cdot n = 0, \quad (\text{curl } B) \times n = 0. \tag{1.45}$$

Then we refer to [43, Theorem 3.1], [45, Theorem 1.2], [44, Theorem 3.1] for maximal regularity up to $T < \infty$. For maximal L^p-regularity up to $T = \infty$, along with exponential stability we quote the subsequent theorems of the same references [43, Theorem 3.2], [45, Theorem 1.2], [44, Theorem 3.2]. Furthermore, to characterize the real interpolation space between $\mathbf{L}^q_\sigma(\Omega)$ and $\mathcal{D}(A_{2,q})$ we refer to [49] and [44, Remark 1.3] and symbolically

$$\begin{aligned}(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p} &= \Big\{\mathbf{B} \in \mathbf{B}^{2-2/p}_{q,p}(\Omega) : \text{ div } \mathbf{B} = 0, \, \mathbf{B} \cdot n|_\Gamma = 0\Big\} \\ &\equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \qquad \text{if } 1 - \frac{2}{p} < \frac{1}{q},\end{aligned} \tag{1.46a}$$

$$\begin{aligned}&(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p} \\ &\quad = \Big\{\mathbf{B} \in \mathbf{B}^{2-2/p}_{q,p}(\Omega) : \text{ div } \mathbf{B} = 0, \, \mathbf{B} \cdot n|_\Gamma = 0, \, (\text{curl } \mathbf{B}) \times n|_\Gamma = 0\Big\} \\ &\qquad \text{if } 1 - \frac{2}{p} > \frac{1}{q}.\end{aligned} \tag{1.46b}$$

From (1.43b), we have for the Stokes case, $(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ for $1 < p < {}^{2q}/_{2q-1}$. This implies that $1 < p < {}^{2q}/_{q-1}$. Then by (1.46a) we have $(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p} \equiv \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$. Similar to the Stokes setting in (1.43b), this does not contain B.C. and fits to the objective of the present paper.

THEOREM 1.5. *Let* $1 < q < \infty, 1 < p < {}^{2q}/_{2q-1}$.

(i) *The magnetic operator* $-A_{2,q}$ *in* (1.21) *generates a s.c. analytic semigroup* $e^{-A_{2,q}t}$ *on the space* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ *defined in* (1.43b) *or* (1.46)*, which moreover is uniformly stable, as in* (1.35)*,*

$$\left\| e^{-A_{2,q}t} \right\|_{\mathcal{L}\left(\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\right)} \le Me^{-\delta t}, \quad t > 0. \tag{1.47}$$

(ii) *The corresponding operator* $\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^{-}_{y_e})$ *in* (1.36b) *generates a s.c. analytic semigroup* $e^{\mathcal{A}_{2,q}t}$ *on the space* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ *in* (1.46)*.*

1.14. Space of maximal $\mathbf{L}^p$ regularity on $\mathbf{L}^q_\sigma(\Omega)$ of the Stokes operator $-A_{1,q}$, $1 < p < \infty$, $1 < q < \infty$ up to $T = \infty$.

We return to the dynamic Stokes problem in $\{\varphi(t,x), \pi(t,x)\}$

$$\begin{cases} \varphi_t - \Delta\varphi + \nabla\pi = F & \text{in } (0,T] \times \Omega \equiv Q, & (1.48a)\\ div\ \varphi \equiv 0 & \text{in } Q, & (1.48b)\\ \varphi|_{\Sigma} \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma, & (1.48c)\\ \varphi|_{t=0} = \varphi_0 & \text{in } \Omega, & (1.48d) \end{cases}$$

rewritten in abstract form, after applying the Helmholtz projection P_q to (1.48a) and recalling $A_{1,q}$ in (1.20) = (1.32) as

$$\varphi' + A_{1,q}\varphi = F_\sigma \equiv P_q F, \quad \varphi_0 \in (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p}. \tag{1.49}$$

Next, we introduce the space of maximal regularity for $\{\varphi, \varphi'\}$ as [19, p. 2; Theorem 2.8.5.iii, p. 17], [16, pp. 1404-5], with T up to ∞ (since $e^{-A_{1,q}t}$ is uniformly stable):

$$\mathbf{X}^T_{p,q,\sigma} = L^p(0,T; \mathcal{D}(A_{1,q})) \cap W^{1,p}(0,T; \mathbf{L}^q_\sigma(\Omega)) \tag{1.50}$$

(recall (1.32) for $\mathcal{D}(A_{1,q})$) and the corresponding space for the pressure as

$$Y^T_{p,q} = L^p(0,T; \widehat{W}^{1,q}(\Omega)), \quad \widehat{W}^{1,q}(\Omega) = W^{1,q}(\Omega)/\mathbb{R}. \tag{1.51}$$

The following embedding, also called trace theorem, holds true [2, Theorem 4.10.2, p. 180, BUC for $T = \infty$], [38].

$$\begin{aligned} \mathbf{X}^T_{p,q,\sigma} \subset \mathbf{X}^T_{p,q} \equiv L^p(0,T; \mathbf{W}^{2,q}(\Omega)) \cap W^{1,p}(0,T; \mathbf{L}^q(\Omega)) \\ \hookrightarrow C\left([0,T]; \mathbf{B}^{2-2/p}_{q,p}(\Omega)\right). \end{aligned} \tag{1.52}$$

For a function $\mathbf{g}$ such that div $\mathbf{g} \equiv 0$, $\mathbf{g}|_\Gamma = 0$ we have $\mathbf{g} \in \mathbf{X}^T_{p,q} \iff \mathbf{g} \in \mathbf{X}^T_{p,q,\sigma}$, by (1.15).
The solution of Eq (1.49) is

$$\varphi(t) = e^{-A_{1,q}t}\varphi_0 + \int_0^t e^{-A_{1,q}(t-\tau)} F_\sigma(\tau) d\tau. \tag{1.53}$$

The following is the celebrated result on maximal regularity on $\mathbf{L}^q_\sigma(\Omega)$ of the Stokes problem due originally to Solonnikov [48] reported in [19, Theorem 2.8.5.(iii) and Theorem 2.10.1, p. 24 for $\varphi_0 = 0$], [39], [16, Proposition 4.1, p. 1405].

THEOREM 1.6. *Let* $1 < p, q < \infty, T \leq \infty$. *With reference to problem* (1.48)=(1.49), *assume*

$$F_\sigma \in L^p(0,T;\mathbf{L}^q_\sigma(\Omega)), \;\; \varphi_0 \in (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p}. \tag{1.54}$$

Then, with reference to (1.47), (1.48), (1.49), *there exists a unique solution* $\varphi \in \mathbf{X}^T_{p,q,\sigma}$, $\pi \in Y^T_{p,q}$ *to the dynamic Stokes problem* (1.48) *or* (1.49), *continuously on the data: there exist constants* C_0, C_1 *independent of* T, F_σ, φ_0 *such that via* (1.52)

$$\begin{aligned} C_0 \left\|\varphi\right\|_{C\left([0,T];\mathbf{B}^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\varphi\|_{\mathbf{X}^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\ &\equiv \|\varphi'\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|A_{1,q}\varphi\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|\pi\|_{Y^T_{p,q}} \\ &\leq C_1 \left\{ \|F_\sigma\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|\varphi_0\|_{(\mathbf{L}^q_\sigma(\Omega),\mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p}} \right\}, \end{aligned} \tag{1.55}$$

$T \leq \infty$. *In particular,*

(i) *With reference to the variation of parameters formula* (1.53) *of problem* (1.49) *arising from the Stokes problem* (1.48), *we have recalling* (1.50)*: the map*

$$F_\sigma \longrightarrow \int_0^t e^{-A_{1,q}(t-\tau)} F_\sigma(\tau) d\tau \;\; : \textit{continuous} \tag{1.56}$$

$$L^p(0,T;\mathbf{L}^q_\sigma(\Omega)) \longrightarrow \mathbf{X}^T_{p,q,\sigma} \equiv L^p(0,T;\mathcal{D}(A_{1,q})) \cap W^{1,p}(0,T;\mathbf{L}^q_\sigma(\Omega)), \quad T \leq \infty. \tag{1.57}$$

(ii) *The s.c. analytic uniformly stable semigroup* $e^{-A_{1,q}t}$ *generated by the Stokes operator* $-A_{1,q}$ *(see* (1.20)*) on the space* $(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p}$ *(see statement below* (1.43b)*) satisfies*

$$\begin{aligned} &e^{-A_{1,q}t} : \textit{ continuous } \quad (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \\ &\longrightarrow \mathbf{X}^T_{p,q,\sigma} \equiv L^p(0,T;\mathcal{D}(A_{1,q})) \cap W^{1,p}(0,T;\mathbf{L}^q_\sigma(\Omega)), \quad T \leq \infty. \end{aligned} \tag{1.58}$$

In particular via (1.43b)*, for future use, for* $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$*, the s.c. analytic uniformly stable semigroup* $e^{-A_{1,q}t}$ *on the space* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$*, satisfies*

$$e^{-A_{1,q}t} : \textit{ continuous } \quad \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \longrightarrow \mathbf{X}^T_{p,q,\sigma}, \quad T \leq \infty. \tag{1.59}$$

(iii) *Moreover, for future use, for* $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$*, then (1.55) specializes to*

$$\|\varphi\|_{\mathbf{X}^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \leq C \left\{ \|F_\sigma\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|\varphi_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \right\}, \quad T \leq \infty. \tag{1.60}$$

1.15. Maximal L^p regularity on $\mathbf{L}^q_\sigma(\Omega)$ of the Oseen operator $\mathcal{A}_{y_e,q}$, $1 < p < \infty$, $1 < q < \infty$, up to $T < \infty$.

We next transfer the maximal regularity of the Stokes operator $(-A_{1,q})$ on $\mathbf{L}^q_\sigma(\Omega)$-asserted in Theorem 1.6 into the maximal regularity of the Oseen operator $\mathcal{A}_{y_e,q} = -\nu_f A_{1,q} - A_{o,q}$ in (1.33) exactly on the same space $\mathbf{X}^T_{p,q,\sigma}$ defined in (1.50), however only up to $T < \infty$.

Thus, consider the dynamic Oseen problem in $\{\psi(t,x), \pi(t,x)\}$ with equilibrium solution y_e:

$$\begin{cases} \psi_t - \Delta\psi + L_e(\psi) + \nabla\pi = F & \text{in } (0,T] \times \Omega \equiv Q, \quad (1.61a) \\ div\ \psi \equiv 0 & \text{in } Q, \quad (1.61b) \\ \psi|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma, \quad (1.61c) \\ \psi|_{t=0} = \psi_0 & \text{in } \Omega, \quad (1.61d) \end{cases}$$

$$L_e(\psi) = (y_e.\nabla)\psi + (\psi.\nabla)y_e \tag{1.62}$$

rewritten in abstract form, after applying the Helmholtz projector P_q to (1.61a) and recalling $\mathcal{A}_{y_e,q}$ in (1.33), as

$$\psi_t = \mathcal{A}_{y_e,q}\psi + P_q F = -\nu_f A_{1,q}\psi - A_{o,y_e,q}\psi + F_\sigma, \quad \psi_0 \in \left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p} \tag{1.63}$$

whose solution is

$$\psi(t) = e^{\mathcal{A}_{y_e,q}t}\psi_0 + \int_0^t e^{\mathcal{A}_{y_e,q}(t-\tau)} F_\sigma(\tau) d\tau. \tag{1.64}$$

$$\psi(t) = e^{-\nu_f A_{1,q}t}\psi_0 + \int_0^t e^{-\nu_f A_{1,q}(t-\tau)} F_\sigma(\tau) d\tau - \int_0^t e^{-\nu_f A_{1,q}(t-\tau)} A_{o,y_e,q}\psi(\tau) d\tau. \tag{1.65}$$

THEOREM 1.7. *Let $1 < p, q < \infty$, $0 < T < \infty$. Assume (as in (1.54))*

$$F_\sigma \in L^p\left(0, T; \mathbf{L}^q_\sigma(\Omega)\right), \quad \psi_0 \in \left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p} \tag{1.66}$$

where $\mathcal{D}(A_{1,q}) = \mathcal{D}(\mathcal{A}_{y_e,q})$, see (1.33). Then there exists a unique solution $\psi \in \mathbf{X}^T_{p,q,\sigma}$ in (1.50), $\pi \in Y^T_{p,q}$ in (1.51) of the dynamic Oseen problem (1.61), continuously on the data: that is, there exist constants C_0, C_1 independent of F_σ, ψ_0 such that

$$\begin{aligned} C_0 \|\psi\|_{C\left([0,T]; \mathbf{B}^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\psi\|_{\mathbf{X}^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\ &\equiv \|\psi'\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|A_{1,q}\psi\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|\pi\|_{Y^T_{p,q}} \\ &\leq C_T \left\{ \|F_\sigma\|_{L^p(0,T;\mathbf{L}^q_\sigma(\Omega))} + \|\psi_0\|_{(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p}} \right\} \end{aligned} \tag{1.67}$$

where $T < \infty$. Equivalently, for $1 < p$, $q < \infty$, (i) and (ii) below:

i. The map

$$F_\sigma \longrightarrow \int_0^t e^{\mathcal{A}_{y_e,q}(t-\tau)} F_\sigma(\tau) d\tau \ \text{ : continuous}$$
$$L^p(0,T; \mathbf{L}^q_\sigma(\Omega)) \longrightarrow L^p\left(0, T; \mathcal{D}(\mathcal{A}_{y_e,q}) = \mathcal{D}(A_{1,q})\right) \tag{1.68}$$

where then automatically, see (1.63)

$$L^p(0,T; \mathbf{L}^q_\sigma(\Omega)) \longrightarrow W^{1,p}(0,T; \mathbf{L}^q_\sigma(\Omega)) \tag{1.69}$$

and ultimately

$$L^p(0,T; \mathbf{L}^q_\sigma(\Omega)) \longrightarrow \mathbf{X}^T_{p,q,\sigma} \equiv L^p\left(0, T; \mathcal{D}(A_{1,q})\right) \cap W^{1,p}(0,T; \mathbf{L}^q_\sigma(\Omega)). \tag{1.70}$$

ii. The s.c. analytic semigroup $e^{\mathcal{A}_{y_e,q}t}$ generated by the Oseen operator $\mathcal{A}_{y_e,q}$ (see (1.33)) on the space $\left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p}$ satisfies for $1 < p, q < \infty$

$$e^{\mathcal{A}_{y_e,q}t} : \text{ continuous } \quad \left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p} \longrightarrow$$
$$L^p\left(0, T; \mathcal{D}(\mathcal{A}_{y_e,q}) = \mathcal{D}(A_{1,q})\right) \tag{1.71}$$

and hence automatically by (1.50)

$$e^{\mathcal{A}_{y_e,q}t} : \text{ continuous } \quad \left(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q})\right)_{1-\frac{1}{p},p} \longrightarrow \mathbf{X}^T_{p,q,\sigma}. \tag{1.72}$$

In particular, for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$, *we have that the s.c. analytic semigroup* $e^{\mathcal{A}_{y_e,q}t}$ *on the space* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$, *defined in* (1.43b) *satisfies*

$$e^{\mathcal{A}_{y_e,q}t}: \ \textit{continuous} \quad \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \longrightarrow L^p\left(0,T;\mathcal{D}(\mathcal{A}_{y_e,q}) = \mathcal{D}(A_{1,q})\right), \quad T < \infty. \tag{1.73}$$

and hence automatically

$$e^{\mathcal{A}_{y_e,q}t}: \ \textit{continuous} \quad \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \longrightarrow \mathbf{X}^T_{p,q,\sigma}, \ T < \infty. \tag{1.74}$$

A proof is given in [27, Appendix A].

1.16. Maximal L^p-regularity on $\mathbf{L}^q_\sigma(\Omega)$ of the magnetic operator $(-A_{2,q})$, up to $T = \infty$; and of the perturbation $\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^-_{y_e})$ up to $T < \infty$; $1 < q < \infty$.

Part A: We begin with the operator $(-A_{2,q})$ in (1.21) up to $T = \infty$. Thus, we consider the problem

$$\begin{cases} \psi_t - \Delta\psi = F & \text{in } (0,T] \times \Omega \equiv Q, & (1.75a) \\ div\ \psi \equiv 0 & \text{in } Q, & (1.75b) \\ \psi \cdot n = 0, \ (\text{curl } \psi) \times n \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma, & (1.75c) \\ \psi|_{t=0} = \psi_0 & \text{in } \Omega, & (1.75d) \end{cases}$$

or in abstract form (refer to the $\mathbb{W}$-equation (1.30a), (1.31))

$$\psi_t = -A_{2,q}\psi + F_\sigma, \quad F_\sigma = P_q F \tag{1.76}$$

or its variation of parameters formula

$$\psi(t) = e^{-A_{2,q}t}\psi_0 + \int_0^t e^{-A_{2,q}(t-\tau)} F_\sigma(\tau) d\tau. \tag{1.77}$$

The following result is the perfect equivalent of Theorem 1.6 for Stokes operator transported to the magnetic operator $(-A_{2,q})$, in the range $1 < p < \dfrac{2q}{2q-1}$ of our interest.

THEOREM 1.8. *Let* $1 < p,\ q < \infty$, $T \leq \infty$. *With reference to problem* (1.75) *or* (1.76), *we have* *(i) the map*

$$F_\sigma \longrightarrow \int_0^t e^{-A_{2,q}(t-\tau)}F_\sigma(\tau)d\tau \ : \quad \text{continuous } L^p(0,T;\mathbf{L}^q_\sigma(\Omega)) \longrightarrow L^p\left(0,T;\mathcal{D}(A_{2,q})\right) \tag{1.78}$$

so that then, by (1.76) *and* (1.78)

$$\psi_t \in L^p(0,T;\mathbf{L}^q_\sigma(\Omega)) \text{ continuously} \tag{1.79}$$

with respect to $F_\sigma \in L^p(0,T;\mathbf{L}^q_\sigma(\Omega))$.

(ii) Let $1 < q < \infty$, $1 < p < \dfrac{2q}{2q-1}$. *The s.c. analytic semigroup* $e^{-A_{2,q}t}$ *generated by the magnetic operator* $-A_{2,q}$ *in the space* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ *(recall* (1.46)*) satisfies*

$$e^{-A_{2,q}t} : \text{continuous } \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \longrightarrow L^p(0,T;\mathcal{D}(A_{2,q}))\cap W^{1,p}(0,T;\mathbf{L}^q_\sigma(\Omega)). \tag{1.80}$$

Part B: We now consider the perturbation $\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^-_{y_e})$, however up to $T < \infty$.

With reference to the $\mathbb{W}$-equation in (1.30a),we consider the uncoupled part $\nu_m \equiv 1$

$$\begin{cases} \psi_t - \Delta\psi + \mathcal{L}^-_{y_e}\psi = F & \text{in } (0,T]\times\Omega \equiv Q, \quad (1.81a)\\ div\ \psi \equiv 0 & \text{in } Q, \quad (1.81b)\\ \psi\cdot n,\ (\text{curl } \psi)\times n \equiv 0 & \text{in } (0,T]\times\Gamma \equiv \Sigma, \quad (1.81c)\\ \psi|_{t=0} = \psi_0 & \text{in } \Omega, \quad (1.81d) \end{cases}$$

or an abstract form (refer to (1.31))

$$\begin{aligned} \psi_t &= -A_{2,q}\psi - L^-_{y_e}\psi + F_\sigma, \quad F_\sigma = P_q F \\ &= \mathcal{A}_{2,q}\psi + F_\sigma, \end{aligned} \tag{1.82a}$$

recalling $A_{2,q}$ in (1.21), $L^-_{y_e}$ in (1.25), $\mathcal{A}_{2,q}$ in (1.36b). Let $\psi_0 = 0$. We write the variation of parameters formula of the problem (1.82) in two ways.

$$\begin{aligned} \psi(t) &= \int_0^t e^{\mathcal{A}_{2,q}(t-\tau)}F_\sigma(\tau)d\tau \\ &= \int_0^t e^{-A_{2,q}(t-\tau)}(-L^-_{y_e}\psi)(\tau)d\tau + \int_0^t e^{-A_{2,q}(t-\tau)}F_\sigma(\tau)d\tau. \end{aligned} \tag{1.83}$$

THEOREM 1.9 ([29, Theorem 1.8]). *Let $1 < p,\ q < \infty$, $T < \infty$. With reference to problem* (1.81)-(1.83)*, we have*

(i) the map (recall Theorem 1.4B)

$$F_\sigma \longrightarrow \int_0^t e^{\mathcal{A}_{2,q}(t-\tau)} F_\sigma(\tau) d\tau \ : \quad \text{continuous } L^p(0,T;\mathbf{L}^q_\sigma(\Omega)) \longrightarrow L^p\left(0,T;\mathcal{D}(\mathcal{A}_{2,q}) = \mathcal{D}(A_{2,q})\right) \tag{1.84}$$

so that then, by (1.82a)

$$\psi_t \in L^p(0,T;\mathbf{L}^q_\sigma(\Omega)) \text{ continuously} \tag{1.85}$$

with respect to $F_\sigma \in L^p(0,T;\mathbf{L}^q_\sigma(\Omega))$.

(ii) Let $1 < q < \infty$, $1 < p < \dfrac{2q}{2q-1}$. The s.c. analytic semigroup $e^{\mathcal{A}_{2,q}t}$ generated by the operator $\mathcal{A}_{2,q} = -(\nu_m A_{2,q} + L^-_{y_e})$ in (1.36b) *in the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ (recall* (1.46)*) satisfies*

$$e^{\mathcal{A}_{2,q}t} : \text{ continuous } \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \longrightarrow L^p(0,T;\mathcal{D}(\mathcal{A}_{2,q}) = \mathcal{D}(A_{2,q})) \cap W^{1,p}(0,T;\mathbf{L}^q_\sigma(\Omega)). \tag{1.86}$$

1.17. Maximal L^p-regularity on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ of the operator $\mathbb{A}_{o,q}$ in (1.37) up to $T = \infty$, and of the operator $\widetilde{\mathbb{A}}_q = \mathbb{A}_{o,q} + \Pi$ in (1.39) up to $T < \infty$, $1 < p,\ q < \infty$.

THEOREM 1.10. *(i) Consider the abstract problem*

$$\frac{d\mathbf{h}}{dt} = \mathbb{A}_{o,q}\mathbf{h} + \mathbf{F},\ \ \mathbb{A}_{o,q} = \begin{bmatrix} -\nu_f A_{1,q} & 0 \\ 0 & -\nu_m A_{2,q} \end{bmatrix} \quad \mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \mathbf{Y}^q_\sigma(\Omega) = \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \tag{1.87}$$

recalling $\mathbb{A}_{o,q}$ from (1.37)*. Then, for $\mathbf{h}_0 = 0$*

$$\begin{cases} \mathbf{F} \longrightarrow \mathbf{h}(t) \equiv \displaystyle\int_0^t e^{\mathbb{A}_{o,q}(t-\tau)} \mathbf{F}(\tau) d\tau : \\ \text{continuous } L^p(0,T;\mathbf{Y}^q_\sigma(\Omega)) \longrightarrow L^p(0,T;\mathcal{D}(\mathbb{A}_{o,q})). \end{cases} \tag{1.88}$$

(ii) Consider the abstract problem

$$\frac{d\mathbf{f}}{dt} = \widetilde{\mathbb{A}}_q \mathbf{f} + \mathbf{F},\ \widetilde{\mathbb{A}}_q = \mathbb{A}_{o,q} + \Pi,\ \Pi = \begin{bmatrix} -A_{o,y_e,q} & A_{o,B_e,q} \\ L^-_{B_e} & -L^-_{y_e} \end{bmatrix}. \tag{1.89}$$

recalling the operator Π *in* (1.39). *Then, for* $\mathbf{f}_0 = 0$ *and* $T < \infty$*,*

$$\begin{cases} \mathbf{F} \longrightarrow \mathbf{f}(t) \equiv \displaystyle\int_0^t e^{\widetilde{\mathbb{A}}_q(t-\tau)}\mathbf{F}(\tau)d\tau : \\ \textit{continuous } L^p(0,T;\mathbf{Y}^q_\sigma(\Omega)) \longrightarrow L^p(0,T;\mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(\mathbb{A}_{o,q})). \end{cases} \tag{1.90}$$

Proof. (i) is an immediate corollary of the maximal L^p-regularity of the Stokes operator $(-A_{1,q})$ up to $T = \infty$ in Theorem 1.6 and of maximal L^p-regularity of the magnetic operator $(-A_{2,q})$ up to $T = \infty$ in Theorem 1.8(i).

(ii) follows from (i) by perturbation [12, 23, 58], as Π is $\mathbb{A}^{1/2}_{o,q}$- bounded in $\mathbf{Y}^q_\sigma(\Omega)$, see (1.39). □

2. Spectral decomposition of the linearized problem

2.1. Preliminaries

We return to the linearized $[w, \mathbb{W}]$-problem (1.40) defined by the operator $\widetilde{\mathbb{A}}_q$ on $\mathbf{Y}^q_\sigma(\Omega) \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$. We have seen in Theorem 1.2(viii) that $\widetilde{\mathbb{A}}_q$ in the generator of a s.c. analytic semigroup $e^{\widetilde{\mathbb{A}}_q t}$ on $\mathbf{Y}^q_\sigma(\Omega)$ and, moreover, it has compact resolvent on $\mathbf{Y}^q_\sigma(\Omega)$. The assumption for the problem investigated in the present paper to be relevant is that: the generator $\widetilde{\mathbb{A}}_q$ of a s.c. analytic compact semigroup is <u>unstable</u> on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \equiv \mathbf{Y}^q_\sigma(\Omega)$, in the sense that there are N unstable eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ of $\widetilde{\mathbb{A}}_q$

$$\dots \leq Re\ \lambda_{N+2} \leq Re\ \lambda_{N+1} < 0 \leq Re\ \lambda_N \leq \dots \leq Re\ \lambda_2 \leq Re\ \lambda_1 \tag{2.1}$$

where the eigenvalues of $\widetilde{\mathbb{A}}_q$ are numbered in order of decreasing real parts. For each unstable eigenvalue $\lambda_i,\ i = 1, \dots, N$, let

$$\{\mathbf{\Phi}_{ij}\}_{j=1}^{\ell_i} = \left\{ \begin{bmatrix} \varphi_{ij} \\ \psi_{ij} \end{bmatrix} \right\}_{j=1}^{\ell_i}, \quad \varphi_{ij}, \psi_{ij} \in \mathbf{L}^q_\sigma(\Omega) \tag{2.2}$$

be the ℓ_i-linearly independent (normalized) eigenfunctions on $\mathbf{Y}^q_\sigma(\Omega) \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$, where ℓ_i denotes the geometric multiplicity of λ_i, and let $\bar{\lambda}_i$ be the

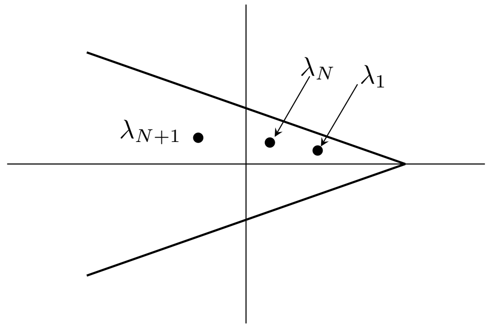

Figure 2: The eigenvalues of $\widetilde{\mathbb{A}}_q$

(unstable) eigenvalues of the $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$-adjoint $\widetilde{\mathbb{A}}^*_q$:

$$\widetilde{\mathbb{A}}_q \mathbf{\Phi}_{ij} = \lambda_i \mathbf{\Phi}_{ij}, \quad \mathbf{\Phi}_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}_q), \; i = 1, \dots, N, \; j = 1, \dots, \ell_i, \tag{2.3a}$$

$$\widetilde{\mathbb{A}}^*_q \mathbf{\Phi}^*_{ij} = \bar{\lambda}_i \mathbf{\Phi}^*_{ij}, \quad \mathbf{\Phi}^*_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}^*_q), \quad \mathbf{\Phi}^*_{ij} = \begin{bmatrix} \varphi^*_{ij} \\ \psi^*_{ij} \end{bmatrix}. \tag{2.3b}$$

We recall the linearized problem (1.31) in the notation of (1.40)

$$\frac{d}{dt} \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} = \widetilde{\mathbb{A}}_q \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} P_q m u \\ P_q m v \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix}. \tag{2.4}$$

The properties of the operator $\widetilde{\mathbb{A}}_q$ are collected in Theorem 1.2(viii). Accordingly, its eigenvalues satisfy their location property in Fig 2. Denote by P_N and P_N^* (which actually depend on q) the projections given explicitly in [22, p. 178]

$$P_N = -\frac{1}{2\pi i} \int_{\mathscr{C}} \left(\lambda I - \widetilde{\mathbb{A}}_q \right)^{-1} d\lambda : \mathbf{Y}^q_\sigma(\Omega) \text{ onto } (\mathbf{Y}^q_\sigma)^u_N \subset \mathbf{Y}^q_\sigma(\Omega) \tag{2.5}$$

$$P_N^* = -\frac{1}{2\pi i} \int_{\bar{\mathscr{C}}} \left(\lambda I - \widetilde{\mathbb{A}}^*_q \right)^{-1} d\lambda : (\mathbf{Y}^q_\sigma(\Omega))^* = \mathbf{Y}^{q'}_\sigma(\Omega) \text{ onto } [(\mathbf{Y}^q_\sigma)^u_N]^* \subset \mathbf{Y}^{q'}_\sigma(\Omega). \tag{2.6}$$

where $\mathscr{C}$ (respectively, its conjugate counterpart $\bar{\mathscr{C}}$) is a smooth closed curve that separates the unstable spectrum from the stable spectrum of $\widetilde{\mathbb{A}}_q$ (respectively, $\widetilde{\mathbb{A}}^*_q$). As in [4, Section 3.4, p. 37], following [50], [51], we decompose the space $\mathbf{Y}^q_\sigma = \mathbf{Y}^q_\sigma(\Omega) \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ into the sum of two complementary subspaces (not necessarily orthogonal):

$$\mathbf{Y}^q_\sigma = (\mathbf{Y}^q_\sigma)^u_N \oplus (\mathbf{Y}^q_\sigma)^s_N;\ (\mathbf{Y}^q_\sigma)^u_N \equiv P_N \mathbf{Y}^q_\sigma;\ (\mathbf{Y}^q_\sigma)^s_N \equiv (I - P_N)\mathbf{Y}^q_\sigma;\quad \dim\,(\mathbf{Y}^q_\sigma)^u_N = N \tag{2.7}$$

(superscript u = unstable; superscript s = stable), where each of the spaces $(\mathbf{Y}^q_\sigma)^u_N$ and $(\mathbf{Y}^q_\sigma)^s_N$ is invariant under $\widetilde{\mathbb{A}}_q$, and let

$$\widetilde{\mathbb{A}}^u_{q,N} = P_N \widetilde{\mathbb{A}}_q = \widetilde{\mathbb{A}}_q|_{(\mathbf{Y}^q_\sigma)^u_N};\quad \widetilde{\mathbb{A}}^s_{q,N} = (I - P_N)\widetilde{\mathbb{A}}_q = \widetilde{\mathbb{A}}_q|_{(\mathbf{Y}^q_\sigma)^s_N} \tag{2.8}$$

be the restrictions of $\widetilde{\mathbb{A}}_q$ to $(\mathbf{Y}^q_\sigma)^u_N$ and $(\mathbf{Y}^q_\sigma)^s_N$ respectively. The original point spectrum (eigenvalues) $\{\lambda_j\}_{j=1}^\infty$ of $\widetilde{\mathbb{A}}_q$ is then split into two sets,

$$\sigma(\widetilde{\mathbb{A}}^u_{q,N}) = \{\lambda_j\}_{j=1}^N;\quad \sigma(\widetilde{\mathbb{A}}^s_{q,N}) = \{\lambda_j\}_{j=N+1}^\infty, \tag{2.9}$$

and $(\mathbf{Y}^q_\sigma)^u_N$ is the generalized eigenspace of $\widetilde{\mathbb{A}}^u_{q,N}$. The system (2.4) on $\mathbf{Y}^q_\sigma \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ can accordingly be decomposed as

$$\boldsymbol{\eta} = \boldsymbol{\eta}_N + \boldsymbol{\zeta}_N,\quad \boldsymbol{\eta}_N = P_N\boldsymbol{\eta},\quad \boldsymbol{\zeta}_N = (I - P_N)\boldsymbol{\eta}. \tag{2.10}$$

After applying P_N and $(I - P_N)$ (which commute with $\widetilde{\mathbb{A}}_q$) on (2.4), we obtain via (2.8)

$$\text{on } (\mathbf{Y}^q_\sigma)^u_N : \boldsymbol{\eta}'_N - \widetilde{\mathbb{A}}^u_{q,N}\boldsymbol{\eta}_N = P_N \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix};\quad \boldsymbol{\eta}_N(0) = P_N \begin{bmatrix} w(0) \\ \mathbb{W}(0) \end{bmatrix} \tag{2.11}$$

$$\text{on } (\mathbf{W}^q_\sigma)^s_N : \boldsymbol{\zeta}'_N - \widetilde{\mathbb{A}}^s_{q,N}\boldsymbol{\zeta}_N = (I - P_N) \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix};\quad \boldsymbol{\zeta}_N(0) = (I - P_N) \begin{bmatrix} w(0) \\ \mathbb{W}(0) \end{bmatrix} \tag{2.12}$$

respectively.

Main Results. We may now state the main feedback stabilization result of the linearized problem (2.4). The proof is constructive. How to construct the finitely many stabilizing vectors will be established in the proof.

We anticipate the fact below that, for $1 < p, q < \infty$:

$$(\mathbf{Y}^q_\sigma)^u_N = \text{space of generalized eigenfunctions of } \widetilde{\mathbb{A}}_q \text{ (i.e. } \widetilde{\mathbb{A}}^u_N) \text{ corresponding to its distinct unstable eigenvalues} \subset \begin{cases} (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \\ [\mathcal{D}(A_{1,q}), \mathbf{L}^q_\sigma(\Omega)]_{1-\alpha} = \mathcal{D}(A^\alpha_{1,q}),\ 0 \le \alpha \le 1 \end{cases} \subset \mathbf{L}^q_\sigma(\Omega). \tag{2.13}$$

recalling (1.41a).

Remark 2.1. The original assumption

$$\ldots \le Re\ \lambda_{N+2} \le Re\ \lambda_{N+1} < 0 \le Re\ \lambda_N \le \ldots \le Re\ \lambda_2 \le Re\ \lambda_1 \tag{2.14}$$

at the beginning of Section 2.1 is intrinsic to the notion of 'stabilization', where by then one seeks to construct a feedback control that transforms an original unstable problem (with no control) into a stable one. However, as is well-known, the same entire procedure [50] can be employed to either stabilize an originally unstable system into a stable one with an arbitrary preassigned decay rate or else to enhance at will the stability of an originally stable one ($Re\ \lambda_1 < 0$).

2.2. Uniform stabilization with an arbitrary decay rate of the $\boldsymbol{\eta}_N$-dynamics (2.11) by a suitable finite-dimensional interior localized feedback control $\mathbf{u}_N = [u_N, v_N]$

Here below in Theorem 2.1 as well as in Theorem 3.2 below, we say the Finite Dimensional Spectral Assumption (FDSA) is satisfied to mean that for each of the distinct eigenvalue $\lambda_1, \ldots, \lambda_M$ of $\widetilde{\mathbb{A}}_q$ *algebraic and geometric multiplicity coincide*. Thus, the restriction $\widetilde{\mathbb{A}}_N$ in (2.11) is diagonalizable. In this case Kato [22, p. 41] calls the operator $\widetilde{\mathbb{A}}_{q,N}$ semisimple.

Theorem 2.1. *Let $1 < q < \infty$. Let $\lambda_1, \ldots, \lambda_M$ be the unstable distinct M eigenvalues of $\widetilde{\mathbb{A}}_q$ and let ω be an arbitrarily small open portion of the interior with smooth boundary $\partial\omega$. As established below in Theorem 3.2 under the FDSA (and in Theorem 4.2 [29] in the general case), there exist constructively infinitely many interior vectors $[\mathbf{u}_1, \ldots, \mathbf{u}_K]$ in $(\mathbf{Y}^q_\sigma)^u_N \subset \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$, $\mathbf{u}_i =$*

$[u_i^1, u_i^2]$ *such that the rank conditions*

$$\text{rank} \begin{bmatrix} (\mathbf{u}_1, \mathbf{\Phi}_{i1}^*)_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}_{i1}^*)_\omega \\ (\mathbf{u}_1, \mathbf{\Phi}_{i2}^*)_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}_{i2}^*)_\omega \\ \vdots & & \vdots \\ (\mathbf{u}_1, \mathbf{\Phi}_{i\ell_i}^*)_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}_{i\ell_i}^*)_\omega \end{bmatrix} = \ell_i; \ \ell_i \times K \text{ for each } i = 1, \dots, M, \tag{2.15a}$$

hold true with $K = \sup \ \{\ell_i : i = 1, \dots, M\}$ *where*

$$(\mathbf{u}_j, \mathbf{\Phi}_{i1}^*)_\omega = \left(\begin{bmatrix} u_j^1 \\ u_j^2 \end{bmatrix}, \begin{bmatrix} \varphi_{ij}^* \\ \psi_{ij}^* \end{bmatrix} \right)_{\mathbf{L}_\sigma^q(\omega) \times \mathbf{L}_\sigma^q(\omega)}. \tag{2.15b}$$

That is, the matrix in (2.15a) *is full rank.*

Then: Given $\gamma > 0$ *arbitrarily large, there exists a* K*-dimensional interior controller* $\mathbf{u} = \mathbf{u}_N = \{u_N^1, u_N^2\}$ *acting on* ω*, of the form given by*

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{u} = \mathbf{u}_N = \sum_{k=1}^{K} \mu_k(t) \mathbf{u}_k,$$
$$\mathbf{u}_k = \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \in (\mathbf{Y}_\sigma^q)_N^u \subset \mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega) \equiv \mathbf{Y}_\sigma^q(\Omega), \mu_k(t) = \text{scalar} \tag{2.15c}$$

see (3.10) *below, with the vectors* $\mathbf{u}_k = \{u_k^1, u_k^2\}$ *given in* (3.10) *below under the FDSA (and in Theorem 4.2 of [29] in the general case) via the rank conditions* (2.15a)*, such that, once inserted in the dynamics* (2.11) *yield the estimate*

$$\|\boldsymbol{\eta}_N(t)\|_{\mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega)} + \|\mathbf{u}_N(t)\|_{\mathbf{L}_\sigma^q(\omega) \times \mathbf{L}_\sigma^q(\omega)} \leq C_\gamma e^{-\gamma t} \|P_N \boldsymbol{\eta}_0\|_{\mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega)}, \ t \geq 0, \tag{2.16a}$$

where the $\mathbf{L}_\sigma^q(\Omega)$*-norm in* (2.16a) *may be replaced by the* $(\mathbf{L}_\sigma^q(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p}$*-norm ,* $1 < p, q < \infty$*; in particular the* $\widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\Omega)$*-norm in* (1.19b)*,* $1 < q < \infty$*,* $1 < p < \dfrac{2q}{2q-1}$*:*

$$\|\boldsymbol{\eta}_N(t)\|_{\widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\Omega) \times \widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\Omega)} + \|\mathbf{u}_N(t)\|_{\widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\omega) \times \widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\omega)} \leq C_\gamma e^{-\gamma t} \|P_N \boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\Omega) \times \widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\Omega)}, \ t \geq 0. \tag{2.16b}$$

Here, $\boldsymbol{\eta}_N$ *is the solution of Eq* (3.12) *under the FDSA (or (4.15) of [29] in the general case), i.e.* (3.7) *corresponding to the control* $\mathbf{u} = \mathbf{u}_N$ *obtained in*

(2.15c). *Moreover, such controller* $\mathbf{u} = \mathbf{u}_N$ *in* (2.15c) *can be chosen in feedback form: that is, with reference to the explicit expression* (2.15c) = (3.10) *for* $\mathbf{u}$, *of the form* $\mu_k(t) = (\boldsymbol{\eta}_N(t), \mathbf{p}_k)_\omega$ *for suitably constructed vectors* $\mathbf{p}_k = \{p_k^1, p_k^2\} \in ((\mathbf{Y}_\sigma^q)_N^u)^* \subset \mathbf{L}_\sigma^{q'}(\Omega) \times \mathbf{L}_\sigma^{q'}(\Omega)$ *depending on* γ. *In conclusion,* $\boldsymbol{\eta}_N$ *in* (2.16) *is the solution of the, feedback equation on* $(\mathbf{Y}_\sigma^q)_N^u$ *(see* (2.7)*)*

$$\boldsymbol{\eta}_N' - \widetilde{\mathbb{A}}_{q,N}^u \boldsymbol{\eta}_N = P_N P_q \left(m \left(\sum_{k=1}^K (\boldsymbol{\eta}_N(t), \mathbf{p}_k)_\omega \mathbf{u}_k \right) \right),$$
$$\mathbf{u}_k \in (\mathbf{Y}_\sigma^q)_N^u \subset \mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega), \;\; \mathbf{p}_k \in ((\mathbf{Y}_\sigma^q)_N^u)^* \subset \mathbf{L}_\sigma^{q'}(\Omega) \times \mathbf{L}_\sigma^{q'}(\Omega) \tag{2.17}$$

rewritten (since it is linear) as

$$\boldsymbol{\eta}_N' = \bar{A}^u \boldsymbol{\eta}_N, \quad \boldsymbol{\eta}_N(t) = e^{\bar{A}^u t} P_N \boldsymbol{\eta}_0, \quad \boldsymbol{\eta}_N(0) = P_N \boldsymbol{\eta}_0. \tag{2.18}$$

The proof will be given in Section 4.

2.3. Feedback stabilization of the original linearized $\boldsymbol{\eta} \equiv \{w, \mathbb{W}\}$-system (2.4) by a finite dimensional feedback controller $\boldsymbol{u} = [u, v]$

THEOREM 2.2. *Let* $1 < q < \infty$. *Let the linearized operator* $\widetilde{\mathbb{A}}_q$ *have N possibly repeated unstable eigenvalues* $\{\lambda_j\}_{j=1}^N$ *of which M are distinct. Let* $\varepsilon > 0$ *and set* $\gamma_0 = |Re\, \lambda_{N+1}| - \varepsilon$, *see Fig 1. Consider the setting of Theorem 2.1 so that, in particular, the feedback finite-dimensional control* $\mathbf{u} = \mathbf{u}_N$ *is given by* $\mathbf{u} = \mathbf{u}_N = \sum_{k=1}^K (\boldsymbol{\eta}_N(t), \mathbf{p}_k)\mathbf{u}_k$ *and satisfies estimates* (2.16a), (2.16b) *with* $\gamma > 0$ *arbitrarily large, for vectors* $\mathbf{p}_1, \dots, \mathbf{p}_k \in ((\mathbf{Y}_\sigma^q)_N^u)^* \subset \mathbf{L}_\sigma^{q'}(\Omega) \times \mathbf{L}_\sigma^{q'}(\Omega)$ *and vectors* $\mathbf{u}_1, \dots, \mathbf{u}_k \in (\mathbf{Y}_\sigma^q)_N^u \subset \mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega)$ *given by Theorem 2.1, as established in Section 4. Thus, the linearized problem* (2.4) *specializes to* (2.19) *with* $\boldsymbol{\eta} = \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix}$ *and* $\boldsymbol{\eta}_N = P_N \boldsymbol{\eta}$

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} &= \widetilde{\mathbb{A}}_q \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix} \\ &= \widetilde{\mathbb{A}}_q \boldsymbol{\eta} + P_q \left(m \left(\sum_{k=1}^K (\boldsymbol{\eta}_N(t), \mathbf{p}_k)_\omega \mathbf{u}_k \right) \right) \equiv \mathbb{A}_{_F} \boldsymbol{\eta} = \mathbb{A}_{_F} \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix}. \end{aligned} \tag{2.19}$$

Here $\mathbb{A}_{_F} = \mathbb{A}_{_{F,q}}$ *is the generator of a s.c. analytic semigroup on either the space* $\mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega)$, $1 < q < \infty$, *or on the space* $(\mathbf{L}_\sigma^q(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}_\sigma^q(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}$, $1 < p, q < \infty$, *in particular on the space* $\widetilde{\mathbf{B}}_{q,p}^{2-2/p}(\Omega) \times$

$\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega), 1 < q, 1 < p < 2q/_{2q-1}$, *recall* (1.19b). *Moreover, such dynamics* $\boldsymbol{\eta}$ *(equivalently, generator* $\mathbb{A}_F$*) in* (2.19) *is uniformly stable in each of these spaces, say*

$$\begin{aligned}
\left\| e^{\mathbb{A}_F t}\boldsymbol{\eta}_0 \right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} & \\
&= \|\boldsymbol{\eta}(t,\boldsymbol{\eta}_0)\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} \\
&= \left\| \begin{bmatrix} w(t,w_0) \\ \mathbb{W}(t,\mathbb{W}_0) \end{bmatrix} \right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} \\
&\leq C_{\gamma_0} e^{-\gamma_0 t} \|\boldsymbol{\eta}_0\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)}, \ \boldsymbol{\eta}_0 = \begin{bmatrix} w_0 \\ \mathbb{W}_0 \end{bmatrix}, \ t \geq 0.
\end{aligned} \tag{2.20}$$

$\gamma_0 = |Re\,\lambda_{N+1}| - \varepsilon$ *or, for* $0 < \theta < 1$, *and* $\delta > 0$ *arbitrarily small*

$$\left\| \begin{bmatrix} A^\theta_{1,q} \\ A^\theta_{2,q} \end{bmatrix} e^{\mathbb{A}_F t}\boldsymbol{\eta}_0 \right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} = \left\| \begin{bmatrix} A^\theta_{1,q} \\ A^\theta_{2,q} \end{bmatrix} \boldsymbol{\eta}(t,\boldsymbol{\eta}_0) \right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)}$$

$$\leq \begin{cases} C_{\gamma_0,\theta} e^{-\gamma_0 t} \left\| \begin{bmatrix} A^\theta_{1,q} \\ A^\theta_{2,q} \end{bmatrix} \boldsymbol{\eta}_0 \right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)}, & \boldsymbol{\eta}_0 \in \mathcal{D}(A^\theta_{1,q})\times\mathcal{D}(A^\theta_{2,q}), \ t \geq 0, \\ C_{\gamma_0,\theta,\delta} e^{-\gamma_0 t} \|\boldsymbol{\eta}_0\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)}, \ t \geq \delta > 0. & \end{cases} \tag{2.21}$$

In (2.21), *we have recalled the fractional powers of the (-Stokes) operator* $A_{1,q}$ *in* (1.20) = (1.32), *and similarly for the magnetic operator* $A_{2,q}$ *in* (1.21). *As in the case of Theorem 2.1, we may replace the* $\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)$*-norm in* (2.20), $1 < q < \infty$, *with the* $(\mathbf{L}^q_\sigma(\Omega),\mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}^q_\sigma(\Omega),\mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}$*-norm* $1 < p, q < \infty$, *in particular with* $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$*-norm, see* (1.19b) *or say*

$$\begin{aligned}
\left\| e^{\mathbb{A}_F t}\boldsymbol{\eta}_0 \right\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} &= \|\boldsymbol{\eta}(t,\boldsymbol{\eta}_0)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\
&= \left\| \begin{bmatrix} w(t,w_0) \\ \mathbb{W}(t,\mathbb{W}_0) \end{bmatrix} \right\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\
&\leq C_{\gamma_0} e^{-\gamma_0 t} \|\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0,
\end{aligned} \tag{2.22}$$

$\gamma_0 = |Re\,\lambda_{N+1}| - \varepsilon$, *see Figure 1.*

The proof is given in Section 5.

3. Algebraic Rank Condition for the $\boldsymbol{\eta}_N$-dynamics in (2.11) on $(\mathbf{Y}^q_\sigma(\Omega))^u_N$ under the (preliminary) Finite-Dimensional Spectral Assumption

Preliminaries. Let M be the number of distinct unstable eigenvalues of $\widetilde{\mathbb{A}}_q$ (or $\widetilde{\mathbb{A}}^*_q$). For each $i = 1, \dots, M$, we denote by

$$\{\mathbf{\Phi}_{ij}\}_{j=1}^{\ell_i} = \left\{ \begin{bmatrix} \varphi_{ij} \\ \psi_{ij} \end{bmatrix} \right\}_{j=1}^{\ell_i}, \{\mathbf{\Phi}^*_{ij}\}_{j=1}^{\ell_i} = \left\{ \begin{bmatrix} \varphi^*_{ij} \\ \psi^*_{ij} \end{bmatrix} \right\}_{j=1}^{\ell_i}$$

the normalized, linearly independent eigenfunctions of $\widetilde{\mathbb{A}}_q$, respectively $\widetilde{\mathbb{A}}^*_q$, say, on

$$\begin{aligned} &\mathbf{Y}^q_\sigma(\Omega) \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \text{ and} \\ &\quad (\mathbf{Y}^q_\sigma(\Omega))^* \equiv (\mathbf{L}^q_\sigma(\Omega))' \times (\mathbf{L}^q_\sigma(\Omega))' = \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1, \end{aligned} \tag{3.1}$$

(where in the last equality we have invoked the identity (A.2) in Appendix A of [27]) corresponding to the M distinct unstable eigenvalues $\lambda_1, \dots, \lambda_M$ of $\widetilde{\mathbb{A}}_q$ and $\overline{\lambda}_1, \dots, \overline{\lambda}_M$ of $\widetilde{\mathbb{A}}^*_q$ respectively,:

$$\begin{aligned} \widetilde{\mathbb{A}}_q \mathbf{\Phi}_{ij} &= \lambda_i \mathbf{\Phi}_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(A_{1,q}) \times \mathcal{D}(A_{2,q}) \\ &= [\mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega)] \times [\mathbf{W}^{2,q}(\Omega) \cap \mathbf{L}^q_\sigma(\Omega)] \end{aligned} \tag{3.2}$$

$$\begin{aligned} \widetilde{\mathbb{A}}^*_q \mathbf{\Phi}^*_{ij} &= \bar{\lambda}_i \mathbf{\Phi}^*_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}^*_q) \\ &\quad = [\mathbf{W}^{2,q'}(\Omega) \cap \mathbf{W}^{1,q'}_0(\Omega) \cap \mathbf{L}^{q'}_\sigma(\Omega)] \times [\mathbf{W}^{2,q'}(\Omega) \cap \mathbf{L}^{q'}_\sigma(\Omega)]. \end{aligned} \tag{3.3}$$

recalling (1.20), (1.21), (1.39):

$$\begin{array}{cccc} & \bar{\lambda}_i & & \\ \swarrow & \downarrow & \cdots & \searrow \\ \mathbf{\Phi}^*_{i1}, & \mathbf{\Phi}^*_{i2}, & \cdots & \mathbf{\Phi}^*_{i\ell_i} \end{array} \qquad \ell_i = \text{geometric multiplicity}$$

The Finite Dimensional Spectral Assumption (FDSA)

As noted at the beginning of Section 2.2, we henceforth assume in this section the Finite Dimensional Spectral Assumption (FDSA). This means that for each of the distinct eigenvalues $\lambda_1, \dots, \lambda_M$ of $\widetilde{\mathbb{A}}_q$, **algebraic and geometric multiplicity coincide**:

$$\begin{aligned} &(\mathbf{Y}^q_\sigma)^u_{N,i} \equiv P_{N,i} \mathbf{Y}^q_\sigma(\Omega) = \operatorname{span}\{\mathbf{\Phi}_{ij}\}_{j=1}^{\ell_i}; \\ &\qquad (\mathbf{Y}^{q'}_\sigma)^u_{N,i} \equiv P^*_{N,i}(\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega))^* = \operatorname{span}\{\mathbf{\Phi}^*_{ij}\}_{j=1}^{\ell_i}; \end{aligned} \tag{3.4}$$

Here $P_{N,i}, P^*_{N,i}$ are the projections corresponding to the eigenvalues λ_i and $\bar{\lambda}_i$, respectively. For instance, $P_{N,i}$ is given by an integral such as the one on the RHS of (2.5), where now $\mathscr{C}$ is a closed smooth curve encircling the eigenvalue λ_i and no other. Similarly for $P^*_{N,i}$. The space $\mathbf{Y}^u_{N,i}$ = range of $P_{N,i}$ is the algebraic eigenspace of the eigenvalues λ_i, and $\ell_i = \dim \mathbf{Y}^u_{N,i}$ is the algebraic = geometric multiplicity of λ_i, so that $\ell_1 + \ell_2 + \cdots + \ell_M = N$. As a consequence of the FDSA, we obtain

$$(\mathbf{Y}^q_\sigma)^u_N \equiv P_N\,[\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)] = \text{span}\{\mathbf{\Phi}_{ij}\}^{M}_{i=1,}{}^{\ell_i}_{j=1}; \tag{3.5a}$$

$$((\mathbf{Y}^q_\sigma)^*)^u_N = (\mathbf{Y}^{q'}_\sigma)^u_N \equiv P^*_N\,[(\mathbf{L}^q_\sigma(\Omega))^* \times (\mathbf{L}^q_\sigma(\Omega))^*] = \text{span}\{\mathbf{\Phi}^*_{ij}\}^{M}_{i=1,}{}^{\ell_i}_{j=1}. \tag{3.5b}$$

Without the FDSA, $(\mathbf{Y}^q_\sigma)^u_N$ is the span of the generalized eigenfunctions of $\widetilde{\mathbb{A}}_q$, corresponding to its unstable distinct eigenvalues $\{\lambda_j\}^M_{j=1}$; and similarly for $((\mathbf{Y}^q_\sigma)^*)^u_N$ (see the subsequent section). In other words, the FDSA says that the restriction $\widetilde{\mathbb{A}}^u_{q,N}$ in (2.11) is *diagonalizable* or that $\widetilde{\mathbb{A}}^u_{q,N}$ is *semisimple* on $(\mathbf{Y}^q_\sigma)^u_N$ in the terminology of [22, p. 41]. Under the FDSA, any vector $\boldsymbol{\eta} \in (\mathbf{Y}^q_\sigma)^u_N$ admits the following unique expansion [22, p. 12, Eq. (2.16)], [6, p. 1453], in terms of the basis $\{\mathbf{\Phi}_{ij}\}^{M}_{i=1,}{}^{\ell_i}_{j=1}$ in $(\mathbf{Y}^q_\sigma)^u_N$ and its adjoint basis [22, p. 12] $\{\mathbf{\Phi}^*_{ij}\}^{M}_{i=1,}{}^{\ell_i}_{j=1}$ in $((\mathbf{Y}^q_\sigma)^*)^u_N = (\mathbf{Y}^{q'}_\sigma)^u_N$:

$$(\mathbf{Y}^q_\sigma)^u_N \ni \boldsymbol{\eta} = \sum_{i,j}^{M,\ell_i} (\boldsymbol{\eta}, \mathbf{\Phi}^*_{ij})\mathbf{\Phi}_{ij}; \quad (\mathbf{\Phi}_{ij}, \mathbf{\Phi}^*_{hk}) = \begin{cases} 1 & \text{if } i = h,\ j = k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

that is, the system consisting of $\{\mathbf{\Phi}_{ij}\}$ and $\{\mathbf{\Phi}^*_{hk}\}$, $i = 1, \dots, M,\ j = 1, \dots, \ell_i$, can be chosen to form bi-orthogonal sequences. Here (,) denotes the scalar product between $(\mathbf{Y}^q_\sigma)^u_N$ and $(\mathbf{Y}^{q'}_\sigma)^u_N$ [22, p. 12]. i.e. ultimately, the duality pairing in Ω between $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ and $(\mathbf{L}^q_\sigma(\Omega))^* \times (\mathbf{L}^q_\sigma(\Omega))^*$. Next, we return to the $\boldsymbol{\eta}_N$-dynamics in (2.11), rewritten here for convenience

$$\text{on } (\mathbf{Y}^q_\sigma)^u_N : \boldsymbol{\eta}'_N - \widetilde{\mathbb{A}}^u_{q,N}\boldsymbol{\eta}_N = P_N \begin{bmatrix} P_q m u \\ P_q m v \end{bmatrix}; \quad \boldsymbol{\eta}_N(0) = P_N \begin{bmatrix} w(0) \\ \mathbb{W}(0) \end{bmatrix}. \tag{3.7}$$

The term $P_N \begin{bmatrix} P_q m u \\ P_q m v \end{bmatrix}$ expressed in terms of adjoint bases.

Let $mu \in \mathbf{L}^q(\omega),\ mv \in \mathbf{L}^q(\omega),\ q > 1$. In the computation below, we notice that $P^*_N \mathbf{\Phi}^*_{ij} = \mathbf{\Phi}^*_{ij}$ as $\mathbf{\Phi}^*_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}^*_q)$, so that $\mathbf{\Phi}^*_{ij}$ is invariant under the projections

P_N^*. Similarly, $P_q^*\varphi_{ij}^* = \varphi_{ij}^*$, $P_q^*\psi_{ij}^* = \psi_{ij}^*$. With $(f,g)_\omega = \displaystyle\int_\omega f\bar{g}\, d\omega$, we obtain

$$\begin{aligned}(\mathbf{Y}_\sigma^q)_N^u \ni P_N \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix} &= \sum_{i,j=1}^{M,\ell_i} \left(P_N \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix}, \begin{bmatrix} \varphi_{ij}^* \\ \psi_{ij}^* \end{bmatrix} \right) \mathbf{\Phi}_{ij}, \\ &= \sum_{i,j=1}^{M,\ell_i} \left(\begin{bmatrix} P_q(mu) \\ umP_q(mv) \end{bmatrix}, \begin{bmatrix} \varphi_{ij}^* \\ \psi_{ij}^* \end{bmatrix} \right) \mathbf{\Phi}_{ij}, \\ &= \sum_{i,j=1}^{M,\ell_i} \left(\begin{bmatrix} mu \\ mv \end{bmatrix}, \begin{bmatrix} \varphi_{ij}^* \\ \psi_{ij}^* \end{bmatrix} \right) \mathbf{\Phi}_{ij}, \\ &= \sum_{i,j=1}^{M,\ell_i} \left(\mathbf{u}, \mathbf{\Phi}_{ij}^*\right)_\omega \mathbf{\Phi}_{ij}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned} \tag{3.8}$$

so that the dynamics (3.7) on $(\mathbf{Y}_\sigma^q)_N^u$ becomes by (3.8)

$$\text{on } (\mathbf{Y}_\sigma^q)_N^u : \boldsymbol{\eta}_N' - \widetilde{\mathbb{A}}_{q,N}^u \boldsymbol{\eta}_N = \sum_{i,j=1}^{M,\ell_i} (\mathbf{u}, \mathbf{\Phi}_{ij}^*)_\omega \mathbf{\Phi}_{ij}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{3.9}$$

Selection of the scalar interior control function $\mathbf{u} = \mathbf{u}_N$ in finite dimensional separated form (with respect to K coordinates).

Next, we select the control $\mathbf{u} = \mathbf{u}_N$ of the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{u} = \sum_{k=1}^{K} \mu_k(t)\mathbf{u}_k, \quad \mathbf{u}_k = \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} \in (\mathbf{Y}_\sigma^q)_N^u \subset \mathbf{L}_\sigma^q(\Omega) \times \mathbf{L}_\sigma^q(\Omega) \equiv \mathbf{Y}_\sigma^q(\Omega),$$
$$\mu_k(t) = \text{scalar} \tag{3.10}$$

so that the term in (3.8) in $(\mathbf{Y}_\sigma^q)_N^u$ specializes to

$$(\mathbf{Y}_\sigma^q)_N^u \ni P_N \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix} = \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} \left(\begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix}, \mathbf{\Phi}_{ij}^* \right)_\omega \mu_k(t) \right\} \mathbf{\Phi}_{ij}. \tag{3.11}$$

Substituting (3.11) on the RHS of (3.7), we finally obtain

$$\text{on } (\mathbf{Y}_\sigma^q)_N^u : \boldsymbol{\eta}_N' - \widetilde{\mathbb{A}}_{q,N}^u \boldsymbol{\eta}_N = \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} \left(\begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix}, \mathbf{\Phi}_{ij}^* \right)_\omega \mu_k(t) \right\} \mathbf{\Phi}_{ij}. \tag{3.12}$$

The dynamics (3.12) in coordinate form on $(\mathbf{Y}^q_\sigma)^u_N$.

Our next goal is to express the finite dimensional dynamics (3.12) on the N-dimensional space $(\mathbf{Y}^q_\sigma)^u_N$ in a component-wise form. To this end, we introduce the following ordered bases β_i and β of length ℓ_i and N respectively:

$$\begin{aligned}\beta_i &= [\boldsymbol{\Phi}_{i1}, \dots, \boldsymbol{\Phi}_{i\ell_i}]: \quad \text{basis on } (\mathbf{Y}^q_\sigma)^u_{N,i}\\ \beta &= \beta_1 \cup \beta_2 \cup \dots \cup \beta_M \\ &= [\boldsymbol{\Phi}_{11}, \dots, \boldsymbol{\Phi}_{1\ell_1}, \boldsymbol{\Phi}_{21}, \dots, \boldsymbol{\Phi}_{2\ell_2}, \dots, \boldsymbol{\Phi}_{M1}, \dots, \boldsymbol{\Phi}_{M\ell_M}]: \text{ basis on } (\mathbf{Y}^q_\sigma)^u_N.\end{aligned} \tag{3.13}$$

Thus, we can represent the N-dimensional vector $\boldsymbol{\eta}_N \in (\mathbf{Y}^q_\sigma)^u_N$ as column vector $\hat{\boldsymbol{\eta}}_N = [\boldsymbol{\eta}_N]_\beta$ as,

$$\boldsymbol{\eta}_N = \sum_{i,j=1}^{M,\ell_i} \boldsymbol{\eta}^{ij}_N \boldsymbol{\Phi}_{ij};$$

and set $\hat{\boldsymbol{\eta}}_N = \text{col}\,[\boldsymbol{\eta}^{1,1}_N, \dots, \boldsymbol{\eta}^{1,\ell_1}_N, \dots, \boldsymbol{\eta}^{i,1}_N, \dots, \boldsymbol{\eta}^{i,\ell_i}_N, \dots, \boldsymbol{\eta}^{M,1}_N, \dots, \boldsymbol{\eta}^{M,\ell_M}_N]$.

REMARK 3.1. The eigenfunction $\boldsymbol{\Phi}_{ij} = \{\varphi_{ij}, \psi_{ij}\}$ belongs to $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ as well as to $\mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(\widetilde{\mathbb{A}}^u_{q,N})$ in (2.8). Thus, by real/complex interpolation, see (1.43)/(1.41a) they also belong to

$$\begin{aligned}&(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}\\ &\qquad\text{as well as to } [\mathcal{D}(A_{1,q}), \mathbf{L}^q_\sigma(\Omega)]_{1-\alpha} \times [\mathcal{D}(A_{2,q}), \mathbf{L}^q_\sigma(\Omega)]_{1-\alpha}\\ &\qquad\qquad = \mathcal{D}(A^\alpha_{1,q}) \times \mathcal{D}(A^\alpha_{2,q}),\ 0 \le \alpha \le 1\end{aligned} \tag{3.14}$$

in particular, $\boldsymbol{\Phi}_{ij} = \{\varphi_{ij}, \psi_{ij}\} \in \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$, see (1.19b) = (1.43b). Thus, exponential decay in $\mathbb{C}^N \times \mathbb{C}^N$ of the $\mathbb{C}^N \times \mathbb{C}^N$-vector $\hat{\boldsymbol{\eta}}_N$ translates at once into exponential decay with the same rate in any of the spaces $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega), (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}, \mathcal{D}(A^\alpha_{1,q}) \times \mathcal{D}(A^\alpha_{2,q})$, in particular, $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ for the vector $\boldsymbol{\eta}_N$, viewed as a vector on any one of these spaces. This remark applies to $\boldsymbol{\eta}_N(t)$ and $\mathbf{u}_N(t)$ in Theorem 2.1, equations (2.16), (2.17) as well as Theorem 2.2, equations (2.20)-(2.22).

LEMMA 3.1. *In $\mathbb{C}^N$, under the FDSA with respect to the ordered basis β : $\{\boldsymbol{\Phi}_{ij}\}^{M\ \ \ell_i}_{i=1,j=1}$ in (3.13) of normalized eigenfunctions of $\widetilde{\mathbb{A}}^u_{q,N}$, we may rewrite system (3.7) = (3.12) recalling (3.10) as*

$$(\hat{\boldsymbol{\eta}}_N)' - \Lambda \hat{\boldsymbol{\eta}}_N = U \hat{\mu}_K \tag{3.15}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & \mathit{0} \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ \mathit{0} & & & \lambda_M I_M \end{bmatrix} : \textit{ of size } N \times N,$$

$$I_i : \textit{ identity matrix of size } \ell_i \times \ell_i \quad (3.16)$$

$$U_i = \begin{bmatrix} (\mathbf{u}_1, \mathbf{\Phi}^*_{i1})_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i1})_\omega \\ (\mathbf{u}_1, \mathbf{\Phi}^*_{i2})_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i2})_\omega \\ \vdots & \ddots & \vdots \\ (\mathbf{u}_1, \mathbf{\Phi}^*_{i\ell_i})_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i\ell_i})_\omega \end{bmatrix} : \ell_i \times K;$$

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} : N \times K; \quad \hat{\mu}_K = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_K \end{bmatrix} : K \times 1; \quad (3.17)$$

where $(f,g)_\omega = \int_\omega f\bar{g}\, d\omega$ *and we take* $K \geq \ell_i,\ i = 1, \dots, M$. *Thus* (3.15) *gives the dynamics on* $(\mathbf{Y}^q_\sigma)^u_N$ *as a linear* N*-dimensional ordinary differential equation in coordinate form in* $\mathbb{C}^N$.

Proof. Recalling the basis β_i in (3.13) and the definitions of U_i in (3.17), we can rewrite the term in (3.11) with respect to this basis as

$$\left\{ P_N \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix} \right\}_{\beta_i} = U_i \hat{\mu}_K : \ell_i \times 1; \quad (3.18)$$

Then with respect to the basis β in (3.13) and recalling the definition U in (3.17), we can rewrite the term (3.11) with respect to this basis as

$$\left\{ P_N \begin{bmatrix} P_q(mu) \\ P_q(mv) \end{bmatrix} \right\}_{\beta} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} \hat{\mu}_K = \begin{bmatrix} U_1 \hat{\mu}_K \\ U_2 \hat{\mu}_K \\ \vdots \\ U_M \hat{\mu}_K \end{bmatrix} = U \hat{\mu}_K : N \times 1. \quad (3.19)$$

Finally, clearly $\widetilde{\mathbb{A}}^u_{q,N}$ in (3.12) becomes the diagonal matrix Λ in (3.16) with respect to the basis β, recalling its eigenvalues in (3.2). □

The following is the main result of the present section.

THEOREM 3.2. *Assume the FDSA. It is possible to select vectors* $\mathbf{u}_1, \dots, \mathbf{u}_K \in \mathbf{L}^q_\sigma(\omega) \times \mathbf{L}^q_\sigma(\omega)$, $\mathbf{u}_i = [u_i^1, u_i^2]$, *see* (3.10), $q > 1$, $K = \sup \{\ell_i : i = 1, \dots, M\}$, *such that the matrix* U_i *of size* $\ell_i \times K$ *in* (3.17) *satisfies*

$$rank\ [U_i] = full = \ell_i \ or \tag{3.20a}$$

$$rank \begin{bmatrix} (\mathbf{u}_1, \mathbf{\Phi}^*_{i1})_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i1})_\omega \\ (\mathbf{u}_1, \mathbf{\Phi}^*_{i2})_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i2})_\omega \\ \vdots & & \vdots \\ (\mathbf{u}_1, \mathbf{\Phi}^*_{i\ell_i})_\omega & \dots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i\ell_i})_\omega \end{bmatrix} = \ell_i;\ \ell_i \times K \ for\ each\ i = 1, \dots, M, \tag{3.20b}$$

$$(\mathbf{u}_j, \mathbf{\Phi}^*_{i1})_\omega = \left(\begin{bmatrix} u_j^1 \\ u_j^2 \end{bmatrix}, \begin{bmatrix} \varphi^*_{ij} \\ \psi^*_{ij} \end{bmatrix} \right)_{\mathbf{L}^q_\sigma(\omega) \times \mathbf{L}^q_\sigma(\omega)} \tag{3.20c}$$

In fact, the vectors $\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}$ *are linearly independent in* $\mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega)$.

Proof. Step 1. By selection, see (3.2) and statement preceding it, the set of vectors $\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}$ is linearly independent in $\mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$, q' is the Hölder conjugate of q, $1/q + 1/q' = 1$, for each $i = 1, \dots, M$. We want to show that the set $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}\}$ remains linearly independent on $\mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega)$, after which the desired conclusion (3.20a) for the matrix U_i to be full rank, would follow for infinitely many choices of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_K \in \mathbf{L}^q_\sigma(\omega) \times \mathbf{L}^q_\sigma(\omega)$.

Claim: The set $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}\}$ is linearly independent on $\mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega)$, for each $i = 1, \dots, M$.

The proof will critically depend on a unique continuation result [54] see also [6, Lemma 3.7, p. 1466]. By contradiction, let us assume that the vectors $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}\}$ are instead linearly dependent on $\mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega)$, so that

$$\mathbf{\Phi}^*_{i\ell_i} = \sum_{j=1}^{\ell_i - 1} \alpha_j \mathbf{\Phi}^*_{i\ell_j} \text{ in } \mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega) \tag{3.21}$$

We shall then conclude by [6, Lemma 3.7] and [53] below, that in fact $\mathbf{\Phi}^*_{i\ell_i} \equiv 0$ on all of Ω as well, thereby making the system $\{\mathbf{\Phi}^*_{ij},\ j = 1, \dots, \ell_i\}$ linearly dependent on Ω, a contradiction. To this end, define the following function (depending on i) in $\mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$

$$\mathbf{\Phi}^* = \left[\sum_{j=1}^{\ell_i - 1} \alpha_j \mathbf{\Phi}^*_{i\ell_j} - \mathbf{\Phi}^*_{i\ell_i} \right] \in \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega), \quad i = 1, \dots, M. \tag{3.22}$$

so that $\mathbf{\Phi}^* \equiv 0$ in ω by (3.21). As each $\mathbf{\Phi}^*_{ij}$ is an eigenvalue of $\widetilde{\mathbb{A}}^*_q$ (or $(\widetilde{\mathbb{A}}^u_{q,N})^*$) corresponding to the eigenvalue $\bar{\lambda}_i$, see (3.3), so is the linear combination $\mathbf{\Phi}^*$. This property, along with $\mathbf{\Phi}^* \equiv 0$ in ω yields that $\mathbf{\Phi}^*$ satisfies the following eigenvalue problem for the operator $\widetilde{\mathbb{A}}^*_q$ (or $(\widetilde{\mathbb{A}}^u_{q,N})^*$):

$$\widetilde{\mathbb{A}}^*_q \mathbf{\Phi}^* = \bar{\lambda}\mathbf{\Phi}^*, \text{ div } \mathbf{\Phi}^* = 0 \text{ in } \Omega; \quad \mathbf{\Phi}^* = 0 \text{ in } \omega, \text{ by (3.21).} \tag{3.23}$$

with the over-determined condition $\mathbf{\Phi}^* \equiv 0$ in ω. But the linear combination $\mathbf{\Phi}^*$ in (3.22) of the eigenfunctions $\mathbf{\Phi}^*_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}^*_q)$ satisfies itself the Dirichlet B.C $\mathbf{\Phi}^*|_{\partial\Omega} = 0$. The crux of the proof consists in showing the following Unique Continuation Property: that statement (3.23) with an over-determined condition implies, in fact, $\mathbf{\Phi}^* \equiv 0$ on all of Ω, so that by (3.22), the vectors $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i},\}$ are linearly dependent in Ω: i.e. on $\mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$, a contradiction.

The proof relies on the explicit PDE-version of statement (3.23). To avoid introducing additional notation for the adjoint problem $\widetilde{\mathbb{A}}^*_q$, we shall provide a proof for the original operator:

$$\widetilde{\mathbb{A}}_q \mathbf{\Phi} = \lambda\mathbf{\Phi}, \text{ div } \mathbf{\Phi} \equiv 0 \text{ in } \Omega, \ \mathbf{\Phi} \equiv \text{ in } \omega \tag{3.24}$$

$$\Longrightarrow \ \mathbf{\Phi} \equiv 0 \text{ on } \Omega. \tag{3.25}$$

The PDE-version of the implication in (3.25) is given by the following result.

Theorem 3.3 ([30]). *Let ω be an arbitrary open, connected smooth subset of Ω, thus of positive measure, see Fig 1. Let $\{\varphi, \xi, p\} \in \mathbf{W}^{2,q}(\Omega) \times \mathbf{W}^{2,q}(\Omega) \times W^{1,q}(\Omega)$, $q > d$, solve the original eigenvalue problem* (3.24) *(static version of problem* (1.6)*)*

$$-\nu_f \Delta\varphi + (y_e \cdot \nabla)\varphi + (\varphi \cdot \nabla)y_e - (\xi \cdot \nabla)B_e - (B_e \cdot \nabla)\xi + \nabla p = \lambda\varphi \quad \text{in } \Omega, \tag{3.26a}$$

$$-\nu_m \Delta\xi + (\varphi \cdot \nabla)B_e + (y_e \cdot \nabla)\xi - (\xi \cdot \nabla)y_e - (B_e \cdot \nabla)\varphi = \lambda\xi \quad \text{in } \Omega, \tag{3.26b}$$

$$\mathit{div}\ \varphi = 0, \quad \mathit{div}\ \xi = 0 \quad \text{in } \Omega, \tag{3.26c}$$

$$\varphi = 0, \ \xi \cdot n = 0, \ (\mathit{curl}\ \xi) \times n = 0 \quad \text{on } \Gamma. \tag{3.26d}$$

along with the over-determination condition

$$\varphi \equiv 0, \quad \xi \equiv 0 \quad \text{in } \omega. \tag{3.27}$$

Then

$$\varphi \equiv 0, \quad \xi \equiv 0, \quad p \equiv \mathit{const} \ \text{in } \Omega. \tag{3.28}$$

The same proof applies, mutatis mutandis, to the adjoint problem (3.23), with the over-determination $\mathbf{\Phi}^* \equiv 0$ in ω. As already noted the conclusion is that problem (3.24) implies

$$\mathbf{\Phi}^* = 0 \text{ in } \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega);$$
$$\text{that is } \mathbf{\Phi}^*_{i\ell_i} = \alpha_1 \mathbf{\Phi}^*_{i1} + \alpha_2 \mathbf{\Phi}^*_{i2} + \cdots + \alpha_{\ell_i - 1} \mathbf{\Phi}^*_{i\ell_i - 1} \text{ in } \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega), \tag{3.29}$$

i.e. the set $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}\}$ in linearly dependent on $\mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$. But this is false, by the very selection of such eigenvectors, see (3.2) and statement preceding it. Thus, the condition (3.29) cannot hold.

The required unique continuation result is established in [30], following the scheme in [53, 56]. The original proof is done in the Hilbert setting but we may invoke the same result because $\mathbf{\Phi}^*$ has more regularity and integrability than required since $\mathbf{\Phi}^*$ is an eigenfunction of $\widetilde{\mathbb{A}}^*_q$. Thus the *claim* is established. In conclusion: it is possible to select, in infinitely many ways, interior functions $\mathbf{u}_1, \dots, \mathbf{u}_K \in \mathbf{L}^q_\sigma(\omega) \times \mathbf{L}^q_\sigma(\omega)$, $\mathbf{u}_i = [u^1_i, u^2_i]$ such that the algebraic full rank condition (3.20) holds true for each $i = 1, \dots, M$. □

REMARK 3.2. The general case without the FDSA uses a controllability characterization of the pair $\{J, B\}$, Jordan form [21, p. 204], [32, Ex. #7, p. 102], [8, p. 212], [9, p. 165], [7, Theorem 3.2.4, p. 148]; and moreover, is computationally intensive. We refer to Section 4 of [29].

4. Proof of Theorem 2.1: uniform stabilization on $(\mathbf{Y}^q_\sigma(\Omega))^u_N$ with arbitrary decay rates of the $\boldsymbol{\eta}_N$-dynamics (2.11) by a suitable finite-dimensional interior localized feedback control $\mathbf{u}_N = [u_N, v_N]$

Step 1: Following [50] the proof consists in testing controllability of the linear, finite-dimensional system (3.7), in short, the pair

$$\{J, B\}, \; B = U: \; N \times K, K = \sup\,\{\ell_i; i = 1 \dots, M\} \tag{4.1}$$

$U = [U_1, \dots, U_M]^{\text{tr}}$, U_i given by (4.12) of [29] in the general case (or by (3.17) under FDSA). J is the Jordan form of $\mathcal{A}^u_N$ with respect to the Jordan basis $\beta = \beta_1 \cup \cdots \cup \beta_M$, β_i being given by (4.6a) of [29]. But the rank conditions (4.13) of [29] in the general case precisely asserts such controllability property of the pair $\{\widetilde{\mathbb{A}}^u_{q,N} = J, B\}$, in light of Theorem 4.1 in [29] in the general case, or Theorem 3.2 under FDSA.

Step 2: Having established the controllability criterion for the pair $\{\widetilde{\mathbb{A}}^u_{q,N} = J, B\}$ then by the well-known Popov's criterion in finite-dimensional theory (arbitrary spectral allocation), there exists a real feedback Wonham's matrix $Q = K \times N$, such that the spectrum of the matrix $(J + BQ) = (J + UQ)$ may be arbitrarily preassigned; in particular, to lie in the left half-plane $\{\lambda : \text{Re } \lambda < -\gamma < -\text{Re } \lambda_{N+1}\}$, as desired. The resulting closed-loop system (corresponding to (3.15) under FDSA)

$$(\hat{\boldsymbol{\eta}}_N)' - J\hat{\boldsymbol{\eta}}_N = U\hat{\mathbf{u}}_N, \tag{4.2}$$

is obtained with $\mathbb{C}^N$-vector $\hat{\mathbf{u}}_N = Q\hat{\boldsymbol{\eta}}_N$, Q being the $K \times N$ matrix with row vectors $[\hat{p}_1, \dots, \hat{p}_K]$, $\mu_N^k = (\hat{\boldsymbol{\eta}}_N, \hat{p}_k)$ in the $\mathbb{C}^N$-inner product and hence decays with an arbitrary preassigned exponential rate $\gamma > 0$

$$|\hat{\boldsymbol{\eta}}_N(t)|_{\mathbb{C}^N} \leq C_\gamma e^{-\gamma t} |\hat{\boldsymbol{\eta}}_N(0)|_{\mathbb{C}^N}, \ t \geq 0. \tag{4.3}$$

But the N-dimensional vector $\boldsymbol{\eta}_N \in (\mathbf{Y}^q_\sigma)^u_N \subset \mathbf{L}^q_\sigma(\Omega)$ is represented by the $\mathbb{C}^N$-vector $\hat{\boldsymbol{\eta}}_N = [\boldsymbol{\eta}_N]_\beta$, where in the general case of Section 4 of [29], β is a Jordan basis of generalized eigenfunctions of $\widetilde{\mathbb{A}}_{q,N}$ $(= \widetilde{\mathbb{A}}^u_N)$ corresponding to its M distinct unstable eigenvalues. Such basis is given by $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_M$, where a representative β_i is given in (4.6a) of [29] in the general case. The whole basis can be read off from (4.15) of [29] in the general case. In the special case of Section 3 where the FDSA holds, the basis β in $(\mathbf{Y}^q_\sigma)^u_N$ is given by the eigenfunctions of the $\widetilde{\mathbb{A}}^u_N$ corresponding to its M distinct eigenvalues, see (3.13). But such eigenfunctions/generalized eigenfunctions are in $\mathcal{D}(\widetilde{\mathbb{A}}_q)$, hence smooth. Thus, the exponential decay in (4.3) of the coordinate vector $\hat{\boldsymbol{\eta}}_N$ in $\mathbb{C}^N$ translates in same exponential decay of the vector $\boldsymbol{\eta}_N(t) \in (\mathbf{Y}^q_\sigma)^u_N$ not only in the $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$-norm but also in the $\mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(A_{1,q}) \times \mathcal{D}(A_{2,q})$-norm, see (1.39), hence in the $(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}$-norm, in particular in the $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$-norm, recall (1.43b). See also Remark 3.1. Thus, returning from $\mathbb{C}^N \times \mathbb{C}^N$ back to $(\mathbf{Y}^q_\sigma)^u_N \times ((\mathbf{Y}^q_\sigma)^u_N)^*$, there exist suitable $\mathbf{p}_1, \dots, \mathbf{p}_K \in ((\mathbf{Y}^q_\sigma)^u_N)^* \subset \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$, such that $\mu^k_N = (\boldsymbol{\eta}_k, \mathbf{p}_k)$, whereby the closed-loop system (2.17) corresponds precisely to (4.15) of [29] via $P_N P_q(m\mathbf{u})$ written in terms of the Jordan basis of eigenvectors β in (4.6a) of [29] in the general case.

Thus not only do we obtain in view of (2.17), (2.18) and (4.3)

$$\begin{aligned}\|\boldsymbol{\eta}_N(t)\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} &= \left\| e^{\bar{A}^u t} P_N \boldsymbol{\eta}_0 \right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} \\ &\leq C_\gamma e^{-\gamma t} \|P_N \boldsymbol{\eta}_0\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)}, \ t \geq 0, \end{aligned} \tag{4.4}$$

$\gamma > 0$ arbitrarily preassigned, but also, say $1 < q < \infty, 1 < p < \dfrac{2q}{2q-1}$

$$\|\boldsymbol{\eta}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} = \left\|e^{\bar{A}^u t}P_N\boldsymbol{\eta}_0\right\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)}$$
$$\leq C_\gamma e^{-\gamma t}\|P_N\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{4.5}$$

Hence with $\mathbf{u}_N = Q\boldsymbol{\eta}_N$, we obtain not only

$$\|\boldsymbol{\eta}_N(t)\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} + \|\mathbf{u}_N(t)\|_{\mathbf{L}^q_\sigma(\omega)\times\mathbf{L}^q_\sigma(\omega)} = \|\boldsymbol{\eta}_N(t)\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} + \|Q\boldsymbol{\eta}_N(t)\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)}$$
$$\leq (|Q|+1)\left\|e^{\bar{A}^u t}P_N\boldsymbol{\eta}_0\right\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} \leq C_\gamma e^{-\gamma t}\|P_N\boldsymbol{\eta}_0\|_{\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)} \tag{4.6}$$

but also, say

$$\|\boldsymbol{\eta}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} + \|\mathbf{u}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\omega)}$$
$$\leq C_\gamma e^{-\gamma t}\|P_N\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{4.7}$$

Remark 4.1. Under the FDSA, checking controllability of the system (3.15) is easier. To this end, we can pursue, as usual, two strategies.

A first strategy invokes the well-known Kalman controllability criterion by constructing the $N \times KN$ Kalman controllability matrix

$$\mathcal{K} = [B, \Lambda B, \Lambda^2 B, \dots, \Lambda^{N-1}B] = \begin{bmatrix} B_1 & J_1B_1 & \dots & J_1^{N-1}B_1 \\ B_2 & J_2B_2 & \dots & J_2^{N-1}B_2 \\ \dots & \dots & \dots & \dots \\ B_M & J_MB_M & \dots & J_M^{N-1}B_M \end{bmatrix}, \tag{4.8}$$

$$B = \text{col}\,[B_1, B_2, \dots, B_M], \quad B_i = U_i : \ell_i \times \ell_i \tag{4.9}$$

of size $N \times KN$, $N = \dim\,(Y^q_\sigma)^u_N$, $J_i = \lambda_i I_i : \ell_i \times \ell_i$, $B_i = U_i : \ell_i \times \ell_i$, and requiring that it be full rank.

$$\text{rank}\,\mathcal{K} = \text{full} = N. \tag{4.10}$$

In view of generalized Vandermond determinants, we then have

$$\text{rank}\,\mathcal{K} = N \quad \text{if and only if rank } U_i = \ell_i \text{ (full) } i = 1, \dots, M, \tag{4.11}$$

precisely as guaranteed by (3.20a). A second strategy invokes the Hautus controllability criterion:

$$\textit{rank}\,[\Lambda - \lambda_i I, B] = \textit{rank}\,[\Lambda - \lambda_i I, U] = N \text{ (full)} \tag{4.12}$$

for all unstable eigenvalues $\lambda_i, 1, \dots, M$, yielding again the condition that rank $[U_i] = \ell_i, 1, \dots, M$, as guaranteed by (3.20a).

5. Proof of Theorem 2.2: Feedback stabilization of the original linearized $\boldsymbol{\eta} = \{w, \mathbb{W}\}$-system (2.4) by a finite dimensional feedback controller $\mathbf{u} = [u, v]$

Step 1: According to Theorem 2.1, the finite-dimensional system $\boldsymbol{\eta}_N$ in (2.11) is uniformly stabilized by the finite dimensional feedback controller $\mathbf{u} = \mathbf{u}_N$ given in the RHS of (2.17) with an arbitrary preassigned decay rate $\gamma > 0$, as given, either in the $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$-norm as in (2.16a) = (4.6), or in the $(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}$-norm, or in particular, in the $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$-norm as in (2.16b) = (4.7).

Step 2: Next, we examine the impact of such constructive feedback control $\mathbf{u}_N$ on the $\boldsymbol{\zeta}_N$-dynamics (2.12), whose explicit solution can be given by a variation of parameter formula,

$$\boldsymbol{\zeta}_N(t) = e^{\widetilde{\mathbb{A}}^s_{q,N} t}\boldsymbol{\zeta}_N(0) + \int_0^t e^{\widetilde{\mathbb{A}}^s_{q,N}(t-r)}(I - P_N)P_q(m\mathbf{u}_N(r))dr. \tag{5.1a}$$

in the notation $\widetilde{\mathbb{A}}^s_{q,N} = (I - P_N)\widetilde{\mathbb{A}}_q$, of (2.8).

We now recall from Theorem 1.2(viii) that the operator $\widetilde{\mathbb{A}}_q$ in (1.39) generates a s.c. analytic semigroup not only on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ but also on $(\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{1,q}))_{1-\frac{1}{p},p} \times (\mathbf{L}^q_\sigma(\Omega), \mathcal{D}(A_{2,q}))_{1-\frac{1}{p},p}$, in particular on $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$. Hence the feedback operator $\mathbb{A}_F = \mathbb{A}_{F,q}$ in (2.19) similarly generates a s.c. analytic semigroup on these spaces, being a bounded perturbation of the operator $\widetilde{\mathbb{A}}_q$. So we can estimate (5.1a) in the norm of either of these spaces. Furthermore, the (point) spectrum of the generator $\widetilde{\mathbb{A}}^s_{q,N}$ on $(\mathbf{Y}^q_\sigma)^s_N$ satisfies $\sup\{Re\ \sigma(\mathbb{A}^s_{q,N})\} < -|\lambda_{N+1}| < -\gamma_0$ by assumption. Thus, we have

$$\left\| e^{\mathbb{A}^s_{q,N} t} \right\|_{\mathcal{L}(\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega))} \leq M e^{-\gamma_0 t}, \ t > 0. \tag{5.1b}$$

We shall carry our the supplemental computations explicitly in the space $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$ for the case of greatest interest in the nonlinear analysis of Sections 9 and 10 of [29]. In the norm of $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$, we obtain from (5.1a) since the operators $(I - P_N)$, P_q are bounded

$$\|\boldsymbol{\zeta}_N(t)\| \leq \left\| e^{\widetilde{\mathbb{A}}^s_{q,N} t}\boldsymbol{\zeta}_N(0) \right\| + C\int_0^t \left\| e^{\widetilde{\mathbb{A}}^s_{q,N}(t-\tau)} \right\| \|\mathbf{u}_N(\tau)\|\, d\tau \tag{5.2}$$

$$\begin{aligned} \|\boldsymbol{\zeta}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} &\leq Ce^{-\gamma_0 t} \|\boldsymbol{\zeta}_N(0)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\ &+ C\int_0^t e^{-\gamma_0(t-r)} e^{-\gamma r} dr \,\|P_N \boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)}. \end{aligned} \tag{5.3}$$

recalling (5.1b) and estimate (2.16b) or (4.7) for $\|\mathbf{u}_N\|$ in the $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\omega) \times \widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\omega)$-norm. Since we may choose $\gamma > \gamma_0$ by Theorem 2.1, we then obtain as $\boldsymbol{\zeta}_N(0) = (I - P_N)\boldsymbol{\eta}_0$ by (2.12):

$$\begin{aligned} &\|\boldsymbol{\zeta}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\ &\leq C\left[e^{-\gamma_0 t} + e^{-\gamma_0 t}\frac{1 - e^{-(\gamma-\gamma_0)t}}{\gamma - \gamma_0}\right]\|\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\ &\leq Ce^{-\gamma_0 t}\|\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)}, \quad \forall t > 0. \end{aligned} \tag{5.4}$$

Then, estimate (5.4) for $\boldsymbol{\zeta}_N(t)$ along with estimate (4.7) for $\boldsymbol{\eta}_N(t)$ with $\gamma > \gamma_0$ yields the desired estimate (2.21) for $\boldsymbol{\eta} = \boldsymbol{\eta}_N + \boldsymbol{\zeta}_N$ in the $\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)$-norm:

$$\begin{aligned} &\|\boldsymbol{\eta}(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\ &\leq \|\boldsymbol{\zeta}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} + \|\boldsymbol{\eta}_N(t)\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\ &\leq \left[\widetilde{C}_{\gamma_0}e^{-\gamma_0 t} + C_\gamma e^{-\gamma t}\right]\|\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \\ &\leq C_{\gamma_0}e^{-\gamma_0 t}\|\boldsymbol{\eta}_0\|_{\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)\times\widetilde{\mathbf{B}}^{2-2/p}_{q,p}(\Omega)} \end{aligned} \tag{5.5}$$

and (2.22) is proved. Computations similar to these from (5.1a) to (5.4) apply also in the $\mathbf{L}^q_\sigma(\Omega)\times\mathbf{L}^q_\sigma(\Omega)$-norm for $\boldsymbol{\zeta}_N(t)$, as the operator $\widetilde{\mathbb{A}}_q$ in (1.39) generates a s.c. analytic semigroup on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$, as noted in Theorem 1.2(vii). This, coupled with estimate (2.16a) for $\boldsymbol{\eta}_N(t)$, yields estimate (2.20) for the $\boldsymbol{\eta} = \boldsymbol{\eta}_N + \boldsymbol{\zeta}_N$ with $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$-norm. Computations such as those in [6, p. 1473] using the analyticity of the semigroup $e^{\mathbb{A}_F t}$ show the alternative estimates (2.21) of Theorem 2.2. Theorem 2.2 is established.

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