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Global bifurcation of double phase problems

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"Dedicated to Professor Enzo Mitidieri on the occasion of his 70th birthday, with high feelings of admiration for his notable contributions in Mathematics and great affection"

Abstract. *Via the global bifurcation theorem due to Rabinowitz, the paper shows bifurcation properties of the solutions of the following nonlinear Dirichlet problem, involving a double phase operator, that is*

$$\begin{cases} -\Delta_p^a u - \nu\Delta_m u = \lambda a(x)|u|^{m-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < m < p < N$, $p/m < 1 + 1/N$ *and* $\lambda, \nu \in \mathbb{R}$.

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1. Introduction

Motivated by the celebrated results contained in [4, 8, 12], we wrote this note on a problem somehow connected with the models studied by Mitidieri and his collaborators in these well known papers. More specifically, this paper deals with bifurcation properties of (weak) solutions of the nonlinear double phase elliptic Dirichlet problem

$$\begin{cases} -\Delta_p^a u - \nu\Delta_m u = \lambda a(x)|u|^{m-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ 1 < m < p < N, \qquad \dfrac{p}{m} < 1 + \dfrac{1}{N}, \qquad \lambda, \nu \in \mathbb{R}, \end{cases} \tag{$\mathcal{P}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary,

$$\Delta_p^a u = \operatorname{div}(a(x)|Du|^{p-2}Du)$$

and a is a positive weight of class $C^{0,1}(\overline{\Omega})$, throughout the paper.

Problems like $(\mathcal{P})$ arise from the prototype equation

$$u_t = \Delta_p^a u + \Delta_m u + f(x,u),$$

where u generally stands for a concentration, $\Delta_p^a u + \Delta_m u$ is the diffusion with coefficient $a|Du|^{p-2} + |Du|^{m-2}$, while $f(x,u)$ stands for the reaction term related to source and loss processes, see Cherfils and Il'yasov [10] and Singer [27] for more details. In order to describe the behavior of strongly anisotropic materials, also known as the Lavrentiev phenomenon, Zhikov first introduced the functional

$$\int_\Omega (a(x)|\nabla u|^p + |\nabla u|^m)dx, \tag{1}$$

where a is an auxiliary tool for regulating the mixture between two different materials by hardening p and m, respectively, see for instance [29]. Moreover, in [19, 20], in view of the Marcellini terminology, the functional (1) appears in the class of the integral functionals, having non-standard growth conditions. In [5, 6, 7], Mingione et al. investigate the interior regularity results primarily for minimizers of (1) and obtained sharp results when $p > m$ and $a \geq 0$ in Ω. A detailed historical survey of the recent developments on the subject as well as its applications can be found in [21] due to Mingione and Rădulescu. Recently, the double phase operator has been widely investigated to describe the steady-state solutions of reaction-diffusion problems in biophysics, plasma physics, and chemical reaction analysis, see [9, 29, 30] and the references cited therein. More precisely, Liu and Dai in [18] study the existence and multiplicity results of the sign-changing ground state solution of the problem

$$\begin{cases} -\Delta_p^a u - \Delta_m u = f(x,u) & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Furthermore, the detailed spectral analysis and the existence and multiplicity of a nonlinear elliptic Dirichlet problem involving a double phase operator are presented in [22] by Papageorgiou, Pudełko and Rădulescu. More recently, the existence of solutions of the critical equation

$$-\Delta_m u - \Delta_p u = \lambda w(x)|u|^{p-2}u + |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N,$$

is proved in [24] by variational methods.

In order to state our results, we need to fix some basic notations for the Musielak-Orlicz space. A convex, left-continuous function $\phi : [0,\infty) \to [0,\infty)$. with

$$\phi(0) = 0 \quad \text{and} \quad \lim_{t\to 0^+} \phi(t) = 0$$

is called a Φ-*function.* A Φ-function is said to be *positive*, if $\phi(t) > 0$ for all $t > 0$.

Furthermore, a function $\phi : \Omega \times [0, \infty) \to [0, \infty)$ is called a *generalized Φ-function*, if

(i) $\phi(x, \cdot)$ is a Φ-function for all $x \in \Omega$;

(ii) $\phi(\cdot, t)$ is measurable for all $t \geq 0$.

From here on, $\Phi(\Omega)$ denotes the set of all generalized Φ-functions. Finally, $\phi \in \Phi(\Omega)$ is said to be *locally integrable*, if $\phi(\cdot, t) \in L^1(\Omega)$ for all $t \geq 0$.

Let $\xi : \Omega \times [0, \infty) \to [0, \infty)$ be the function $(x, t) \mapsto t^m + a(x)t^p$ for $x \in \Omega$ and $t \geq 0$, with $1 < m < p$ and $0 < a \in L^1(\Omega)$. It is clear that ξ is in $\Phi(\Omega)$, that ξ is locally integrable and that ξ satisfies the condition (Δ_2), that is,

$$\xi(x, 2t) \leq 2^p \xi(x, t) \quad \text{for a.e. } x \in \Omega \text{ and } t \geq 0.$$

Recall that the Musielak-Orlicz space $L^\xi(\Omega)$ is

$$L^\xi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is measurable} : \ \rho_\xi(u) < \infty\},$$

where ρ_ξ is the modular function defined by

$$\rho_\xi(u) = \int_\Omega \xi(x, |u|)dx = \int_\Omega [a(x)|u|^p + |u|^m]dx.$$

The space $L^\xi(\Omega)$ is equipped with the Luxemburg norm

$$\|u\|_\xi = \inf \{\ell > 0 : \ \rho_\xi(u/\ell) \leq 1\}.$$

The Musielak-Orlicz space $L^\xi(\Omega)$ is proved to be a separable and uniformly convex (and so reflexive) Banach space.

The corresponding Musielak-Orlicz-Sobolev space is

$$W^{1,\xi}(\Omega) = \left\{u \in L^\xi(\Omega) : \ |Du| \in L^\xi(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,\xi} = \|u\|_\xi + \|Du\|_\xi,$$

where $\|Du\|_\xi = \||Du|\|_\xi$.

It is well known that $W^{1,\xi}(\Omega)$ is a separable and uniformly convex (and so reflexive) Banach space. Moreover,

$$W_0^{1,\xi}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\xi}}$$

is again a separable and uniformly convex (and so reflexive) Banach space. For more details on the Musielak-Orlicz theory we refer to [11, 14, 18] and to the references therein.

Thanks to the fact that a is a positive weight of class $C^{0,1}(\overline{\Omega})$ and to the Poincaré inequality, in $W_0^{1,\xi}(\Omega)$ we consider the equivalent norm

$$\|u\| = \|Du\|_\xi, \quad \text{for any } u \in W_0^{1,\xi}(\Omega).$$

The dual space of $W_0^{1,\xi}(\Omega)$ is simply denoted by $W_0^{1,\xi}(\Omega)^*$.

In order to state the main result of the paper we need to use the p-Muckenhoupt class $\widetilde{A}_p(\Omega)$, as introduced by Muckenhoupt in 1972 in connection with the properties of the Hardy-Littlewood maximal operators. Following [1, Definition 1.4.3], we say that a function w is a *weight* in the open set Ω, if $w \in L^1_{\text{loc}}(\Omega)$ and $w > 0$ a.e. in Ω. Moreover, we say that a weight w is a *p-Muckenhoupt weight*, $p > 1$, and we write $w \in \widetilde{A}_p(\Omega)$, if w satisfies the condition

$$\sup_{B\subseteq\Omega} \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-p)} dx\right)^{p-1} < \infty.$$

We are now ready to state the main assumptions of the paper.

(H) $a \in C^{0,1}(\overline{\Omega}) \cap \widetilde{A}_p(\Omega)$, $a > 0$ in Ω;

(f_1) $f = f(x,t)$ satisfies the Carathéodory condition;

(f_2) $|f(x,t)| = o(|t|^{m-1})$ as $t \to 0$ uniformly a.e. with respect to $x \in \Omega$;

(f_3) There exist a constant $C > 0$ and an exponent r, with $m < r < m^*$, such that for all $t \in \mathbb{R}$

$$|f(x,t)| \le C|t|^{r-1}$$

uniformly a.e. for $x \in \Omega$, where

$$m^* = \frac{mN}{N-m}.$$

Let us note in passing that $p < m^*$, since throughout the paper we require that $1 < m < p < N$ and $p/m < 1 + 1/N$. We are now able to state the main results of the paper.

THEOREM 1.1. *Let the assumptions (H), (f_1)–(f_3) hold and let $\nu = 1$. Then, when $a \equiv 1$, the positive (weak) solutions of problem $(\mathcal{P})$ have a bifurcation point at $(0,0)$. Moreover, there exists a component $\mathcal{C}_0$ in $\mathbb{R} \times W_0^{1,\xi}(\Omega)$ of the positive (weak) solutions of $(\mathcal{P})$, such that its closure contains $(0,0)$, and $\mathcal{C}_0$ is unbounded.*

Let us also consider

$$\begin{cases} -\Delta_p^a u = \mu a(x)|u|^{p-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \\ 1 < m < p < N, \qquad \dfrac{p}{m} < 1 + \dfrac{1}{N}, \end{cases} \tag{EP}$$

and let λ_1 denote the first eigenvalue of (EP), see the next Section 3 for details.

THEOREM 1.2. *Assume that* (H) *holds and let* λ *in* $(\mathcal{P})$ *be such that* $0 < \lambda < \lambda_1$. *Let* f *satisfy* (f_1) *and*

(f_4) *There exists* κ *such that* $0 < \kappa < \lambda_1 - \lambda$ *and for all* $t \in \mathbb{R}$

$$|f(x,t)| \le \kappa a(x)|t|^{p-1}$$

uniformly a.e. in $x \in \Omega$.

If $(u_k)_k$ *is an unbounded sequence in* $W_0^{1,\xi}(\Omega)$ *consisting of (weak) solutions of problem* $(\mathcal{P})$ *corresponding to* $(\nu_k)_k \subset \mathbb{R}^+$, *then* $\nu_k \to 0$ *as* $k \to \infty$.

The organization of this paper is as follows. Section 2 contains some definitions and key lemmas useful in what follows. Section 3 deals with the basic bifurcation properties of the fundamental operator of $(\mathcal{P})$ via the topological degree. In particular, Section 3 contains the proof of bifurcation at $(0,0)$ for a problem related to $(\mathcal{P})$. Finally. Section 3 presents also the proof of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

We assume, throughout the paper and without further mentioning, that

$$1 < m < p < N, \qquad \frac{p}{m} < 1 + \frac{1}{N},$$

and that assumptions (H) and (f_1) hold.

In this section, we first introduce some notations, definitions, and properties of the functional setting for $(\mathcal{P})$, useful for the proofs of the main results of the paper.

Let us introduce the operators A_p^a, $A_m : W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ defined pointwise for all u, $v \in W_0^{1,\xi}(\Omega)$ by

$$\begin{aligned} \langle A_p^a(u), v\rangle &= \int_\Omega (a(x)|Du|^{p-2}Du, Dv)dx \\ \langle A_m(u), v\rangle &= \int_\Omega (|Du|^{m-2}Du, Dv)dx. \end{aligned} \tag{3}$$

Put $J = A_p^a + \nu A_m$, then

Definition 2.1. Let G, $F : W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ be defined pointwise for any u, $v \in W_0^{1,\xi}(\Omega)$ by

$$\langle G(u), v\rangle = \int_\Omega a(x)|u|^{m-2}uvdx, \quad \langle F(u), v\rangle = \int_\Omega f(x,u)vdx.$$

A function $u \in W_0^{1,\xi}(\Omega)$ is called a (weak) *solution* of $(\mathcal{P})$ if

$$J(u) - \lambda G(u) - F(u) = 0 \quad \text{in } W_0^{1,\xi}(\Omega)^*. \tag{4}$$

If $(\lambda_n)_n \subset \mathbb{R}$ and $(u_n)_n \subset W_0^{1,\xi}(\Omega)$ is a sequence of nontrivial solutions of $(\mathcal{P})$ such that $(\lambda_n, u_n) \to (0,0)$ as $n \to \infty$, then $(0,0)$ is called to be a *bifurcation point* of $(\mathcal{P})$. Furthermore, if

$$\mathcal{C} = \{(\lambda, u) \in \mathbb{R} \times W_0^{1,\xi}(\Omega) : \ u \not\equiv 0 \text{ and } (\lambda, u) \text{ solves } (\mathcal{P})\}$$

is a connected set in $\mathbb{R} \times W_0^{1,\xi}(\Omega)$, then $\mathcal{C}$ is called a *component of nontrivial solutions* of $(\mathcal{P})$.

For more details, we refer for instance to [26]. The next embedding results are particularly useful in what follows. Let us note in passing that the next two lemmas continue to hold under the weaker request that a is a positive weight of class $C^{0,1}(\overline{\Omega})$.

Lemma 2.2 ([17, Chapter 6]). *The following properties hold true.*

(a) *$L^\xi(\Omega) \hookrightarrow L^\wp(\Omega)$ and $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,\wp}(\Omega)$ continuously for all $\wp \in [1, m]$;*

(b) *The embedding $W_0^{1,\xi}(\Omega) \hookrightarrow L^\wp(\Omega)$ is continuous for all $\wp \in [1, m^*]$ and compact for all $\wp \in [1, m^*)$.*

Lemma 2.3 ([23, Lemma 2 and Proposition 1]). *Consider $\xi_0(x,t) = a(x)t^p$. Let $L^{\xi_0}(\Omega)$ be the corresponding Banach space, equipped with the Luxemburg norm $\|\cdot\|_{\xi_0}$ associated to the modular function*

$$\rho_{\xi_0}(u) = \int_\Omega a(x)|u|^p dx.$$

Let $W_0^{1,\xi_0}(\Omega)$ be the Banach space, endowed with the norm $\|Du\|_{\xi_0}$. Then the embedding $W_0^{1,\xi_0}(\Omega) \hookrightarrow L^{\xi_0}(\Omega)$ is compact. Moreover,

(a) *If $|Du| \in L^{\xi_0}(\Omega)$, then $\rho_{\xi_0}(|Du|) < 1$ (resp. $= 1$, > 1) $\Leftrightarrow \||Du|\|_{\xi_0} < 1$ (resp. $= 1$, > 1).*

(b) *$\||Du|\|_{\xi_0} \to 0 \Leftrightarrow \rho_{\xi_0}(|Du|) \to 0$ and $\||Du|\|_{\xi_0} \to \infty \Leftrightarrow \rho_{\xi_0}(|Du|) \to \infty$.*

Finally, the embedding $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,\xi_0}(\Omega)$ *is continuous.*

LEMMA 2.4 ([17, Section 3.2]). *The following properties hold.*

(*a*) $\|u\| < 1$ *(resp.* $= 1; > 1$*)* $\Leftrightarrow \rho_\xi(|Du|) < 1$ *(resp.* $= 1; > 1$*);*

(*b*) $\|u\| \to 0 \Leftrightarrow \rho_\xi(|Du|) \to 0$ *and* $\|u\| \to \infty \Leftrightarrow \rho_\xi(|Du|) \to \infty$.

LEMMA 2.5 ([22, Proposition 10]). *The operator* $A_p^a : W_0^{1,\xi_0}(\Omega) \to W_0^{1,\xi_0}(\Omega)^*$ *is of type* $(S)_+$.

LEMMA 2.6. *The operator* $A_m : W_0^{1,m}(\Omega) \to W_0^{1,m}(\Omega)^*$ *is of type* $(S)_+$.

Proof. Consider a sequence $(u_n)_n \subset W_0^{1,m}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,m}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_m(u_n), u_n - u\rangle \le 0.$$

We claim that $\langle A_m(u), u_n - u\rangle \to 0$ as $n \to \infty$. In fact, since $u_n \rightharpoonup u$ in $W_0^{1,m}(\Omega)$, in particular $Du_n \rightharpoonup Du$ in $[L^m(\Omega)]^N$ as $n \to \infty$ and clearly $|Du|^{m-1}$ is in $L^{m'}(\Omega)$. This gives at once that as $n \to \infty$

$$\langle A_m(u), u_n - u\rangle = \int_\Omega (|Du|^{m-2}Du, Du_n - Du)dx \to 0.$$

Therefore, the convexity and the fact that $\limsup_{n\to\infty}\langle A_m(u_n), u_n - u\rangle \le 0$ imply

$$0 \le \limsup_{n\to\infty}\langle A_m(u_n) - A(u), u_n - u\rangle \le 0.$$

In other words,

$$\lim_{n\to\infty}\langle A_m(u_n) - A(u), u_n - u\rangle = 0, \tag{5}$$

that is the sequence $n \mapsto (|Du_n|^{m-2}Du_n - |Du|^{m-2}Du, Du_n - Du) \ge 0$ converges to 0 in $L^1(\Omega)$. Hence, up to a subsequence, still denoted in the same way,

$$(|Du_n|^{m-2}Du_n - |Du|^{m-2}Du, Du_n - Du) \to 0 \quad \text{a.e. in } \Omega.$$

By virtue of [13, Lemma 3], we also have $Du_n \to Du$ a.e. in Ω. Furthermore, the Brézis-Lieb theorem gives as $n \to \infty$

$$\|Du\|_m^m = \|Du_n\|_m^m - \|Du_n - Du\|_m^m + o(1).$$

and

$$\lim_{n\to\infty}\langle A_m(u_n), u_n - u\rangle = 0.$$

Consequently, combining all the above facts, we get

$$\begin{aligned}
o(1) &= \langle A_m(u_n), u_n - u\rangle \\
&= \int_\Omega |Du_n|^{m-2}(Du_n, Du_n - Du)dx \\
&= \|Du_n\|_m^m - \int_\Omega |Du_n|^{m-2}(Du_n, Du)dx \\
&= \|Du\|_m^m + \|Du_n - Du\|_m^m - \|Du\|_m^m + o(1) \\
&= \|Du_n - Du\|_m^m + o(1)
\end{aligned}$$

being $|Du_n|^{m-2}Du_n \rightharpoonup |Du|^{m-2}Du$ in $[L^{m'}(\Omega)]^N$. Thus, $\|Du_n - Du\|_m^m \to 0$ as $n \to \infty$, that is $u_n \to u$ in $W_0^{1,m}(\Omega)$, as required. □

We are now in a position to prove the next result.

LEMMA 2.7. *Let either (f_4) or (f_2) and (f_3) hold. Then, the operators G, F, given in Definition 2.1, are continuous and compact in $W_0^{1,\xi}(\Omega)$.*

Furthermore, if (f_2) and (f_3) hold, then F satisfies

$$\lim_{\|u\|\to 0} \frac{\|F(u)\|_{W_0^{1,\xi}(\Omega)^*}}{\|u\|^{m-1}} = 0 \quad \text{and} \quad \lim_{\|Du\|_m\to 0} \frac{\|F(u)\|_{W^{1,m}(\Omega)^*}}{\|Du\|_m^{m-1}} = 0. \tag{6}$$

Proof. Let us first prove that G and F are compact. To this aim, fix φ in $W_0^{1,\xi}(\Omega)$, with $\|\varphi\| \le 1$ and a bounded sequence $(w_n)_n \subset W_0^{1,\xi}(\Omega)$. Clearly, $m < p < m^*$, since $p/m < 1 + 1/N$. Hence Lemma 2.2 guarantees that there exists a function $w \in W_0^{1,\xi}(\Omega)$, such that $w_n \rightharpoonup w$ in $W_0^{1,\xi}(\Omega)$, $w_n \to w$ in $L^\wp(\Omega)$, with $\wp \in [1, m^*)$, and $w_n \to w$ a.e. in Ω, up to sequences if necessary. Take any subsequence $(w_{n_k})_k \subset (w_n)_n$. Of course, $w_{n_k} \to w$ a.e. in Ω. Thus,

$$a(x)|w_{n_k}|^{m-2}w_{n_k}\varphi - a(x)|w|^{m-2}w\varphi \to 0 \quad \text{a.e. in } \Omega.$$

Furthermore, for each measurable subset $E \subset \Omega$, the assumption (H), the Hölder inequality and Lemma 2.2 imply that

$$\begin{aligned}
\int_E a(x)\left||w_{n_k}|^{m-2}w_{n_k}\varphi\right| dx &\le \|a\|_{C^{0,1}(E)}\|w_{n_k}\|_{L^m(E)}^{m-1}\|\varphi\|_{L^m(E)} \\
&\le C_m\|a\|_{C^{0,1}(E)}\|w_{n_k}\|^{m-1}\|\varphi\| \\
&\le C_m\|a\|_{C^{0,1}(E)}
\end{aligned}$$

being $\|\varphi\| \le 1$ and $a \in C^{0,1}(\Omega)$. Consequently, $(a(x)|w_{n_k}|^{m-2}w_{n_k}\varphi)_n$ is equi-integrable and uniformly bounded in $L^1(\Omega)$. Hence, the Vitali convergence theorem implies at once that for any φ, with $\|\varphi\| \le 1$, as $k \to \infty$

$$\langle G(w_{n_k}) - G(w), \varphi\rangle = \int_\Omega a(x)(|w_{n_k}|^{m-2}w_{n_k} - |w|^{m-2}w)\varphi dx \to 0$$

and so $\langle G(w_n) - G(w), \varphi\rangle \to 0$, since the sequence $(w_{n_k})_k$ is arbitrary. Therefore, as $n \to \infty$

$$\|G(w_n) - G(w)\|_{W_0^{1,\xi}(\Omega)^*} = \sup_{\|\varphi\|\le 1} |\langle G(w_n) - G(w), \varphi\rangle| \to 0.$$

This shows that the operator G is compact, as required. Proceeding in a similar way, we prove that G is continuous in $W_0^{1,\xi}(\Omega)$.

Next we prove that F is compact in $W_0^{1,\xi}(\Omega)$. To see this, fix φ in $W_0^{1,\xi}(\Omega)$ and a sequence $(u_n)_n \subset W_0^{1,\xi}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,\xi}(\Omega)$. Lemma 2.2 implies that $u_n \to u$ in $L^{\wp}(\Omega)$, with $\wp \in [1, m^*)$. Moreover, passing eventually to a subsequence, we can assume that $u_n \to u$ a.e. in Ω and that there exists $g \in L^{\wp}(\Omega)$, such that $|u_n| \le g$ a.e. in Ω for all n, thanks to [3, Theorem 2.3]. Hence, (f_1) gives that

$$f(x, u_n)\varphi \to f(x, u)\varphi \text{ a.e. in } \Omega.$$

Fix $\varepsilon > 0$. By (f_2) there exists a positive number $\delta = \delta(\varepsilon)$, such that uniformly for a.e. $x \in \Omega$,

$$|f(x,t)| \le \varepsilon |t|^{m-1} \quad \text{for all } t, \text{ with } |t| \le \delta.$$

In particular, assumption (f_3) guarantees that uniformly for a.e. $x \in \Omega$,

$$|f(x,t)| \le C|t|^{r-1} \quad \text{for all } t, \text{ with } |t| \ge \delta.$$

Moreover, by (f_2) and (f_3), there exist $g_1 \in L^m(\Omega)$ and $g_2 \in L^r(\Omega)$, such that for all n and a.e. in Ω

$$|f(x, u_n)\varphi| \le (\varepsilon|u_n|^{m-1} + C|u_n|^{r-1})|\varphi| \le C(|g_1|^{m-1} + |g_2|^{r-1})|\varphi|. \tag{7}$$

When (f_4) holds, there exists $g_3 \in L^p(\Omega)$ such that for all n and a.e. in Ω

$$|f(x, u_n)\varphi| \le a(x)|u_n|^{p-1}|\varphi| \le a(x)|g_3|^{p-1}|\varphi|. \tag{8}$$

Since

$$\int_\Omega (|g_1|^{m-1} + |g_2|^{r-1})|\varphi| dx \le \|g_1\|_{L^m(\Omega)}^{m-1}\|\varphi\|_{L^m(\Omega)} + \|g_2\|_{L^r(\Omega)}^{r-1}\|\varphi\|_{L^r(\Omega)} < \infty$$

by (7), and

$$\int_\Omega a(x)|g_3|^{p-1}|\varphi| dx \le \|a\|_{C^{0,1}(\Omega)}\|g_3\|_{L^p(\Omega)}^{p-1}\|\varphi\|_{L^p(\Omega)} < \infty$$

by (8), when either (f_2) and (f_3) or (f_4) hold, the Lebesgue dominated convergence theorem yields that

$$\lim_{n\to\infty} \int_\Omega |f(x, u_n)\varphi| dx = \int_\Omega |f(x, u)\varphi| dx. \tag{9}$$

Set $h_n(x) = |f(x,u_n)\varphi| - |(f(x,u_n) - f(x,u))\varphi|$. Obviously, $h_n \to |f(\cdot,u)\varphi|$ a.e. in Ω. Moreover, from the trivial fct that $|h_n| \le |f(\cdot,u_n)\varphi|$ in Ω, from (7) and (8), we are able to apply once again the Lebesgue dominated convergence theorem and get

$$\lim_{n\to\infty} \int_\Omega \big(|f(x,u_n)\varphi| - |(f(x,u_n) - f(x,u))\varphi|\big)dx = \int_\Omega |f(x,u)\varphi|dx.$$

Therefore, as $n \to \infty$,

$$\sup_{\|\varphi\|\le 1} |\langle F(u_n) - F(u), \varphi\rangle| = \sup_{\|\varphi\|\le 1} \int_\Omega |(f(x,u_n) - f(x,u))\varphi|dx \to 0,$$

that is F is a compact operator. Similarly, we can show that F is continuous.

Finally, let us prove (6) under assumptions (f_2) and (f_3). Suppose first that $(v_n)_n \subset W_0^{1,\xi}(\Omega)$, with $v_n \not\equiv 0$ for all n and with $\|v_n\| \to 0$ as $n \to \infty$. Set $\overline{v}_n = v_n\|v_n\|^{-1}$. The assumptions (f_2) and (f_3) guarantee that for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that for any n

$$\begin{aligned}\int_\Omega |f(x,v_n)\varphi|dx &\le \varepsilon \int_{\Omega_{\delta,n}} ||v_n|^{m-1}\varphi|dx + C\int_{\Omega\setminus\Omega_{\delta,n}} ||v_n|^{r-1}\varphi|dx\\ &\le \varepsilon \int_\Omega ||v_n|^{m-1}\varphi|dx + C\int_\Omega ||v_n|^{r-1}\varphi|dx,\end{aligned}$$

where $\Omega_{\delta,n} = \{x \in \Omega : |v_n| \le \delta\}$.

Fix $\varphi \in W_0^{1,\xi}(\Omega)$, with $\|\varphi\| \le 1$. Lemma 2.2 and the Hölder inequality give for all n

$$\begin{aligned}\frac{1}{\|v_n\|^{m-1}}\left|\int_\Omega f(x,v_n)\varphi dx\right| &\le \frac{1}{\|v_n\|^{m-1}}\int_\Omega \big|f(x,v_n)\varphi\big|dx\\ &\le \varepsilon\int_\Omega \big||\overline{v}_n|^{m-1}\varphi\big|dx + C\int_\Omega \big||\overline{v}_n|^{m-1}|v_n|^{r-m}\varphi\big|dx\\ &\le \varepsilon\|\overline{v}_n\|^{m-1}_{L^m(\Omega)}\|\varphi\|_{L^m(\Omega)} + C\|\overline{v}_n\|^{m-1}_{L^r(\Omega)}\|v_n\|^{r-m}_{L^r(\Omega)}\|\varphi\|_{L^r(\Omega)}\\ &\le C_m\varepsilon + C_r\|v_n\|^{r-m}.\end{aligned} \tag{10}$$

Consequently, since $\varepsilon > 0$ is arbitrary and $m < r$ by (f_3), we have

$$\frac{1}{\|v_n\|^{m-1}}\left|\int_\Omega f(x,v_n)\varphi dx\right| \to 0, \qquad \text{as } n \to \infty.$$

This shows that the first part of(6) holds true.

To show the second part of(6), fix $(u_n)_n \subset W_0^{1,m}(\Omega)$, with $u_n \not\equiv 0$ for all n and with $\|Du_n\|_m \to 0$ as $n \to \infty$. Set $\widehat{u}_n = u_n\|Du_n\|_m^{-1}$. Since the embedding

$W_0^{1,m}(\Omega) \hookrightarrow L^{\wp}(\Omega)$, with $\wp \in [1, m^*)$, is compact, arguing as in (10), we get for any $\phi \in W_0^{1,m}(\Omega)$, with $\|D\phi\|_m \leq 1$, as $n \to \infty$

$$\begin{aligned}\frac{1}{\|Du_n\|_m^{m-1}} \left| \int_\Omega f(x, u_n)\varphi dx \right| &\leq \frac{1}{\|Du_n\|_m^{m-1}} \int_\Omega \big|f(x, u_n)\varphi\big| dx \\ &\leq \varepsilon \int_\Omega \big||\widehat{u}_n|^{m-1}\varphi\big| dx + C \int_\Omega \big||\widehat{u}_n|^{m-1}|u_n|^{r-m}\varphi\big| dx \\ &\leq \varepsilon \|\widehat{u}_n\|_{L^m(\Omega)}^{m-1} \|\varphi\|_{L^m(\Omega)} + C\|\widehat{u}_n\|_{L^r(\Omega)}^{m-1} \|u_n\|_{L^r(\Omega)}^{r-m} \|\varphi\|_{L^r(\Omega)} \\ &\leq C_m\varepsilon \|D\widehat{u}_n\|_m^{m-1} \|D\varphi\|_m + C_r \|D\widehat{u}_m\|_m^{m-1} \|Du_n\|_m^{r-m} \|D\varphi\|_m^m \\ &\leq C_m\varepsilon + C_r\|Du_n\|^{r-m}.\end{aligned}$$

Since $m < r$ by (f_3), we get

$$\limsup_{n\to\infty} \frac{1}{\|Du_n\|_m^{m-1}} \left| \int_\Omega f(x, u_n)\varphi dx \right| \leq C_m\varepsilon.$$

The fact that $\varepsilon > 0$ is arbitrary completes the proof of (6) and of the lemma. □

3. Bifurcation results

Before proving Theorems 1.1, we introduce some preliminary results. We recall that assumptions (H) and (f_1) are assumed throughout the paper. By (3) the main operator $J : W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ is defined pointwise for u, $v \in W_0^{1,\xi}(\Omega)$ by

$$\langle J(u), v \rangle = \int_\Omega a(x)|Du|^{p-2}(Du, Dv)dx + \nu \int_\Omega |Du|^{m-2}(Du, Dv)dx.$$

Lemma 3.1 ([18, Proposition 3.1]). *Then, the operator J has the following properties.*

(*a*) *The operator J is bounded (that is, J maps any bounded set into a bounded set), continuous and strictly monotone;*

(*b*) *J satisfies condition $(S)_+$, that is, if $(u_n)_n \subset W_0^{1,\xi}(\Omega)$ weakly converges in $W_0^{1,\xi}(\Omega)$ to some u of $W_0^{1,\xi}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle J(u_n), u_n - u \rangle \leq 0,$$

then $u_n \to u$ strongly in $W_0^{1,\xi}(\Omega)$;

(*c*) *J is a homeomorphism.*

Properties (a) and (c) of Lemma 3.1 yield that for any $g \in W_0^{1,\xi}(\Omega)^*$ there exists a unique $u = K(g) \in W_0^{1,\xi}(\Omega)$ of $J(u) = g$, that is $u = K(g)$ is a weak solution of the equation

$$-\Delta_p^a u - \nu \Delta_m u = g. \tag{11}$$

Obviously, thanks to Lemma 3.1, the operator

$$K : W_0^{1,\xi}(\Omega)^* \to W_0^{1,\xi}(\Omega) \quad \text{is a homeomorphism} \tag{12}$$

from the Banach space $W_0^{1,\xi}(\Omega)^*$ to the Banach space $W_0^{1,\xi}(\Omega)$. Additionally, for any $\lambda \in \mathbb{R}$, define the operator $T_\lambda :\ W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ pointwise for all $u,\, v \in W_0^{1,\xi}(\Omega)$ by

$$\langle T_\lambda(u), v\rangle = \int_\Omega [\lambda |u|^{m-2}u - f(x,u)]v dx, \tag{13}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $W_0^{1,\xi}(\Omega)$ and $W_0^{1,\xi}(\Omega)^*$.

Now, let $\widehat{\lambda}_1$ denote the first eigenvalue of $\Big(-\Delta_m, W_0^{1,m}(\Omega)\Big)$, that is the problem

$$\begin{cases} -\Delta_m u = \widehat{\lambda}_1 |u|^{m-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{14}$$

admits a nontrivial solution in $W_0^{1,m}(\Omega)$ and

$$\widehat{\lambda}_1 = \min_{\substack{u \in W_0^{1,m}(\Omega) \\ u \not\equiv 0}} \frac{\|Du\|_m^m}{\|u\|_{L^m(\Omega)}^m}. \tag{15}$$

Let us also consider the eigenvalue problem

$$\begin{cases} -\Delta_p^a u = \mu b(x) a(x) |u|^{p-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \\ 1 < m < p < N, \qquad \dfrac{p}{m} < 1 + \dfrac{1}{N}, \end{cases} \tag{16}$$

where a is a positive weight of class $C^{0,1}(\overline{\Omega})$, the coefficient b is positive and of class $L^\infty(\Omega)$ and $\mu \in \mathbb{R}$. We refer to [22, Propositions 5–10] for the proof of the next results.

Lemma 3.2 (Eigenvalues and eigenfunctions of (16) [22, Propositions 5–10]). *Let the functions a and b be as above.*

(a) *There exists the smallest eigenvalue $\lambda_{1,b} > 0$ of problem* (16) *and moreover the corresponding eigenfunction $u_{1,b} \in W_0^{1,\xi_0}(\Omega)$ satisfies*

$$u_{1,b} \in L^\infty(\Omega), \quad \textit{and either} \quad u_{1,b} > 0 \quad \textit{or} \quad u_{1,b} < 0 \quad \textit{in } \Omega.$$

(*b*) *The first eigenvalue* $\lambda_{1,b} > 0$ *is simple and isolated, i.e. the first positive eigenfunction corresponding to* $\lambda_{1,b}$ *is unique up to a multiplicative constant and there exists a neighborhood of* $\lambda_{1,b}$ *in which no other eigenvalues lie.*

Let us state the global bifurcation theorem in what follows. Suppose that $E = \big(E, \|\cdot\|\big)$ is a real Banach space. Let $\mathcal{F} : \mathbb{R} \times E \to E$ be a continuous and compact operator, and let $L : E \to E$ be a linear compact operator. Take $\lambda \in \mathbb{R}$. Assume that

$$\mathcal{F}(\lambda, u) = \lambda L u + H(\lambda, u),$$

where $H(\lambda, u) = o(\|u\|)$ as $\|u\| \to 0$ uniformly in λ, as λ varies on bounded real intervals $I \subset \mathbb{R}$. Consider the parametric operator equation

$$u = \mathcal{F}(\lambda, u), \quad u \in E. \tag{17}$$

Let Ψ be the closure of the set consisting of the couples (λ, u), where u is a nontrivial (weak) solution of (17). Let $r(L)$ denote the set of $\lambda \in \mathbb{R}$, such that there exists $v \in E \setminus \{0\}$, with $v = \lambda L v$, i.e. $r(L)$ consists of the reciprocals of the real nonzero eigenvalues of L.

From now on we denote by B_r and $\mathcal{B}_r$ any open ball of E and $\mathbb{R} \times E$ of radius $r > 0$ centered at $0 \in E$, and $(\lambda, 0) \in \mathbb{R} \times E$, respectively.

Lemma 3.3 ([25, Lemma 1.2]). *Let* $\lambda \in r(L)$. *Suppose that there does not exist a sub-component* $\mathcal{C}$ *of* $\Psi \cup \{(\lambda, 0)\}$, *which meets* $(\lambda, 0)$, *and such that either*

(i) $\mathcal{C}$ *is unbounded, or*

(ii) $(\lambda, 0) \in \mathcal{C}$ *whenever* $\lambda \in r(L)$ *and* $\lambda \neq \lambda$.

Then there exists a bounded open set $\mathcal{O} \subset \mathbb{R} \times E$, *such that* $\partial\mathcal{O} \cap \Psi = \emptyset$, $(\lambda, 0) \in \mathcal{O}$, *and* $\mathcal{O}$ *contains no trivial solutions other than those in* $\mathcal{B}_\epsilon$, *where* $0 < \epsilon < \epsilon_0$, *and* ϵ_0 *is the distance from* λ *to* $(r(L) - \{\lambda\})$.

Lemma 3.4 ([25, Lemma 1.3]). *If* $\lambda \in r(L)$ *is of odd multiplicity, then* Ψ *possesses a maximal sub-component* $\mathcal{C}$ *such that* $(\lambda, 0) \in \mathcal{C}$ *and either*

(i) $\mathcal{C}$ *is unbounded, or*

(ii) $(\lambda, 0) \in \mathcal{C}$ *whenever* $\lambda \in r(L)$ *and* $\lambda \neq \lambda$.

We are now ready to prove the first result, applying the above results when $E = W_0^{1,\xi}(\Omega)$.

Proof of Theorem 1.1. Let us now split the proof into the following two steps.

Step 1. We claim that $(0, 0)$ is a bifurcation point of the positive (weak) solutions of problem $(\mathcal{P})$.

Otherwise, if $(0,0)$ is not a bifurcation point of problem $(\mathcal{P})$, then there exist $\alpha_0 > 0$ and ε, with $0 < \varepsilon < \widehat{\lambda}_1$, where $\widehat{\lambda}_1$ is defined by (15), such that for any λ, with $|\lambda| \le \varepsilon$ and any α, with $0 < \alpha < \alpha_0$, there exist no nontrivial (weak) solutions of problem $(\mathcal{P})$. Namely, from the invariance of the topological degree,

$$\deg(I - K \circ T_\lambda, B_\alpha, 0) = \text{constant}, \quad \text{for } \lambda \in [-\varepsilon, \varepsilon], \tag{18}$$

where $I - K \circ T_\lambda : W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)$ and K and T_λ are defined in (12) and (13), respectively.

First fix λ, with $-\varepsilon < \lambda < 0$. Then, define the operator

$$H_\lambda(t,u) := K \circ tT_\lambda(u) : [0,1] \times W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega).$$

We assert that there exists r, with $0 < r < \alpha_0$, such that

$$\text{if } u \in B_r \setminus \{0\}, \text{ then } u \neq H_\lambda(t,u) \text{ for all } t \in [0,1]. \tag{19}$$

Otherwise, there exist sequences $(u_n)_n$, $(t_n)_n$, with $(u_n)_n \subset W_0^{1,\xi}(\Omega)$ and $(t_n)_n \subset [0,1]$, such that $u_n > 0$ a.e. in Ω, $\|u_n\| \to 0$ in $W_0^{1,\xi}(\Omega)$ and $u_n = H_\lambda(t_n, u_n)$ for all n. Hence,

$$\int_\Omega (a(x)|Du_n|^p + |Du_n|^m)dx = t_n \int_\Omega (\lambda|u_n|^m + f(x,u_n)u_n)dx. \tag{20}$$

Fix $\overline{u}_n = u_n\|u_n\|^{-1}$. Multiplying (20) by $\|u_n\|^{-m}$, by virtue of (6) and Lemma 3.1, we get as $n \to \infty$

$$\begin{aligned}
0 &\le \|u_m\|^{p-m} \int_\Omega a(x)|D\overline{u}_n|^p dx + \int_\Omega |D\overline{u}_n|^m dx \\
&= \lambda t_n \int_\Omega |\overline{u}_n|^m dx + t_n\|u_n\|^{1-m} \int_\Omega f(x,u_n)\overline{u}_n dx \\
&\le \lambda t_n \int_\Omega |\overline{u}_n|^m dx + o(1) < 0,
\end{aligned}$$

being $1 < m < p$. This is impossible and so (19) holds.

Now, choosing $\epsilon \in (0,r)$ and using the homotopy invariance of H_λ, we deduce that

$$\begin{aligned}
\deg(\mathbb{I} - K \circ T_\lambda, B_\epsilon, 0) &= \deg(\mathbb{I} - H_\lambda(1,\cdot), B_\epsilon, 0) \\
&= \deg(\mathbb{I} - H_\lambda(0,\cdot), B_\epsilon, 0) \\
&= \deg(\mathbb{I}, B_\epsilon, 0) = 1.
\end{aligned} \tag{21}$$

Fix now $0 < \lambda < \varepsilon$. Take $\psi \in L^\infty(\Omega)$, with $\psi < 0$ a.e. in Ω, and define the operator $T_f : [0,1] \times W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ pointwise for all $t \in [0,1]$ and u,

$v \in W_0^{1,\xi}(\Omega)$ by

$$\langle T_f(t,u), v\rangle = \int_\Omega (\lambda|u|^{m-2}u + f(x,u) + t\psi)v dx.$$

Put

$$\mathcal{H}_\lambda(t,u) = K \circ T_f(t,u) : [0,1] \times W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega).$$

In order to show the invariance of the topological degree, we furthermore claim that there exists r_1, such that $0 < r_1 < \alpha_0$ and

$$\text{if } u \in B_{r_1} \setminus \{0\}, \text{ then } u \neq \mathcal{H}_\lambda(t,u), \text{ for any } t \in [0,1]. \tag{22}$$

Otherwise, there exist sequences $(u_j)_j$, $(t_j)_j$, with $(u_j)_j \subset W_0^{1,\xi}(\Omega)$, $u_j > 0$ a.e. in Ω, and $(t_j)_j \subset [0,1]$, such that $\|u_j\| \to 0$ as $j \to \infty$ in $W_0^{1,\xi}(\Omega)$ and $u_j = \mathcal{H}_\lambda(t_j,u_j)$ for all j. Hence, for any $v \in W_0^{1,\xi}(\Omega)$,

$$\langle J(u_j), v\rangle = \langle T_f(t_j,u_j), v\rangle. \tag{23}$$

Indeed, as $u_j \to 0$ in $W_0^{1,\xi}(\Omega)$, there are two possibilities either

1 $\|Du_j\|_m = 0$ for all j sufficiently large; or

2 there is a subsequence $(u)_{j_k}$ such that $\|Du_{j_k}\|_m \neq 0$ for all k.

In case 1, assumptions (f_2) and (f_3), Lemma 2.2–(a) and the Poincaré inequality yield that

$$\int_\Omega f(x,u_j)u_j dx \leq C(\|u_j\|^m_{L^m(\Omega)} + \|u_j\|^r_{L^r(\Omega)}) \leq C(C_m\|Du_j\|^m_m + C_r\|Du_j\|^r_m) = 0.$$

Therefore,

$$\begin{aligned}\langle A_p^a(u_j), u_j\rangle &= \lambda\|u_j\|^m_{L^m(\Omega)} + \int_\Omega \big(f(x,u_j)u_j + t\psi u_j\big)dx \\ &\leq \lambda\|u_j\|^m_{L^m(\Omega)} + \int_\Omega f(x,u_j)u_j dx = 0\end{aligned}$$

which implies that $u_j = 0$ in $W_0^{1,\xi}(\Omega)$ by Lemma 2.4. This contradicts the definition of $(u_j)_j$.

Put $\widehat{u}_{j_k} = u_{j_k}\|Du_{j_k}\|_m^{-1}$. The reflexivity of $W_0^{1,m}(\Omega)$ and the continuity of embedding $W_0^{1,m}(\Omega) \hookrightarrow L^\wp(\Omega)$, with $\wp \in [1, m^*)$, and of the embedding $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,m}(\Omega)$, by Lemma 2.2–$(a)$, yield that there exists $\widehat{u} \in W_0^{1,m}(\Omega)$, such that

$$\widehat{u}_{j_k} \rightharpoonup \widehat{u} \text{ in } W_0^{1,m}(\Omega) \quad \text{and} \quad \widehat{u}_{j_k} \to \widehat{u} \text{ in } L^\wp(\Omega), \quad \text{with } \wp \in [1, m^*),$$

passing eventually to subsequences, if necessary. Taking $v = u_{j_k}$ and dividing $\|Du_{j_k}\|_m^m$ into both sides of (23), by virtue of Lemma 2.7, the variational characterization of $\widehat{\lambda}_1$ and the fact that $0 < \lambda < \widehat{\lambda}_1$, we get as $k \to \infty$

$$\begin{aligned}
&\|Du_{j_k}\|_m^{p-m}\langle A_p^a(\widehat{u}_{j_k}), \widehat{u}_{j_k}\rangle \\
&\qquad \le \|Du_{j_k}\|_m^{p-m}\langle A_p^a(\widehat{u}_{j_k}), \widehat{u}_{j_k}\rangle + \langle A_m(\widehat{u}_{j_k}), \widehat{u}_{j_k}\rangle - \lambda\|\widehat{u}_{j_k}\|_{L^m(\Omega)}^m \\
&\qquad = \|Du_{j_k}\|_m^{-m}\int_\Omega \big(f(x, u_{j_k})u_{j_k} + t\psi u_{j_k}\big)dx \qquad (24)\\
&\qquad \le \|Du_{j_k}\|_m^{-m}\int_\Omega f(x, u_{j_k})u_{j_k}dx \to 0.
\end{aligned}$$

Hence, taking $v = \widehat{u}$ and multiplying (23) by $\|Du_{j_k}\|_m^{1-m}$, Lemma 2.7 and the fact that $1 < m < p$ guarantee that as $k \to \infty$

$$\begin{aligned}
\langle A_m(\widehat{u}), \widehat{u}\rangle &= \lim_{j\to\infty}\big(\|Du_{j_k}\|_m^{p-m}\langle A_p^a(\widehat{u}_{j_k}), \widehat{u}\rangle + \langle A_m(\widehat{u}_{j_k}), \widehat{u}\rangle\big) \\
&= \lim_{j\to\infty}\bigg(\int_\Omega \big\{\lambda|\widehat{u}_{j_k}|^{m-2}\widehat{u}_{j_k}\widehat{u} + \|Du_{j_k}\|_m^{1-m}f(x, u_{j_k})\widehat{u} \qquad (25)\\
&\qquad\qquad + t_{j_k}\psi|\widehat{u}_{j_k}|^{m-1}|u_{j_k}|^{1-m}\widehat{u}\big\}dx\bigg) \\
&\le \lambda\|\widehat{u}\|_m^m.
\end{aligned}$$

Now $0 < \lambda < \varepsilon < \widehat{\lambda}_1$. Hence, the variational characterization of $\widehat{\lambda}_1$ and equation (25) imply at once that $\widehat{u} = 0$.

Moreover, in (23), putting $v = \widehat{u}_{j_k} - \widehat{u} = \widehat{u}_{j_k}$ and multiplying by $\|Du_{j_k}\|_m^{1-m}$, using the same argument of (24), by (6), the Hölder inequality and the fact that $m < p < m^*$, since $1 < m < p < N$ and $p/m < 1 + 1/N$, we obtain as $k \to \infty$

$$\begin{aligned}
\langle A_m(\widehat{u}_{j_k}), \widehat{u}_{j_k}\rangle &= -\|Du_{j_k}\|_m^{p-m}\langle A_p^a(\widehat{u}_{j_k}), \widehat{u}_{j_k}\rangle + \lambda\|\widehat{u}_{j_k}\|_{L^m(\Omega)}^m \\
&\qquad + \int_\Omega \big(\|Du_{j_k}\|_m^{1-m}f(x, u_{j_k})\widehat{u}_{j_k} + t_{j_k}\psi|\widehat{u}_{j_k}|^m|u_{j_k}|^{1-m}\big)dx \\
&\le \lambda\|\widehat{u}_{j_k}\|_{L^m(\Omega)}^m + o(1) \\
&\to 0.
\end{aligned}$$

The fact that by Lemma 2.6 the operator A_m is of type (S_+) in $W_0^{1,m}(\Omega)$ implies that

$$\widehat{u}_{j_k} \to 0 \text{ strongly in } W_0^{1,m}(\Omega). \qquad (26)$$

This is impossible with $\|D\widehat{u}_{j_k}\|_m = 1$ for all k. Thus, the claim (22) is shown, when $0 < \lambda < \varepsilon < \widehat{\lambda}_1$.

Hence, since $\mathcal{H}_\lambda(0,\cdot) = K\circ T_\lambda$, choosing $\epsilon \in (0, r_1)$, the homotopy invariance of $\mathcal{H}_\lambda$ yields that

$$\deg(\mathbb{I} - K \circ T_\lambda, B_\epsilon, 0) = \deg(\mathbb{I} - \mathcal{H}_\lambda(0,\cdot), B_\epsilon, 0) \tag{27}$$
$$= \deg(\mathbb{I} - \mathcal{H}_\lambda(1,\cdot), B_\epsilon, 0) = 0. \tag{28}$$

Thus, (21)and (27) contradict (18) and so $(0,0)$ is the bifurcation point of equation $(\mathcal{P})$ in all the cases.

The classical global bifurcation theorem in Lemma 3.4 cannot be directly applied to the equation $u = K \circ T_\lambda(u)$ because of the lack of differentiability at $u = 0$ and also because of the lack of the odd-multiplicity eigenvalues of the "double operator". However, minor modifications and the topological degree results (21) and (27) allow us to apply the global bifurcation theorem given in [2, Proposition 3.5]. Consequently, the assertion of the global bifurcation theorem for the problem $(\mathcal{P})$ is still valid. Thus, $(0,0)$ is the bifurcation point of the positive (weak) solutions of $(\mathcal{P})$, as stated. This proves the first part of the theorem.

Step 2. Now, let us turn to the proof of the existence of the unbounded component $\mathcal{C}_0$.

For any $\lambda \neq 0$, we first claim that $(\lambda, 0)$ is an isolated solution of $(\mathcal{P})$. If $\lambda < 0$, similarly to the analysis of (19), we are able to show that there are no (weak) nontrivial solutions of $(\mathcal{P})$.

Fix $\lambda > 0$. Assume that there exist sequences $(\lambda_n)_n \subset \mathbb{R}^+$ and positive (weak) nontrivial solutions $(u_n)_n \subset W_0^{1,\xi}(\Omega)$, such that $\|u_n\| \to 0$ and $\lambda_n \to \lambda$ as $n \to \infty$. Therefore, multiplying the equation $(\mathcal{P})$ by $\|Du_n\|_m^{1-m}$, for any $\eta > 0$ there exists $N = N(\eta) > 0$, such that for any $n \geq N(\eta)$

$$\begin{aligned}\|Du_n\|_m^{1-m}(-\Delta_p^a u_n - \Delta_m u_n) &= \lambda_n |\widehat{u}_n|^{m-2}\widehat{u}_n + \|Du_n\|_m^{1-m} f(x,u_n)\\ &\leq (\lambda+\eta)|\widehat{u}_n|^{m-2}\widehat{u}_n + \|Du_n\|_m^{1-m} f(x,u_n).\end{aligned}$$

Similar arguments as in (25), (26) yield a contradiction. Consequently, $(\lambda, 0)$ cannot meet $\mathcal{C}_0$.

Moreover, if $\mathcal{C}_0$ is bounded in $\mathbb{R}\times W_0^{1,\xi}(\Omega)$, it follows from Lemma 3.3 that there is a bounded open set $\mathcal{O} \subset \mathbb{R}\times W_0^{1,\xi}(\Omega)$ such that $(0,0) \in \mathcal{O}$ and $\mathcal{O}$ contains no trivial solutions other than those in $\mathcal{B}_\eta \subset \mathbb{R}\times W_0^{1,\xi}(\Omega)$, with $\eta > 0$ sufficiently small.

Let us now argue as in the proof of (1.11) in Lemma 1.3 in [25] (see also the proof of [25]) to conclude that there exist $\eta > 0$ and the values $\underline{\lambda}$, $\overline{\lambda}$, such that $-\eta < \underline{\lambda} < 0 < \overline{\lambda} < \eta$ and $i(\mathbb{I} - K\circ T_{\underline{\lambda}}, 0) = i(\mathbb{I} - K\circ T_{\overline{\lambda}}, 0)$. Consequently, (21) and (27) imply that

$$1 = i(\mathbb{I} - K\circ T_{\underline{\lambda}}, 0) = i(\mathbb{I} - K\circ T_{\overline{\lambda}}, 0) = 0,$$

which is an obvious contradiction. Then, $\mathcal{C}_0$ is an unbounded component bifurcating from $(0,0)$. The proof of theorem is so completed. $\square$

Under assumption (f_4), it follows from Lemma 2.7 and the definition of the operator K that $K \circ T_\lambda$ is also a compact operator in $W_0^{1,\xi}(\Omega)$.

Proof of Theorem 1.2. Let $(u_k)_k$ be a fixed unbounded sequence in $W_0^{1,\xi}(\Omega)$ consisting of (weak) solutions of problem $(\mathcal{P})$ corresponding to $(\nu_k)_k \subset \mathbb{R}^+$. For any $v \in W_0^{1,\xi}(\Omega)$, it follows that

$$\langle A_p^a(u_k), v\rangle + \nu_k \langle A_m(u_k), v\rangle = \int_\Omega \big(\lambda_1 a(x)|u_k|^{p-2}u_k + f(x,u_k)\big)v dx. \tag{29}$$

We assert that $\nu_k \to 0$ as $k \to \infty$. Otherwise, there exist a subsequence, still denoted for simplicity by $(\nu_k)_k$ and $\alpha > 0$, such that $\nu_k \geq \alpha$ for all k.

Now, we claim $(u_k)_k$ is also unbounded in $W_0^{1,\xi_0}(\Omega)$.

Otherwise, if $(u_k)_k$ is bounded in $W_0^{1,\xi_0}(\Omega)$, Lemma 2.3–(a) implies that $(\langle A_p^a(u_k), u_k\rangle)_k$ is bounded. Thus, (29) and assumption (f_4) yield that

$$\begin{aligned}\alpha\|Du_k\|_m^m = \alpha\langle A_m(u_k), u_k\rangle &\leq -\langle A_p^a(u_k), u_k\rangle + \int_\Omega (\lambda a(x)|u_k|^p + f(x,u_k)u_k)dx \\ &\leq -\langle A_p^a(u_k), u_k\rangle + 2\lambda_1 \int_\Omega a(x)|u_k|^p dx \\ &\leq \langle A_p^a(u_k), u_k\rangle.\end{aligned} \tag{30}$$

This implies that $(\|Du_k\|_m)_k$ is bounded, which is impossible by Lemma 2.4–(b), since $(u_k)_k$ is an unbounded sequence in $W_0^{1,\xi}(\Omega)$.

Hence, we may assume that $\|Du_k\|_{\xi_0} > 0$ for all k and that $\|Du_k\|_{\xi_0} \to \infty$, going possibly to a subsequence if necessary. Put $\widetilde{u}_k = u_k\|Du_k\|_{\xi_0}^{-1}$ for any k. Then, $\|\widetilde{u}_k\|_{\xi_0} = 1$. By virtue of the reflexivity of $W_0^{1,\xi_0}(\Omega)$ and Lemma 2.3, there exists $\widetilde{u} \in W_0^{1,\xi_0}(\Omega)$, such that

$$\widetilde{u}_k \rightharpoonup \widetilde{u} \text{ in } W_0^{1,\xi_0}(\Omega) \quad \text{and} \quad \widetilde{u}_k \to \widetilde{u} \text{ in } L^{\xi_0}(\Omega). \tag{31}$$

Dividing by $\|Du_k\|_{\xi_0}^{1-p}$ and taking as test function $\widetilde{u}$ in (29), from Lemma 2.7, (30), assumptions (H), (f_4) and the fact that $0 < \lambda < \lambda_1$ we get that

$$\begin{aligned}&\langle A_p^a(\widetilde{u}_k), \widetilde{u}\rangle + \nu_k\|u_k\|_{1,\xi_0}^{m-p}\langle A_m(\widetilde{u}_k), \widetilde{u}\rangle \\ &\qquad = \|Du_k\|_{\xi_0}^{1-p}\int_\Omega \big(\lambda a(x)|u_k|^{p-2}u_k\widetilde{u} + f(x,u_k)\widetilde{u}\big)dx \\ &\qquad < \lambda_1 \int_\Omega a(x)|\widetilde{u}_k|^{p-1}\widetilde{u}dx.\end{aligned} \tag{32}$$

Hence,

$$\langle A_p^a(\widetilde{u}), \widetilde{u}\rangle < \lambda_1 \int_\Omega a(x)|\widetilde{u}|^p dx$$

and so $\widetilde{u} = 0$ by Lemma 3.2. Then, similar to the argument of (32), taking $v = \widetilde{u}_k - \widetilde{u} = \widetilde{u}_k$ in (29), by (31), we get as $k \to \infty$

$$\langle A_p^a(\widetilde{u}_k), \widetilde{u}_k\rangle \leq \lambda_1 \int_\Omega a(x)|\widetilde{u}_k|^p dx \to 0.$$

Consequently, $\widetilde{u}_k \to 0$ in $W_0^{1,\xi_0}(\Omega)$, since the operator A_p^a is of type $(S)_+$ by Lemma 2.5. Of course, this contradicts the fact that $\|D\widetilde{u}_k\|_{\xi_0} = 1$ for all k and shows the claim. Hence $(u_k)_k$ is a bounded sequence in $W_0^{1,\xi_0}(\Omega)$.

Then, we conclude that if $(u_k)_k$ is unbounded (weak) solutions of $(\mathcal{P})$, it follows that $\nu_k \to 0$ as $k \to \infty$. □

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