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Group analysis of the generalized radial Liouville-Bratu-Gelfand problem, I: the group classification

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To Professor Enzo Mitidieri on his 70th birhday

Abstract. *We classify completely the equivalence groups and the classical Lie point symmetry groups of generalized radial Liouville-Bratu-Gelfand problems.*

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1. Introduction

The classical *Liouville-Bratu-Gelfand Problem*

$$\begin{cases} \Delta u + \lambda e^u = 0 \text{ in } B, \\ u > 0 \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases} \tag{1}$$

where B is a ball of radius R centered in the origin and λ is a real number, has been extensively studied in the recent years by many authors due to its versatility and usefulness which make it a valuable tool in various scientific and engineering disciplines. Its rich history begins with the paper [16] by Liouville published in 1853 where he proposed and studied the partial differential equation:

$$u_{xy}(x,y) \pm \frac{1}{2a^2} e^{u(x,y)} = 0, \ a \in \mathbb{R}^*, \tag{2}$$

known in the literature, along its two alternative forms

$$u_{xx} \pm u_{yy} = \lambda e^u,$$

as the *Liouville equation.*

In [16], Liouville obtained an exact solution for $n = 1$ and a solution in terms of an arbitrary harmonic function for $n = 2$. More than half a century later, in 1914, Bratu, in [4], found two solutions of the problem for the case $0 < \lambda < 2/R^2$ when $n = 2$. In [8], Gelfand considered the problem of thermal self-ignition of a chemically active mixture of gases for plane, cylindrical and spherical vessels. In addition, for the three-dimensional case, he looked into the possible values of the parameter λ for which the problem admits solution and studied their multiplicity.

In mathematics, Liouville's equation appears in the study of isothermal coordinates in differential geometry. It describes the conformal factor of a metric on a surface of constant Gaussian curvature. This equation is used to analyse the geometry of surfaces and is related to the uniformization problem of Riemann surfaces.

Since there is a huge number of applications of the *Liouville-Bratu-Gelfand Problem* and its generalizations in physics and engineering, we mention just a few fields where it can be encountered: chemical reactor theory; electrospinning process for the fabrication of nano-fibers (nano-technology); radiative heat transfer; thermal reaction processes in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction; modeling of electrically conducting solids, analysis of Joule losses in electrically conducting solids; thermo-electro-hydrodynamics models; elasticity theory; modeling of the formation and evolution of planetary nebulae in astrophysics; Liouville quantum field theory which appears in the context of string theory, and which is a two-dimensional conformal field theory whose classical equation of motion is a generalization of Liouville's equation, etc. See [1, 6, 15, 21, 22] and the references therein for further details.

An important result of Gidas, Ni and Nirenberg, [9], implies that that any solution of (1) — if it exists — must be a radial function. That is, (1) reduces to the following problem for an ordinary differential equation:

$$\begin{cases} y''(r) + \frac{n-1}{r} y'(r) + \lambda e^{y(r)} = 0 & r \in (0, R), \\ y'(0) = y(R) = 0, & \\ y > 0, & r \in [0, R). \end{cases} \tag{3}$$

Clément, de Figueiredo and Mitidieri, in the seminal paper [5], proposed and studied the following generalization of (3):

$$\begin{cases} -(x^\alpha |y'|^\beta y')' = \lambda\, x^\gamma e^y & \text{in} \quad (0, R), \\ y'(0) = y(R) = 0, & \\ y > 0 & \text{in} \quad [0, R), \end{cases} \tag{4}$$

assuming that the constants involved satisfy the constraints

$$\alpha - \beta - 1 = 0, \quad \beta > -1, \quad \gamma > -1. \tag{5}$$

One can observe that the equation in (4) contains the radial forms of Laplace, $p-$Laplace and $k-$Hessian operators.

They explicitly identified a constant $\lambda^* > 0$ for problem (4), establishing that there is a unique solution when $\lambda = \lambda^*$ and exactly two solutions when $0 < \lambda < \lambda^*$. Furthermore, by employing the method of first integrals, they derived explicit formulas for these solutions, assuming that conditions (5) are satisfied. Subsequently, Bozhkov and Martins, [3], reached the same conclusion using symmetry methods.

In the present work, we investigate a more general, than that in (3), radial ODE involving variable coefficients,

$$u'' + \nu(r)u' + \lambda(r)e^{\mu(r)u} = 0, \tag{6}$$

under the prism of enhanced group analysis. That is, we first perform the group classification of (6) employing its equivalence transformations and then, we classify the non equivalence classes found using their Lie point symmetries.

The article is shaped as follows: in section 2 we introduce the main concepts used in our analysis, the next section contains the main body of results. Finally, we close the present work with some comments and concluding remarks.

2. Methodology

2.1. Symmetries

In the heart of our analysis resides the concept of *symmetry*. Symmetry, loosely put, is a transformation — a *diffeomorphism* — between elements of a differential equation that leaves it invariant. These symmetries form a *group*, and for our purposes we will restrict ourselves to its connected component, which forms a *Lie group*. This means that the kind of symmetries that we shall use will depend on a continuous variable, ε, and we shall identify the symmetry that corresponds to $\varepsilon = 0$ with the identity transformation — which is a symmetry of any differential equation. But this is not the only reason for making this assumption: we can now elegantly represent any such symmetry with an element of a *Lie algebra*, that is, with a special type of *vector space*.

There is one more restriction to make: we shall assume that the transformations, our symmetries, involve only the independent and dependent variables. In other words we shall work with *Lie point symmetries*. In particular, for the study of (6) we shall work with symmetries of the form

$$\Upsilon = \xi(r,u)\frac{\partial}{\partial r} + \eta(r,u)\frac{\partial}{\partial u}. \tag{7}$$

Having a symmetry as an element of a Lie algebra of the form (7) we can retrieve the corresponding continuous transformation, an element of a (local)

Lie group, by solving the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\bar{r} = \xi(\bar{r},\bar{u}), & \bar{r}(0) = r, \\ \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\bar{u} = \eta(\bar{r},\bar{u}), & \bar{u}(0) = u, \end{cases}$$

for this reason $\mathcal{X}$ is also called *infinitesimal generator*. We call this procedure *exponentiation*.

But how we shall obtain the symmetries of a differential equation? Under certain conditions that the differential equation must satisfy — and for the case of (6) there are indeed satisfied — $\mathcal{X}$ is a symmetry of (6) if, and only if, it satisfies the linearised symmetry condition:

$$\mathcal{X}^{(2)}[u'' + \nu(r)u' + \lambda(r)e^{\mu(r)u}]\Big|_{u''+\nu(r)u'+\lambda(r)e^{\mu(r)u}=0} = 0, \tag{8}$$

where $\mathcal{X}^{(2)}$ is the second *prolongation* of the vector field $\mathcal{X}$,

$$\mathcal{X}^{(2)} = \xi(r,u)\frac{\partial}{\partial r} + \eta(r,u)\frac{\partial}{\partial u} + \eta^1(r,u)\frac{\partial}{\partial u'} + \eta^2(r,u)\frac{\partial}{\partial u''},$$

where $\eta^1(r,u) = D_r\eta - u'D_r\xi$, $\eta^2(r,u) = D_r^2\eta - u'D_r^2\xi$ and D_r denotes the total derivative with respect to r.

Considering r, u, u' as independent variables, (8) breaks down to an overdetermined — in our case — system of (linear) partial differential equations which is called the *determining equations*. Its general solution will provide us with the general form of the Lie algebra of Lie point symmetries of (6).

Getting the Lie point symmetries of a differential equation is only the beginning. Like the DNA for a living organism, symmetries help us identify and unearth important properties, reduce the order and even obtain solutions. We have just skimmed the surface of this remarkably beautiful theory, for more details see also [18, 17, 2, 11, 20, 10, 12].

Evidently, getting the Lie algebra from (8) on the one hand involves a great deal of copious and error-prone calculations and on the other hand is a completely algorithmic procedure. These facts render the use of *computer algebra systems* essential. For our purposes we employed the symbolic package SYM for Wolfram Mathematica™, [7, 14].

2.2. Equivalence transformations

When the differential equation involves arbitrary elements, for instance parameters, we deal in fact with a family, or a set, of differential equations. For each element of this family we can associate the Lie algebra of its Lie point symmetries. Due to the structural importance of symmetries it merits our attention to try to classify the family with respect to the possible Lie algebras that can be admitted: enter the *group classification*!

An efficient way to perform the group classification of a family of differential equations is by employing its *equivalence transformations*. Equivalence transformations can be considered as *meta-symmetries* of the family, being defined as *transformations* that leave invariant this set of differential equations as a whole, and as such, they contain the Lie point symmetries of the family.

Hence, following the same reasoning as in the previous subsection we shall restrict ourselves to continuous equivalence transformations that involve only the independent and dependent variables. Therefore, the algorithmic procedure previously described remains the same with the only differences that now the infinitesimal generator has the form

$$\begin{aligned}\hat{\mathrm{X}} = \xi(r,u;\nu,\lambda,\mu)\frac{\partial}{\partial r} + \eta(r,u;\nu,\lambda,\mu)\frac{\partial}{\partial u} + \omega_\nu(r,u;\nu,\lambda,\mu)\frac{\partial}{\partial \nu} \\ + \omega_\lambda(r,u;\nu,\lambda,\mu)\frac{\partial}{\partial \lambda} + \eta_\mu(r,u;\nu,\lambda,\mu)\frac{\partial}{\partial \mu},\end{aligned} \tag{9}$$

where $\alpha = \alpha(r,u)$, $\alpha = \nu, \lambda, \mu$, and the first prolongations of $\omega_\nu, \omega_\lambda, \omega_\mu$ follow the formulas, see also [13],

$$\begin{aligned}\omega_\alpha^r =& D_r\omega_\alpha - \alpha_r D_r\xi - \alpha_u D_r\eta,\\ \omega_\alpha^u =& D_u\omega_\alpha - \alpha_r D_u\xi - \alpha_u D_u\eta.\end{aligned}$$

By construction, the equivalence transformations split the family of differential equations into *equivalence classes* and all the elements of a class will have the same Lie algebra of Lie point symmetries. As a result, through the equivalence classification of the family we achieve also a first take on its group classification. We choose as the *canonical representative* of each equivalence class the simplest possible differential equation — in form — of that class.

3. Enhanced modern group analysis of (6)

By enhanced modern group analysis we mean a toolkit of analytical tools, that enhances the use of modern group analysis — that is the use of symmetries — for studying differential equations. Amongst them are, equivalence transformations for group classification, optimal systems for classifying the invariant solutions found, the concept of adjointness, or cosymmetry, for obtaining (local) conservation laws and others.

3.1. Preliminary group classification

We start the study of (6) by its group classification. As previously mentioned, we shall employ its equivalence transformations. A basis of the Lie algebra of equivalence transformations is

$$\begin{aligned}
\hat{\mathsf{X}}_1 =& u\frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial \lambda} - \mu\frac{\partial}{\partial \mu},\\
\hat{\mathsf{X}}_2 =& \frac{\partial}{\partial u} - \mu\lambda\frac{\partial}{\partial \mu},\\
\hat{\mathsf{X}}_3 =& \mathsf{F}(r)\frac{\partial}{\partial r} - 2\lambda\mathsf{F}'(r)\frac{\partial}{\partial \lambda} + \left(\mathsf{F}''(r) - \nu\mathsf{F}'(r)\right)\frac{\partial}{\partial \nu},
\end{aligned}$$

where $\mathsf{F}(r)$ is an arbitrary function. To obtain the *principal algebra*, that is the maximal Lie algebra that all the possible cases will include, we need to apply an arbitrary element of the Lie algebra of equivalence transformations found, $c_1\hat{\mathsf{X}}_1 + c_2\hat{\mathsf{X}}_2 + \hat{\mathsf{X}}_3$, to the relations $\lambda = f(r),\ \mu = g(r)$ and $\nu = h(r)$:

$$\begin{aligned}
c_1\lambda - 2\lambda\mathsf{F}'(r) - \mathsf{F}(r)f'(r) =& 0,\\
-c_1\mu(r) - c_2\lambda\mu - \mathsf{F}(r)g'(r) =& 0,\\
\left(\mathsf{F}''(r) - \nu\mathsf{F}'(r)\right) - \mathsf{F}(r)h'(r) =& 0.
\end{aligned}$$

These equations when $\lambda = f(r),\ \mu = g(r)$ and $\nu = h(r)$, for any given function f, g, h, yield

$$\mathsf{F}(r) = 0,\ c_1 = c_2 = 0.$$

Thus, the principal algebra for equation (6) is spanned by the zero element, that is, in general equation (6) admits only the identity transformation.

Now, exponentiating the vector $\hat{\mathsf{X}}_3$ we get the equivalence transformation

$$\bar{r} = \phi(r), \qquad \bar{u} = u, \qquad \bar{\lambda} = \frac{\lambda}{(\phi')^2}, \qquad \bar{\nu} = \frac{\nu}{\phi'} + \frac{\phi''}{(\phi')^2}.$$

Hence, by choosing

$$\phi(r) = \int e^{-\int \nu(r)\,\mathrm{d}r}\,\mathrm{d}r$$

$\bar{\nu} = 0$ and thus the transformation

$$\bar{r} = \int e^{-\int \nu(r)\,\mathrm{d}r}\,\mathrm{d}r$$

turns (6) to

$$\bar{u}'' + \bar{\lambda}(r)e^{\mu(r)\bar{u}} = 0 \tag{10}$$

where $\bar{\lambda} = \dfrac{\lambda}{(\phi')^2} = e^{2\int \nu(r)\,\mathrm{d}r}\lambda$. Turning to equation (3), where $\nu(r) = \dfrac{n-1}{r}$, the transformation found has the form

$$\phi(r) = \int r^{1-n}\,\mathrm{d}x = \begin{cases} \dfrac{r^{2-n}}{2-n}, & n \neq 2\\ \ln r, & n = 2 \end{cases}.$$

We repeat the same process for (10), after dropping the bars for clarity. A basis of the Lie algebra of equivalence transformations now is

$$
\begin{aligned}
\hat{\mathrm{X}}_1 &= \frac{\partial}{\partial r}, & \hat{\mathrm{X}}_2 &= r\frac{\partial}{\partial r} - 2\lambda\frac{\partial}{\partial \lambda},\\
\hat{\mathrm{X}}_3 &= \frac{\partial}{\partial u} - \lambda\mu\frac{\partial}{\partial \lambda}, & \hat{\mathrm{X}}_4 &= r\frac{\partial}{\partial u} - \lambda\mu r\frac{\partial}{\partial \lambda},\\
\hat{\mathrm{X}}_5 &= u\frac{\partial}{\partial u} + \lambda\frac{\partial}{\partial \lambda} - \mu\frac{\partial}{\partial \mu}, & \hat{\mathrm{X}}_6 &= r^2\frac{\partial}{\partial r} + ru\frac{\partial}{\partial u} - 3\lambda r\frac{\partial}{\partial \lambda} - \mu r\frac{\partial}{\partial \mu}.
\end{aligned}
$$

Again, by exponentiation we get the corresponding equivalence transformations:

$$
\begin{aligned}
&T_1(\varepsilon): \bar{r} = r + \varepsilon, && \bar{u} = u, && \bar{\lambda} = \lambda, && \bar{\mu} = \mu,\\
&T_2(\varepsilon): \bar{r} = \varepsilon r, && \bar{u} = u, && \bar{\lambda} = \frac{1}{\varepsilon^2}\lambda, && \bar{\mu} = \mu,\\
&T_3(\varepsilon): \bar{r} = r, && \bar{u} = u + \varepsilon, && \bar{\lambda} = e^{-\varepsilon\mu}\lambda, && \bar{\mu} = \mu,\\
&T_4(\varepsilon): \bar{r} = r, && \bar{u} = u + \varepsilon r, && \bar{\lambda} = e^{-\varepsilon r\mu}\lambda, && \bar{\mu} = \mu,\\
&T_5(\varepsilon): \bar{r} = r, && \bar{u} = \varepsilon u, && \bar{\lambda} = \varepsilon\lambda, && \bar{\mu} = \frac{1}{\varepsilon}\mu,\\
&T_6(\varepsilon): \bar{r} = \frac{r}{1-\varepsilon r}, && \bar{u} = \frac{u}{1-\varepsilon r}, && \bar{\lambda} = (1-\varepsilon r)^3\lambda, && \bar{\mu} = (1-\varepsilon r)\mu.
\end{aligned}
$$

Looking at the structure of the equivalence transformations we can see that the lack of arbitrary functions will limit their usefulness. Indeed, combining the last five transformations we get the five-parameter equivalence transformation

$$
\begin{aligned}
T(\mathrm{A},\mathrm{B},\Gamma,\Delta,\mathrm{E}): R &= \frac{\Gamma r}{1-\mathrm{B}r}, & U &= \mathrm{A}\frac{u + \mathrm{E}r + \Delta}{1-\mathrm{B}r},\\
\Lambda &= \frac{\mathrm{A}(1-\mathrm{B}r)^3}{\Gamma^2}e^{-(\mathrm{E}r+\Delta)\mu}\lambda, & M &= \frac{1-\mathrm{B}r}{\mathrm{A}}\mu,
\end{aligned}
$$

where $\mathrm{A}, \Gamma \neq 0$. Therefore, they are relevant only when μ is of the form $\dfrac{\alpha}{1-\beta r}$ and/or λ is of the form $\dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{(1-\tilde{\beta}r)^3}$, where $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, \gamma, \delta$ are constants. The equivalence classification of (10) is presented in the table 1.

Alas, our job is far from over! Our canonical representatives still include arbitrary functions. For that reason we called this group classification preliminary: our classification needs further refinement. We shall accomplish that by studying the determining equations of (10).

$\lambda(r)$	$\mu(r)$	Canonical representative
$=\dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{(1-\beta r)^3},\ \tilde{\alpha}\neq 0$	$=\dfrac{\alpha}{1-\beta r},\ \alpha\neq 0$	$u''+\operatorname{sign}(\alpha\tilde{\alpha})e^u=0$
$\neq\dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{(1-\beta r)^3},0$		$u''+\overbrace{\dfrac{\alpha}{\Gamma^2}(1-\beta r)^3 e^{-\alpha\frac{\mathrm{E}r+\Delta}{1-\beta r}}\lambda(r)}^{\overline{\lambda}(r)}\,e^u=0$
$=\dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{(1-\beta r)^3},\ \tilde{\alpha}\neq 0$	$\neq\dfrac{\alpha}{1-\beta r},0$	$u''+e^{\overbrace{\frac{\tilde{\alpha}}{\Gamma^2}(1-\beta r)\mu(r)}^{\overline{\mu}(r)}\,u}=0$
$\neq\dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{(1-\tilde{\beta}r)^3},0$		$u''+\lambda(r)e^{\mu(r)u}=0$
$=0$	$\neq 0$	$u''=0$
$\neq 0$	$=0$	

Table 1: Preliminary group classification of (10).

3.2. Group classification

We continue with the complete group classification of (10) by employing the *direct method*. That is, by examining the determining equations we shall try to find special cases of the arbitrary functions that extend the solution space of the determining equations. The canonical representatives already found will help to simplify the calculations.

We proceed as follows: using (8) with $\nu=0$ and the infinitesimal generator given in (7) we arrive at the determining equations. We solve the ones that does not involve the two arbitrary functions λ and μ until we arrive at the *classification equations*:

$$(c_3+c_4r)\mu(r)+(c_6+c_5r+c_4r^2)\mu'(r)=0, \tag{11a}$$

$$(2c_5-c_3+3c_4r+(c_1+c_2r)\mu(r))\lambda(r)+(c_6+c_5r+c_4r^2)\lambda'(r)=0. \tag{11b}$$

On top of that, the infinitesimal generator, up to this point, has the form

$$\mathfrak{X}=(c_6+c_5r+c_4r^2)\frac{\partial}{\partial r}+(c_1+c_2r+(c_3+c_4r)u)\frac{\partial}{\partial u}.$$

Now it is time to use the canonical representatives as promised.

First of all, for the canonical representatives $u''+\operatorname{sign}(\alpha\tilde{\alpha})e^u=0$ and $u''=0$ we already know their general solution: they are

$$u(r)=\ln\left(\operatorname{sign}(\alpha\tilde{\alpha})\frac{c_1}{2}\left(1-\tanh^2\frac{\sqrt{c_1}|x+c_2|}{2}\right)\right)$$

and

$$u(r) = c_1 r + c_2,$$

respectively. So, we shall examine the three central cases of table 1 that remain.

$\bar{\mu}(r) = 1$: First, from (11a) we have that $c_3 = c_4 = 0$. Solving (11b) under these restrictions we obtain

$$\bar{\lambda}(r) = \begin{cases} c_7 e^{-\dfrac{c_2 r}{c_6}} (c_5 + c_6 r)^{\dfrac{c_2 c_5 - c_1 c_6}{c_6^2} - 2}, & c_6 \neq 0 \\ c_7 e^{-\dfrac{r(2c_1 + c_2 r)}{2c_5}}, & c_6 = 0 \text{ and } c_5 \neq 0 \end{cases}.$$

Tidying the constants up we have that for

$$\bar{\lambda}(r) = p_1 e^{p_2 r} (r + p_3)^{p_4},\ p_2^2 + p_4^2, p_1 \neq 0,$$

the Lie algebra of Lie point symmetries is spanned by

$$(p_3 + r)\frac{\partial}{\partial r} - (2 + p_2 p_3 + p_4 + p_2 r)\frac{\partial}{\partial u},$$

while for

$$\bar{\lambda}(r) = p_1 e^{p_2 r + p_3 r^2},\ p_2^2 + p_3^2, p_1 \neq 0,$$

the Lie algebra of Lie point symmetries is spanned by

$$\frac{\partial}{\partial r} - (p_2 + 2p_3 r)\frac{\partial}{\partial u}.$$

$\bar{\lambda}(r) = 1$: Similarly, from (11b) we have that

$$\bar{\mu}(r) = \frac{c_3 - 2c_5 - 3c_4 r}{c_1 + c_2 r},$$

or

$$c_1 = c_2 = c_4 = 0 \text{ and } c_3 = 2c_5.$$

Substituting each of the two options to the first equation, (11a), we can easily see that

$$\bar{\mu}(r) = \begin{cases} -\dfrac{c_5}{c_1 + c_2 r}, & c_2 \neq 0, \\ \dfrac{c_7}{(c_6 + c_5 r)^2}, & c_5 \neq 0. \end{cases}$$

Tidying the constants up we have that for

$$\bar{\mu}(r) = \frac{p_1}{p_2 + r},\ p_1 \neq 0,$$

the Lie algebra of Lie point symmetries is spanned by

$$p_1(p_2+r)\frac{\partial}{\partial r}+(p_1u-p_2-r)\frac{\partial}{\partial u},$$

while for

$$\bar{\mu}(r)=\frac{p_1}{(p_2+r)^2},\, p_1\neq 0,$$

the Lie algebra of Lie point symmetries is spanned by

$$(p_2+r)\frac{\partial}{\partial r}+2u\frac{\partial}{\partial u}.$$

$\boldsymbol{\lambda(r)\neq\dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{\left(1-\tilde{\beta}r\right)^3}}$ **and** $\boldsymbol{\mu(r)\neq\dfrac{\alpha}{1-\beta r}}$**:** Accordingly, in the Tables 2, 3 and 4 we give all the possible pairs of solutions of (11) that have a non empty Lie algebra.

4. Comments and concluding remarks

Thirty years ago, Professor Enzo Mitidieri introduced Yuri Bozhkov to the Liouville-Bratu-Gelfand Problem, highlighting its fundamental concepts and significance for both mathematics and physics. This is one of the primary reasons to choose this topic for a research paper dedicated to him.

In this work we have initiated the study of the generalized radial Liouville type equation (6), classifying its equivalence and Lie point symmetry groups. In order to reduce (6) to a quadrature we need a two-dimensional Lie algebra of Lie point symmetries. Unfortunately, our group classification showed that at most we shall get an one-dimensional Lie algebra. That is, by using the Lie point symmetries it is possible to reduce (6) to a first order ode — which in fact will be an *Abel ode of the first, or second, kind*. For these type of odes we have at disposal numerous *ad hoc* and algorithmic methods for obtaining their general solution, see for instance [19]. Obtaining all the possible general solutions for the subcases of (6) that our group classification highlighted, using the available methods or by looking for other kinds of symmetries, shall be the main focus of the second part of the current work.

Subcase	$\mu(r)$	$\lambda(r)$	restrictions
I	$p_1 \dfrac{\exp\left(p_2 \tan^{-1}\left(\frac{p_3+(k^2+p_3^2)r}{k}\right)\right)}{\sqrt{1+2p_3r+(k^2+p_3^2)r^2}}$	$p_4 \dfrac{\exp\left(-p_2 \tan^{-1} \frac{kr}{1+p_3r}\right)}{\sqrt{(1+2p_3r+(k^2+p_3^2)r^2)^3}}$	$k>0$ and $p_1p_4 \neq 0$
II	$\dfrac{p_1}{\sqrt{1+2p_3r+(k^2+p_3^2)r^2}}$	$p_4 \dfrac{\exp\left(\frac{p_5+p_6r}{\sqrt{1+2p_3r+(k^2+p_3^2)r^2}}\right)}{\sqrt{(1+2p_3r+(k^2+p_3^2)r^2)^3}}$	$k>0$ and $p_1p_4 \neq 0$
III	$p_1 \dfrac{\exp\left(p_2 \tanh^{-1}\left(\frac{p_3+(p_3^2-k^2)r}{k}\right)\right)}{\sqrt{1+2p_3r+(p_3^2-k^2)r^2}}$	$p_4(1+(p_3-k)r)^{-\frac{3+p_2}{2}}(1+(p_3+k)r)^{\frac{p_2-3}{2}} \times \exp\left(\dfrac{(k+p_3+(p_3^2-k^2)r)^{p_2/2}(1+(k+p_3)r)^{-\frac{1+p_2}{2}}(p_5+p_6r)}{\sqrt{1+(p_3-k)r}}\right)$	$k>0, p_1 \neq 0$ and $p_2 \neq \pm 3$ or $p_5^2+p_6^2 \neq 0$

Table 2: The possible solutions of system (11) when $\lambda(r) \neq \dfrac{\tilde{\alpha}e^{(\gamma+\delta r)\mu(r)}}{\left(1-\tilde{\beta}r\right)^3}$ and $\mu(r) \neq \dfrac{\alpha}{1-\beta r}$. *Part I.*

Subcase	$\mu(r)$	$\lambda(r)$	restrictions
IV	$p_1 \dfrac{e^{\frac{p_2}{p_3+r}}}{p_3+r}$	$p_4 \dfrac{e^{(p_6+p_5 r)\mu(r)}}{(p3+r)^4 \mu(r)}$	$p_1 p_2 p_4 \neq 0$
V	$p_1 e^{p_2 r}$	$p_3 \dfrac{e^{(p4+p_5 r)\mu(r)}}{\mu(r)}$	$p_1 p_2 p_3 \neq 0$
VI	$p_1 \dfrac{r^{p_2}}{(1+p_3 r)^{1+p_2}}$	$p_4 \dfrac{e^{(p_5+p_6 r)\mu(r)}}{r^2 (1+p_3 r)^2 \mu(r)}$	$p_1 p_2 (p_2+2) p_4 \neq 0$
VII	$\dfrac{p_1}{r}$	$p_2 e^{p_3 \mu(r)} \dfrac{r^{p_1 p_4 - 1}}{(1+p_5 r)^{2+p_1 p_4}}$	$p_1 (p_1 p_4 - 1) p_4 \neq 0$
VIII	$p_1 \dfrac{e^{p_2/r}}{r}$	$p_3 \dfrac{e^{(p_4+p_5 r)\mu(r)}}{r^4 \mu(r)}$	$p_1 p_3 \neq 0$
IX	$\dfrac{p1}{r}$	$\dfrac{p_2}{r^3} e^{\frac{p_4+p_3 r}{r}\mu(r)}$	$p_1 p_2 \neq 0$

Table 3: The possible solutions of system (11) when $\lambda(r) \neq \dfrac{\tilde{\alpha} e^{(\gamma+\delta r)\mu(r)}}{\left(1-\tilde{\beta} r\right)^3}$ and $\mu(r) \neq \dfrac{\alpha}{1-\beta r}$. *Part II.*

Subcase	span$\{\cdot\}$
I	$(1+2p_3r+(k^2+p_3^2)r^2)\dfrac{\partial}{\partial r}+(p_3-p_2k+(k^2+p_3^2)r)u\dfrac{\partial}{\partial u}$
II	$p_1(1+2p_3r+(k^2+p_3^2)r^2)\dfrac{\partial}{\partial r}+$ $(p_3p_5-p_6+((k^2+p_3^2)p_5-p_3p_6)r+p_1(p_3+(k^2+p_3^2)r)u)\dfrac{\partial}{\partial u}$
III	$p_1(1+2p_3r+(p_3^2-k^2)r^2)\dfrac{\partial}{\partial r}+\big((k-p_3)^{p_2/2}((p_2k+p_3)p_5-p_6)$ $+(k-p_3)^{p_2/2}(p_6(p_2k-p_3)+p_5(p_3^2-k^2))r+$ $(p_1(p_2k+p_3)+p_1(p_3^2-k^2)r)u\big)\dfrac{\partial}{\partial u}$
IV	$(p_3+r)^2\dfrac{\partial}{\partial r}+\big((p_2+p_3)p_6-p_5p_3^2$ $+((p_2-p_3)p_5+p_6)r+(p_2+p_3+r)u\big)\dfrac{\partial}{\partial u}$
V	$\dfrac{\partial}{\partial r}-(p_5+p_2p_4+p_2p_5r+p_2u)\dfrac{\partial}{\partial u}$
VI	$(1+p_3r)r\dfrac{\partial}{\partial r}-(p_2p_5+(p_6(1+p_2)-p_3p_5)r+(p_2-p_3r)u)\dfrac{\partial}{\partial u}$
VII	$(1+p_5r)r\dfrac{\partial}{\partial r}+(p_3+(p_3p_5-p_4)r+(1+p_5r)u)\dfrac{\partial}{\partial u}$
VIII	$r^2\dfrac{\partial}{\partial r}+(p_2p_4+(p_2p_5+p_4)r+(p_2+r)u)\dfrac{\partial}{\partial u}$
IX	$r^2\dfrac{\partial}{\partial r}+(2p_4+p_3r+ru)\dfrac{\partial}{\partial u}$

Table 4: The Lie algebra of Lie point symmetries that corresponds to each subcase.

Furthermore, we intent to extend the methods used in the present work for two generalizations of the Liouville-Bratu-Gelfand Problem that have not yet been thoroughly examined using symmetry methods:

a) The problem

$$\begin{cases} \Delta u+\lambda(x)e^{\mu(x)u}=0,\ x\in\Omega \\ u(x)=0,\ x\in\partial\Omega, \end{cases}$$

where $\lambda(x),\mu(x)$ are continuous functions and Ω is a bounded domain in $\mathbb{R}^n$. There are a few intriguing scenarios that emerge when λ, μ are constants and Ω is a cube $[0,1]^n \subset \mathbb{R}^n$ — especially for $n = 2$ — where one cannot apply the Gidas-Ni-Nirenberg result!

b) The Liouville type pde

$$\Delta^m u + \lambda(x) e^{\mu(x) u} = 0$$

involving the *polyharmonic operator* Δ^m.

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References

[1] M. R. Ali and W.-X. Ma, *New exact solutions of Bratu Gelfand model in two dimensions using Lie symmetry analysis*, Chinese J. Phys. **65** (2020), 198–206.
[2] G. W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer, New York, 1989.
[3] Y. Bozhkov and A. C. Gilli Martins, *On the symmetry group of a differential equation and the Liouville-Gelfand problem*, Rend. Istit. Mat. Univ. Trieste **34** (2002), 103–120.
[4] G. Bratu, *Sur les équations intégrales non linéaires*, Bull. Soc. Math. France **42** (1914), 113–142.
[5] P. Clément, D. G. De Figueiredo, and E. Mitidieri, *Quasilinear elliptic equations with critical exponents*, Topol. Methods Nonlinear Anal. **7** (1996), 133–170.
[6] T. L. Curtright and C. B. Thorn, *Conformally invariant quantization of the Liouville theory*, Phys. Rev. Lett. **48** (1982), no. 19, 1309.
[7] S. Dimas and D. Tsoubelis, *SYM: A new symmetry-finding package for Mathematica*, The 10th International Conference in MOdern GRoup ANalysis (Nicosia) (N.H. Ibragimov, C. Sophocleous, and P.A. Damianou, eds.), University of Cyprus, 2005, pp. 64–70.
[8] I. M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. Ser. 2 **29** (1963), 295–381.
[9] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243.

[10] P. E. Hydon, *Symmetry methods for differential equations*, 1st ed., Cambridge Texts Appl. Math., Cambridge Univ. Press, Cambridge, 2000.
[11] N. H. Ibragimov, *Transformation groups applied to mathematical physics*, D. Reidel Publishing Co., 1985, Translated from Math. Appl. (Soviet Ser.).
[12] N. H. Ibragimov, *CRC handbook of Lie group analysis of differential equations*, 1st ed., vol. 1–3, CRC Press, Boca Raton, Fl., 1995.
[13] N. H. Ibragimov, Selected Works, vol. IV, 57–58, ALGA publications, Blekinge Institute of Technology, S-371 79 Karlskrona, Sweden., 2009, pp. 57–58.
[14] Wolfram Research, Inc., *Mathematica, Version 14.x*, Champaign, IL, 2024.
[15] S. Iqbal and P. A. Zegeling, *A numerical study of the higher-dimensional Gelfand-Bratu model*, Comput. Math. Appl. **79** (2020), no. 6, 1619–1633.
[16] J. Liouville, *Sur l'équation aux différences partielles* $\frac{\partial^2 \log \lambda}{\partial u \partial v} \pm \frac{\lambda}{2a^2} = 0$, J. Math. Pures Appl. **18** (1853), 71–72.
[17] P. J. Olver, *Applications of Lie groups to differential equations*, 2nd ed., Grad. Texts in Math., vol. 107, Springer, New York, 2000.
[18] L. V. Ovsiannikov, *Group analysis of differential equations*, 1st ed., Academic Press, June 1982.
[19] F. Schwarz, *Algorithmic solution of Abel's equation*, Computing **61** (1998), no. 1, 39–46.
[20] H. Stephani, *Differential equations: Their solution using symmetries*, 1st ed., Cambridge Univ. Press, Cambridge, 1990.
[21] G. Swaminathan, G. Hariharan, V. Selvaganesan, and S. Bharatwaja, *A new spectral collocation method for solving Bratu-type equations using Genocchi polynomials*, J. Math. Chem. **59** (2021), 1837–1850.
[22] A.-M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Bratu-type equations*, Appl. Math. Comput. **166** (2005), no. 3, 652–663.

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