Rend. Istit. Mat. Univ. Trieste
Vol. 57 (2025), Art. No. 10, 30 pages
DOI: 10.13137/2464-8728/37126

A note on the Fermi Golden Rule constant for the pure power NLS

Scipio Cuccagna and Masaya Maeda

Dedicated to Enzo Mitidieri in Honour of his 70th birthday

Abstract. *We provide a detailed proof that the Nonlinear Fermi Golden Rule coefficient that appears in our recent proof of the asymptotic stability of ground states for the pure power Nonlinear Schrödinger equations in* $\mathbb{R}$ *with exponent* $0 < |p-3| \ll 1$ *is nonzero.*

Keywords: Fermi Golden rule, nonlinear Schroedinger equation.
MS Classification 2020: 35Q55.

1. Introduction

We consider the pure power Nonlinear Schrödinger equation on the line

$$\mathrm{i}\dot{u} - u'' - |u|^{p-1}u = 0, \ u : \mathbb{R}^{1+1} \to \mathbb{C}, \tag{1}$$

where $\dot{u} = \partial_t u$ and $u' = \partial_x u$. In this paper we only consider p near 3. It is well known that Equation (1) has standing waves, solutions of the form $u(t,x) = e^{\mathrm{i}\omega t}\phi_\omega(x)$. They are obtained from $\phi_p[\omega](x) = \omega^{\frac{1}{p-1}}\phi_p(\sqrt{\omega}x)$ with the explicit formula

$$\phi_p(x) := \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right),$$

see formula (3.1) in Chang et al. [5]. For $p = 3$ we have

$$\phi_3 = \sqrt{2}\operatorname{sech}(x).$$

It is well known that the *linearization* of (1) at $\phi_p[\omega]$, is given by

$$\partial_t \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathcal{L}_{p\omega} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \text{ with } \mathcal{L}_{p\omega} := \begin{pmatrix} 0 & L_{-,p,\omega} \\ -L_{+,p,\omega} & 0 \end{pmatrix}, \tag{2}$$

where

$$L_{+,p,\omega} := -\partial_x^2 + \omega - p\phi_p^{p-1}[\omega] \text{ and } L_{-,p,\omega} := -\partial_x^2 + \omega - \phi_p^{p-1}[\omega].$$

The essential spectrum is $\sigma_e(\mathcal{L}_{p\omega}) = (-\mathrm{i}\infty, -\mathrm{i}\omega] \cup [\mathrm{i}\omega, \mathrm{i}\infty)$ and $0 \in \sigma_p(\mathcal{L}_{p\omega})$. For $p = 3$ there is no other spectrum, indeed for $p = 3$ all eigenfunctions and generalized eigenfunctions have been explicitly known have been known explicitly since Kaup [13]. Notice that the operator in (2) is obtained by a simple scaling from

$$\mathcal{L}_p = \begin{pmatrix} 0 & L_{+,p} \\ -L_{-,p} & 0 \end{pmatrix},$$

where

$$L_{-,p} = -\partial_x^2 + 1 - p\phi_p^{p-1} \text{ and } L_{+,p} = -\partial_x^2 + 1 - \phi_p^{p-1}.$$

For $0 < |p-3| \ll 1$, Coles and Gustafson [6] proved $\mathcal{L}_{p\omega}$ has exactly one eigenvalue of the form $\mathrm{i}\lambda(p,\omega)$ near $\mathrm{i}\omega$ where with $0 < \lambda(p,\omega) = \omega\lambda(p,1) < \omega$ with $\dim\ker(\mathcal{L}_{p\omega} - \mathrm{i}\lambda(p,\omega)) = 1$, thus giving a partial rigorous confirmation of the numerical results of Chang et al. [5]. For $\xi_p[\omega] \in H^1(\mathbb{R}, \mathbb{C}^2)$ a generator of $\ker(\mathcal{L}_{p\omega} - \mathrm{i}\lambda(p,\omega))$ in [8] we stated the following result involving only radial functions in $H^1_{\mathrm{rad}}(\mathbb{R}, \mathbb{C})$.

THEOREM 1.1. *There exists $p_1 < 3 < p_2$ s.t. for any $p \in (p_1, p_2) \setminus \{3\}$ and any $\omega_0 > 0$, any $a > 0$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in D_{H^1_{\mathrm{rad}}(\mathbb{R})}(\phi_{\omega_0}, \delta)$ there exist functions $\vartheta, \omega \in C^1(\mathbb{R}, \mathbb{R})$, $z \in C^1(\mathbb{R}, \mathbb{C})$ and $\omega_+ > 0$ s.t. the solution of (1) with initial datum u_0 can be written as*

$$\begin{aligned}
&u(t) = e^{\mathrm{i}\vartheta(t)}\left(\phi_{\omega(t)} + z(t)\xi_p[\omega(t)] + \overline{z}(t)\overline{\xi}_p[\omega(t)] + \eta(t)\right) \text{ with} \\
&\int_{\mathbb{R}} \|e^{-a\langle x\rangle}\eta(t)\|^2_{H^1(\mathbb{R})} dt < \epsilon \text{ where } \langle x\rangle := \sqrt{1+x^2}, \\
&\lim_{t\to\infty} \|e^{-a\langle x\rangle}\eta(t)\|_{L^2(\mathbb{R})} = 0, \\
&\lim_{t\to\infty} z(t) = 0, \\
&\lim_{t\to\infty} \omega(t) = \omega_+.
\end{aligned}$$

REMARK 1.2. Theorem 1.1 and the fact that the $H^1(\mathbb{R})$ norm of η is uniformly bounded for all times, guaranteed by classically known orbital stability results of Cazenave and Lions [4], Shatah [21] and Weinstein [24], imply the following local in space asymptotic convergence up to the phase,

$$\lim_{t\to+\infty} u(t)e^{-\mathrm{i}\vartheta(t)} = \phi_{\omega_+} \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}).$$

A result similar to Theorem 1.1 was obtained completely independently, unbeknownst to the two sets of authors during the process of writing the respective papers and posted on Arxiv the same day, with a different proof by

Rialland [20], which remains closer than us to the theory in Martel [19], which in turn has a similar result but for a cubic quintic NLS.

We set

$$\xi_p[\omega] = (\xi_{p,1}[\omega], \xi_{p,2}[\omega])^\intercal$$

and let furthermore $g_p^{(\omega)} = \left(g_{p,1}^{(\omega)}, g_{p,2}^{(\omega)}\right)^\intercal \in L^\infty\left(\mathbb{R}, \mathbb{C}^2\right)$ be an nonzero solution of

$$\mathcal{L}_{p\omega} g_p^{(\omega)} = 2\mathrm{i}\lambda(p,\omega) g_p^{(\omega)}, \tag{3}$$

see Buslaev and Perelman [1] and Krieger and Schlag [15]. Notice that if $g_p = g_p^{(1)}$, then $g^{(\omega)}(x) = g_p\left(\sqrt{\omega}x\right)$ solves (3). Similarly if $\xi_p = \xi_p[1]$, then $\xi_p[\omega](x) = \xi_p\left(\sqrt{\omega}x\right)$ is a generator of $\ker(\mathcal{L}_{p\omega} - \mathrm{i}\lambda(p,\omega))$. Here, for later reference, we record that

$$\mathcal{L}_p \xi_p = \mathrm{i}\lambda(p,1)\xi_p. \tag{4}$$

Using now the notation

$$\langle f, g\rangle := \operatorname{Re}\int_{\mathbb{R}} f(x)\overline{g(x)}dx,$$

we consider the constant

$$\gamma(p,\omega) := \left\langle \phi_p^{p-2}[\omega]\left(p\xi_{p,1}^2[\omega] + \xi_{p,2}^2[\omega]\right), g_{p,1}^{(\omega)}\right\rangle + 2\left\langle \phi_p^{p-2}[\omega]\xi_{p,1}[\omega]\xi_{p,2}[\omega], g_{p,2}^{(\omega)}\right\rangle, \tag{5}$$

which plays an important role in the proof of Theorem 1.1. Let $\gamma(p) := \gamma(p,1)$. In [8] we proved that it is possible to choose g_p and ξ_p so that there is a constant $\Gamma \in \mathbb{R}$ such that

$$\gamma(p) = (p-3)\Gamma + O\left((p-3)^2\right).$$

The purpose of the present paper is to prove the following.

Theorem 1.3. *We have*

$$\Gamma = \frac{\pi}{\sqrt{2}\cosh(\pi/2)}.$$

This implies automatically that the corresponding constant (5) is nonzero for any $\omega > 0$ and for $0 < |p-3| \ll 1$.

The proof of Theorem 1.1 in [8] utilizes in an essential way that $\gamma(p) \neq 0$ for $0 < |p-3| \ll 1$ and gives only a sketch of Theorem 1.3. A similar sketch is given in Rialland [20]. Both [8, 20] follow closely Martel [19] for their proof of their own versions of Theorem 1.3 and all three papers omit to write a full proof, on account of the fact that this would add many pages of very elementary computations to papers devoted to solving longstanding problems in the context of classical models like Equation (1). Our aim in this paper is to fill in the details of the proof of Theorem 1.3, following the succinct description of the analogous computation in Martel [19]. While the computations added here and missing in [8] are elementary, we think that writing the complete proof might give a useful reference for fellow researches. Besides, the computation of the constant Γ raises at least one interesting question. Indeed, it is rather surprising that both for the pure power equation (1) and for the cubic–quintic equation in Martel [19], the constant Γ has a very simple form. We ignore why this is the case, but certainly this merits further investigation. Rialland [20] has also some interesting numerical computations. A partial extension of $\gamma(p) \neq 0$ for exponents p further away from 3 is in [9] which proves the 3rd order FGR for generic p's and in [10]. Some analogues of Theorem 1.3 are proved in the case of the Nonlinear Klein Gordon Equation in Delort and Masmoudi [12], Li and Lührmann [16] and Lührmann and Schlag [17], while a numerical verification in that setting is in Kowalczyk et al. [14], but these papers deal with a much simpler situation than here and [8, 19, 20], because their linearization is a scalar Schrödinger operator with as potential a sech^2 function and can be reduced to the flat Schrödinger operator using two Darboux standard transformations, see Chang et al. [5] and Deift and Trubowitz [11].

We explain now, briefly, why it is important that these constants be different from 0. In the study of stability problems like in Theorem 1.1 it is quite natural to linearize Hamiltonian systems (1) at ground states. The linearized operators have both discrete and continuous modes and it is quite natural to use these modes to study also the nonlinear systems. At a linear level the different modes are uncoupled and so the discrete modes tend to describe harmonic oscillators. However there is mechanism, first introduced by Sigal [22], the so called nonlinear Fermi Golden Rure (from now on FGR), which explains why, by nonlinear interaction, the discrete modes, specifically related to excited states of the linearization, lose energy. The discrete modes are like a finite dimensional Hamiltonian system embedded in a sea of radiation represented by the continuous modes, with energy spilling out of the finite dimensional system and then, essentially by linear dispersion, scattered to infinity. Sigal's idea was developed in relatively simple contexts by Buslaev and Perelman [2], see also Buslaev and Sulem [3], and by Soffer and Weinstein [23]. Later [7] described the FGR for general spectral configurations, illustrating the deep relation between the FGR and the Hamiltonian structure of the systems. The FGR involves the fact that certain coefficients of the mixed system of discrete and continuous modes, have nonzero restriction on appropriate sphere of the *distorted* phase space associated to the linearization (2). In dimension 1 the spheres reduce to pair of points and the FGR reduces to having certain single frequencies different from 0, and that is exactly the meaning of the condition $\gamma(p,\omega) \neq 0$.

2. A first formula for the constant Γ

The main purpose of this section is to prove Proposition 2.4. Before stating and proving it, we review briefly some results on $\gamma(p)$ proved in [8]. First of all we proved the following.

LEMMA 2.1. *There exists a small $\delta_1 > 0$ and a function $\alpha \in C^\infty(D_{\mathbb{R}}(3,\delta_1),\mathbb{R})$ such that*

$$\lambda(p,1) = 1 - \alpha^2(p) \text{ with}$$
$$\alpha(p) = (p-3)^2 c_0 + O\left((p-3)^3\right) \text{ where } c_0 = 2^{-2} + 2^{-5} 2^{-\frac{1}{2}} \left\langle \phi_3^2, T \right\rangle,$$

where $T := e^{-\sqrt{2}|\cdot|} * \operatorname{sech}^2(x)$.

Notice that the above was a different proof, simpler thanks to the Darboux transformation due to Martel [19], of the result on the eigenvalue by Coles and Gustafson [6]. The specific value of the constant $c_0 > 0$ does not play a role here and [8].

Next, in [8] we proved the following.

LEMMA 2.2. *There exists an open interval $\mathcal{I}$ containing 3 and for each $p \in \mathcal{I}$ it is possible to choose $\xi_p = (\xi_{p,1}, \xi_{p,2})^\intercal$ so that (4) holds, we have*

$$\xi_3 = \left(1 - \phi_3^2, \mathrm{i}\right)^\intercal$$

and

$$\xi_{p,1} = 1 - \phi_3^2 + (p-3)R_1 + (p-3)^2 \tilde{\xi}_{p,1},$$
$$\xi_{p,2} = \mathrm{i}\left(1 + (p-3)R_2 + (p-3)^2 \tilde{\xi}_{p,2}\right),$$

where

$$R_1 = -x\phi_3\phi_3' - \frac{1}{4\sqrt{2}}(3-\phi_3^2)T - \frac{\phi_3'}{2\sqrt{2}\phi_3}T' \text{ and}$$
$$R_2 = \frac{1}{2}\phi_3^2 + \frac{3}{4\sqrt{2}}T + \frac{\phi_3'}{2\sqrt{2}\phi_3}T'$$

and where furthermore, for any $k \geq 0$ there exists a constant C_k such that

$$|\widetilde{\xi}_{p,j}^{(k)}(x)| \leq C_k \langle x \rangle^3 \text{ for all } x \in \mathbb{R} \text{ and all } p \in \mathcal{I}.$$

Here R_j and $\tilde{\xi}_{p,j}$ for $j = 1, 2$ are real valued.

Next, we set

$$g_3 = (h_{3,1}, \mathrm{i}h_{3,2})^\intercal \ \text{ with } \ (h_{3,1}, h_{3,2})^\intercal = \left(\frac{1}{2}\phi_3^2\cos(x) + \frac{\phi_3'}{\phi_3}\sin(x), \frac{\phi_3'}{\phi_3}\sin(x)\right)^\intercal.$$

Then in [8] we stated without proof the following variant of Lemma 19 of Martel [19], which can be proved similarly.

Lemma 2.3. *There exists an open interval $\mathcal{I}$ containing 3 and for each $p \in \mathcal{I}$ it is possible to choose g_p so that*

$$\|g_p - \left(\frac{1}{2}\phi_3^2\cos(\tau x) + \frac{\phi_3'}{\phi_3}\sin(\tau x), \mathrm{i}\frac{\phi_3'}{\phi_3}\sin(\tau x)\right)\|_{L^\infty} \lesssim |p-3|,$$

where $\tau = \sqrt{1-\lambda(p,1)^2}$ *and*

$$|\partial_x^k g_p(x)| \le C_k \ \textit{and} \ |\partial_x^k \partial_p g_p(x)| \le C_k(1+|x|) \ \textit{for all } x \in \mathbb{R} \ \textit{and all } p \in \mathcal{I}.$$

An elementary differentiation yields

$$\begin{aligned} E &:= \partial_p|_{p=3}\, \phi_p = \frac{1}{2}\phi_3\left(\frac{1}{4} - \log\phi_3\right) + \frac{1}{2}x\phi_3', \\ F &:= \partial_p|_{p=3}\, \phi_p^{p-2} = E + \phi_3\log\phi_3. \end{aligned}$$

Recalling

$$\gamma(p) = \left\langle \phi_p^{p-2}(p\xi_{p,1}^2 + \xi_{p,2}^2), g_{p,1}\right\rangle + 2\left\langle \phi_p^{p-2}\xi_{p,1}\xi_{p,2}, g_{p,2}\right\rangle,$$

we set, following the notation and the argument in Martel [19],

$$G_{p,1} := \phi_p^{p-2}(p\xi_{p,1}^2 + \xi_{p,2}^2), \qquad G_{p,2} := -2\mathrm{i}\phi_p^{p-2}\xi_{p,1}\xi_{p,2}.$$

Then by elementary computations

$$\begin{aligned} G_{p,1} &= \phi_3\left(3(1-\phi_3^2)^2 - 1\right) + (p-3)\Delta_1 + \tilde{\Delta}_1, \\ \frac{1}{2}G_{p,2} &= \phi_3(1-\phi_3^2) + (p-3)\Delta_2 + \tilde{\Delta}_2, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= F\left(3(1-\phi_3^2)^2 - 1\right) + \phi_3(1-\phi_3^2)^2 + 6\phi_3(1-\phi_3^2)R_1 - 2\phi_3 R_2, \\ \Delta_2 &= F(1-\phi_3^2) + \phi_3 R_1 + \phi_3(1-\phi_3^2)R_2 \end{aligned}$$

and $\tilde{\Delta}_1, \tilde{\Delta}_2$ are remainder terms of $(p-3)^2$ order. The following formula is proved in [8],

$$\Gamma = \langle \Delta_1, h_{3,1} \rangle + 2 \langle \Delta_2, h_{3,2} \rangle + \frac{1}{2} \left\langle \left(6x \tanh \operatorname{sech}^2 - \tfrac{7}{2} \operatorname{sech}^2 \right) G_{3,2}, h_{3,1} \right\rangle \\ - 2 \langle E, h_{3,1} \rangle =: \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \quad (6)$$

where sech = sech(x), log = log(x), tanh = tanh(x) etc. With this notation, denoting similarly $\log \circ \operatorname{sech} = \log(\operatorname{sech}(x))$, $T = T(x)$, we consider the following constants, similar to analogous ones introduced in Martel [19],

$$\begin{aligned} p_k &= \int \operatorname{sech}^k \cos, & b_k &= \int \operatorname{sech}^k \tanh \sin \\ q_k &= \int \operatorname{sech}^k \log \circ \operatorname{sech} \cos & c_k &= \int \operatorname{sech}^k \log \circ \operatorname{sech} \tanh \sin \\ r_k &= \int \operatorname{sech}^k T \cos & d_k &= \int x \operatorname{sech}^k \sin \\ s_k &= \int \operatorname{sech}^k T \tanh \sin & e_k &= \int \operatorname{sech}^k \tanh T' \cos \\ a_k &= \int x \operatorname{sech}^k \tanh \cos & f_k &= \int \operatorname{sech}^k T' \sin . \end{aligned}$$

The first step in our computations is the following.

Proposition 2.4. *The following formula holds,*

$$\begin{aligned} \Gamma = \sqrt{2} \Bigg(& \left(\frac{1}{2} \log 2 + \frac{17}{4} \right) p_1 + (-13 \log 2 - 71) p_3 \\ & + \left(28 \log 2 + \frac{311}{2} \right) p_5 + \left(-15 \log 2 - \frac{173}{2} \right) p_7 \Bigg) \\ & + \sqrt{2} \left(3q_1 - 28q_3 + 56q_5 - 30q_7 - a_1 + 93a_3 - 336a_5 + 252a_7 \right) \\ & + 4r_1 - 2r_3 - 30r_5 + 30r_7 + 2s_1 + 18s_3 - 20s_5 . \end{aligned}$$

In order to prove the proposition we will examine separately each term in (6).

Lemma 2.5. *We have*

$$\begin{aligned}
\gamma_1 = \sqrt{2}\left(\left(-3\log 2 - \frac{15}{2}\right)p_5 + \left(3\log 2 + \frac{11}{2}\right)p_7\right) \\
+ \sqrt{2}\,(q_3 - 6q_5 + 6q_7 - a_3 + 18a_5 - 30a_7) \\
+ \sqrt{2}\left(\left(3\log 2 + \frac{15}{2}\right)b_3 - \left(3\log 2 + \frac{11}{2}\right)b_5\right) \\
+ \sqrt{2}\,(-c_1 + 6c_3 - 6c_5 + d_1 - 19d_3 + 48d_5 - 30d_7) \\
- 6r_3 + 12r_5 - 6r_7 + 6s_1 - 12s_3 + 6s_5 + 4e_3 - 6e_5 - 4f_1 + 10f_3 - 6f_5.
\end{aligned}$$

Proof. We write

$$\begin{aligned}
\gamma_1 &= \langle \Delta_1, h_{3,1}\rangle \\
&= \left\langle F(3\xi_{3,1}^2 - \xi_{3,2}^2) + \phi_3\xi_{3,1}^2 + 6\phi_3\xi_{3,1}R_1 - 2\phi_3\xi_{3,2}R_2, h_{3,1}\right\rangle \\
&= 3\left\langle F\xi_{3,1}^2, h_{3,1}\right\rangle - \left\langle F\xi_{3,2}^2, h_{3,1}\right\rangle + \left\langle \phi_3\xi_{3,1}^2, h_{3,1}\right\rangle \\
&\quad + 6\left\langle \phi_3\xi_{3,1}R_1, h_{3,1}\right\rangle - 2\left\langle \phi_3\xi_{3,2}R_2, h_{3,1}\right\rangle \\
&=: \gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{15}.
\end{aligned}$$

Claim 2.6. *We have*

$$\gamma_{11} = \gamma_{111} + \gamma_{112}$$

with

$$\begin{aligned}
\gamma_{111} = -\frac{3}{\sqrt{2}}(2\log 2 + 1)p_5 + \frac{3}{\sqrt{2}}(2\log 2 + 1)p_7 + \frac{3}{\sqrt{2}}q_3 \\
- \frac{12}{\sqrt{2}}q_5 + \frac{12}{\sqrt{2}}q_7 - \frac{3}{\sqrt{2}}a_3 + \frac{12}{\sqrt{2}}a_5 - \frac{12}{\sqrt{2}}a_7
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\gamma_{112} = \frac{3}{\sqrt{2}}(2\log 2 + 1)b_3 - \frac{3}{\sqrt{2}}(2\log 2 + 1)b_5 - \frac{3}{\sqrt{2}}c_1 + \frac{12}{\sqrt{2}}c_3 - \frac{12}{\sqrt{2}}c_5 \\
+ \frac{3}{\sqrt{2}}(d_1 - d_3) - \frac{12}{\sqrt{2}}(d_3 - d_5) + \frac{12}{\sqrt{2}}(d_5 - d_7).
\end{aligned} \tag{8}$$

Proof. We have

$$
\begin{aligned}
\gamma_{11} &= 3\left\langle F\xi_{3,1}^2, h_{3,1}\right\rangle \\
&= 3\left\langle\left(\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right)(1-2\mathrm{sech}^2)^2, \mathrm{sech}^2\cos - \tanh\sin\right\rangle \\
&= 3\left\langle\left(\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right)(1-2\mathrm{sech}^2)^2, \mathrm{sech}^2\cos\right\rangle \\
&\quad - 3\left\langle\left(\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right)(1-2\mathrm{sech}^2)^2, \tanh\sin\right\rangle \\
&=: \gamma_{111} + \gamma_{112}.
\end{aligned}
$$

where we warn the reader that $\mathrm{sech}\cdot x$, here and in analogous expressions below, stands for the product of the functions $\sinh(x)$ and x. We have

$$
\begin{aligned}
\gamma_{111} &= \frac{3}{\sqrt{2}}\left\langle\left(\frac{1}{4}(2\log 2+1)\mathrm{sech} + \mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \mathrm{sech}\cdot x\tanh\right)(1-2\mathrm{sech}^2)^2, \mathrm{sech}^2\cos\right\rangle \\
&= 3\left\langle\left(\boxed{\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech}} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right), \mathrm{sech}^2\cos\right\rangle \\
&\quad - 12\left\langle\left(\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right)\mathrm{sech}^2, \mathrm{sech}^2\cos\right\rangle \\
&\quad + 12\left\langle\left(\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right. \\
&\qquad\qquad\left.\left. - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right)\mathrm{sech}^4, \mathrm{sech}^2\cos\right\rangle
\end{aligned}
$$

and so

$$\begin{aligned}\gamma_{111} = & -\frac{3}{\sqrt{2}}(2\log 2+1)\int \operatorname{sech}^5 \cos + \frac{3}{\sqrt{2}}(2\log 2+1)\int \operatorname{sech}^7 \cos \\ & + \frac{3}{\sqrt{2}}\int \operatorname{sech}^3 \log\circ\operatorname{sech} \cos - \frac{12}{\sqrt{2}}\int \operatorname{sech}^5 \log\circ\operatorname{sech} \cos \\ & + \frac{12}{\sqrt{2}}\int \operatorname{sech}^7 \cdot \log\circ\operatorname{sech} \cos - \frac{3}{\sqrt{2}}\int \operatorname{sech}^3 \cdot x\tanh \cos \\ & + \frac{12}{\sqrt{2}}\int \operatorname{sech}^5 \cdot x\tanh \cos - \frac{12}{\sqrt{2}}\int \operatorname{sech}^7 \cdot x\tanh \cos\end{aligned}$$

where we get (7) if we redefine γ_{111} as the last term in the formula, where, by an abuse of notation, in the last term we did not rewrite the boxed part, for reasons of space and because by $\langle \phi_3, h_{3,1}\rangle = 0$ it cancels when added to the boxed part in the next formula. The next formula is, applying the same convention,

$$\begin{aligned}\gamma_{112} = & -3\left\langle \left(\frac{1}{4\sqrt{2}}(2\log 2+1)\operatorname{sech} + \frac{1}{\sqrt{2}}\operatorname{sech}\log\circ\operatorname{sech} \right.\right. \\ & \left.\left. - \frac{1}{\sqrt{2}}\operatorname{sech}\cdot x\tanh \right)(1-2\operatorname{sech}^2)^2, \tanh\sin \right\rangle \\ = & -3\left\langle \left(\boxed{\frac{1}{4\sqrt{2}}(2\log 2+1)\operatorname{sech}} + \frac{1}{\sqrt{2}}\operatorname{sech}\log\circ\operatorname{sech} \right.\right. \\ & \left.\left. - \frac{1}{\sqrt{2}}\operatorname{sech}\cdot x\tanh \right), \tanh\sin \right\rangle \\ & + 12\left\langle \left(\frac{1}{4\sqrt{2}}(2\log 2+1)\operatorname{sech} + \frac{1}{\sqrt{2}}\operatorname{sech}\log\circ\operatorname{sech} \right.\right. \\ & \left.\left. - \frac{1}{\sqrt{2}}\operatorname{sech}\cdot x\tanh \right)\operatorname{sech}^2, \tanh\sin \right\rangle \\ & - 12\left\langle \left(\frac{1}{4\sqrt{2}}(2\log 2+1)\operatorname{sech} + \frac{1}{\sqrt{2}}\operatorname{sech}\log\circ\operatorname{sech} \right.\right. \\ & \left.\left. - \frac{1}{\sqrt{2}}\operatorname{sech}\cdot x\tanh \right)\operatorname{sech}^4, \tanh\sin \right\rangle \\ = & \frac{3}{\sqrt{2}}(2\log 2+1)\int \operatorname{sech}^3 \tanh\sin - \frac{3}{\sqrt{2}}(2\log 2+1)\int \operatorname{sech}^5 \tanh\sin \\ & - \frac{3}{\sqrt{2}}\int \operatorname{sech}\log\circ\operatorname{sech}\tanh\sin + \frac{12}{\sqrt{2}}\int \operatorname{sech}^3\log\circ\operatorname{sech}\tanh\sin \\ & - \frac{12}{\sqrt{2}}\int \operatorname{sech}^5\log\circ\operatorname{sech}\tanh\sin + \frac{3}{\sqrt{2}}\int \operatorname{sech}\cdot x(1-\operatorname{sech}^2)\sin \\ & - \frac{12}{\sqrt{2}}\int \operatorname{sech}^3\cdot x(1-\operatorname{sech}^2)\sin + \frac{12}{\sqrt{2}}\int \operatorname{sech}^5\cdot x(1-\operatorname{sech}^2)\sin,\end{aligned}$$

where we get (8). □

Claim 2.7. *We have*

$$\gamma_{12} = \gamma_{121} + \gamma_{122}$$

with

$$\begin{aligned}\gamma_{121} &= -\frac{1}{\sqrt{2}}q_3 + \frac{1}{\sqrt{2}}a_3, \\ \gamma_{122} &= \frac{1}{\sqrt{2}}c_1 - \frac{1}{\sqrt{2}}(d_1 - d_3).\end{aligned}$$

Proof. We have

$$\begin{aligned}\gamma_{12} &= -\left\langle F\xi_{3,2}^2, h_{3,1}\right\rangle \\ &= -\Bigg\langle \cancel{\frac{1}{4\sqrt{2}}(2\log 2 + 1)\mathrm{sech}} + \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech} \\ &\qquad\qquad - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh, \mathrm{sech}^2\cos - \tanh\sin\Bigg\rangle \\ &= -\left\langle \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech} - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh, \mathrm{sech}^2\cos\right\rangle \\ &\quad + \left\langle \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech} - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh, \tanh\sin\right\rangle \\ &=: \gamma_{121} + \gamma_{122},\end{aligned}$$

where the canceled term is null by $\langle \phi_3, h_{3,1}\rangle = 0$. Then from

$$\begin{aligned}\gamma_{121} &= -\left\langle \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech} - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh, \mathrm{sech}^2\cos\right\rangle \\ &= -\frac{1}{\sqrt{2}}\int \mathrm{sech}^3\log\circ\mathrm{sech}\cos + \frac{1}{\sqrt{2}}\int \mathrm{sech}^3\cdot x\tanh\cos\end{aligned}$$

and

$$\begin{aligned}\gamma_{122} &= \left\langle \frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech} - \frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh, \tanh\sin\right\rangle \\ &= \frac{1}{\sqrt{2}}\int \mathrm{sech}\log\circ\mathrm{sech}\tanh\sin - \frac{1}{\sqrt{2}}\int \mathrm{sech}(1 - \mathrm{sech}^2)x\sin,\end{aligned}$$

we obtain the desired claim. □

Claim 2.8. *We have*

$$\gamma_{13} = \gamma_{131} + \gamma_{132}$$

with

$$\gamma_{131} = -4\sqrt{2}p_5 + 4\sqrt{2}p_7,$$
$$\gamma_{132} = 4\sqrt{2}b_3 - 4\sqrt{2}b_5.$$

Proof. We have

$$\begin{aligned}
\gamma_{13} &= \left\langle \phi_3 \xi_{3,1}^2, h_{3,1} \right\rangle \\
&= \sqrt{2} \left\langle \mathrm{sech}(1-2\mathrm{sech}^2)^2, \mathrm{sech}^2 \cos - \tanh \sin \right\rangle \\
&= \sqrt{2} \left\langle \cancel{\mathrm{sech}} - 4\mathrm{sech}^3 + 4\mathrm{sech}^5, \mathrm{sech}^2 \cos - \tanh \sin \right\rangle \\
&= \sqrt{2} \left\langle -4\mathrm{sech}^3 + 4\mathrm{sech}^5, \mathrm{sech}^2 \cos \right\rangle - \sqrt{2} \left\langle -4\mathrm{sech}^3 + 4\mathrm{sech}^5, \tanh \sin \right\rangle \\
&=: \gamma_{131} + \gamma_{132},
\end{aligned}$$

where the canceled term is null by $\langle \phi_3, h_{3,1} \rangle = 0$. Then from

$$\begin{aligned}
\gamma_{131} &= \sqrt{2} \left\langle -4\mathrm{sech}^3 + 4\mathrm{sech}^5, \mathrm{sech}^2 \cos \right\rangle \\
&= -4\sqrt{2} \int \mathrm{sech}^5 \cos + 4\sqrt{2} \int \mathrm{sech}^7 \cos
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{132} &= -\sqrt{2} \left\langle -4\mathrm{sech}^3 + 4\mathrm{sech}^5, \tanh \sin \right\rangle \\
&= 4\sqrt{2} \int \mathrm{sech}^3 \tanh \sin - 4\sqrt{2} \int \mathrm{sech}^5 \tanh \sin,
\end{aligned}$$

we obtain the claim. □

Claim 2.9. *We have*

$$\gamma_{14} = \gamma_{141} + \gamma_{142}$$

with

$$\begin{aligned}
\gamma_{141} &= 12\sqrt{2}(a_5 - 2a_7) - \frac{3}{2}(3r_3 - 8r_5 + 4r_7) + 3(e_3 - 2e_5) \\
\gamma_{142} &= -12\sqrt{2}(d_3 - 3d_5 + 2d_7) + \frac{3}{2}(3s_1 - 8s_3 + 4s_5) - 3(f_1 - 3f_3 + 2f_5).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\gamma_{14} &= 6\left\langle \phi_3 \xi_{3,1} R_1, h_{3,1}\right\rangle \\
&= 6\sqrt{2}\left\langle \operatorname{sech}(1-2\operatorname{sech}^2)\left(2x\tanh\operatorname{sech}^2 - \frac{1}{4\sqrt{2}}(3-2\operatorname{sech}^2)T \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{2\sqrt{2}}\tanh T'\right), \operatorname{sech}^2\cos - \tanh\sin\right\rangle \\
&= 6\sqrt{2}\left\langle \operatorname{sech}(1-2\operatorname{sech}^2)\left(2x\tanh\operatorname{sech}^2 - \frac{1}{4\sqrt{2}}(3-2\operatorname{sech}^2)T \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{2\sqrt{2}}\tanh T'\right), \operatorname{sech}^2\cos\right\rangle \\
&\quad - 6\sqrt{2}\left\langle \operatorname{sech}(1-2\operatorname{sech}^2)\left(2x\tanh\operatorname{sech}^2 - \frac{1}{4\sqrt{2}}(3-2\operatorname{sech}^2)T \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{2\sqrt{2}}\tanh T'\right), \tanh\sin\right\rangle \\
&=: \gamma_{141} + \gamma_{142}.
\end{aligned}$$

From

$$\begin{aligned}
\gamma_{141} &= 6\sqrt{2}\left\langle \operatorname{sech}(1-2\operatorname{sech}^2)\left(2x\tanh\operatorname{sech}^2 - \frac{1}{4\sqrt{2}}(3-2\operatorname{sech}^2)T \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{2\sqrt{2}}\tanh T'\right), \operatorname{sech}^2\cos\right\rangle \\
&= 12\sqrt{2}\int (\operatorname{sech}^5 - 2\operatorname{sech}^7)x\tanh\cos \\
&\quad - \frac{3}{2}\int (3\operatorname{sech}^3 - 8\operatorname{sech}^5 + 4\operatorname{sech}^7)T\cos \\
&\quad + 3\int (\operatorname{sech}^3 - 2\operatorname{sech}^5)\tanh T'\cos
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{142} &= -6\sqrt{2}\left\langle \operatorname{sech}(1-2\operatorname{sech}^2)\left(2x\tanh\operatorname{sech}^2 - \frac{1}{4\sqrt{2}}(3-2\operatorname{sech}^2)T \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{2\sqrt{2}}\tanh T'\right), \tanh\sin\right\rangle \\
&= -12\sqrt{2}\int (\operatorname{sech}^3 - 3\operatorname{sech}^5 + 2\operatorname{sech}^7)x\sin \\
&\quad + \frac{3}{2}\int (3\operatorname{sech} - 8\operatorname{sech}^3 + 4\operatorname{sech}^5)T\tanh\sin \\
&\quad - 3\int \left(\operatorname{sech} - 3\operatorname{sech}^3 + 2\operatorname{sech}^5\right) T'\sin,
\end{aligned}$$

we obtain the claim. □

Claim 2.10. *We have*

$$\gamma_{15} = \gamma_{151} + \gamma_{152}$$

with

$$\gamma_{151} = -2\sqrt{2}p_5 - \frac{3}{2}r_3 + e_3,$$
$$\gamma_{152} = 2\sqrt{2}b_3 + \frac{3}{2}s_1 - f_1 + f_3.$$

Proof. We have

$$\begin{aligned}
\gamma_{15} &= -2\left\langle \phi_3\xi_{3,2}R_2, h_{3,1}\right\rangle \\
&= -2\sqrt{2}\left\langle \operatorname{sech}\left(\operatorname{sech}^2 + \frac{3}{4\sqrt{2}}T - \frac{\tanh}{2\sqrt{2}}T'\right), \operatorname{sech}^2\cos - \tanh\sin\right\rangle \\
&= -2\sqrt{2}\left\langle \operatorname{sech}\left(\operatorname{sech}^2 + \frac{3}{4\sqrt{2}}T - \frac{\tanh}{2\sqrt{2}}T'\right), \operatorname{sech}^2\cos\right\rangle \\
&\quad + 2\sqrt{2}\left\langle \operatorname{sech}\left(\operatorname{sech}^2 + \frac{3}{4\sqrt{2}}T - \frac{\tanh}{2\sqrt{2}}T'\right), \tanh\sin\right\rangle \\
&=: \gamma_{151} + \gamma_{152}.
\end{aligned}$$

From

$$\begin{aligned}
\gamma_{151} &= -2\sqrt{2}\left\langle \operatorname{sech}\left(\operatorname{sech}^2 + \frac{3}{4\sqrt{2}}T - \frac{\tanh}{2\sqrt{2}}T'\right), \operatorname{sech}^2\cos\right\rangle \\
&= -2\sqrt{2}\int \operatorname{sech}^5\cos - \frac{3}{2}\int \operatorname{sech}^3 T\cos + \int \operatorname{sech}^3\tanh T'\cos
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{152} &= 2\sqrt{2}\left\langle \operatorname{sech}\left(\operatorname{sech}^2 + \frac{3}{4\sqrt{2}}T - \frac{\tanh}{2\sqrt{2}}T'\right), \tanh\sin\right\rangle \\
&= 2\sqrt{2}\int \operatorname{sech}^3\tanh\sin + \frac{3}{2}\int \operatorname{sech}T\tanh\sin - \int \operatorname{sech}(1-\operatorname{sech}^2)T'\sin,
\end{aligned}$$

we obtain the claim. □

From

$$\gamma_1 = \gamma_{111} + \gamma_{112} + \gamma_{121} + \gamma_{122} + \gamma_{131} + \gamma_{132} + \gamma_{141} + \gamma_{142} + \gamma_{151} + \gamma_{152},$$

we obtain

$$\begin{aligned}
\gamma_1 = &-\frac{3}{\sqrt{2}}(2\log 2+1)p_5 + \frac{3}{\sqrt{2}}(2\log 2+1)p_7 + \frac{3}{\sqrt{2}}q_3 - \frac{12}{\sqrt{2}}q_5 + \frac{12}{\sqrt{2}}q_7 \\
&- \frac{3}{\sqrt{2}}a_3 + \frac{12}{\sqrt{2}}a_5 - \frac{12}{\sqrt{2}}a_7 + \frac{3}{\sqrt{2}}(2\log 2+1)b_3 - \frac{3}{\sqrt{2}}(2\log 2+1)b_5 \\
&- \frac{3}{\sqrt{2}}c_1 + \frac{12}{\sqrt{2}}c_3 - \frac{12}{\sqrt{2}}c_5 + \frac{3}{\sqrt{2}}(d_1-d_3) - \frac{12}{\sqrt{2}}(d_3-d_5) \\
&+ \frac{12}{\sqrt{2}}(d_5-d_7) \underbrace{-\frac{1}{\sqrt{2}}q_3 + \frac{1}{\sqrt{2}}a_3}_{=\gamma_{121}} + \underbrace{\frac{1}{\sqrt{2}}c_1 - \frac{1}{\sqrt{2}}(d_1-d_3)}_{=\gamma_{122}} \\
&\underbrace{-4\sqrt{2}p_5 + 4\sqrt{2}p_7}_{=\gamma_{131}} + \underbrace{4\sqrt{2}b_3 - 4\sqrt{2}b_5}_{=\gamma_{132}} \\
&\underbrace{+\,12\sqrt{2}(a_5-2a_7) - \frac{3}{2}(3r_3-8r_5+4r_7) + 3(e_3-2e_5)}_{=\gamma_{141}} \\
&\underbrace{-12\sqrt{2}(d_3-3d_5+2d_7) + \frac{3}{2}\left(3s_1-8s_3+4s_5\right) - 3\left(f_1-3f_3+2f_5\right)}_{=\gamma_{142}} \\
&\underbrace{-2\sqrt{2}p_5 - \frac{3}{2}r_3 + e_3}_{=\gamma_{151}} + \underbrace{2\sqrt{2}b_3 + \frac{3}{2}s_1 - f_1 + f_3}_{=\gamma_{152}}
\end{aligned}$$

Summing up corresponding terms and tracking them to make the computations simpler to read, we obtain

$$\begin{aligned}
\gamma_1 = &\sqrt{2}\Bigg(\underbrace{\left(-\frac{3}{2}(2\log 2+1)-4-2\right)}_{=-3\log 2-\frac{15}{2}}p_5 + \underbrace{\left(\frac{3}{2}(2\log 2+1)+4\right)}_{=3\log 2+\frac{11}{2}}p_7\Bigg) \\
&+\sqrt{2}\Bigg(\underbrace{\left(\frac{3}{2}-\frac{1}{2}\right)}_{=1}q_3 - 6q_5 + 6q_7 + \underbrace{\left(-\frac{3}{2}+\frac{1}{2}\right)}_{=-1}a_3 + \underbrace{(6+12)}_{=18}a_5 + \underbrace{(-6-24)}_{=-30}a_7\Bigg) \\
&+\sqrt{2}\Bigg(\underbrace{\left(\frac{3}{2}(2\log 2+1)+4+2\right)}_{=3\log 2+\frac{15}{2}}b_3 + \underbrace{\left(-\frac{3}{2}(2\log 2+1)-4\right)}_{=-3\log 2-\frac{11}{2}}b_5\Bigg) \\
&+\sqrt{2}\Bigg(\underbrace{\left(-\frac{3}{2}+\frac{1}{2}\right)}_{=-1}c_1 + 6c_3 - 6c_5\Bigg)
\end{aligned}$$

↪ *The formula continues on the next page*

$$+\sqrt{2}\Bigg(\underbrace{\left(\frac{3}{2}-\frac{1}{2}\right)}_{=1}d_1+\underbrace{\left(-\frac{3}{2}-6+\frac{1}{2}-12\right)}_{-19}d_3+\underbrace{(6+6+36)}_{=48}d_5+\underbrace{(-6-24)}_{=-30}d_7\Bigg)$$
$$+\underbrace{\left(-\frac{9}{2}-\frac{3}{2}\right)}_{=-6}r_3+12r_5-6r_7+\underbrace{\left(\frac{9}{2}+\frac{3}{2}\right)}_{=6}s_1-12s_3+6s_5$$
$$+\underbrace{(3+1)}_{=4}e_3-6e_5+\underbrace{(-3-1)}_{=-4}f_1+\underbrace{(9+1)}_{=10}f_3-6f_5$$

which concludes the proof of Lemma 2.5. □

LEMMA 2.11. *We have*

$$\gamma_2=\sqrt{2}\left(-\frac{1}{4}(2\log 2+1)b_1+\left(\log 2-\frac{3}{2}\right)b_3+4b_5\right.$$
$$\left.-c_1+2c_3+d_1-7d_3+6d_5\right)+2s_3-2f_3+2f_5.$$

Proof. We write

$$\begin{aligned}\gamma_2&=2\langle\Delta_2,h_{3,2}\rangle=2\langle F\xi_{3,1}\xi_{3,2}+\phi_3R_1\xi_{3,1}+\phi_3\xi_{3,1}R_2,h_{3,2}\rangle\\&=2\langle F\xi_{3,1}\xi_{3,2},h_{3,2}\rangle+2\langle\phi_3R_1\xi_{3,2},h_{3,2}\rangle+2\langle\phi_3\xi_{3,1}R_2,h_{3,2}\rangle\\&=:\gamma_{21}+\gamma_{22}+\gamma_{23}.\end{aligned}$$

CLAIM 2.12. *We have*

$$\gamma_{21}=-\frac{1}{2\sqrt{2}}(2\log 2+1)(b_1-2b_3)-\sqrt{2}\,(c_1-2c_3)+\sqrt{2}\,(d_1-3d_3+2d_5)\,.$$

Proof. It is a consequence of

$$\begin{aligned}\gamma_{21}&=2\langle F\xi_{3,1}\xi_{3,2},h_{3,2}\rangle\\&=2\left\langle\left(\frac{1}{4\sqrt{2}}(2\log 2+1)\mathrm{sech}+\frac{1}{\sqrt{2}}\mathrm{sech}\log\circ\mathrm{sech}\right.\right.\\&\qquad\left.\left.-\frac{1}{\sqrt{2}}\mathrm{sech}\cdot x\tanh\right)(1-2\mathrm{sech}^2),-\tanh\sin\right\rangle\\&=-\frac{1}{2\sqrt{2}}(2\log 2+1)\int\mathrm{sech}(1-2\mathrm{sech}^2)\tanh\sin\\&\quad-\frac{2}{\sqrt{2}}\int\mathrm{sech}(1-2\mathrm{sech}^2)\log\circ\mathrm{sech}\tanh\sin\\&\quad+\sqrt{2}\int\mathrm{sech}(1-3\mathrm{sech}^2+2\mathrm{sech}^4)x\sin\,.\end{aligned}$$

□

Claim 2.13. *We have*

$$\gamma_{22} = -4\sqrt{2}(d_3 - d_5) + \frac{1}{2}(3s_1 - 2s_3) - f_1 + f_3.$$

Proof. It is a consequence of

$$\begin{aligned}\gamma_{22} &= 2\left\langle \phi_3 R_1 \xi_{3,2}, h_{3,2}\right\rangle \\ &= -2\sqrt{2}\left\langle \operatorname{sech}\left(2x\tanh\operatorname{sech}^2 - \frac{1}{4\sqrt{2}}(3-2\operatorname{sech}^2)T \right.\right. \\ &\qquad \left.\left. + \frac{1}{2\sqrt{2}}\tanh T'\right), \tanh\sin\right\rangle \\ &= -4\sqrt{2}\int \operatorname{sech}^3(1-\operatorname{sech}^2)x\sin + \frac{1}{2}\int(3\operatorname{sech} - 2\operatorname{sech}^3)T\tanh\sin \\ &\quad -\int \operatorname{sech}(1-\operatorname{sech}^2)T'\sin.\end{aligned}$$

□

Claim 2.14. *We have*

$$\gamma_{23} = -2\sqrt{2}(b_3 - 2b_5) - \frac{3}{2}(s_1 - 2s_3) + f_1 - 3f_3 + 2f_5.$$

Proof. It is a consequence of

$$\begin{aligned}\gamma_{23} &= 2\left\langle \phi_3 \xi_{3,1} R_2, h_{3,2}\right\rangle \\ &= -2\sqrt{2}\left\langle \operatorname{sech}(1-2\operatorname{sech}^2)\left(\operatorname{sech}^2 + \frac{3}{4\sqrt{2}}T - \frac{\tanh}{2\sqrt{2}}T'\right), \tanh\sin\right\rangle \\ &= -2\sqrt{2}\int(\operatorname{sech}^3 - 2\operatorname{sech}^5)\tanh\sin - \frac{3}{2}\int(\operatorname{sech} - 2\operatorname{sech}^3)T\tanh\sin \\ &\quad + \int(\operatorname{sech} - 3\operatorname{sech}^3 + 2\operatorname{sech}^5)T'\sin.\end{aligned}$$

□

Summing up the quantities computed in the three claims we get Lemma 2.11. □

Lemma 2.15. *We have*

$$\gamma_3 = 6\sqrt{2}(a_5 - 2a_7) - 6\sqrt{2}(d_3 - 3d_5 + 2d_7) - \frac{7}{\sqrt{2}}(p_5 - 2p_7) + \frac{7}{\sqrt{2}}(b_3 - 2b_5).$$

Proof. We have

$$\begin{aligned}\gamma_3 &= \left\langle (6x\tanh\operatorname{sech}^2 - \tfrac{7}{2}\operatorname{sech}^2)(\phi_3\xi_{3,1}v_{3,2}), h_{3,1}\right\rangle \\ &= 6\left\langle x\tanh\operatorname{sech}^2\phi_3\xi_{3,1}\xi_{3,2}, h_{3,1}\right\rangle - \tfrac{7}{2}\left\langle \operatorname{sech}^2\phi_3\xi_{3,1}\xi_{3,2}, h_{3,1}\right\rangle \\ &=: \gamma_{31} + \gamma_{32}.\end{aligned}$$

CLAIM 2.16. *We have*

$$\gamma_{31} = 6\sqrt{2}(a_5 - 2a_7) - 6\sqrt{2}(d_3 - 3d_5 + 2d_7).$$

Proof. It is a consequence of

$$\begin{aligned}\gamma_{31} &= 6\left\langle x \tanh \operatorname{sech}^2 \phi_3 \xi_{3,1}\xi_{3,2}, h_{3,1}\right\rangle \\ &= 6\sqrt{2}\left\langle x \tanh \operatorname{sech}^3(1-2\operatorname{sech}^2), \operatorname{sech}^2\cos - \tanh\sin\right\rangle \\ &= 6\sqrt{2}\int \operatorname{sech}^5(1-2\operatorname{sech}^2) x \tanh\cos \\ &\quad - 6\sqrt{2}\int \operatorname{sech}^3(1-3\operatorname{sech}^2+2\operatorname{sech}^4) x \sin .\end{aligned}$$

□

CLAIM 2.17. *We have*

$$\gamma_{32} = -\frac{7}{\sqrt{2}}(p_5 - 2p_7) + \frac{7}{\sqrt{2}}(b_3 - 2b_5).$$

Proof. It is a consequence of

$$\begin{aligned}\gamma_{32} &= -\frac{7}{2}\left\langle \operatorname{sech}^2 \phi_3 v_{1,3} v_{2,3}, h_{3,1}\right\rangle \\ &= -\frac{7}{\sqrt{2}}\left\langle \operatorname{sech}^3(1-2\operatorname{sech}^2), \operatorname{sech}^2\cos - \tanh\sin\right\rangle \\ &= -\frac{7}{\sqrt{2}}\int \operatorname{sech}^5(1-2\operatorname{sech}^2)\cos + \frac{7}{\sqrt{2}}\int \operatorname{sech}^3(1-2\operatorname{sech}^2)\tanh\sin .\end{aligned}$$

□

Summing up the formulas in the Claims we get the proof of Lemma 2.15. □

LEMMA 2.18. *We have*

$$\gamma_4 = \sqrt{2}q_3 - \sqrt{2}c_1 + \sqrt{2}a_3 - \sqrt{2}(d_1 - d_3).$$

Proof. We have

$$\begin{aligned}\gamma_4 &= -2\left\langle E, h_{3,1}\right\rangle \\ &= -2\left\langle \frac{1}{\sqrt{2}}\operatorname{sech}\left(\cancel{-\frac{1}{2}\log 2 + \frac{1}{4}} - \log\operatorname{sech} - x\tanh\right), h_{3,1}\right\rangle \\ &= \sqrt{2}\left\langle \operatorname{sech}\log\circ\operatorname{sech}, h_{3,1}\right\rangle + \sqrt{2}\left\langle \operatorname{sech}\cdot x\tanh, h_{3,1}\right\rangle \\ &=: \gamma_{42} + \gamma_{43},\end{aligned}$$

where the canceled term is null by $\langle \phi_3, h_{3,1}\rangle = 0$. Then the statement follows from

$$\begin{aligned}\gamma_{42} &= \sqrt{2}\,\langle \operatorname{sech} \log \circ \operatorname{sech}, h_{3,1}\rangle \\ &= \sqrt{2}\,\big\langle \operatorname{sech} \log \circ \operatorname{sech}, \operatorname{sech}^2 \cos - \tanh \sin \big\rangle \\ &= \sqrt{2} \int \operatorname{sech}^3 \log \circ \operatorname{sech} \cos - \sqrt{2} \int \operatorname{sech} \log \circ \operatorname{sech} \tanh \sin \\ &= \sqrt{2} q_3 - \sqrt{2} c_1\end{aligned}$$

and

$$\begin{aligned}\gamma_{43} &= \sqrt{2}\,\langle \operatorname{sech} \cdot x \tanh, h_{3,1}\rangle \\ &= \sqrt{2}\,\big\langle \operatorname{sech} \cdot x \tanh, \operatorname{sech}^2 \cos - \tanh \sin \big\rangle \\ &= \sqrt{2} \int \operatorname{sech}^3 \cdot x \tanh \cos - \sqrt{2} \int \operatorname{sech}(1 - \operatorname{sech}^2) x \sin \\ &= \sqrt{2} a_3 - \sqrt{2}(d_1 - d_3),\end{aligned}$$

where we remind the reader that our convention is that $\operatorname{sech}^n \cdot x$ is the product of the function $\operatorname{sech}^n(x)$ with the function x. □

We now consider the following reduction formulas.

LEMMA 2.19. *We have the following relations:*

$$\begin{aligned}b_k &= (k+1) p_{k+2} - k p_k; \\ c_k &= (k+1) q_{k+2} - k q_k + p_{k+2} - p_k; \\ d_k &= -k a_k + p_k; \\ e_k &= s_k + k r_k - (k+1) r_{k+2}; \\ f_k &= -r_k + k s_k.\end{aligned}$$

Proof. The formulas follow from the following ones:

$$\begin{aligned}b_k &= \int \operatorname{sech}^k \tanh \sin \\ &= -k \int \operatorname{sech}^k (1 - \operatorname{sech}^2) \cos + \int \operatorname{sech}^{k+2} \cos \\ &= -k p_k + k p_{k+2} + p_{k+2};\end{aligned}$$

$$
\begin{aligned}
c_k &= \int \operatorname{sech}^k \log \circ \operatorname{sech} \tanh \sin \\
&= -k \int \operatorname{sech}^k (1-\operatorname{sech}^2) \log \circ \operatorname{sech} \cos - \int \operatorname{sech}^k (1-\operatorname{sech}^2) \cos \\
&\quad + \int \operatorname{sech}^{k+2} \log \circ \operatorname{sech} \cos \\
&= -k(q_k - q_{k+2}) - p_k + p_{k+2} + q_{k+2};
\end{aligned}
$$

$$
\begin{aligned}
d_k &= \int \operatorname{sech}^k \cdot x \sin \\
&= -k \int \operatorname{sech}^k x \tanh \cos + \int \operatorname{sech}^k \cos \\
&= -k a_k + p_k;
\end{aligned}
$$

$$
\begin{aligned}
e_k &= \int \operatorname{sech}^k T' \tanh \cos \\
&= \int \operatorname{sech}^k T \tanh \sin - \int \operatorname{sech}^{k+2} T \cos \\
&\quad + k \int \operatorname{sech}^k (1-\operatorname{sech}^2) T \cos \\
&= s_k - r_{k+2} + k(r_k - r_{k+2});
\end{aligned}
$$

$$
\begin{aligned}
f_k &= \int \operatorname{sech}^k T' \sin \\
&= -\int \operatorname{sech}^k T \cos + k \int \operatorname{sech}^k T \tanh \sin . \qquad \square
\end{aligned}
$$

Thanks to Lemma 2.19 we can eliminate some of the variables.

LEMMA 2.20. *We have the following formulas:*

$$
\begin{aligned}
\gamma_1 = \sqrt{2} \Bigg(2p_1 + \left(-9\log 2 - \frac{97}{2} \right) p_3 + (24 \log 2 + 110)\, p_5 & \\
+ \left(-15 \log 2 - \frac{127}{2} \right) p_7 \Bigg) & \\
+ \sqrt{2}\, (q_1 - 19 q_3 + 48 q_5 - 30 q_7 - a_1 + 56 a_3 - 222 a_5 + 180 a_7) & \\
+ 4r_1 - 4r_3 - 28 r_5 + 30 r_7 + 2s_1 + 22 s_3 - 30 s_5; &
\end{aligned} \tag{9}
$$

$$\gamma_2 = \sqrt{2}\left(\left(\frac{1}{2}\log 2 + \frac{9}{4}\right)p_1 + (-4\log 2 - 6)\,p_3 + (4\log 2 - 18)\,p_5 + 24p_7\right)$$
$$+ \sqrt{2}\,(q_1 - 8q_3 + 8q_5 - a_1 + 21a_3 - 30a_5)$$
$$+ 2r_3 - 2r_5 - 4s_3 + 10s_5;$$
$$\gamma_3 = \sqrt{2}\left(-\frac{33}{2}p_3 + \frac{127}{2}p_5 - 47p_7 + 18a_3 - 84a_5 + 72a_7\right); \tag{10}$$
$$\gamma_4 = \sqrt{2}\,(q_1 - q_3 + a_1 - 2a_3)\,. \tag{11}$$

Proof. Starting from the simplest formulas we have

$$\gamma_4 = \sqrt{2}q_3 - \sqrt{2}c_1 + \sqrt{2}a_3 - \sqrt{2}d_1 + \sqrt{2}d_3$$
$$= \sqrt{2}q_3 - \sqrt{2}(2q_3 - q_1 + \cancel{p_3 - p_1}) + \sqrt{2}a_3 - \sqrt{2}(-a_1 + \cancel{p_1}) + \sqrt{2}(-3a_3 + \cancel{p_3})$$
$$= \sqrt{2}q_1 - \sqrt{2}q_3 + \sqrt{2}a_1 - 2\sqrt{2}a_3 = \sqrt{2}\,(q_1 - q_3 + a_1 - 2a_3)\,,$$

which yields (11). Then we consider

$$\gamma_3 = 6\sqrt{2}(a_5 - 2a_7) - 6\sqrt{2}(d_3 - 3d_5 + 2d_7) - \frac{7}{\sqrt{2}}(p_5 - 2p_7) + \frac{7}{\sqrt{2}}(b_3 - 2b_5)$$
$$= \sqrt{2}\left(6a_5 - 12a_7 - 6d_3 + 18d_5 - 12d_7 - \frac{7}{2}p_5 + 7p_7 + \frac{7}{2}b_3 - 7b_5\right)$$
$$= \sqrt{2}\Big(6a_5 - 12a_7 - 6(-3a_3 + p_3) + 18(-5a_5 + p_5) - 12(-7a_7 + p_7) - \frac{7}{2}p_5$$
$$+7p_7 + \frac{7}{2}(4p_5 - 3p_3) - 7(6p_7 - 5p_5)\Big)$$
$$= \sqrt{2}\Big(\underbrace{(-6 - \frac{21}{2})}_{=-\frac{33}{2}}p_3 + \underbrace{(18 - \frac{7}{2} + 14 + 35)}_{=\frac{127}{2}}p_5 + \underbrace{(-12 + 7 - 42)}_{=-47}p_7$$
$$+ 18a_3 + \underbrace{(6 - 90)}_{=-84}a_5 + \underbrace{(-12 + 84)}_{=72}a_7\Big)$$
$$= \sqrt{2}\left(-\frac{33}{2}p_3 + \frac{127}{2}p_5 - 47p_7 + 18a_3 - 84a_5 + 72a_7\right).$$

Using Lemma 2.11 we have

$$\gamma_2 = \sqrt{2}\Big(-\frac{1}{4}(2\log 2 + 1)b_1 + \left(\log 2 - \frac{3}{2}\right)b_3 + 4b_5 - c_1 + 2c_3 + d_1$$
$$+ (-3 - 4)d_3 + (2 + 4)d_5\Big) + 2s_3 - 2f_3 + 2f_5$$
$$= \sqrt{2}\Big(-\frac{1}{4}(2\log 2 + 1)(2p_3 - p_1) + \left(\log 2 - \frac{3}{2}\right)(4p_5 - 3p_3) + 4(6p_7 - 5p_5)$$
$$- (2q_3 - q_1 + p_3 - p_1)\Big) + \sqrt{2}\Big(2(4q_5 - 3q_3 + p_5 - p_3) + (-a_1 + p_1)$$
$$- 7(-3a_3 + p_3) + 6(-5a_5 + p_5)\Big) + 2s_3 - 2(-r_3 + 3s_3) + 2(-r_5 + 5s_5).$$

Collecting together similar terms we have the following, which yields (10),

$$\begin{aligned}\gamma_2 = \sqrt{2}\Bigg(&\underbrace{\left(\frac{1}{4}(2\log 2+1)+1+1\right)}_{=\frac{1}{2}\log 2+\frac{9}{4}} p_1 \\ &+ \underbrace{\left(-\frac{1}{2}(2\log 2+1)-3\left(\log 2-\frac{3}{2}\right)-1-2-7\right)}_{=-4\log 2-6} p_3\Bigg) \\ &+ \sqrt{2}\Bigg(\underbrace{\left(4\left(\log 2-\frac{3}{2}\right)-20+2+6\right)}_{=4\log 2-18} p_5 + 24p_7 + q_1 \\ &\qquad + (-2-6)q_3 + 8q_5 - a_1 + 21a_3 - 30a_5\Bigg) \\ &+ 2r_3 - 2r_5 + (2-6)s_3 + 10s_5.\end{aligned}$$

Using Lemma 2.5 and Lemma 2.19 we have

$$\begin{aligned}\gamma_1 = \sqrt{2}\Bigg(&\left(-3\log 2-\frac{15}{2}\right)p_5 + \left(3\log 2+\frac{11}{2}\right)p_7 + q_3 - 6q_5 + 6q_7 - a_3 \\ &\qquad + 18a_5 - 30a_7\Bigg) \\ &+ \sqrt{2}\left(\left(3\log 2+\frac{15}{2}\right)(-3p_3+4p_5) + \left(-3\log 2-\frac{11}{2}\right)(-5p_5+6p_7)\right) \\ &+ \sqrt{2}\Big(-(-p_1+p_3-q_1+2q_3) + 6(-p_3+p_5-3q_3+4q_5) \\ &\qquad - 6(-p_5+p_7-5q_5+6q_7)\Big) \\ &+ \sqrt{2}\left((p_1-a_1) - 19(p_3-3a_3) + 48(p_5-5a_5) - 30(p_7-7a_7)\right) \\ &- 6r_3 + 12r_5 - 6r_7 + 6s_1 - 12s_3 + 6s_5 + 4(3r_3-4r_5+s_3) - 6(5r_5-6r_7+s_5) \\ &- 4(-r_1+s_1) + 10(-r_3+3s_3) - 6(-r_5+5s_5).\end{aligned}$$

Collecting similar terms we have the following, which yields (9),

$$\begin{aligned}\gamma_1 = &\sqrt{2}\Bigg(\underbrace{(1+1)}_{=2}p_1 + \underbrace{\left(-3\left(3\log 2+\frac{15}{2}\right)-1-6-19\right)}_{=-9\log 2-\frac{97}{2}} p_3\Bigg) \\ &+ \sqrt{2}\Bigg(\underbrace{\left(\left(-3\log 2-\tfrac{15}{2}\right)+4\left(3\log 2+\tfrac{15}{2}\right)-5\left(-3\log 2-\tfrac{11}{2}\right)+6+6+48\right)}_{=24\log 2+110} p_5\Bigg) \\ &+ \sqrt{2}\Bigg(\underbrace{\left(\left(3\log 2+\frac{11}{2}\right)+6\left(-3\log 2-\frac{11}{2}\right)-6-30\right)}_{=-15\log 2-\frac{127}{2}} p_7\Bigg) \\ &+ \sqrt{2}\left(q_1 + (1-2-18)\,q_3 + (-6+24+30)\,q_5 + (6-36)\,q_7\right) \\ &+ \sqrt{2}\left(-a_1 + (-1+57)a_3 + (18-240)a_5 + (-30+210)a_7\right) \\ &+ 4r_1 + (-6+12-10)r_3 + (12-16-30+6)r_5 + (36-6)r_7 \\ &+ (6-4)s_1 + (-12+4+30)s_3 + (\cancel{6-6}-30)s_5.\end{aligned}$$

□

Proof of Proposition 2.4. Summing up the formulas in Lemma 2.20, we obtain

$$\begin{aligned}
\Gamma &= \gamma_1+\gamma_2+\gamma_3+\gamma_4\\
&= \sqrt{2}\left(2p_1+\left(-9\log 2-\frac{97}{2}\right)p_3+(24\log 2+110)\,p_5+\left(-15\log 2-\frac{127}{2}\right)p_7\right)\\
&\quad+\sqrt{2}\,(q_1-19q_3+48q_5-30q_7-a_1+56a_3-222a_5+180a_7)\\
&\quad+4r_1-4r_3-28r_5+30r_7+2s_1+22s_3-30s_5\\
&\quad+\sqrt{2}\Bigg(\left(\frac{1}{2}\log 2+\frac{9}{4}\right)p_1+(-4\log 2-6)\,p_3+(4\log 2-18)\,p_5\\
&\qquad\qquad+24p_7+q_1-8q_3+8q_5\Bigg)\\
&\quad+\sqrt{2}\,(-a_1+21a_3-30a_5)+2r_3-2r_5-4s_3+10s_5\\
&\quad+\sqrt{2}\left(-\frac{33}{2}p_3+\frac{127}{2}p_5-47p_7+18a_3-84a_5+72a_7\right)\\
&\quad+\sqrt{2}\,(q_1-q_3+a_1-2a_3)
\end{aligned}$$

where collecting together similar terms we have the following which yields Proposition 2.4

$$\begin{aligned}
\Gamma =&\sqrt{2}\Bigg(\underbrace{\left(\frac{1}{2}\log 2+2+\frac{9}{4}\right)}_{=\frac{1}{2}\log 2+\frac{17}{4}}p_1+\underbrace{\left((-9-4)\log 2-\frac{97}{2}-6-\frac{33}{2}\right)}_{=-13\log 2-71}p_3\Bigg)\\
&+\sqrt{2}\Bigg(\underbrace{\left((24+4)\log 2+110-18+\frac{127}{2}\right)}_{=28\log 2+\frac{311}{2}}p_5\\
&\qquad+\underbrace{\left(-15\log 2-\frac{127}{2}+24-47\right)}_{=-15\log 2-\frac{173}{2}}p_7\Bigg)\\
&+\sqrt{2}\Bigg(\underbrace{(1+1+1)}_{=3}q_1+\underbrace{(-19-8-1)}_{=-28}q_3+\underbrace{(48+8)}_{=56}q_5-30q_7\Bigg)\\
&+\sqrt{2}\Bigg(\underbrace{(-1-1+1)}_{=-1}a_1+\underbrace{(56+21+18-2)}_{=93}a_3+\underbrace{(-222-30-84)}_{=-336}a_5\\
&\qquad+\underbrace{(180+72)}_{252}a_7\Bigg)\\
&+4r_1+\underbrace{(-4+2)}_{=-2}r_3+\underbrace{(-28-2)}_{=-30}r_5+30r_7+2s_1+\underbrace{(22-4)}_{=18}s_3+\underbrace{(-30+10)}_{=-20}s_5.
\end{aligned}$$

□

3. Reductions and cancellations

Now we want to perform further reductions and express Γ in terms of p_1, q_1, a_1, r_1 and s_1.

LEMMA 3.1. *We have the formulas*

$$p_3 = p_1, \qquad p_5 = \frac{5}{6}p_1, \qquad p_7 = \frac{13}{18}p_1.$$

Proof. We have

$$\begin{aligned} p_k &= \int \operatorname{sech}^k (\sin)' = k\int \operatorname{sech}^k \tan(-\cos)' = k\int (\operatorname{sech}^k \tan)' \cos \\ &= -k^2 \int \operatorname{sech}^k (1-\operatorname{sech}^2) \cos + k\int \operatorname{sech}^{k+2} \cos \\ &= -k^2 p_k + (k^2+k) p_{k+2}. \end{aligned}$$

Thus,

$$p_{k+2} = \frac{1+k^2}{k(k+1)} p_k$$

and

$$\begin{aligned} p_3 &= \frac{1+1}{1\cdot(1+1)} p_1 = p_1, \\ p_5 &= \frac{1+3^2}{3\cdot 4} p_3 = \frac{10}{12} p_1 = \frac{5}{6} p_1, \\ p_7 &= \frac{1+5^2}{5\cdot 6} p_5 = \frac{26}{30}\frac{5}{6} p_1 = \frac{13}{18} p_1. \end{aligned}$$ □

LEMMA 3.2. *We have the formulas*

$$\begin{aligned} q_3 &= q_1 - \frac{1}{2} p_1, \\ q_5 &= \frac{5}{6} q_1 - \frac{29}{72} p_1, \\ q_7 &= \frac{13}{18} q_1 - \frac{121}{360} p_1. \end{aligned}$$

Proof. Using $(\log \operatorname{sech})' = \frac{1}{\operatorname{sech}}(-\operatorname{sech}\tanh) = -\tanh$, we have

$$\begin{aligned} q_k &= \int \operatorname{sech}^k \log\circ\operatorname{sech}(\sin)' = k\int \operatorname{sech}^k \log\circ\operatorname{sech}\tanh\sin + \int \operatorname{sech}^k \tanh\sin \\ &= kc_k + b_k \\ &= k\left((k+1)q_{k+2} - kq_k + p_{k+2} - p_k\right) + (k+1)p_{k+2} - kp_k \\ &= k(k+1)q_{k+2} - k^2 q_k + (2k+1)p_{k+2} - 2kp_k. \end{aligned}$$

Thus,

$$q_{k+2} = \frac{1}{k(k+1)}\big((1+k^2)q_k - (2k+1)p_{k+2} + 2kp_k\big).$$

In particular,

$$\begin{aligned}
q_3 &= \frac{1}{2}\left((1+1)q_1 - (2+1)p_3 + 2p_1\right) = q_1 - \frac{1}{2}p_1,\\
q_5 &= \frac{1}{3\cdot 4}\left((1+9)q_3 - (2\cdot 3+1)p_5 + 2\cdot 3p_3\right) = \frac{5}{6}q_3 - \frac{7}{12}p_5 + \frac{1}{2}p_3\\
&= \frac{5}{6}\left(q_1 - \frac{1}{2}p_1\right) - \frac{7}{12}\frac{5}{6}p_1 + \frac{1}{2}p_1 = \frac{5}{6}q_1 - \frac{29}{72}p_1,\\
q_7 &= \frac{1}{5\cdot 6}\big((1+5^2)q_5 - (10+1)p_7 + 2\cdot 5p_5\big)\\
&= \frac{13}{15}q_5 - \frac{11}{30}p_7 + \frac{1}{3}p_5 = \frac{13}{15}\left(\frac{5}{6}q_1 - \frac{29}{72}p_1\right) - \frac{11}{30}\frac{13}{18}p_1 + \frac{1}{3}\frac{5}{6}p_1\\
&= \frac{13}{18}q_1 - \frac{121}{360}p_1.
\end{aligned}$$

□

LEMMA 3.3. *We have the formulas*

$$\begin{aligned}
r_3 &= -r_1 + s_1 + \sqrt{2}p_1, & r_5 &= -r_1 + \frac{2}{3}s_1 + \frac{37}{36}\sqrt{2}p_1,\\
s_3 &= \frac{1}{3}s_1 - r_1 + \frac{7}{9}\sqrt{2}p_1, & s_5 &= -\frac{2}{5}r_1 + \frac{1}{15}s_1 + \frac{13}{36}\sqrt{2}p_1,\\
& & r_7 &= -\frac{13}{15}r_1 + \frac{23}{45}s_1 + \frac{83\sqrt{2}}{90}p_1.
\end{aligned}$$

Proof. By $T'' = 2T - 2\sqrt{2}\mathrm{sech}^2$ and $e_k = kr_k - (k+1)r_{k+2} + s_k$.

$$\begin{aligned}
r_k &= \int \mathrm{sech}^k T(\sin)' = -\int (\mathrm{sech}^k T)'\sin\\
&= k\int \mathrm{sech}^k \tanh T\sin - \int \mathrm{sech}^k T'\sin\\
&= ks_k - \int \mathrm{sech}^k T'(-\cos)' = ks_k - \int (\mathrm{sech}^k T')'\cos\\
&= ks_k + k\int \mathrm{sech}^k \tanh T'\cos - \int \mathrm{sech}^k(2T - 2\sqrt{2}\mathrm{sech}^2)\cos\\
&= ks_k + ke_k - 2r_k + 2\sqrt{2}p_{k+2}\\
&= ks_k + k(kr_k - (k+1)r_{k+2} + s_k) - 2r_k + 2\sqrt{2}p_{k+2}\\
&= -k(k+1)r_{k+2} + (k^2-2)r_k + 2ks_k + 2\sqrt{2}p_{k+2}.
\end{aligned}$$

Thus,

$$r_{k+2} = \frac{1}{k(k+1)}\left((k^2-3)r_k + 2ks_k + 2\sqrt{2}p_{k+2}\right).$$

We have

$$\begin{aligned}
s_k &= \int \operatorname{sech}^k T \tanh(-\cos)' \\
&= \int (\operatorname{sech}^k T \tanh)' \cos \\
&= -k\int \operatorname{sech}^k(1-\operatorname{sech}^2)T\cos + \int \operatorname{sech}^{k+2} T\cos + \int \operatorname{sech}^k T' \tanh\cos \\
&= -kr_k + (k+1)r_{k+2} - \int (\operatorname{sech}^k T'\tanh)'\sin \\
&= -kr_k + (k+1)r_{k+2} + k\int \operatorname{sech}^k(1-\operatorname{sech}^2)T'\sin \\
&\quad - \int \operatorname{sech}^{k+2}T'\sin - \int \operatorname{sech}^k(2T - 2\sqrt{2}\operatorname{sech}^2)\tanh\sin \\
&= -kr_k + (k+1)r_{k+2} + k(f_k - f_{k+2}) - f_{k+2} - 2s_k + 2\sqrt{2}b_{k+2} \\
&= -kr_k + (k+1)r_{k+2} + k(-r_k + ks_k) - (k+1)(-r_{k+2} + (k+2)s_{k+2}) \\
&\quad - 2s_k + 2\sqrt{2}((k+3)p_{k+4} - (k+2)p_{k+2}).
\end{aligned}$$

Thus,

$$\begin{aligned}
s_{k+2} = \frac{1}{(k+1)(k+2)}\Bigg(&(k^2-3)s_k + 2(k+1)r_{k+2} - 2kr_k \\
&+ 2\sqrt{2}(k+3)p_{k+4} - 2\sqrt{2}(k+2)p_{k+2}\Bigg).
\end{aligned}$$

Hence

$$\begin{aligned}
r_3 &= \frac{1}{2}\left((1-3)r_1 + 2s_1 + 2\sqrt{2}p_3\right) = -r_1 + s_1 + \sqrt{2}p_1, \\
s_3 &= \frac{1}{2\cdot 3}\left((1-3)s_1 + 2\cdot 2r_3 - 2r_1 + 2\sqrt{2}(1+3)p_5 - 2\sqrt{2}(1+2)p_3\right) \\
&= -\frac{1}{3}s_1 + \frac{2}{3}(-r_1 + s_1 + \sqrt{2}p_1) - \frac{1}{3}r_1 + \frac{4\sqrt{2}}{3}\cdot\frac{5}{6}p_1 - \sqrt{2}p_1 \\
&= \frac{1}{3}s_1 - r_1 + \frac{7}{9}\sqrt{2}p_1,
\end{aligned}$$

$$\begin{aligned}
r_5 &= \frac{1}{12}\left((9-3)r_3 + 6s_3\right) \\
&= \frac{1}{3\cdot 4}\left((9-3)r_3 + 2\cdot 3s_3 + 2\sqrt{2}p_5\right) = \frac{1}{12}\left(6r_3 + 6s_3 + 2\sqrt{2}p_5\right) \\
&= \frac{1}{2}\left(-r_1 + s_1 + \sqrt{2}p_1\right) + \frac{1}{2}\left(\frac{1}{3}s_1 - r_1 + \frac{7}{9}\sqrt{2}p_1\right) + \frac{\sqrt{2}}{6}\cdot\frac{5}{6}p_1 \\
&= -r_1 + \frac{2}{3}s_1 + \frac{37}{36}\sqrt{2}p_1,
\end{aligned}$$

$$\begin{aligned}
s_5 &= \frac{1}{4\cdot 5}\left(6s_3 + 8r_5 - 6r_3 + 12\sqrt{2}p_7 - 10\sqrt{2}p_5\right) \\
&= \frac{3}{10}\left(\frac{1}{3}s_1 - r_1 + \frac{7}{9}\sqrt{2}p_1\right) + \frac{2}{5}\left(-r_1 + \frac{2}{3}s_1 + \frac{37}{36}\sqrt{2}p_1\right) \\
&\quad - \frac{3}{10}\left(-r_1 + s_1 + \sqrt{2}p_1\right) + \frac{3}{5}\sqrt{2}\frac{13}{18}p_1 - \frac{1}{2}\sqrt{2}\frac{5}{6}p_1 \\
&= -\frac{2}{5}r_1 + \frac{1}{15}s_1 + \frac{13}{36}\sqrt{2}p_1
\end{aligned}$$

and

$$\begin{aligned}
r_7 &= \frac{1}{5\cdot 6}\left((25-3)r_5 + 10s_5 + 2\sqrt{2}p_7\right) \\
&= \frac{11}{15}\left(-r_1 + \frac{2}{3}s_1 + \frac{37}{36}\sqrt{2}p_1\right) + \frac{1}{3}\left(-\frac{2}{5}r_1 + \frac{1}{15}s_1 + \frac{13}{36}\sqrt{2}p_1\right) + \frac{\sqrt{2}}{15}\frac{13}{18}p_1 \\
&= -\frac{13}{15}r_1 + \frac{23}{45}s_1 + \frac{83\sqrt{2}}{90}p_1.
\end{aligned}$$

□

Proof of Theorem 1.3. We substitute in the formula of Γ in Proposition 2.4 the formulas in Lemmas 3.1, 3.2 and 3.3. We obtain

$$\begin{aligned}
\Gamma = \sqrt{2}\Bigg(&\left(\frac{1}{2}\log 2 + \frac{17}{4}\right)p_1 - (13\log 2 + 71)\,p_1 + \left(28\log 2 + \frac{311}{2}\right)\frac{5}{6}p_1 \\
&\qquad - \left(15\log 2 + \frac{173}{2}\right)\frac{13}{18}p_1\Bigg) \\
&+ \sqrt{2}\left(3q_1 - 28\left(q_1 - \frac{1}{2}p_1\right) + 56\left(\frac{5}{6}q_1 - \frac{29}{72}p_1\right) - 30\left(\frac{13}{18}q_1 - \frac{121}{360}p_1\right)\right) \\
&+ \sqrt{2}\left(-a_1 + 93\left(\frac{1}{3}a_1 + \frac{1}{3}p_1\right) - 336\left(\frac{1}{6}a_1 + \frac{1}{5}p_1\right) + 252\left(\frac{13}{126}a_1 + \frac{83}{630}p_1\right)\right) \\
&+ 4r_1 - 2\left(-r_1 + s_1 + \sqrt{2}p_1\right) - 30\left(-r_1 + \frac{2}{3}s_1 + \frac{37}{36}\sqrt{2}p_1\right) \\
&+ 30\left(-\frac{13}{15}r_1 + \frac{23}{45}s_1 + \frac{83\sqrt{2}}{90}p_1\right) + 2s_1 + 18\left(\frac{1}{3}s_1 - r_1 + \frac{7\sqrt{2}}{9}p_1\right) \\
&- 20\left(-\frac{2}{5}r_1 + \frac{1}{15}s_1 + \frac{13\sqrt{2}}{36}p_1\right).
\end{aligned}$$

Collecting together similar terms and cancelling the null ones, we obtain

$$\begin{aligned}\Gamma &= \sqrt{2}\alpha p_1 + \sqrt{2}\cancel{\left(3-28+56\frac{5}{6}-30\frac{13}{18}\right)}q_1 + \cancel{\left(4+2+\frac{63}{2}-\frac{30\cdot 13}{15}-18+8\right)}r_1 \\ &+ \cancel{\left(-2-\frac{63\cdot 2}{2\cdot 3}+30\cdot\frac{23}{45}+2+\frac{18}{3}-20\cdot\frac{1}{15}\right)}s_1 \\ &+ \sqrt{2}\cancel{\left(-1+\frac{93}{3}-\frac{336}{6}+\frac{252\cdot 13}{126}\right)}a_1\end{aligned}$$

where

$$\begin{aligned}\alpha = \log 2 &\cancel{\left(\frac{1}{2}-13+28\frac{5}{6}-15\frac{13}{18}\right)} \\ &+\frac{17}{4}-71+\frac{311}{2}\frac{5}{6}-\frac{173\cdot 13}{2\cdot 18}+14-56\cdot\frac{29}{72}+30\frac{121}{360} \\ &+\frac{93}{3}-\frac{336}{5}+252\frac{83}{630}-2-30\frac{37}{36}+30\frac{83}{90}+18\frac{7}{9}-20\frac{13}{36}=\frac{1}{2}\end{aligned}$$

Hence $\Gamma = \frac{1}{\sqrt{2}}p_1$. Finally, $p_1 = \pi \mathrm{sech}(\pi/2)$ by an application of the Residue Theorem. $\square$

We remark that we do not have a conceptual justification of why the computations give such a simple formula for Γ which we would not expect from the outset.

Acknowledgments

C. was supported by the Prin 2020 project *Hamiltonian and Dispersive PDEs* N. 2020XB3EFL. M. was supported by the JSPS KAKENHI Grant Number 19K03579, G19KK0066A, 23H01079 and 24K06792.

References

[1] V. Buslaev and G. Perelman, *Scattering for the nonlinear Schrödinger equation: states close to a soliton*, St. Petersburg Math. J. **4** (1993), 1111–1142.

[2] V. Buslaev and G. Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, editor N.N. Uraltseva, Transl. Ser. 2, 164, Amer. Math. Soc., 75–98, Amer. Math. Soc., Providence (1995).

[3] V. Buslaev and C. Sulem, *On asymptotic stability of solitary waves for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **20** (2003), 419–475.

[4] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[5] S. M. Chang, S. Gustafson, K. Nakanishi, and T. P. Tsai, *Spectra of linearized operators for NLS solitary waves*, SIAM J. Math. Anal. **39** (2007/08), 1070–1111.

[6] M. Coles and S. Gustafson, *A degenerate edge bifurcation in the 1D linearized nonlinear Schrödinger equation*, Discrete Contin. Dyn. Syst. **36** (2016), 2991–3009.

[7] S. Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, Comm. Math. Phys. **305** (2011), 279–331.

[8] S. Cuccagna and M. Maeda, *The asymptotic stability on the line of ground states of the pure power NLS with* $0 < |p-3| \ll 1$, arXiv:2404.14287.

[9] S. Cuccagna and M. Maeda, *On the asymptotic stability of ground states of the pure power NLS on the line at 3rd and 4th order Fermi Golden Rule*, arXiv:2405.01922.

[10] S. Cuccagna and M. Maeda, *On the asymptotic stability on the line of ground states of the pure power NLS with* $0 \le 2-p \ll 1$, arXiv:2410.01546 .

[11] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251.

[12] J.-M.Delort and N.Masmoudi, *Long time Dispersive Estimates for perturbations of a kink solution of one dimensional wave equations*, Mem. Eur. Math. Soc., Vol. 1, EMS Press, Berlin, 2022.

[13] D. J. Kaup, *Perturbation theory for solitons in optical fibers*, Phys. Rev. A **42** (1990), 5689–5694.

[14] M. Kowalczyk, Y. Martel, and C. Muñoz, *Kink dynamics in the* ϕ^4 *model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. **30** (2017), 769–798.

[15] J. Krieger and W. Schlag, *Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension*, J. Amer. Math. Soc. **19** (2006), 815–920.

[16] Y. Li and J. Lührmann, *Soliton dynamics for the 1D quadratic Klein-Gordon equation with symmetry,* J. Differential Equations **344** (2023), 172–202.

[17] J. Lührmann and W.Schlag, *On codimension one stability of the soliton for the 1D focusing cubic Klein–Gordon equation*, Commun. Amer. Math. Soc. **4** (2024), 230–356.

[18] Y. Martel, *Asymptotic stability of solitary waves for the 1D cubic-quintic Schrödinger equation with no internal mode*, Prob. Math. Phys. **3** (2022), 839–867.

[19] Y. Martel, *Asymptotic stability of small standing solitary waves of the one-dimensional cubic-quintic Schrödinger equation*, arXiv:2312.11016.

[20] G. Rialland, *Asymptotic stability of solitons for near–cubic NLS equation with an internal mode*, arXiv:2404.13980.

[21] J. Shatah, *Stable standing waves of nonlinear Klein–Gordon equations*, Comm. Math. Phys. **91** (1983), 313–327.

[22] I. M. Sigal, *Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions*, Comm. Math. Phys. **153** (1993), 297–320.

[23] A. Soffer and M. I. Weinstein, *Resonances, radiation damping and instabil-*

ity in Hamiltonian nonlinear wave equations, Invent. Math. **136** (1999), 9–74.
[24] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive equations*, Comm. Pure Appl. Math. **39** (1986), 51–68.

Authors' addresses:

Scipio Cuccagna
Dipartimento di Matematica, Informatica e Geoscienze
Università degli Studi di Trieste
via Valerio 12/1
Trieste, 34127 Italy
E-mail: scuccagna@units.it

Masaya Maeda
Department of Mathematics and Informatics
Graduate School of Science
Chiba University
Chiba 263-8522, Japan
E-mail: maeda@math.s.chiba-u.ac.jp

Received May 15, 2024
Accepted January 22, 2025