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Regularity properties of solutions to a sixth order Kirchhoff-Love's type model for nanoplates

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"Dedicated to Enzo Mitidieri for his 70th birthday"

Abstract. *We prove advanced regularity results for solutions to a sixth order equation arising in the mechanical Kirchhoff-Love's type model of the static equilibrium of a nanoplate in bending. Such regularity properties play a crucial role in the treatment, among others, of the inverse problem consisting in the determination of the Winkler coefficient of a nanoplate.*

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1. Introduction

This work is within a line of research that intends to further the investigation of recent models for two-dimensional nanomechanical systems, which we will refer as nanoplates. In particular, we analyse the regularity properties of the solutions of the direct problem describing the static equilibrium of nanoplates as fundamental tools for the subsequent study of related inverse problems. The modelling of nanostructures has specific challenges due to the presence of small-scale phenomena. Indeed, classical continuum mechanics lacks its predictive capability. In recent years, many theories have been proposed in the field of linear elasticity to model nanostructures. Among these, we mention the Simplified Strain Gradient Elasticity Theory (SSGET) introduced by Lam [5] and some recent developments that address the study of the Kirchhoff-Love nanoplate using SSGET [4, 7].

In our recent paper [2], we consider the issue of the identification of the Winkler coefficient k of the elastic foundation for a nanoplate from the measurement of the deflection produced by a given concentrated force $f\delta(P_0)$ at an internal point P_0. We also assume that the nanoplate is clamped at the boundary and we set $\Omega \subset \mathbb{R}^2$ to be the middle surface of the nanoplate, having constant thickness t. According to the Winkler model and working in the framework of the Kirchhoff-Love theory in infinitesimal deformation, the transversal displacement w of the nanoplate satisfies the following boundary value problem

$$\begin{cases} \frac{\partial^2}{\partial x_i \partial x_j}\left((P_{ijlm}+P^h_{ijlm})\frac{\partial^2 w}{\partial x_l \partial x_m} \right. \\ \qquad \left. -\frac{\partial}{\partial x_k}\left(Q_{ijklmn}\frac{\partial^3 w}{\partial x_l \partial x_m \partial x_n}\right)\right) + kw = f\delta_{P_0}, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \\ w_{,n} = 0, & \text{on } \partial\Omega, \\ w_{,nn} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where n is the unit outer normal to $\partial\Omega$ and the summation over repeated indexes $i, j, k, l, m, n = 1, 2$ is assumed. Here, $f \in \mathbb{R}, f > 0$ and P_{ijlm}, P^h_{ijlm} are the Cartesian components of the fourth-order tensors $\mathbb{P}, \mathbb{P}^h$ respectively, whereas Q_{ijklmn} are the components of the sixth-order tensor $\mathbb{Q}$. Let us also observe that the fourth order tensor $\mathbb{P}$ describes the material response in the classical Kirchhoff-Love theory, while $\mathbb{P}^h, \mathbb{Q}$ take into account the parameters peculiar to the small size effect.

As main result in [2], we prove that for $\mathbb{P}, \mathbb{P}^h \in W^{2,\infty}(\Omega) \cap H^{2+s}(\Omega)$, and $\mathbb{Q} \in W^{3,\infty}(\Omega) \cap H^{3+s}(\Omega)$, satisfying some suitable isotropic and strong convexity conditions (see the assumptions ii), iii) in Section 2), if $w_i \in H^3_0(\Omega),\ i = 1, 2$, is the solution to (1) for Winkler coefficient $k = k_i \in L^\infty \cap H^s(\Omega),\ i = 1, 2$ and if for a given $\varepsilon > 0$

$$\|w_1 - w_2\|_{L^2(\Omega)} \le \varepsilon\ , \tag{2}$$

then for every $\sigma > 0$ the following Hölder estimate holds

$$\|k_1 - k_2\|_{L^2(\Omega_\sigma)} \le C\varepsilon^\beta\ ,$$

where $\Omega_\sigma = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \sigma\}$ and $C > 0, \beta \in (0, 1)$ are constants depending on the a priori data and on σ only.

As is common when tackling inverse problems, the preliminary study of the fine properties of the solutions to governing equations are instrumental to theoretical results such as unique continuation estimates as well as quantitative stability estimates for inverse problems. In this respect, in the present paper we consider the following sixth order partial differential equation

$$\frac{\partial^2}{\partial x_i \partial x_j}\Big(-(P_{ijlm}+P^h_{ijlm})\frac{\partial^2 u}{\partial x_l \partial x_m} \\ + \frac{\partial}{\partial x_k}\left(Q_{ijklmn}\frac{\partial^3 u}{\partial x_l \partial x_m \partial x_n}\right)\Big) = g \quad \text{in } \Omega, \tag{3}$$

and we are interested in analyzing the regularity properties of its solutions under suitable assumptions on the coefficients and the source term g. More precisely, our main purpose is to prove the following.
Let $g \in H^s(\Omega)$, for some $0 < s < 1$ and let $w \in H^3_0(\Omega)$ be a weak solution to (3), where $\mathbb{P}, \mathbb{P}^h \in W^{2,\infty}(\Omega) \cap H^{2+s}(\Omega)$, $\mathbb{Q} \in W^{3,\infty}(\Omega) \cap H^{3+s}(\Omega)$, satisfy some suitable isotropic and strong convexity conditions (see (8)-(9) and (14)-(15)) . Then for every $\sigma > 0$, we have that

$$\|u\|_{H^{6+s}(\Omega_\sigma)} \leq C\left(\|u\|_{H^3(\Omega_{\frac{\sigma}{2}})} + \|g\|_{H^s(\Omega)}\right), \tag{4}$$

where $C > 0$ is a constant depending on the a priori data and on σ only.

The proof of the above mentioned stability result fundamentally relies on the smoothness property at hand. Without ambitions of completeness, we try to give the reader an idea of the argument adopted in [2] and where the use of regularity strongly comes into play.

It is easy to observed that if we set $u = w_1 - w_2$, then u is a solution to

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial x_j}\Big(&- (P_{ijlm} + P^h_{ijlm})\frac{\partial^2 u}{\partial x_l \partial x_m} \\ &+ \frac{\partial}{\partial x_k}\left(Q_{ijklmn}\frac{\partial^3 u}{\partial x_l \partial x_m \partial x_n}\right)\Big) - k_2 u = (k_2 - k_1)w_1 \quad \text{in } \Omega,\end{aligned} \tag{5}$$

or rather a solution to (3) with $g = k_2 u + (k_2 - k_1)w_1 \in H^s(\Omega)$. Thanks to the regularity result, we may infer that $u \in H^{6+s}(\Omega_\sigma)$ and, combining the well known interpolation inequality

$$\|u\|_{H^6(\Omega_\sigma)} \leq C\|u\|^{\frac{6}{6+s}}_{H^{6+s}(\Omega_\sigma)}\|u\|^{\frac{s}{6+s}}_{L^2(\Omega_\sigma)}$$

with (2) and (4), we obtain the following control on the sixth order terms by means of the lower order ones and of measurement error, namely

$$\|u\|_{H^6(\Omega_\sigma)} \leq C\left(\|u\|_{H^3(\Omega_{\frac{\sigma}{2}})} + \|g\|_{H^s(\Omega)}\right)^{\frac{6}{6+s}} \cdot \varepsilon^{\frac{s}{6+s}} \tag{6}$$

By exploiting again the equation (5), the estimate (6) and standard energy bounds, we can obtain the following estimate

$$\int_{\Omega_\sigma} (k_1 - k_2)^2 w_1^2 \leq C\varepsilon^{\frac{2s}{6+s}} .$$

Finally, thanks to strong unique continuation estimates, which in turn rely on preliminary regularity results of the solutions, we end up with

$$\int_{\Omega_\sigma} (k_1 - k_2)^2 \leq C\varepsilon^{2\beta} .$$

Our goal in this notes is to provide a detailed proof of Proposition 4.1 in [2] .

Let us observe that, although the method of proof is based on the path traced in [1], [3], our result improves upon the more classical ones because of the less restrictive conditions on the smoothness of the coefficients and because we are considering fractional exponent Sobolev spaces. Another new feature of the present result is that we keep constructive track of the dependence of constants on a priori data, which is a fundamental aspect when dealing with quantitative estimate of stability in inverse problem.

2. The nanoplate model

Let us consider a nanoplate $\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$ with middle surface Ω represented by a bounded domain of $\mathbb{R}^2$ and having constant thickness t, $t << diam(\Omega)$. We assume that the boundary $\partial\Omega$ of Ω is of class $C^{2,1}$ with constants ρ_0, M_0 and that

$$|\Omega| \leq M_1 \rho_0^2,$$

where M_1 is a positive constant.

We consider the following equation

$$\operatorname{div}\left(\operatorname{div}\left((\mathbb{P} + \mathbb{P}^h)\nabla^2 w\right)\right) - \operatorname{div}\left(\operatorname{div}\left(\operatorname{div}\left(\mathbb{Q}\nabla^3 w\right)\right)\right) = g, \quad \text{in } \Omega, \tag{7}$$

where, for the sake of simplicity, the compact notation in the left hand side denotes the following sixth order elliptic operator

$$\frac{\partial^2}{\partial x_i \partial x_j}\left((P_{ijlm} + P^h_{ijlm})\frac{\partial^2 w}{\partial x_l \partial x_m} - \frac{\partial}{\partial x_k}\left(Q_{ijklmn}\frac{\partial^3 w}{\partial x_l \partial x_m \partial x_n}\right)\right),$$

where the summation over repeated indexes $i, j, k, l, m, n = 1, 2$ is implied.

We shall denote by $\mathbb{M}^2, \mathbb{M}^3$ the Banach spaces of second order and third order tensors and by $\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^3$ the corresponding subspaces of tensors having components invariant with respect to permutations of all the indexes. Moreover, the space of bounded linear operators between Banach spaces X and Y will be denoted by $\mathcal{L}(X, Y)$.

On the elasticity tensors $\mathbb{P}$, $\mathbb{P}^h$, $\mathbb{Q}$ we make the following assumptions:

i) Regularity

$$\begin{aligned}
\|\mathbb{P}\|_{W^{2,\infty}(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))} &\leq A_1\rho_0^3, & \|\mathbb{P}\|_{H^{2+s}(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))} &\leq A_2\rho_0^3,\\
\|\mathbb{P}^h\|_{W^{2,\infty}(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))} &\leq A_1\rho_0^3, & \|\mathbb{P}^h\|_{H^{2+s}(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))} &\leq A_2\rho_0^3,\\
\|\mathbb{Q}\|_{W^{3,\infty}(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))} &\leq A_1\rho_0^5, & \|\mathbb{Q}\|_{H^{3+s}(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))} &\leq A_2\rho_0^5,
\end{aligned}$$

where A_1, A_2 are positive constants.

ii) Isotropy

$$
\begin{aligned}
P_{\alpha\beta\gamma\delta} &= B((1-\nu)\delta_{\alpha\gamma}\delta_{\beta\delta} + \nu\delta_{\alpha\beta}\delta_{\gamma\delta}), \qquad (8)\\
P^h_{\alpha\beta\gamma\delta} &= (2a_2 + 5a_1)\delta_{\alpha\gamma}\delta_{\beta\delta} + (-a_1 - a_2 + a_0)\delta_{\alpha\beta}\delta_{\gamma\delta},\\
Q_{ijklmn} &= \frac{1}{3}(b_0 - 3b_1)\delta_{ij}\delta_{kn}\delta_{lm} + + \frac{1}{6}(b_0 - 3b_1)\Big(\delta_{ik}(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln})\\
&\qquad + \delta_{jk}(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln})\Big) + Q_8(\delta_{kn}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})) +\\
&\quad + Q_9(\delta_{jn}(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})), \qquad (9)
\end{aligned}
$$

where $2(Q_8 + 2Q_9) = 5b_1$.

The bending stiffness (per unit length) $B = B(x)$ is given by the function

$$B(x) = \frac{t^3 E(x)}{12(1-\nu^2(x))}, \quad \text{a.e. in } \Omega,$$

where the Young's modulus E and the Poisson's coefficient ν of the material can be written in terms of the Lamé moduli μ and λ as follows

$$E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \quad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))}.$$

The coefficients $a_i(x)$, $i = 0, 1, 2$, are given by

$$a_0(x) = 2\mu(x)tl_0^2, \quad a_1(x) = \frac{2}{15}\mu(x)tl_1^2, \quad a_2(x) = \mu(x)tl_2^2 \quad \text{a.e. in } \Omega, \qquad (10)$$

where the material length scale parameters l_i are assumed to be positive constants. We denote

$$l = \min\{l_0, l_1, l_2\}.$$

The coefficients $b_i(x)$, $i = 0, 1$, are given by

$$b_0(x) = 2\mu(x)\frac{t^3}{12}l_0^2, \quad b_1(x) = \frac{2}{5}\mu(x)\frac{t^3}{12}l_1^2 \quad \text{a.e. in } \Omega. \qquad (11)$$

iii) Strong convexity for $\mathbb{P} + \mathbb{P}^h$, $\mathbb{Q}$.

We assume the following ellipticity conditions on μ and λ:

$$\mu(x) \geq \alpha_0 > 0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0 > 0 \quad \text{a.e. in } \Omega, \qquad (12)$$

where α_0, γ_0 are positive constants. By (10), (11) and (12) we also have

$$
\begin{aligned}
a_i(x) &\geq tl^2\alpha_0^h > 0, \quad i = 0, 1, 2,\\
b_j(x) &\geq t^3 l^2 \beta_0^h > 0, \quad j = 0, 1,
\end{aligned}
\qquad \text{a.e. in } \Omega, \qquad (13)
$$

where $\alpha_0^h = \frac{2}{15}\alpha_0$ and $\beta_0^h = \frac{1}{30}\alpha_0$.

By (12), (13) we obtain the following strong convexity conditions on $\mathbb{P}+\mathbb{P}^h$ and $\mathbb{Q}$. For every $A \in \widehat{\mathbb{M}}^2$ we have

$$(\mathbb{P}+\mathbb{P}^h)A \cdot A \geq t(t^2+l^2)\xi_{\mathbb{P}}|A|^2 \quad \text{a.e. in} \quad \Omega; \tag{14}$$

for every $B \in \widehat{\mathbb{M}}^3$ we have

$$\mathbb{Q}B \cdot B \geq t^3 l^2 \xi_{\mathbb{Q}}|B|^2 \quad \text{a.e. in} \quad \Omega; \tag{15}$$

where $\xi_{\mathbb{P}}$, $\xi_{\mathbb{Q}}$ are positive constants only depending on α_0 and γ_0.

In the sequel, we will refer to the set of parameters

$$\rho_0, \quad M_0, \quad M_1, \quad \alpha_0, \quad \gamma_0, \quad t, \quad l, \quad A_1, \quad A_2$$

as the *a priori data*, whereas the dimensional parameter ρ_0 shall appear explicitly in our estimates.

3. Main Result

THEOREM 3.1. *Let $w \in H^3(\Omega)$ be a weak solution to* (7) *with $g \in H^s(\Omega)$. For any $\sigma > 0$, we have that*

$$\|w\|_{H^{6+s}(\Omega_{\sigma\rho_0})} \leq C\left(\|w\|_{H^3(\Omega_{\frac{\sigma}{2}\rho_0})} + \|g\|_{H^s(\Omega)}\right),$$

where $C > 0$ depends on the a priori data, on σ and on s only.

Before proving Theorem 3.1, let us state the following lemma whose proof can be carried out by slightly adapting the proof of Theorem 3.9 in [6].

LEMMA 3.2. *Let $w \in H^3(B_\sigma)$ be a weak solution to* (7) *with $g \in L^2(B_\sigma)$. We have that $w \in H^6(B_{\frac{\sigma}{8}})$ and*

$$\|w\|_{H^6(B_{\frac{\sigma}{8}})} \leq C\left(\|w\|_{H^3(B_{\frac{\sigma}{2}})} + \|g\|_{L^2(B_\sigma)}\right)$$

where $C > 0$ depends only on α_0, γ_0, $\frac{t}{\rho_0}$, $\frac{l}{\rho_0}$, A_1.

Proof of Theorem 3.1. By straightforward computation,

$$\operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w)) = B\Delta^2 w + \sum_{|\alpha|=2}^{3} C_\alpha D^\alpha w, \tag{16}$$

$$\operatorname{div}(\operatorname{div}(\mathbb{P}^h\nabla^2 w)) = B^h\Delta^2 w + \sum_{|\alpha|=2}^{3} C_\alpha^h D^\alpha w,$$

$$\operatorname{div}(\operatorname{div}(\operatorname{div}(\mathbb{Q}\nabla^3 w))) = \bar{B}\Delta^3 w + \sum_{|\alpha|=3}^{5} \bar{C}_\alpha D^\alpha w, \tag{17}$$

where $\bar{B} = b_0 + 2b_1$, $B^h = a_0 + 4a_1 + a_2$; C_α, C_α^h involve up to the second order derivatives of B, ν, a_0, a_1, a_2 and $\bar{C}_\alpha$ involve up to the third order derivatives of b_0, b_1.

Let us denote

$$\mathcal{L}w = -\mathrm{div}\,(\mathrm{div}\,((\mathbb{P} + \mathbb{P}^h)\nabla^2 w)) + \mathrm{div}\,(\mathrm{div}\,(\mathrm{div}\,(\mathbb{Q}\nabla^3 w))).$$

In view of (16)–(17), we may rewrite $\mathcal{L}w$ in the form

$$\mathcal{L}w = \bar{B}\Delta^3 w + \mathcal{L}_0(w),$$

with

$$\mathcal{L}_0(w) = \sum_{|\alpha|=2}^{5} d_\alpha D^\alpha w,$$

where d_α involve up to third order derivative of B, ν, μ.

Therefore we may rewrite (7) as follows

$$\bar{B}\Delta^3 w + \mathcal{L}_0(w) = -g. \tag{18}$$

We now localize the equation by considering a cut-off function φ with the following properties. We assume without loss of generality that $0 \in \Omega, B_{6R}(0) \subset \Omega$. Let $\varphi \in C_0^\infty(\Omega)$ such that $\mathrm{supp}(\varphi) \subset B_\rho(0)$ with $\rho = \frac{3}{4}R$ and

$$\begin{aligned} &0 \le \varphi(x) \le 1\ , \quad \varphi \equiv 1 \ \text{ for } \ |x| \le \frac{R}{2}, \\ &|\nabla^k \varphi| \le \frac{C}{R^k}\ , \quad k = 1, \dots, 6\ . \end{aligned}$$

Let us consider the function

$$v = w\varphi\ .$$

We have that

$$\Delta^3(v) = \varphi\Delta^3 w + F_0(w, \varphi) \tag{19}$$

where

$$\begin{aligned} F_0(w,\varphi) = \Big\{ &[2\nabla\varphi\cdot\nabla(\Delta^2 w) + \Delta\varphi\Delta^2 w] \\ &+ 4[\nabla(\Delta\varphi)\cdot\nabla(\Delta w) + 2\nabla^2\varphi\cdot\nabla^2(\Delta w) + \nabla\varphi\cdot\nabla(\Delta^2 w)] \\ &+ 2[\Delta^2 w\Delta\varphi + 2\nabla(\Delta w)\cdot\nabla(\Delta\varphi) + \Delta w\Delta^2\varphi] \\ &+ 4[\nabla^2(\Delta w)\cdot\nabla^2\varphi + 2\nabla^3 w\cdot\nabla^3\varphi + \nabla^2 w\cdot\nabla^2(\Delta\varphi)] + \\ &+ 4[\nabla(\Delta w)\cdot\nabla(\Delta\varphi) + 2\nabla^2 w\cdot\nabla^2(\Delta\varphi) + \nabla w\cdot\nabla(\Delta^2\varphi)] \\ &+ [\Delta w\Delta^2\varphi + 2\nabla w\cdot\nabla(\Delta^2\varphi) + w\Delta^3\varphi]\Big\} = \sum_{|\alpha|=0}^{5} e_\alpha D^{6-|\alpha|}\varphi D^\alpha w. \end{aligned}$$

By (18) and (19), we have that

$$\begin{aligned}\mathcal{L}(v) &= \bar{B}\Delta^3 v + \mathcal{L}_0(v)\\ &= \bar{B}[\varphi\Delta^3 w + F_0(w,\varphi)] + \mathcal{L}_0(v) - \varphi\mathcal{L}_0(w) + \varphi\mathcal{L}_0(w)\\ &= \varphi(\bar{B}\Delta^3 w + \mathcal{L}_0(w)) + \mathcal{L}_0(v) - \varphi\mathcal{L}_0(w) + \bar{B}F_0(w,\varphi)\\ &= -\varphi g + \mathcal{L}_0(v) - \varphi\mathcal{L}_0(w) + \bar{B}F_0(w,\varphi),\end{aligned}$$

which leads to

$$\Delta^3 v = -\frac{\varphi g}{\bar{B}} - \frac{\varphi\mathcal{L}_0(w)}{\bar{B}} + F_0(w,\varphi).$$

Let us set

$$F(x) = -\frac{\varphi(x)g(x)}{\bar{B}(x)} - \frac{\varphi(x)\mathcal{L}_0(w)(x)}{\bar{B}(x)} + F_0(w,\varphi)(x).$$

We notice that

$$\begin{aligned}\|w\|_{H^{6+s}(B_{\frac{R}{2}})} = \|v\|_{H^{6+s}(B_{\frac{R}{2}})} &\leq \|v\|_{H^{6+s}(B_{\mathbb{R}^n})}\\ &\leq C\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^{6+s}|\hat{v}(\xi)|^2 \mathrm{d}\xi\right)^{1/2},\end{aligned}$$

where with $\hat{v}$ we denote the Fourier transform of v. We wish to obtain a bound on the last term on the right hand side of the above inequality.

In this respect we recall that

$$\widehat{\Delta^3 v}(\xi) = (2\pi i)^6|\xi|^6\hat{v}(\xi)\ ,$$

which leads to

$$(2\pi i)^6|\xi|^6\hat{v}(\xi) = \hat{F}(\xi)\,. \tag{20}$$

After straightforward computation, since $(1+x)^6 \leq 32(1+x^6)$ for $x \geq 0$, we have

$$(1+|\xi|^2)^{6+s}|\hat{v}(\xi)|^2 \leq 32[(1+|\xi|^2)^s|\hat{v}(\xi)|^2 + (1+|\xi|^2)^s(|\xi|^6|\hat{v}(\xi)|)^2].$$

Combining the above inequality and (20) we obtain that

$$(1+|\xi|^2)^{6+s}|\hat{v}(\xi)|^2 \leq C[(1+|\xi|^2)^s|\hat{v}(\xi)|^2 + (1+|\xi|^2)^s(|\hat{F}(\xi)|)^2]\ ,$$

where C is an absolute positive constant.

Integrating the above inequality over $\mathbb{R}^2$ we get

$$\|\hat{v}\|^2_{H^{6+s}(\mathbb{R}^2)} \leq C\left(\|\hat{v}\|^2_{H^s(\mathbb{R}^2)} + \|\hat{F}\|^2_{H^s(\mathbb{R}^2)}\right).$$

From the Plancherel identity we deduce that

$$\|v\|^2_{H^{6+s}(\mathbb{R}^2)} \le C\left(\|v\|^2_{H^s(\mathbb{R}^2)} + \|F\|^2_{H^s(\mathbb{R}^2)}\right). \tag{21}$$

From one hand we have

$$\|v\|^2_{H^{6+s}(\mathbb{R}^2)} \ge \|v\|^2_{H^{6+s}(B_{\frac{R}{2}}(0))} = \|w\|^2_{H^{6+s}(B_{\frac{R}{2}}(0))},$$

on the other hand

$$\|v\|^2_{H^s(\mathbb{R}^2)} \le \|v\|^2_{H^3(\mathbb{R}^2)} \le \|w\|^2_{H^3(B_\rho(0))}. \tag{22}$$

Hence, by combining (21)-(22), we have that

$$\|w\|^2_{H^{6+s}(B_{\frac{R}{2}}(0))} \le C\left(\|w\|^2_{H^3(B_\rho(0))} + \|F\|^2_{H^s(\mathbb{R}^2)}\right). \tag{23}$$

Next, we wish to bound the term $\|F\|_{H^s(\mathbb{R}^2)}$. We trivially observe that

$$\|F\|_{H^s(\mathbb{R}^2)} \le \left\|\frac{g\varphi}{\bar{B}}\right\|_{H^s(\mathbb{R}^2)} + \|F_0(w,\varphi)\|_{H^s(\mathbb{R}^2)} + \left\|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right\|_{H^s(\mathbb{R}^2)}. \tag{24}$$

We bound each term on the right hand side of (24) separately starting from $\|\frac{\varphi g}{\bar{B}}\|_s = \|\frac{\varphi g}{\bar{B}}\|_{H^s(\mathbb{R}^2)}$. In this respect we recall that $\bar{B}(x) = (b_0(x) + 2b_1(x))$, so that, by (11), (12), there exists a positive constant B_0, only depending on l, t, α_0, such that $\bar{B}(x) \ge B_0$ in Ω. Hence we can deduce that

$$\begin{aligned}
&\left\|\frac{\varphi}{\bar{B}}\right\|_{L^\infty(\Omega)} \le \frac{1}{B_0},\\
&\left\|\nabla\left(\frac{\varphi}{\bar{B}}\right)\right\|_{L^\infty(\Omega)} \le \frac{C}{B_0^2}(\|\nabla\varphi\|_{L^\infty(\Omega)} + \|\nabla\bar{B}\|_{L^\infty(\Omega)}) \le C,
\end{aligned}$$

where $C > 0$ is a constant depending on the a priori data.

We recall that

$$\left\|\frac{\varphi g}{\bar{B}}\right\|_s = \left\|\frac{\varphi g}{\bar{B}}\right\|_{L^2(\mathbb{R}^n)} + \left[\frac{\varphi g}{\bar{B}}\right]_{s,\mathbb{R}^2},$$

where

$$\left[\frac{\varphi g}{\bar{B}}\right]_{s,\mathbb{R}^2} = \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\left|\frac{\varphi g}{\bar{B}}(x) - \frac{\varphi g}{\bar{B}}(y)\right|}{|x-y|^{2+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

We have that

$$\left\|\frac{\varphi g}{\bar{B}}\right\|_{L^2(\mathbb{R}^2)} = \left\|\frac{\varphi g}{\bar{B}}\right\|_{L^2(B_\rho)} \le \frac{1}{B_0}\|g\|_{L^2(B_\rho)}. \tag{25}$$

Moreover, we have that for any $x, y \in B_\rho$

$$\begin{aligned}
&\left|\left(\frac{\varphi}{\bar{B}}g\right)(x) - \left(\frac{\varphi}{\bar{B}}g\right)(y)\right|^2 \\
&\qquad \le 2\left[\frac{\varphi(x)}{\bar{B}}(g(x)-g(y))\right]^2 + 2\left[\left(\frac{\varphi}{\bar{B}}(x) - \frac{\varphi}{\bar{B}}(y)\right)g(y)\right]^2 \\
&\qquad\qquad = 2\left(\frac{\varphi}{\bar{B}}(x)\right)^2[g(x)-g(y)]^2 + 2\left(\frac{\varphi}{\bar{B}}(x) - \frac{\varphi}{\bar{B}}(y)\right)^2|g(y)|^2
\end{aligned}$$

In view of the regularity of φ and $\bar{B}$ we may bound

$$\left|\frac{\varphi}{\bar{B}}(x) - \frac{\varphi}{\bar{B}}(y)\right| \le \left\|\nabla\left(\frac{\varphi}{\bar{B}}\right)\right\|_{L^\infty(B_\rho)}|x-y| \le C|x-y| \ . \tag{26}$$

Hence from the two formulas above we deduce that

$$\left|\left(\frac{\varphi g}{\bar{B}}\right)(x) - \left(\frac{\varphi g}{\bar{B}}\right)(y)\right|^2 \le 2\left|\frac{\varphi}{\bar{B}}(x)\right|^2|g(x)-g(y)|^2 + 2C^2|x-y|^2|g(y)|^2 \ .$$

It follows that

$$\begin{aligned}
\left[\frac{\varphi g}{\bar{B}}\right]^2_{s,\mathbb{R}^2} &\le C\int_{B_\rho}\mathrm{d}x\int_{B_\rho}\frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}}\mathrm{d}y + C\int_{B_\rho}\mathrm{d}x\int_{B_\rho}\frac{|g(y)|^2}{|x-y|^{2s}}\mathrm{d}y \\
&\le C[g]^2_{s,\mathbb{R}^2} + C\int_{B_\rho}\mathrm{d}x\int_{B_\rho}\frac{|g(y)|^2}{|x-y|^{2s}}\mathrm{d}y \ .
\end{aligned} \tag{27}$$

By Fubini's formula we have that

$$\int_{B_\rho}\mathrm{d}x\int_{B_\rho}\frac{|g(y)|^2}{|x-y|^{2s}}\mathrm{d}y = \int_{B_\rho}|g(y)|^2\left(\int_{B_\rho}\frac{1}{|x-y|^{2s}}\mathrm{d}x\right)\mathrm{d}y.$$

Let us now show that there exists a contant $C > 0$ such that for any $y \in B_\rho$ we have that

$$\int_{B_\rho}\frac{1}{|x-y|^{2s}}\mathrm{d}x \le C.$$

For a fixed $y \in B_\rho$, we set $z = y - x$. Hence, by a change of variable

$$\int_{B_\rho}\frac{1}{|x-y|^{2s}}\mathrm{d}x = \int_{B_\rho(y)}\frac{1}{|z|^{2s}}\mathrm{d}z \ .$$

By noticing that $B_\rho(y) \subset B_{2\rho}$ and by standard computations based on the use polar coordinates we have that

$$\int_{B_\rho}\frac{1}{|x-y|^{2s}}\mathrm{d}x \le \int_{B_{2\rho}(y)}\frac{1}{|z|^{2s}}\mathrm{d}z \le \frac{\pi}{1-s}2^{2(1-s)}\rho^{2(1-s)}.$$

Hence we have that

$$\int_{B_\rho} \mathrm{d}x \int_{B_\rho} \frac{|g(y)|^2}{|x-y|^{2s}} \mathrm{d}y \leq C\|g\|_{L^2(B_\rho)} \ . \tag{28}$$

Combining (27) and (28) we have

$$\left[\frac{\varphi g}{\bar{B}}\right]^2_{s,\mathbb{R}^2} \leq C([g]^2_{s,B_\rho} + \|g\|^2_{L^2(B_\rho)}) \ . \tag{29}$$

Namely, by (25) and (29),

$$\left\|\frac{\varphi g}{\bar{B}}\right\|_s \leq C\|g\|_{H^s(B_\rho)} \ .$$

We now handle the term $\|F_0\|_s$ appearing in the right hand side of (24) . We observe that, in view of the regularity of φ, and since the expression of F_0 involves at most fifth order derivatives of w, reasoning as above we get

$$\|F_0\|_s \leq C\|w\|_{H^{5+s}(B_\rho)} \ .$$

We now analyze the term $\left\|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right\|_s$ and in this respect we recall that

$$\mathcal{L}_0(w) = \sum_{|\alpha|=2}^{5} d_\alpha D^\alpha w \ .$$

We recall that

$$\left\|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right\|_s = \left\|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right\|_{L^2(\mathbb{R}^2)} + \left[\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right]_{s,\mathbb{R}^2} . \tag{30}$$

In view of the regularity of the coefficients we easily bound the first term on the right hand side of (30), namely we have

$$\left\|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right\|_{L^2(\mathbb{R}^2)} = \left\|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right\|_{L^2(B_\rho)} \leq \frac{1}{\bar{B}}\|\mathcal{L}_0(w)\|_{L^2(B_\rho)} \leq C\|w\|_{H^5(B_\rho)}.$$

For what concern the second term on the right hand side of (30), we first observe that in view of (26) we have that

$$\left|\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}(x) - \frac{\varphi\mathcal{L}_0(w)}{\bar{B}}(y)\right| \leq C|\mathcal{L}_0(w)(x) - \mathcal{L}_0(w)(y)| + C|x-y||\mathcal{L}_0(w)(y)|.$$

Hence we obtain

$$\begin{aligned}\left[\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}\right]^2_{s,\mathbb{R}^2} &= \int_{\mathbb{R}^2}\mathrm{d}x\int_{\mathbb{R}^2}\frac{|(\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}(x))-(\frac{\varphi\mathcal{L}_0(w)}{\bar{B}}(y))|^2}{|x-y|^{2+2s}}\mathrm{d}y\\ &\le C\int_{B_\rho}\mathrm{d}x\int_{B_\rho}\frac{|(\mathcal{L}_0(w)(x))-(\mathcal{L}_0(w)(y))|^2}{|x-y|^{2+2s}}\mathrm{d}y\\ &\qquad + C\int_{B_\rho}\mathrm{d}x\int_{B_\rho}\frac{|(\mathcal{L}_0(w)(y))|^2}{|x-y|^{2s}}\mathrm{d}y = I_1+I_2.\end{aligned}$$

We notice that, arguing as above, we can obtain the following bound for the integral I_2, namely

$$I_2 \le C\int_{B_\rho}|(\mathcal{L}_0(w)(y))|^2 \le C\|w\|^2_{H^5(B_\rho)}\ .$$

We now handle the integral I_1. In this respect we notice that

$$\begin{aligned}&|(\mathcal{L}_0(w)(x))-(\mathcal{L}_0(w)(y))|^2 = \left|\sum_{|\alpha|=2}^{5}d_\alpha(x)D^\alpha w(x)-\sum_{|\alpha|=2}^{5}d_\alpha(y)D^\alpha w(y)\right|^2\\ &=\left|\sum_{|\alpha|=2}^{5}(d_\alpha(x)D^\alpha w(x)+d_\alpha(x)D^\alpha w(y)-d_\alpha(x)D^\alpha w(y)+d_\alpha(y)D^\alpha w(y))\right|^2\\ &\le C\left|\sum_{|\alpha|=2}^{5}(d_\alpha(x)(D^\alpha w(x)-D^\alpha w(y))\right|^2+C\left|\sum_{|\alpha|=2}^{5}(d_\alpha(x)-d_\alpha(y))D^\alpha w(y)\right|^2 \qquad (31)\end{aligned}$$

where $C>0$ is an absolute constant. In view of the regularity of d_α we have that

$$|d_\alpha(x)-d_\alpha(y)| \le \|\nabla d_\alpha\|_{L^\infty(B_\rho)}|x-y|, \quad \forall\ x,y\in\ B_\rho.$$

Hence, by (30)-(31) we have that

$$\begin{aligned}&|(\mathcal{L}_0(w)(x))-(\mathcal{L}_0(w)(y))|^2\\ &\qquad\le C\sum_{|\alpha|=2}^{5}|D^\alpha w(x)-D^\alpha w(y)|^2+C\sum_{|\alpha|=2}^{5}|x-y|^2|D^\alpha w(y)|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned} I_1 &\leq C \sum_{|\alpha|=2}^{5} \int_{B_\rho} \mathrm{d}x \int_{B_\rho} \frac{|D^\alpha w(x) - D^\alpha w(y)|^2}{|x-y|^{2+2s}} \mathrm{d}y \\ &\qquad + C \sum_{|\alpha=2}^{5} \int_{B_\rho} \mathrm{d}x \int_{B_\rho} \frac{|D^\alpha w(y)|^2}{|x-y|^{2s}} \mathrm{d}y \\ &\leq C \sum_{|\alpha|=2}^{5} [D^\alpha w]^2_{s,B_\rho} + C \sum_{|\alpha|=2}^{5} \int_{B_\rho} |D^\alpha w(y)|^2 \left(\int_{B_\rho} \frac{1}{|x-y|^{2s}} \mathrm{d}x \right) \mathrm{d}y \\ &\leq C \sum_{|\alpha|=2}^{5} [D^\alpha w]^2_{s,B_\rho} + C \sum_{|\alpha|=2}^{5} \|D^\alpha w\|^2_{L^2(B_\rho)} \leq C \sum_{|\alpha|=2}^{5} \|D^\alpha w\|^2_{s,B_\rho}. \end{aligned}$$

It follows that

$$\left[\frac{\varphi \mathcal{L}_0(w)}{\bar{B}}\right]^2_s \leq C \sum_{|\alpha|=2}^{5} \|D^\alpha w\|^2_{s,B_\rho} + C\|w\|^2_{H^5(B_\rho)} \leq C\|w\|^2_{H^6(B_\rho)},$$

and hence

$$\left\| \frac{\varphi \mathcal{L}_0(w}{\bar{B}} \right\|_s \leq C\|w\|_{H^6(B_\rho)} \ .$$

Finally, we have that $\|F\|_{H^s(\mathbb{R}^2)} \leq C\left(\|g\|_{H^s(B_\rho)} + \|w\|_{H^6(B_\rho)}\right)$.

From (23) we have that

$$\|w\|_{H^{6+s}(B_{\frac{R}{2}})} \leq C\left(\|w\|_{H^3(B_\rho)} + \|g\|_{H^s(B_\rho)} + \|w\|_{H^6(B_\rho)}\right).$$

We now use Lemma 3.2 and we get

$$\|w\|_{H^6(B_\rho)} \leq C\left(\|w\|_{H^3(B_{3R})} + \|g\|_{L^2(B_{6R})}\right) \ .$$

It follows that $\|w\|_{H^{6+s}(B_{\frac{R}{2}})} \leq C(\|g\|_{H^s(B_{6R})} + \|w\|_{H^3(B_{3R})})$. □

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