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Non-orientable 3-manifolds of cubic-complexity one

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Abstract. *We classify all closed non-orientable $\mathbb{P}^2$-irreducible 3-manifolds obtained by identifying the faces of a cube, i.e. those with cubic-complexity one. We show that they are the four flat ones.*

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1. Introduction

The study of closed 3-manifolds constructed by identifying the faces of a cube started with Poincaré [17] in 1895 to produce examples of manifolds in the study of the fundamental group and of the Betti numbers. In this paper we will deal with the non-orientable case by starting a classification process.

Non-orientable 3-manifolds seem to be much more sporadic than orientable ones. For instance, among the 8 three-dimensional geometries, only 5 have non-orientable representatives [18]. Moreover, among cusped hyperbolic 3-manifolds of Matveev complexity up to nine, only 14045 of 75956 are non-orientable, as shown in the Callahan-Hildebrand-Weeks-Thistlethwaite-Burton census [7, 9, 20]. Also, among closed $\mathbb{P}^2$-irreducible 3-manifolds of Matveev complexity up to seven, only 8 of 318 are non-orientable [5, 6]. Eventually, all three closed $\mathbb{P}^2$-irreducible 3-manifolds of surface-complexity zero are orientable [1]. Here we show that, among closed $\mathbb{P}^2$-irreducible 3-manifolds of cubic-complexity one, only 4 of 15 are non-orientable.

We refer to three different complexities on 3-manifolds, used to carry out the classification processes. The Matveev complexity was defined in [15], and in the cases described above equals the minimum number of tetrahedra needed to triangulate the manifold if it is distinct from the sphere S^3, the projective space $\mathbb{RP}^3$ and the Lens space $L_{3,1}$ (having Matveev complexity zero) [14]. The cubic-complexity is the minimum number of cubes needed to cubulate the manifold (i.e. to construct the manifold by gluing cubes along the boundary squares) [19]. The surface-complexity is the minimum number of triple points needed by the image of a transverse immersion of a closed surface to divide the

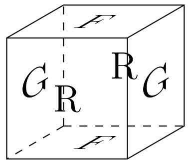


Figure 1: A cubulation of the 3-dimensional torus $S^1 \times S^1 \times S^1$ with one cube (the letters show that the identification of each pair of opposite faces is the obvious one, i.e. the one without twists and reflections).

manifold into balls, and is equal to the cubic-complexity under some hypotheses on the manifold, but it is more flexible [1, 3]. For the sake of completeness, we recall that analogous interesting definitions involving surface immersions are the Montesinos complexity and the triple point spectrum, given by Vigara [21] and studied by Lozano and Vigara [11, 12, 13].

In this paper, we classify all closed non-orientable $\mathbb{P}^2$-irreducible 3-manifolds with cubic-complexity (and hence surface-complexity) one: in some way they are the "simplest" ones, because it turns out that they are the four flat ones.

Usually the classification process is computer-aided. In this case one could study all $8^3 = 512$ possible gluings for the boundary squares of the cube and identify the object obtained (which may not be a manifold), but we have preferred to use some simple theoretical results to simplify the search among the manifolds with Matveev complexity up to six, avoiding hence the complete enumeration. We plan for a subsequent paper to continue the classification process with the aid of a computer.

2. Definitions

Throughout this paper, all 3-manifolds are assumed to be connected and closed. By M, we will always denote such a (connected and closed) 3-manifold. Using the *Hauptvermutung*, we will freely intermingle the differentiable, piecewise linear and topological viewpoints.

A *cubulation* of M is a cell-decomposition of M such that

- each 2-cell (called a *square*) is glued along 4 edges,
- each 3-cell (called a *cube*) is glued along 6 faces arranged like the boundary of a cube.

Note that self-adjacencies and multiple adjacencies are allowed. In the figures we have used (non-symmetric) letters to show the gluing information. In Fig. 1 we have shown a cubulation of the 3-dimensional torus $S^1 \times S^1 \times S^1$ with one cube.

Definition 2.1. *The* cubic-complexity *of M is equal to c if M possesses a cubulation with c cubes and has no cubulation with less than c cubes.*

The classification in the orientable case has been carried out by Korablev and Kazakov [10].

Theorem 2.2. *There are 11 (connected and closed) $\mathbb{P}^2$-irreducible orientable 3-manifolds with cubic-complexity one, and 80 with cubic-complexity two.*

Remark 2.3. A more flexible definition, strictly related to cubic-complexity, is *surface-complexity*, which is the minimum number of triple points of the image of a transverse immersion of a closed surface dividing the manifold into balls. It satisfies some properties, but for the purpose of this paper we recall only that the surface-complexity of a $\mathbb{P}^2$-irreducible 3-manifold, different from the sphere S^3, the projective space $\mathbb{RP}^3$ and the Lens space $L_{4,1}$, is equal to the cubic-complexity of M, and that the three manifolds S^3, $\mathbb{RP}^3$ and $L_{4,1}$ have surface-complexity zero [1, 2]. For the sake of completeness, we mention that the three manifolds S^3, $\mathbb{RP}^3$ and $L_{4,1}$ have cubic-complexity one [10].

3. The classification

The main result of this paper is the following.

Theorem 3.1. *There are 4 (connected and closed) $\mathbb{P}^2$-irreducible non-orientable 3-manifolds with cubic-complexity one: they are the four flat ones.*

The definitions for the flat and the Sol geometries (the latter being mentioned below) and the relations with Seifert fibrations can be found in [18]. We just recall that each of the four flat manifolds has three Seifert fibrations up to fibration-preserving diffeomorphism, which can be visualised by means of the cubulation (see Table 2 below).

Remark 3.2. Since there is no non-orientable $\mathbb{P}^2$-irreducible 3-manifold with surface-complexity or cubic-complexity zero, the surface-complexity of such a manifold equals its cubic-complexity (see Remark 2.3), so Theorem 3.1 on cubic-complexity applies also to surface-complexity.

Matveev complexity and triangulations In the proof we will use the *Matveev complexity* of M, defined in [15]. We do not need all details (see [16] for a comprehensive treatise); we only need the fact that for a $\mathbb{P}^2$-irreducible M distinct from $S^3, \mathbb{RP}^3, L_{3,1}$ the Matveev complexity equals the minimum number of tetrahedra needed to triangulate M. In [4, 5, 6] the list of non-orientable $\mathbb{P}^2$-irreducible 3-manifolds with Matveev complexity at most 7 is given.

REMARK 3.3. For the aim of this paper, we need only the list up to complexity 6: i.e.

- no non-orientable $\mathbb{P}^2$-irreducible 3-manifold has Matveev complexity less than 6, and
- the non-orientable $\mathbb{P}^2$-irreducible 3-manifolds with Matveev complexity 6 are the four flat ones and the torus bundle with monodromy $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, which is a Sol manifold.

From cubulations to triangulations Triangulations and cubulations are related to each other. There are a few ways to obtain a triangulation from a cubulation of a manifold. A first simple construction is shown in [1], while a cheaper one is shown in [19]. If we start from a cubulation with c cubes, the former construction leads to a triangulation with a number of tetrahedra between $5c$ and $8c$, while the latter to a triangulation with exactly $6c$ tetrahedra. We will need a finer analysis of the triangulation obtained if we start from a cubulation with one cube, and we will use the ideas of both constructions.

Let us start from a cubulation of M with one cube. It can be constructed by starting from the abstract cube and by identifying the boundary squares in pairs. Consider now a triangulation of the abstract cube such that the induced triangulation of each boundary square is composed of two triangles: we will call such a triangulation a *block*. In each square the triangulation is unambiguously defined by the diagonal that is the common edge of the two triangles. We will call the set of these diagonals a *diagonal pattern*. When we identify two squares to get M, either the diagonal (and hence the two triangles) match or not. If the three pairs of diagonals match, we get a triangulation of M. Otherwise, we will change the block.

We will use the four blocks that are described in Table 1. Note that the name chosen for the flipped block underlines that a diagonal is flipped with respect to the 5-tetrahedron block; however, like the 4-valent block, the flipped block also has a 4-valent internal edge ("the flipped diagonal"), the star of the 4-valent internal edge is an octrahedron, and the block is obtained by gluing two tetrahedra to the octahedron (but not along two opposite triangles). Note also that the number of tetrahedra of the four blocks is at most 6.

LEMMA 3.4. *Each 3-manifold with a cubulation with one cube has a triangulation obtained by gluing the squares of one of the four blocks described above.*

Proof. Consider a cubulation of a 3-manifold M with one cube. Identify the cube with the 5-tetrahedron block and glue the squares to get M. The gluing pairs the six boundary squares into three pairs (in each of which the two squares are identified to each other). If we consider the diagonal pattern given by the 5-tetrahedron block (Fig. 2), the pairing is inherited by the diagonal pattern. If

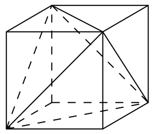
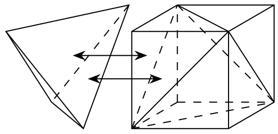
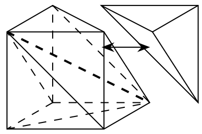
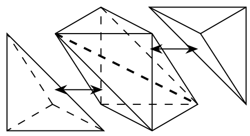
Name	Description	Number of tetrahedra	Picture
5-tetrahedron block	Four tetrahedra glued along the faces of a central one	5	
flipped block	Obtained from the previous one by gluing a tetrahedron along the two triangles of a square	6	
5-valent block	Obtained from the star of a 5-valent edge by gluing a tetrahedron along one of the triangles of the boundary	6	
4-valent block	Obtained from the star of a 4-valent edge (which is an octahedron) by gluing two tetrahedra along two opposite triangles of the boundary	6	

Table 1: Four blocks.

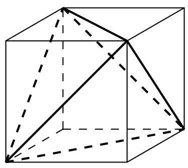


Figure 2: The diagonal pattern of the 5-tetrahedron block.

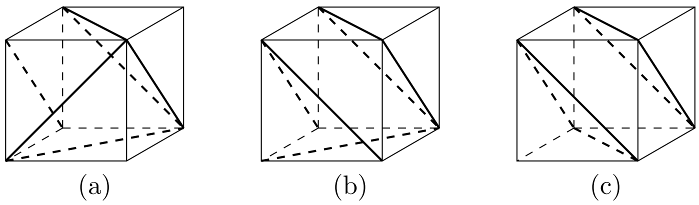


Figure 3: The diagonal pattern if one (a), two (b) or three (c) pairs of diagonals do not match.

the three pairs of the diagonals match, we get a triangulation of M. Otherwise, one, two or three of them do not match.

If one pair of diagonals does not match, we consider the flipped block (i.e. we add a tetrahedron) in order to change one of the two non-matching diagonals, getting a diagonal pattern (shown, up to symmetry, in Fig. 3-a) whose pairs of diagonals match, and hence getting a triangulation of M.

If two pairs of diagonals do not match, it is easy to prove that there are two diagonals, one for each non-matching pair, in adjacent squares. If we change these two diagonals, we get (up to symmetry) the diagonal pattern shown in Fig. 3-b. Therefore, if we consider the 5-valent block, whose pairs of diagonals match, we get a triangulation of M.

Finally, if all of the three pairs of diagonals do not match, it is easy to prove that there are three diagonals, one for each non-matching pair, in squares that share a vertex of the cube. There are two possibilities: either the vertex belong to all of the three diagonals, or it does not belong to any of them. In the former case we change the three diagonals, in the latter case we change the other three diagonals. In both cases we get (up to symmetry) the diagonal pattern shown in Fig. 3-c. Therefore, if we consider the 4-valent block, whose pairs of diagonals match, we get a triangulation of M.

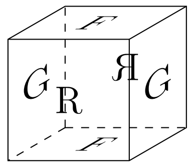
In all cases, we have got a triangulation obtained by gluing the squares of one of the four blocks described above, and the proof is complete. □

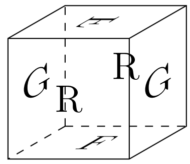
Proof of Theorem 3.1 We can now prove the main result of the paper.

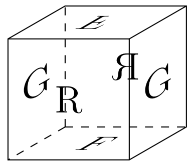
Proof of Theorem 3.1. Consider a non-orientable $\mathbb{P}^2$-irreducible 3-manifold M with a cubulation with one cube. By Lemma 3.4, we have that the Matveev complexity of M is at most 6, so, by Remark 3.3, M is either one of the four flat manifolds or the torus bundle with monodromy $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In Table 2 we have shown a cubulation of the four flat manifolds with one cube, so they have cubic-complexity one.

Instead, the torus bundle with monodromy $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ does not have a cubulation with one cube, so it has cubic-complexity greater than one. In order to prove

Burton notation [6]	Regina's notation for Seifert fibrations [8]	Cubulation

$K^2 \times S^1$	`KB x S1`
	`A= x S1`
	`T x~ S1`
		

$T^2 \times I/\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	`SFS [KB: (1,1)]`
	`M_ x S1`
	`SFS [T/o2: (1,1)]`
		

$K^2 \times I/\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$	`KB/n3 x~ S1`
	`A=/o2 x~ S1`
	`SFS [D_: (2,1) (2,1)]`
		

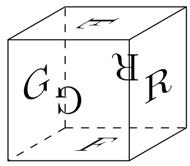
$K^2 \times I/\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$	`SFS [KB/n3: (1,1)]`
	`M_/n2 x~ S1`
	`SFS [RP2: (2,1) (2,1)]`
		

Table 2: A cubulation of the four flat manifolds.

this (and hence to conclude the proof), we suppose by contradiction that a cubulation with one cube exists. By Lemma 3.4 there exists a triangulation obtained by gluing the squares of one of the four blocks shown in Table 1. First of all we can rule out the 5-tetrahedron block because the manifold has Matveev complexity 6 (see Remark 3.3). In order to rule out the other three blocks, we will analyse the valences of the edges of the triangulations that can be obtained by means of them. The valences of the internal edge and of the edges corresponding to the diagonals in the three blocks are listed in Table 3. In each triangulation obtained with these three blocks there is at least one edge with valence 4: the internal one in the case of the flipped block and in the case of the the 4-valent block, and the edge corresponding to a diagonal in the case

Block	Valence of the internal edge	Valences of the diagonal edges
flipped block	4	1, 3, 3, 3, 3, 3
5-valent block	5	2, 2, 2, 2, 3, 3
4-valent block	4	2, 2, 3, 3, 3, 3

Table 3: The valences of the internal and diagonal edges of the three blocks.

of the 5-valent block (indeed two of the four diagonals whose corresponding edge has valance 2 must be glued together). In [6] the unique triangulation with 6 tetrahedra of the torus bundle with monodromy $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ is shown; for the sake of the clarity, we mention that, as a matter of fact, the matrix used is $\left(\begin{smallmatrix}-1&1\\1&0\end{smallmatrix}\right)$, but the resulting manifold is the same (as shown in [5, Corollary A.6]). It has no edge with valence 4 (see also [8]), so we have got a contradiction and the theorem is proved. $\square$

Remark 3.5. The proof that the torus bundle with monodromy $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ has not cubic-complexity one can be also given by means of the orientable census of manifolds with cubic-complexity at most two, given in [10]. Indeed a cubulation with one cube of the torus bundle with monodromy $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$ would lift to a cubulation with two cubes of its orientable double covering, which is the torus bundle with monodromy $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, but this manifold does not appear in the list of [10].

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