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Local Vs Nonlocal De Giorgi Classes:
A brief guide in the homogeneous case.

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We devote this paper to celebrate Enzo's 70th birthday. A dear friend who loves beauty in all its aspects: beautiful sport cars (his legendary Saabs), beautiful music, fine wines and, above all, beautiful Mathematics such as his splendid results with Pohozaev.

Abstract. *We give a brief and concise guide for the analysis of the local behavior of the elements of local and nonlocal homogeneous De Giorgi classes: local boundedness, local Hölder continuity and Harnack-type inequalities. In the local case, we promote a simplified itinerary in the classic theory, propaedeutic for the successive part; while in the nonlocal case, we gather recent new developments into an unitary and concise framework. Employing a suitable definition of De Giorgi classes, we show a new proof of the Harnack inequality, way easier than in the local case, that bypasses any sort of Krylov-Safonov argument or cube decomposition.*

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1. Introduction

Since the work of De Giorgi [8], that answered positively to the 19th Hilbert's Problem on the regularity of minima of the calculus of variations, those sets of weakly differentiable functions in L^p satisfying the energy inequalities

$$\int_{B_r} |\nabla(u-k)_\pm|^p \, dx \le \left(\frac{\gamma}{R-r}\right)^p \int_{B_R} |(u-k)_\pm|^p \, dx \tag{1}$$

have been called De Giorgi classes, see the pioneering book [25], and have been object of intense study ever since. The most notable advantage of studying the local properties of elements of the De Giorgi classes relies in the fact that these same properties are hence shown for functions satisfying an energy inequality, instead of minimizing a functional or satisfying an equation. In this respect, the method is very flexible (see for instance [11] for a relaxed definition and its links with Moser's method [35]), and can be used to encompass a theory of regularity for solutions to equations and minima of functionals.

Issues of Definition

When different growths are into play, a definition of De Giorgi class, and the subsequent development of a regularity theory encumbers, as soon as the homogeneity in (1) is lost. For instance, when an equation/functional has an unbalanced growth, as in the case of parabolic equations (see for instance [6, 9]), or elliptic non-standard/Orlicz growth functionals (see for instance [24, 31, 32, 34, 37]), the sole energy inequalities parallel to (1) are not sufficient to give a complete regularity theory, since solutions may be unbounded; see [10] for the parabolic case with $1 < p < 2$, and [19, 33] for the elliptic cases. This calls for a new definition of De Giorgi class, or energy class (see [5] for the non-standard case); and this is exactly the case of the nonlocal De Giorgi classes (see [7] for a complete account). Roughly speaking, a crucial ingredient in the theory of regularity for functions satisfying (1) is the assumption $W^{1,p}_{loc}(\Omega)$, that bestows a fundamental tool called Discrete Isoperimetric Inequality (see (2.8)). Now, when considering the set of functions $u \in W^{s,p}_{loc}(\Omega)$ satisfying the natural parallel estimate to (1)

$$[(u-k)_\pm]^p_{W^{s,p}(B_{\tau\rho})} \leq \frac{\gamma}{(1-\tau)^p \rho^{sp}} \|(u-k)_\pm\|^p_{L^p(B_\rho)} + \\ + \frac{\gamma}{(1-\tau)^{N+sp}} \|(u-k)_\pm\|_{L^1(B_\rho)} \int_{\mathbb{R}^N \setminus B_\rho} \frac{|(u-k)_\pm|^{p-1}}{|x|^{N+ps}}\, dx\,, \quad (2)$$

even when all the quantities on the right-hand side are bounded, the aforementioned Discrete Isoperimetric Inequality isn't anymore at stake for every $s \in (0,1)$, $p > 1$, and functions satisfying (2) are only proved to be bounded (and Hölder continuous for s close to 1, see [7]). Nevertheless, local weak solutions and minima of the respective equations/functionals, enjoy stronger energy estimates than the sole (2), see (34). The presence of an additional term on the left-hand side (see Section 3) replaces the use of the aforementioned Discrete Isoperimetric inequality, and allows for a complete theory of regularity. In this work we start from this new definition, formerly given in [7] (see (6.1) at page 4792 with $R_0 = \infty$ there) and we describe our itinerary to study the local behavior of the elements of such a nonlocal De Giorgi class. In particular, we show that with the tools provided in [29] for the weak Harnack inequality, it is possible to have a full Harnack inequality for elements of this generalized class.

What about boundary data?

Roughly speaking, since an inequality as (1) can be given with balls B_r intersecting the boundary of a domain $\Omega \subset \mathbb{R}^N$, then it is possible to define boundary De Giorgi classes. Solutions to Dirichlet/Neumann/Mixed problems with elliptic PDEs of p-Laplacian type and minima of functionals can then be embodied in their global fashion into this new formulation (see for instance [10] chap. X, or [20]), and the regularity of the boundary $\partial\Omega$ plays a pivotal role in the scaling of the estimates, and therefore in the theory of regularity. In this work we refrain from describing global problems, in order to focus on the essence of the method of De Giorgi irrespective of any boundary condition. Regarding the nonlocal case: here the elements of De Giorgi classes need to be elements defined in all $\mathbb{R}^N$, but the class is defined in a way to encompass solutions and minima of "local" formulations of the respective problems; see subsection 3.1 for more details.

Structure and style of the paper

In Section 2 we study the local properties of elements of local De Giorgi classes, starting from the local boundedness, then Hölder continuity and finally the Harnack inequality and its consequences. Along the same track, in Section 3 we carry on an analysis of the local regularity for elements of the nonlocal De Giorgi classes. Conversely to the usual way, in Section 2 we give less preliminary details (being a more classical subject) and we construct the various tools needed inside the proofs themselves; while in Section 3 we stress the details in the computation of the estimates, in order to clarify the novel method.

Notation

- We say that a constant γ depends only on the data if it depends only on $\{N, p, s, \hat{\gamma}\}$, where $\hat{\gamma}$ are given in the definitions of De Giorgi classes. When a constant γ depends on a quantity l which is different from the aforementioned, we will write $\gamma(l)$. Constants may be different from line to line.

- When considering an open bounded set $\Omega \subset \mathbb{R}^N$, we will denote its Lebesgue measure by $|\Omega|$. To say that Ω is an open bounded set, we adopt the notation $\Omega \subset\subset \mathbb{R}^N$. For $a, b \in \mathbb{R}$ we use the short notation $(a-b)^{p-1} := |a-b|^{p-2}(a-b)$.

- For a measurable function u, we define the essential infimum and supremum as $\inf u$ and $\sup u$ respectively, in the set of consideration. Given a function $u : \Omega \to \mathbb{R}$, a number $a \in \mathbb{R}$, we will omit the domain when considering sub or super level sets, denoted by $[u \le a] = \{x \in \Omega : u(x) \le a\}$, and when there is no risk of misunderstanding. Finally, if $u \in W^{1,p}_{loc}(\Omega)$ for $\Omega \subset\subset \mathbb{R}^N$, we denote the partial weak derivatives with $\partial_i u = \partial u/\partial x_i$, and its weak gradient with ∇u.

2. Local De Giorgi Classes

Definition 2.1. *A function $u \in W^{1,p}_{loc}(\Omega)$ is an element of the set $DG^{\pm}_p(\hat{\gamma}, \Omega)$ if there exists a positive constant $\hat{\gamma}$ such that for any level $k \in \mathbb{R}$, the inequality*

$$\int_{B_{\sigma\rho}(x_0)} |\nabla(u-k)_{\pm}|^p dx \le \frac{\hat{\gamma}}{(1-\sigma)^p \rho^p} \int_{B_\rho(x_0)} |(u-k)_{\pm}|^p dx, \tag{3}$$

is satisfied for all balls $B_{\sigma\rho}(x_0) \subset B_\rho(x_0) \subset \Omega$. The De Giorgi class $DG_p(\hat{\gamma}, \Omega)$ is the intersection

$$DG_p(\hat{\gamma}, \Omega) = DG^+_p(\hat{\gamma}, \Omega) \cap DG^-_p(\hat{\gamma}, \Omega).$$

Remark 2.2. If $u \in DG^-_p(\hat{\gamma}, \Omega)$, then $-u \in DG^+_p(\hat{\gamma}, \Omega)$.

Now we prove the set inclusion

$$DG_p(\hat{\gamma}, \Omega) \subseteq L^\infty_{loc}(\Omega)\,,$$

that is to say, elements of the De Giorgi classes are locally bounded "functions". The underlying idea is that the membership $u \in DG_p(\hat{\gamma}, \Omega)$ provides a reverse-Poincaré-inequality. Hence, by chaining this reverse-Poincaré-inequality with the embedding $W^{1,p} \hookrightarrow L^{p^*}$ it is possible to obtain a precise decay on the L^p-norms of the truncations of u. See diagram in Figure 1 for a sketch of the idea.

Theorem 2.3. *Let $u \in DG^{\pm}_p(\hat{\gamma}, \Omega)$ and $\sigma \in (0,1)$. Then, there exists a constant γ, depending only on the data, such that for every pair of balls $B_{\sigma\rho}(x_0) \subset B_\rho(x_0)$ contained in Ω, we have*

$$\sup_{B_{\sigma\rho}(x_0)} u_{\pm} \le \gamma \left(\frac{1}{(1-\sigma)^N} \fint_{B_\rho(x_0)} u^p_{\pm} dx \right)^{\frac{1}{p}}. \tag{4}$$

Proof. Without loss of generality we assume x_0 is the origin, since estimate (4) is invariant under translations. Moreover, once (4) is proven for $u \in DG^+_p(\hat{\gamma}, \Omega)$, then by Remark 2.2 the statement for $u \in DG^-_p(\hat{\gamma}, \Omega)$ is recovered from (4) since $\sup(-u)_+ = \sup u_-$.

Let $u \in DG^+_p(\hat{\gamma}, \Omega)$ and for a number $k \in \mathbb{R}^+$ to be chosen later, we define for $n = 1, 2, \dots$ the sequences of nested concentric balls $\{B_n\}$ and $\{\tilde{B}_n\}$, and increasing levels $\{k_n\}$ such that

$$\begin{aligned} B_n = B_{\rho_n} \quad &\text{where} \quad \rho_n = \sigma\rho + \frac{1-\sigma}{2^{n-1}}\rho \\ \tilde{B}_n = B_{\tilde{\rho}_n} \quad &\text{where} \quad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2} \\ k_n &= k - \frac{1}{2^{n-1}} k\,. \end{aligned}$$

We next introduce the Lipschitz cut-off function

$$\xi_n(x) = \begin{cases} 1 & \text{for } |x| \le \rho_{n+1}, \\ \frac{\tilde{\rho}_n - |x|}{\tilde{\rho}_n - \rho_{n+1}} = \frac{2^{n+1}}{(1-\sigma)\rho}\left(\tilde{\rho}_n - |x|\right) & \text{for } \rho_{n+1} \le |x| \le \tilde{\rho}_n, \\ 0 & \text{for } |x| \ge \tilde{\rho}_n\,. \end{cases} \tag{5}$$

We observe that

$$||\nabla \xi_n||_{\infty, \tilde{B}_n} \le \frac{2^{n+1}}{(1-\sigma)\rho}\,.$$

Observe further that (3) for such balls and levels displays as

$$||\nabla(u-k_{n+1})_+||^p_{p,\tilde{B}_n} \le \frac{2^{(n+1)p}\gamma}{(1-\sigma)^p\rho^p}||(u-k_{n+1})_+||^p_{p,B_n}.$$

Since $[(u-k_n)_+\xi_n] \in W^{1,p}_o(\tilde{B}_n)$ we can extend this function by zero outside $\tilde{B}_n$ and apply the Gagliardo–Nirenberg-Sobolev embedding

$$||v||_{p^*,\mathbb{R}^N} \le \gamma ||Dv||_{p,\mathbb{R}^N}\,, \quad \text{with } p^* = \frac{Np}{N-p}\,, \quad \forall v \in W^{1,p}(\mathbb{R}^N)\,, \tag{6}$$

in the aforementioned ball. We let $A^{\pm}_{k,\rho} = [(u-k)_\pm > 0] \cap B_\rho$ and apply Hölder's inequality to get the chain

$$\begin{aligned} ||(u-k_{n+1})_+||^p_{p,B_{n+1}} &\le ||(u-k_{n+1})_+\xi_n||^p_{p,\tilde{B}_n} \\ &\le ||(u-k_{n+1})_+\xi_n||^{p/p^*}_{p^*,\tilde{B}_n}|A^+_{k_{n+1},\tilde{\rho}_n}|^{p/N} \\ &\le \gamma\,||\nabla[(u-k_{n+1})_+\xi_n]||^p_{p,\tilde{B}_n}|A^+_{k_{n+1},\tilde{\rho}_n}|^{p/N} \\ &\le \gamma\left(\frac{2^{np}}{(1-\sigma)^p\rho^p}||(u-k_{n+1})_+||^p_{p,B_n}\right)|A^+_{k_{n+1},\rho_n}|^{p/N}\,. \end{aligned}$$

Next, on the right-hand side we aim to bound with terms involving again the L^p-norm of u: hence we estimate

$$\begin{aligned} \|(u-k_n)_+\|^p_{p,B_n} &= \int_{B_n}(u-k_n)^p_+dx \\ &\geqslant \int_{B_n\cap[u>k_{n+1}]}(k_n-k_{n+1})^p dx \geqslant \frac{k^p}{2^{np}}|A^+_{k_{n+1},\rho_n}|. \end{aligned}$$

Hence,

$$||(u-k_{n+1})_+||^p_{p,B_{n+1}} \le \gamma\frac{2^{np\left(\frac{N+p}{N}\right)}}{(1-\sigma)^p\rho^p}\frac{1}{k^{\frac{p^2}{N}}}\left\|(u-k_n)_+\right\|^{\left(1+\frac{p}{N}\right)}_{p,B_n}\,. \tag{7}$$

If we set

$$Y_n := \frac{1}{k^p} \fint_{B_n} (u - k_n)_+^p \, dx, \qquad b = 2^{\frac{N+p}{N}}, \quad \text{and} \quad \alpha = p/N \,,$$

from (7) we obtain

$$Y_{n+1} \leq \gamma \frac{b^{pn}}{(1-\sigma)^p} Y_n^{1+\alpha} \,. \tag{8}$$

Now, we show that each time we have a recursive relation as (8) above, the logical implications

$$\left\{ Y_0 \leq \frac{1}{\gamma^{\frac{1}{\alpha}} \, b^{\frac{1}{\alpha^2}}} \text{ and } \gamma b^{\frac{1}{\alpha}} \geqslant 1 \right\} \;\Rightarrow\; Y_n \leq b^{-\frac{n}{\alpha}} Y_0 \;\Rightarrow\; \lim_{n \uparrow \infty} Y_n = 0, \tag{9}$$

hold true. In our case, observe that this will imply (4).

The fact that $\{Y_n\}$ is infinitesimal follows from the first implication in (9), that we deduce by induction. Case $n = 0$ being trivially satisfied, we apply (8) to evaluate

$$Y_{n+1} \leq \gamma b^n Y_n^{1+\alpha} \leq \gamma b^n \left(b^{-\frac{n}{\alpha}} Y_0 \right)^{1+\alpha} \leq \left(\gamma b^{\frac{1}{\alpha}} Y_0^\alpha \right) b^{-\frac{(n+1)}{\alpha}} Y_0 .$$

Last term in parenthesis is smaller than one if $Y_0 \leq \gamma^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}}$.

This concludes the argument. Hence, finally getting back to estimate (8) with $\alpha = p/N$, we will have $Y_\infty = 0$ if

$$Y_0 = \frac{1}{k^p} \fint_{B_\rho} u_+^p \, dx \leq b^{-\frac{N^2}{p}} \gamma^{-\frac{N}{p}} (1-\sigma)^N \,,$$

which is satisfied as soon as we set

$$k = \left(\frac{b^{\frac{N^2}{p}} \gamma^{\frac{N}{p}}}{(1-\sigma)^N} \fint_{B_\rho} u_+^p \, dx \right)^{1/p} .$$

Therefore, $\lim_{n \to \infty} Y_n = 0$ and thus $(u - k)_+ = 0$ in $B_{\sigma\rho}$, as required from (4). The scheme of the proof is exemplified in the following diagram. □

The next Lemma transforms a certain information in measure into a precise bound (almost everywhere). Roughly speaking it asserts that, in a ball B_ρ, if the relative measure of the set where u is greater than a certain level k is sufficiently small, then u is smaller than $k/2$ in half ball. It is usually referred to as Critical Mass Lemma (see [3]) or De-Giorgi type Lemma (see [10]). We begin by fixing a ball $B_{2\rho}(x_0) \subset \Omega$ and numbers

$$\mu^+ = \sup_{B_{2\rho}(x_0)} u, \qquad \mu^- = \inf_{B_{2\rho}(x_0)} u, \qquad \omega = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{B_{2\rho}(x_0)} u \,.$$

The Chain to L^∞ bounds

$\|(u - k_{n+1})_+\|_{p,B_{n+1}}$

$\downarrow$

Embedding

$\downarrow$

Energy Estimates

$\downarrow$

Recurrence relation for $\|(u-k_n)_+\|_{p,B_n}$:

$Y_{n+1} \quad \leq \quad Cb^n Y_n^{1+\alpha}$

$\downarrow$

$\lim_{n\uparrow\infty} \|(u - k_n)_+\|_{p,B_n}^p \;=\; 0$

Figure 1: A general scheme for L^∞ estimates.

LEMMA 2.4 (Critical mass lemma). *Let $u \in DG_p^+(\hat{\gamma},\Omega)$, and $B_{2\rho}(x_0) \subset \Omega$. For every $a \in (0,1)$, there exists $\nu(a) \in (0,1)$ depending only on the data and a, specified in* (14)*, such that if for some number $M \in (0,\omega)$ the measure information*

$$\big|\big[u > \mu^+ - M\big] \cap B_\rho(x_0)\big| \leq \nu(a)\,|B_\rho| \tag{10}$$

is at stake, then

$$u \leq \mu^+ - aM \qquad a.e.\ in\ B_{\frac{\rho}{2}}(x_0). \tag{11}$$

Similarly, if $u \in DG_p^-(\hat{\gamma},\Omega)$ and the measure information

$$\big|\big[u < \mu^- + M\big] \cap B_\rho(x_0)\big| \leq \nu(a)\,|B_\rho| \tag{12}$$

is valid, then

$$u \geq \mu^- + aM \qquad a.e.\ in\ B_{\frac{\rho}{2}}(x_0). \tag{13}$$

Proof. We start by proving (10)-(11), as usual assuming x_0 is the origin. Let us consider the sequence of balls $\{B_n\}$, $\{\tilde{B}_n\}$ and the cut-off function ξ_n with $\sigma = \frac{1}{2}$, as introduced in (5). We define the increasing levels $\{k_n\}$ and the nested sets $\{A_n\}$ along with their relative measures Y_n as follows

$$k_n = \mu^+ - aM - \frac{(1-a)M}{2^n}, \quad A_n = [u > k_n] \cap B_n, \quad Y_n = \frac{|A_n|}{|B_n|}.$$

The membership $u \in DG_p^+(\hat{\gamma}, \Omega)$ implies, since $B_n \subset \Omega, \forall n \in \mathbb{N}$,

$$||\nabla(u-k_n)_+||^p_{p,\tilde{B}_n} \leq \frac{2^{np}\gamma}{\rho^p}||(u-k_n)_+||^p_{p,B_n}.$$

Similarly to the proof of Theorem 2.3 we chain estimates (3)with the embedding (6), but this time we look for a recurrence relation for the super-level sets A_n:

$$\begin{aligned}
\left[\frac{(1-a)M}{2^{n+1}}\right]^p |A_{n+1}| &= (k_{n+1}-k_n)^p|A_{n+1}| \\
&\leq ||(u-k_n)_+\xi_n||^p_{p,\tilde{B}_n} \\
&\leq ||(u-k_n)_+\xi_n||^p_{p^*,\tilde{B}_n}|A_n|^{p/N} \\
&\leq ||\nabla[(u-k_n)_+\xi_n]||^p_{p,\tilde{B}_n}|A_n|^{p/N} \\
&\leq \left(\frac{\gamma 2^{np}}{\rho^p}||(u-k_n)_+\xi_n||^p_{p,B_n}\right)|A_n|^{p/N} \\
&\leq \frac{\gamma 2^{np}}{\rho^p}\left(\frac{M}{2^n}\right)^p|A_n|^{1+p/N}
\end{aligned}$$

The previous estimate provides, in relative measure,

$$\begin{aligned}
Y_{n+1} = \frac{|A_{n+1}|}{|B_{n+1}|} &\leq \frac{\gamma 2^{np}}{\rho^p}\frac{M^p|A_n|^{1+p/N}}{|B_{N+1}|}\frac{2^{(n+1)p}}{[(1-a)M]^p} \\
&\leq \frac{\gamma 2^{np}}{\rho^p(1-a)^p}\frac{|A_n|^{1+p/N}}{|B_n|^{1+p/N}}|B_n|^{p/N} = \frac{\gamma 4^{np}Y_n^{1+p/N}}{(1-a)^p}.
\end{aligned}$$

Hence, as in (8) the limit $\{Y_n\} \to 0$ is valid as

$$Y_0 = \frac{|[u > \mu^+ - M] \cap B_\rho|}{|B_\rho|} \leq \frac{(1-a)^{N/p}}{\gamma^{N/p}4^{p\left(\frac{N}{p}\right)^2}} =: \nu \in (0,1) \tag{14}$$

thanks to the assumption (10). This shows (11). Finally, in order to show (13) we denote by $\mu^-(-u) = \inf(-u)$ and we observe that

$$|[u > \mu^+ - M]| = |[-u < -\mu^+ + M]| = |[-u < \mu^-(-u) + M]|.$$

Now, we consider Remark 2.2 and the first part of the Lemma ((10)⇒(11)) to obtain

$$-u \leq \mu^+(-u) - aM \qquad \text{a.e. in} \quad B_{\rho/2}(y),$$

which in turn implies (13). $\square$

Lemma 2.4 furnishes an information almost everywhere. Here below we show, using the method of [28], that we can select a representative of $u \in DG_p(\hat{\gamma}, \Omega)$ that is lower semi-continuous.

DEFINITION 2.5. *For $\Omega \subset \mathbb{R}^N$ open, let $u : \Omega \to \mathbb{R}^N$ be measurable and essentially bounded below. The lower semi-continuous regularization of u is*

$$u_*(x) = \text{ess}\ \liminf_{y \to x} u(y) = \lim_{r \downarrow 0} \inf_{y \in B_r(x_0)} u$$

where $x \in \Omega$.

As usual, for $u \in L^1_{loc}(\Omega)$ we can denote the set of Lebesgue points of u by

$$\mathcal{L} := \left\{ x \in \Omega : |u(x)| < \infty,\ \lim_{r \downarrow 0} \fint_{B_r(x)} |u(x) - u(y)| dy = 0 \right\},$$

and apply the Lebesgue Differentiation Theorem (see [17]) to state that $|\mathcal{L}| = |\Omega|$.

THEOREM 2.6. *Let $u \in DG_p^-(\hat{\gamma}, \Omega)$. Then $u(x) = u_*(x)$ for almost every $x \in \Omega$. In particular, u_* is a lower semi-continuous representative of u.*

Proof. It is simple to see that for each point $x \in \mathcal{L}$ the inequality $u_*(x) \leq u(x)$ is valid, as

$$u_*(x) = \lim_{r \downarrow 0} \operatorname*{ess\,inf}_{B_r(x_0)} u \leq \lim_{r \downarrow 0} \fint_{B_r(x_0)} u(y) dy = u(x).$$

Now, to show the reversed inequality, let us pick $x_0 \in \mathcal{L}$, let us define the value

$$u(x_0) = \lim_{r \downarrow 0} \fint_{B_r(x_o)} u(x)\, dx,$$

and let suppose by contradiction that $u_*(x_0) < u(x_0)$. Fix $R > 0$ such that $B_r(x_0) \subset \Omega$ and let μ^- and M be two numbers satisfying

$$\operatorname*{ess\,inf}_{B_r(x_0)} u := \mu^- \leq u_*(x_0) < \mu^- + M < u(x_0).$$

Next, referring to Lemma 2.4, we choose $a \in (0, 1)$ such that

$$\mu^- + aM > u_*(x_0).$$

Since a is fixed, Lemma 2.4 determines the number ν depending only on a, M, μ^- and other geometric data such as N, but independently of ρ. We claim that there exists a radius $\rho \in (0, r)$ such that

$$0 < \left| \left[u \leq \mu^- + M \right] \cap B_\rho(x_0) \right| \leq \nu \, |B_\rho|,$$

because otherwise for all $0<\rho<r$ we have

$$\begin{aligned}\int_{B_\rho(x_0)} |u(x_0)-u(y)|dy &\geqslant \int_{[u<\mu^-+M]\cap B_\rho(x_0)} u(x_0)-(\mu^-+M)dy\\ &\geqslant \nu|[u(x_0)-(\mu^-+M)]||B_\rho|>0,\end{aligned}$$

contradicting the membership $x_0\in\mathcal{L}$. Finally, our choice of a and (12) results in the point-wise bound $u\geqslant \mu^-+aM>u_*(x_0)$ for almost every $x\in B_{c\rho(x_0)}$. This implies

$$u_*(x_0)<\lim_{r\downarrow 0}\operatorname*{ess\,inf}_{B_r(x_0)} u = u_*(x_0). \qquad \square$$

REMARK 2.7. Similarly, if $u\in DG_p^+(\hat\gamma,\Omega)$ then there exists an upper semi-continuous representative of u. It is enough to observe that Lemma 2.4 implies a property (10)-(11), and by defining the upper semi-continuous regularization of $u\in DG_p(\hat\gamma,\Omega)$, it is possible to run the same machinery of Theorem 2.6. In general, for a $u\in DG_p(\hat\gamma,\Omega)$, it is not given for granted that the lower semi-continuous representative of u coincides with the upper semi-continuous representative of u. This will be the aim of the next Theorem. We will show indeed much more: elements of $DG_p(\hat\gamma,\Omega)$ have an Hölder continuous representative. We denote this property by the arrow

$$DG_p(\hat\gamma,\Omega)\hookrightarrow C^{0;\alpha}_{loc}(\Omega)\,.$$

Our first main tool to show the announced inclusion is a crucial inequality, that is called in literature the De Giorgi Discrete Isoperimetric Inequality (see [10] chap X for a simple proof, and [8] for the original). Differently from [20] and [10], here we use the approach of [25]- that is-, to derive this inequality from the Poincaré inequality

$$\int_{B_\rho}|v-(v)_{B_\rho}|\,dx\le\gamma\rho\int_{B_\rho}|\nabla v|\,dx,\quad \forall v\in W^{1,1}(B_\rho),\ \text{with}\,\gamma=\gamma(N)>0\,. \tag{15}$$

This approach is more flexible than the one in [10], in those metric contexts where one postulates the validity of a Poincaré inequality as (15); see for instance [1] for the general theory and [2] for an application in the context of mixed boundary conditions.

LEMMA 2.8 (Discrete Isoperimetric Inequality). *Let $\Omega\subset\mathbb{R}^N$ be an open set, and $B_\rho\subset\Omega$. Let $u\in W^{1,1}_{loc}(\Omega)$ and $0\le k<h$ two real numbers. Then, there exists a constant $\gamma>0$ depending only on N, such that*

$$(h-k)\,|A_{h,\rho}|\le\frac{\gamma\rho^{N+1}}{|A_{k,\rho}|}\int_{A_{k,\rho}/A_{h,\rho}}|\nabla u|\,dx,$$

where $A_{h,\rho}=B_\rho\cap[u(x)>h]$.

Proof. We consider the truncation

$$w = \begin{cases} 0 & u < k, \\ u-k & k < u < h, \\ h-k & u > h, \end{cases} \qquad\qquad w \in W^{1,1}(B_\rho)\,.$$

We observe that

$$(w)_{B_\rho} = \frac{1}{|B_\rho|}\int_{B_\rho} w\,dx \le \frac{(h-k)|A_{k,\rho}|}{|B_\rho|},$$

so that we can estimate from below

$$\begin{aligned}\int_{B_\rho} |w-(w)_{B_\rho}|\,dx &\geqslant \int_{A_{h,\rho}} |w-(w)_{B_\rho}|\,dx \\ &= \int_{A_{h,\rho}} |(h-k)-(w)_{B_\rho}|\,dx. \\ &\geqslant \int_{A_{h,\rho}} \left|(h-k) - \frac{(h-k)|A_{k,\rho}|}{|B_\rho|}\right| dx, \\ &= (h-k)\left[1-\frac{|A_{k,\rho}|}{|B_\rho|}\right]|A_{h,\rho}|.\end{aligned}$$

Applying Poincaré's inequality (15) to w, we obtain

$$\gamma\rho\int_{A_{k,\rho}/A_{h,\rho}} |\nabla w|\,dx \geqslant \gamma\rho\int_{B_\rho}|\nabla w|\,dx \ge (h-k)|A_{h,\rho}|\left[\frac{|B_\rho|-|A_{k,\rho}|}{|B_\rho|}\right],$$

and since $\nabla w = \nabla u$ in the set $[k < u < l]$ we obtain the desired result

$$(h-k)|A_{h,\rho}| \le \gamma\rho\left[\frac{|B_\rho|}{|B_\rho|-|A_{k,\rho}|}\right]\int_{A_{k,\rho}/A_{h,\rho}} |\nabla u|\,dx. \qquad \square$$

The aforementioned discrete isoperimetric inequality is an essential tool, for functions in $DG_p(\hat\gamma,\Omega)$, to prove the Growth Lemma (see [3], [26]) or Shrinking Lemma (see [10]). This Lemma roughly states that if the relative measure of the set where u is smaller than a level k is greater than some given constant $\theta \in (0,1)$, then, by shrinking k to ϵk, the relative measure where u is greater than ϵk can be reduced as much as we wish.

Lemma 2.9 (Shrinking lemma). *Let $u \in DG_p^+(\hat\gamma,\Omega)$, $B_{2\rho}(x_0) \subset \Omega$ and assume that, for some $\theta \in (0,1)$*

$$|[u \le \mu^+ - \frac{\omega}{2}] \cap B_\rho(x_0)| \geqslant \theta |B_\rho|, \tag{16}$$

Then, for all $\nu \in (0,1)$, there exists $\epsilon \in (0,1)$ that can be determined a priori in terms of the data and θ, independently of ω and ρ, such that

$$\left|\left[u > \mu^+ - \epsilon\omega\right] \cap B_\rho(x_0)\right| < \nu \left|B_\rho\right|. \tag{17}$$

Similarly, if $u \in DG_p^-(\hat{\gamma}, \Omega)$ and

$$|[u \geqslant \mu^- + \frac{\omega}{2}] \cap B_\rho(x_0)| \geqslant \tilde{\theta}|B_\rho|, \tag{18}$$

for some $\tilde{\theta} \in (0,1)$, then for all $\tilde{\nu} > 0$ there exists $\tilde{\epsilon} \in (0,1)$ such that

$$\left|\left[u < \mu^- + \tilde{\epsilon}\omega\right] \cap B_\rho(x_0)\right| \leq \tilde{\nu} \left|B_\rho\right|. \tag{19}$$

Proof. As before, we assume x_0 is the origin and we address first the case (16)-(17). Set

$$k_s = \mu^+ - \frac{\omega}{2^s}, \quad A_s = [u > k_s] \cap B_\rho \quad \text{for } s \in \{1, \dots, s^*\}$$

with s^* to be chosen later. We start by applying the Discrete Isoperimetric Inequalities 2.8 for the levels $k = k_s < l = k_{s+1}$, using (16) we have for every $s > 1$

$$|[u \leq k_s] \cap B_\rho| \geqslant |[u \leq \mu^+ - \frac{\omega}{2}] \cap B_\rho| \geqslant \theta |B_\rho|,$$

as $k_1 = \mu^+ - \frac{\omega}{2}$. Hence the Discrete Isoperimetric Inequalities 2.8 gives and then Hölder inequality we get

$$\begin{aligned}
\frac{\omega}{2^{s+1}} \cdot |A_{s+1}| &\leq \frac{\gamma\rho}{\theta} \int_{A_s \setminus A_{s+1}} |\nabla u| dx \\
&\leq \frac{\gamma\rho}{\theta} \left(\int_{B_\rho} |\nabla (u - k_s)_+|^p \, dx \right)^{\frac{1}{p}} \cdot |A_s - A_{s+1}|^{\frac{p-1}{p}}.
\end{aligned}$$

Now, by taking the p-power of both sides and using $(3)_+$ with $(1 - \sigma) = \frac{1}{2}$, we obtain

$$\begin{aligned}
\frac{\omega^p}{2^{sp}} \cdot |A_{s+1}|^p &\leq \frac{\gamma^p \rho^p}{\theta^p} \left(\frac{||(u - k_s)_+||^p_{p, B_{2\rho}}}{\rho^p} \right) |A_s - A_{s+1}|^{p-1} \\
&\leq \frac{\gamma^p \rho^N \omega^p}{\theta^p 2^{sp}} |A_s - A_{s+1}|^{p-1}.
\end{aligned}$$

Dividing by $\frac{\omega^p}{2^{sp}}$ and taking the $\frac{1}{p-1}$-power of both sides to get

$$|A_{s+1}|^{\frac{p}{p-1}} \leq \left(\frac{\gamma}{\theta}\right)^{\frac{p}{p-1}} \rho^{\frac{N}{p-1}} |A_s - A_{s+1}|, \quad \forall s \in \{1, \dots, s^*\}.$$

Next, we sum over s. We observe that the right-hand side can be controlled by a telescoping series, which in turn is controlled by $|B_\rho|$.

$$s^* \, |A_{s^*+1}|^{\frac{p}{p-1}} \leq \sum_{s=1}^{s^*} |A_{s+1}|^{\frac{p}{p-1}} \leq \left(\frac{\gamma}{\theta}\right)^{\frac{p}{p-1}} \rho^{\frac{N}{p-1}} \sum_{s=1}^{s^*} |A_s - A_{s+1}|$$
$$\leq \left(\frac{\gamma}{\theta}\right)^{\frac{p}{p-1}} |B_\rho| \cdot \rho^{\frac{N}{p-1}} \leq \left(\frac{\gamma}{\theta}\,|B_\rho|\right)^{\frac{p}{p-1}}.$$

From this, we have

$$|A_{s^*+1}| = \left|[u > \mu^+ - \frac{\omega}{2^{s^*+1}}] \cap B_\rho\right| \leq (s^*)^{\frac{1-p}{p}} \left(\frac{\gamma}{\theta}\right) |B_\rho| \leq \nu |B_\rho|$$

for s^* small enough and $\epsilon = 2^{-(s^*+1)}$. Finally, if $u \in DG_p^-(\hat{\gamma}, \Omega)$ and we assume (18) using Remark 2.2 we can directly see that $-u \in DG_p^+(\hat{\gamma}, \Omega)$ satisfies (16) and thus

$$\left|\left[-u > \mu^+(-u) - \epsilon\omega\right] \cap B_\rho\right| < \nu \, |B_\rho| \, .$$

which is (19) as $\mu^+(-u) = \sup(-u) = -\inf u$. □

Now we have all the necessary tools to prove that elements of $DG_p(\hat{\gamma}, \Omega)$ "are" Hölder continuous.

THEOREM 2.10. *Let $u \in DG_p(\hat{\gamma}, \Omega)$. Then there exist constants $\gamma > 1$ and $\alpha \in (0, 1)$ depending only on the data such that for every pair of balls $B_\rho(x_0) \subset B_R(x_0) \subset \Omega$,*

$$\operatorname*{ess\,osc}_{B_\rho(x_0)} u \leq \gamma \left(\frac{\rho}{R}\right)^\alpha \operatorname*{ess\,osc}_{B_R(x_0)} u. \tag{20}$$

Proof. We center x_0 at the origin as usual, and we have the following dichotomy: either

$$|[u \leq \mu^+ - \frac{\omega}{2}] \cap B_\rho| > \frac{1}{2}|B_\rho| \tag{21}$$

or

$$|[u \geqslant \mu^- + \frac{\omega}{2}] \cap B_\rho| > \frac{1}{2}|B_\rho|,$$

because $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$. Assuming the validity of (21), Critical Mass Lemma 2.4 determines $\nu > 0$ depending only on the data. By the Shrinking Lemma 2.9 there exists a number $\epsilon > 0$ depending on the data such that (21) implies

$$|[u > \mu^+ - 2^{-(\epsilon+1)}\omega] \cap B_\rho| < \nu |B_\rho|.$$

Now, Lemma 2.4 provides the measure-to-point information

$$u < \mu^+ - 2^{-(\epsilon+2)}\omega, \quad \text{almost everywhere in } B_{\rho/2}.$$

Hence,

$$\sup_{B_{\rho/2}} u \le \sup_{B_{2\rho}(x_0)} u - 2^{-(\varepsilon+2)}\omega(2\rho)$$

and as $-\inf_{B_{\rho/2}} u \le -\inf_{B_{2\rho}} u$ we have

$$\omega(\rho/2) \le (1-2^{-(\varepsilon+2)})\omega(2\rho) := \delta\omega(2\rho), \qquad \text{being} \quad \delta \in (0,1),$$

which is, in turn, equivalent to

$$\omega(\rho/4) \le \delta\omega(\rho). \tag{22}$$

In general, if $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying for $\tau, \delta \in (0,1)$ the relation

$$\omega(\tau\rho) \le \delta\omega(\rho), \qquad \forall \rho > 0, \tag{23}$$

then there exists a number $\alpha \in (0,1)$ depending only on δ such that for any $0 < \rho < R$ the inequality

$$\omega(\rho) \le \frac{1}{\delta}\left(\frac{\rho}{R}\right)^{\alpha}\omega(R) \tag{24}$$

holds true. We prove this last assertion, since (24) with $\omega = \operatorname{ess\,osc} u$ will imply the claim within the assumption (22). Let us choose $n \in \mathbb{N}$ such that

$$\tau^{n+1}R \le \rho < R\tau^n\,,$$

and let us remark that the right-hand inequality implies that

$$\tau^n < \frac{1}{\tau}\left(\frac{\rho}{R}\right).$$

Now, since ω is nondecreasing, iterating through (23) we obtain

$$\omega(\rho) \le \omega(\tau^n R) \le \delta^n\omega(R).$$

Setting $\alpha = \frac{ln(\gamma)}{ln(\tau)}$ we have $\delta = \tau^\alpha$ and thus,

$$\delta^n\omega(R) = \tau^{\alpha n}\omega(R) \le \frac{1}{\delta}\left(\frac{\rho}{R}\right)^{\alpha}\omega(R),$$

from the previous remark on τ^n. This implies our claim. For an illustration of the main steps, see the “Shrinking Machine” in Figure 2 here below. □

The Shrinking Machine

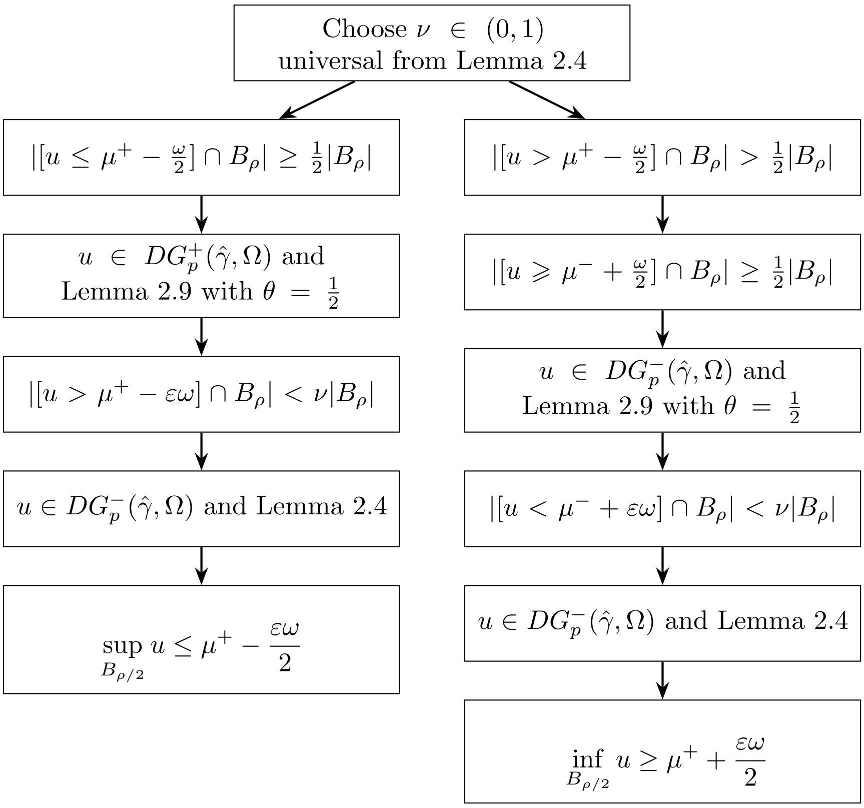


Figure 2: The working principle of the proof of Theorem 2.10.

2.1. Selecting a continuous representative.

Theorem 2.10 is, usually, the end of the story. However, it does not imply that every element $u \in DG_p(\hat\gamma,\Omega)$ is locally Hölder continuous, but rather than it is possible to *select* locally (in the equivalence class of u) a continuous representative. In order to clarify this point, we can simply consider the function

$$v(x) = \begin{cases} 1, & x \in (-1,0) \cup (0,1), \\ 0, & x = 0\,. \end{cases}$$

Function v is a local weak solution to $-\Delta v = 0$ in $(-1,1)$, and therefore it is a member of $DG_2(1,(-1,1))$. It is clear that, up to redefinition on the zero-measure set $\{x = 0\}$, we can define a continuous representative $\tilde v$, that is also C^∞, as claimed by the classical theory.

Here below we show how the estimates of Theorem 2.10 can be used to construct, for each $u \in DG_p(\hat\gamma, \Omega)$, its continuous representative. We will follow the strategy of [18] in the context of Campanato spaces.

Let $x_0 \in \Omega$ be an arbitrary point, let $B_\rho(x_o) \subset B_R(x_0) \subset B_{\tilde R}(x_0) \subset \Omega$ be such that $R \le d(\partial B_{\tilde R}(x_0), \partial\Omega)/4$ and let us fix

$$u_\infty = \|u\|_{L^\infty(B_{\tilde R}(x_0))}\,.$$

Now, the estimate (20) provides (for each $x_0 \in \Omega$ and) for each $0 < r < \rho$ the bound

$$\left| \fint_{B_\rho(x_0)} u(y)\,dy - \fint_{B_r(x_0)} u(y)\,dy \right| \le \operatorname*{ess\,osc}_{B_\rho(x_0)} u \le \gamma(u_\infty)\,(\rho/R)^\alpha\,. \tag{25}$$

Let us set, for $k < h$ natural numbers,

$$\rho = R\,2^{-k}, \qquad r = R\,2^{-h}, \qquad f_k(x_0) = \fint_{B_{R2^{-k}}(x_0)} u(y)\,dy\,.$$

First we observe that the functions $f_k : B_{\tilde R}(x_0) \to \mathbb{R}^+$ are continuous, since the integral is absolutely continuous w.r.t. to the domain of integration. Now, (25) implies

$$|f_k(x_0) - f_h(x_0)| \le \gamma(u_\infty) 2^{-k\alpha}\,,$$

that is, for each assigned $x_0 \in \Omega$, the sequence $\{f_k(x_0)\}$ is Cauchy. Hence, by pointwise convergence, we can define a function $\tilde u : \Omega \to \mathbb{R}$ with

$$\tilde u(x_0) = \lim_{k\uparrow\infty} \fint_{B_{2^{-k}R}(x_0)} u(y)\,dy = \lim_{k\uparrow\infty} f_k(x_0)\,.$$

It is not hard to show that this definition is purely local, and it does not depend on the number R chosen. Indeed, if $R_2 < R$ is another radius and $i \in \mathbb{N}$, there exists $k \in \mathbb{N}$, $k \geqslant i$ such that $2^{-(k+1)}R \le R_2 \le 2^{-k}R$ and

$$\begin{aligned} &\left| \fint_{B_{2^{-i}R}(x_0)} u(y)\,dy - \fint_{B_{2^{-i}R_2}(x_0)} u(y)\,dy \right| \\ &\qquad \le |f_i(x_0) - f_k(x_0)| + \left| f_k(x_0) - \fint_{B_{2^{-i}R_2}(x_0)} u(y)\,dy \right| \le \gamma(u_\infty) 2^{-i\alpha}\,, \end{aligned}$$

using again (25). Actually, the convergence of $\{f_k\}$ is uniform, and therefore the limit $\tilde u$ is a continuous function. Indeed, it is enough now to let r tend to zero in (25), in order to get

$$\left| \fint_{B_\rho(x)} u(y)\,dy - \tilde u(x) \right| \le \gamma(u_\infty, \tilde R)\,\rho^\alpha\,.$$

Recalling the relation of k with ρ, besides passing to the supremum over x on the left-hand side, we infer that the convergence of $\{f_k\}$ is uniform. Finally, we invoke Lebesgue's Theorem, the uniqueness of the limit, and conclude that the constructed continuous function $\tilde{u}$ coincides almost everywhere u in Ω. The results of Lemmata 2.4-2.9, and therefore Theorem 2.10 hold true locally for $\tilde{u}$ without the prefix "essential", and (20) describes, for $\tilde{u}$ the decay of its oscillation.

In this final part of Section 2 we take advantage of the oscillation estimates and the phenomenon of *expansion of positivity* to show that elements of $DG_p(\hat{\gamma}, \Omega)$ satisfy a Harnack inequality. This inequality was first proved in [13], see [16] for a survey on the topic. The version we report here is the one of [10]. See also the pioneering papers [35] and [38], in case of solutions to divergent form equations.

THEOREM 2.11. *Let $u \in DG_p(\hat{\gamma}, \Omega)$ be nonnegative. Then, there exists a constant $\gamma > 1$ depending only on the data, such that for every ball $B_{4\rho}(x_0) \subset \Omega$, we have*

$$u(y) \leq \gamma \inf_{B_\rho(x_0)} u. \tag{26}$$

Proof. Let $x_0 \in \Omega$ such that $u(x_0) > 0$. Consider the following change of variables

$$v = \frac{u}{u(x_0)}, \quad x \to \frac{x - x_0}{\rho},$$

so that $v(0) = 1$ and $v \in DG_p(\hat{\gamma}, B_4)$. In particular, the following estimate holds true for all $B_r(x^*) \subset B_4$,

$$\|\nabla(v-k)\pm\|^p_{p,B_{\sigma r}(x^*)} \leq \frac{\gamma}{(1-\sigma)^p r^p} \|(v-k)\pm\|^p_{p,B_r(x^*)}\,.$$

Thanks to this transformation, in order to prove (26), we just need to find a constant $\gamma > 1$ such that $v \geqslant \gamma$ in B_1.

The Trick of Krylov-Safonov [23]

We perform a stratagem that will allow us to find a (small) ball around an unknown point, where the supremum of v is bounded above by a power of the radius of the ball itself.

Let $s \in [0, 1)$ and let us consider the increasing families of numbers

$$M_s = \sup_{B_s} v, \quad N_s = (1-s)^{-\beta}$$

where, $\beta > 0$ is chosen later. Since v is bounded over B_2, then the set $\{M_s\}$ is also bounded and satisfies

$$M_0 = \sup_{B_0} v = 1 = N_0, \quad \lim_{s\to 1} M_s < \infty, \quad \lim_{s\to 1} N_s = \infty.$$

Therefore, the equation $M_s = N_s$ has roots, and we denote the largest root by s^*. Now, since v is lower-semicontinuous in B_2 we can find a point $x^* \in B_{s^*}$ such that

$$M_{s^*} = v(x^*) = (1 - s^*)^{-\beta}$$

Also, as s^* is the largest root, we can notice that for $R = \frac{1-s^*}{2}$ we have

$$M_{s^*} \leq \sup_{B_{\frac{1+s^*}{2}}(x^*)} v \leq N_{\frac{1+s^*}{2}} = 2^\beta (1 - s^*)^{-\beta}. \tag{27}$$

A lower bound, somewhere

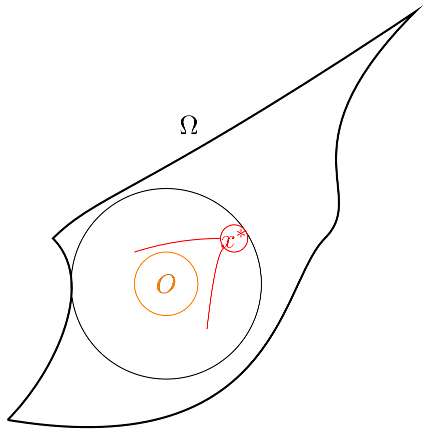
Now we use the oscillation estimates 2.10 to find a lower bound for the function v somewhere inside B_1.

Using Theorem 2.10 for all $0 < r < R$ and for all $x \in B_r(x^*)$ we have

$$\begin{aligned} v(x) - v(x^*) &\leq c\left[\operatorname*{osc}_{B_R(x^*)} v \left(\frac{r}{R}\right)^\alpha\right] \\ &\leq c\left[2^\beta (1 - s^*)^{-\beta}\left(\frac{r}{R}\right)^\alpha\right], \quad \text{using (27)} \end{aligned}$$

We choose $r = \epsilon^* R$ with ϵ^* independent of s^* and small enough such that $c2^\beta(1 - s^*)^{-\beta}\epsilon^{*\alpha} \leq \frac{1}{2}(1 - s^*)^{-\beta}$ to obtain, from the previous inequality,

$$v(x) \geqslant -v(x) \geqslant \frac{1}{2}(1-s^*)^{-\beta} - v(x^*) = \frac{1}{2}(1-s^*)^{-\beta} := M, \quad \forall x \in B_r(x^*)\,. \tag{28}$$



Propagation of Positivity

Now that we have a lower bound in a small ball centered in x^*, our interest is to propagate this lower bound (at the price of a suitable decay) until we cover completely B_1. The strategy goes through Lemma 2.9: first we recover from (28) a measure-theoretical information in $B_r(x^*)$, that is

$$|B_r(x^*)| = |[v \geqslant M] \cap B_r(x^*)| \geqslant \frac{1}{2}|B_r|.$$

Then we observe that (29) implies for a ball of four times that radius

$$|[v \geqslant M] \cap B_{4r}(x^*)| \geqslant |[v \geqslant M] \cap B_r(x^*)| \geqslant \frac{1}{2}|B_r| = 2^{-(2N+1)}|B_{4r}|. \quad (29)$$

As number θ in Lemma 2.9 is arbitrary, we apply it with $\rho = 4r$ and $\tilde{\theta} = \frac{1}{24^N}$ to (29), then there exists $\tilde{\epsilon} \in (0,1)$ such that

$$|[v < 2\tilde{\epsilon}M] \cap B_{4r}(x^*)| \leq \tilde{\nu}|B_{4r}|\,.$$

In order to get a point-wise estimate, we apply the Critical Mass Lemma 2.4 to obtain

$$v(x) \geqslant \tilde{\epsilon}M, \qquad \forall x \in B_{2r}(x^*)\,.$$

Repeating this process on the balls from $B_{2^j r}(x^*)$ to $B_{2^{j+1}r}(x^*)$ we get

$$v > \tilde{\epsilon}^j M \quad \text{a.e. in } B_{2^{j+1}r}(x^*).$$

After n iterations, the balls $B_{2^{n+1}r}(x^*)$ expand to cover B_1 for n large enough such that

$$2 \leq 2^{n+1}r = 2^{n+1}\epsilon^* \frac{1-s^*}{2} \leq 4$$

for which,

$$\begin{aligned} 2\tilde{\epsilon}^n M = 2\tilde{\epsilon}^n \left[\frac{1}{2}(1-s^*)^{-\beta} - v(x^*)\right] &= \tilde{\epsilon}^n (1-s^*)^{-\beta} \\ &\leq (2^\beta \tilde{\epsilon})^n \epsilon^{*\beta} \leq 2^\beta \tilde{\epsilon}^n \left(1-s_*\right)^{-\beta} = 2^{\beta+1}\tilde{\epsilon}^n M. \end{aligned}$$

The remainder of the proof consists in freeing up the lower bound $\tilde{\epsilon}^j M$ from the qualitative parameter s^*, that originated in the Krylov-Safonov argument. The constant ϵ^* is independent of s^* but, in fact, dependent on β. We select β to be big enough that $\tilde{\epsilon}^n(1-s_*)^{-\beta} = 1$ and

$$\tilde{\epsilon}^n M = \tilde{\epsilon}^n \frac{1}{2}\left(1-s_*\right)^{-\beta} = \gamma$$

Therefore, $v \geqslant \tilde{\epsilon}^n M \geqslant \gamma$ in B_1, which leads to the desired result. $\square$

REMARK 2.12. It is possible to prove the Harnack inequality (26) without using the fact that elements $u \in DG_p(\hat{\gamma}, \Omega)$ admit an Hölder continuous representative satisfying the oscillation estimates of Theorem 2.10, see [10] Chap X. The Hölder continuity is used only to ensure the lower bound (28), that can be achieved with the help of the combined use of a suitable Local Clustering Lemma (see [12]) and the measure-to estimate Lemma 2.4.

Here we have preferred to use Theorem 2.10 for the sake of its simplicity, and to show more approaches: in the next Section 3, we will derive a Harnack inequality for the fractional counterpart of $DG_p(\hat{\gamma}; \Omega)$ without the use of the oscillation estimates.

Finally, for the purpose of applications, inequality (26) contains all the information needed. The usual formula indeed, can be recovered easily from the right-hand side inequality (26) and lower-semicontinuity. An attentive read might reveal that these two ingredients are actually all we need, disregarding the membership to the De Giorgi class.

COROLLARY 2.13. *Let $u \in DG_p(\hat{\gamma}, \Omega)$ be a non-negative function. Then, there exists a constant $\gamma > 1$ depending only on the data, such that*

$$\frac{1}{\gamma} \sup_{B_\rho(x_0)} u \le u(x_0) \le \gamma \inf_{B_\rho(x_0)} u$$

Proof. By contradiction, assume that

$$\sup_{B_\rho(x_0)} u > \gamma u(x_0) > 0.$$

Now, by lower- semicontinuity of u (2.10), there exists $x_* \in \overline{B_\rho(x_0)}$ such that

$$u(x_*) > \gamma u(x_0).$$

We apply the Harnack Inequality (26) on $u(x_*)$ and obtain

$$u(x_0) < \frac{1}{\gamma} u(x_*) \le \inf_{B_\rho(x_*)} u \le u(x_0)$$

as $x_0 \in B_\rho(x_*)$. □

COROLLARY 2.14. *Let $u \in DG_p(\hat{\gamma}, \mathbb{R}^N)$ be bounded below. Then, u is constant.*

Proof. If we consider the function $v = u - \inf_{\mathbb{R}^N} u \ge 0$, then $v \in DG(\hat{\gamma}, \mathbb{R}^N)$. Let $x_0 \in \mathbb{R}^N$ be such that $v(x_0) > 0$ and an application of (26) leads us to

$$v(x_0) \le \gamma \inf_{B_\rho(x_0)} v.$$

Now, we can take $\rho \to \infty$ and obtain

$$u(x_0) - \inf_{\mathbb{R}^N} u \le \gamma \inf_{\mathbb{R}^N} \left[u - \inf_{\mathbb{R}^N} u \right] = 0. \qquad \square$$

2.2. How big is the class $DG_p(\hat{\gamma}, \Omega)$?

In this brief subsection we show that our treatment embodies homogeneous local weak solutions to p-Laplacian type equations with measurable and bounded coefficients, and quasi-minima of the Calculus of Variations whose prototype functional is

$$\mathcal{F}(u, \Omega) = \int_{\Omega} |\nabla u|^p \, dx \, .$$

To prove the first assertion, let us consider equations as

$$-\operatorname{div}\big(\mathbf{A}(x, u, \nabla u)\big) = 0, \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{30}$$

where, $\Omega \subset\subset \mathbb{R}^N$ and for $u \in W^{1,p}_{loc}(\Omega)$, the functions $\mathbf{A}(\cdot, u, \nabla u(\cdot))$ are measurable and satisfy the structure conditions

$$\begin{cases} \mathbf{A}(x, u, \nabla u) \cdot \nabla u \geq \lambda |\nabla u|^p, \\ |\mathbf{A}(x, u, \nabla u)| \leq \Lambda |\nabla u|^{p-1} \end{cases} \tag{31}$$

with the ellipticity constants $0 < \lambda \leq \Lambda$. We are interested in the local behavior of solutions to (30)-(31), irrespectively of possibly prescribed data. This motivates the following definition.

Definition 2.15. *A function $u \in W^{1,p}_{loc}(\Omega)$ is a local weak sub(super)-solution of* (30)-(31)*, if*

$$\int_{\Omega} \mathbf{A}(x, u, \nabla u) \nabla \phi \, dx \leq (\geq) 0$$

for all non-negative test functions $\phi \in W^{1,p}_o(\mathcal{K})$, for every open set $\mathcal{K}$ such that $\mathcal{K} \subset\subset \Omega$.

Lemma 2.16. *Let u be a local weak sub(super) solution of* (30)-(31)*. Then there exists a constant $\hat{\gamma} > 0$ depending only on the data $\{N, p, \lambda, \Lambda\}$ such that $u \in DG^{\pm}_p(\hat{\gamma}, \Omega)$.*

Now we turn our attention to the quasi-minima of functionals. Let us consider the functional

$$\mathcal{F}(u, \Omega) = \int_{\Omega} F(x, u, \nabla u) \, dx \, , \tag{32}$$

where $F(x, u, z)$ ia s Carathéodory function satisfying

$$\lambda |z|^p \leq F(x, u, \nabla u) \leq \Lambda |z|^p \, , \qquad 0 < \lambda \leq \Lambda \, . \tag{33}$$

DEFINITION 2.17. *A function $u \in W^{1,p}_{loc}(\Omega)$ is a sub-quasiminimum for the functional $\mathcal{F}$ if for every function $0 \geqslant \varphi \in W^{1,p}(\Omega)$ with support $supp(\varphi) = K \subset \Omega$, we have*

$$\mathcal{F}(u, K) \leq Q\mathcal{F}(u + \varphi, K), \qquad Q \geqslant 1\,.$$

Similarly u is a super-quasiminimum for $\mathcal{F}$ is the previous relation holds true for every $0 \leq \varphi \in W^{1,p}(\Omega)$. A quasi minimum is at the same time a sub-quasiminimum and a super-quasiminimum.

LEMMA 2.18. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a sub(super)-quasiminimum for the functional $\mathcal{F}$ above. Then there exists $\hat{\gamma} > 0$ depending only on the data $\{N, p, \lambda, \Lambda\}$ such that $u \in DG^{\pm}_{p}(\hat{\gamma}, \Omega)$.*

REMARK 2.19. Finally, local weak solutions to equations as (30)-(31) are quasi minima of suitable functionals, as (32)-(33), see [20] Chap VI; while, when the functional $\mathcal{F}$ is differentiable in a suitable sense, the equation satisfied by the minima, called Euler-Lagrange equation, is of the same kind of (30) (see for instance [20]).

3. Fractional De Giorgi classes

In this section, we consider nonlocal De Giorgi classes: a particular subset of the fractional Sobolev space, whose elements satisfy a fractional Caccioppoli inequality. We will show that the elements of these classes are locally bounded and have an Hölder continuous representive which satisfies an Harnack inequality.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $p \geq 1$ and $s \in (0, 1)$ such that $sp < N$. In order to consider fractional De Giorgi classes, the candidate members have to belong to a suitable space of functions (similarly as in Section 2 for the Sobolev spaces), called the fractional Sobolev space and denoted as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \quad \text{and} \quad \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} \in L^p(\Omega \times \Omega) \right\}.$$

For $u \in W^{s,p}(\Omega)$, the term

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dxdy \right)^{\frac{1}{p}}$$

is called the Gagliardo semi norm of u. Similarly to the local case, we can define $W^{s,p}_o(\Omega)$ as the closure of $C^\infty_o(\Omega)$ in $W^{s,p}(\Omega)$. One can prove that $C^\infty_0(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, but it is not true for a general open subset Ω. For the proof and more information about fractional Sobolev spaces, we refer to [15] and [27].

Many estimates derived for the Section 2 have a similar analogue in the fractional case, the main difference being the presence of a new term, called the "tail" of the operator, which takes into account the long-distance behavior of the function. For a point $x_0 \in \mathbb{R}^N$, a radius $R > 0$ and a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, its tail is defined as

$$\mathrm{Tail}(u, x_0, R)^{p-1} = R^{ps} \int_{\mathbb{R}^N \setminus B_R(x_0)} \frac{|u(x)|^{p-1}}{|x - x_0|^{N+ps}}\, dx\,.$$

The quantity at the right hand side can be infinite, in general. As we aim at precise quantitative estimates, we introduce the space

$$L_s^{p-1}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \quad \text{measurable } : \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1 + |x|)^{N+ps}}\, dx < \infty \right\},$$

so that every such tail as above is a finite number.

Definition 3.1 (Fractional De Giorgi classes). *A function $u \in L_s^{p-1}(\mathbb{R}^N) \cap W^{s,p}(\Omega)$ is a member of $DG_{s,p}^{\pm}(\hat{\gamma}, \Omega)$ if there exists a constant $\hat{\gamma} = \hat{\gamma}(N, p, s) > 0$ such that for every $k \in \mathbb{R}$, $\tau \in (0, 1)$ and ball $B_\rho(x_0) \subset \Omega$ the estimate*

$$\begin{aligned}
&[(u-k)_\pm]^p_{W^{s,p}(B_{\tau\rho}(x_0))} + \int_{B_{\tau\rho}(x_0)} \int_{\mathbb{R}^N} \frac{(u(x)-k)_\pm (u(y)-k)_\mp^{p-1}}{|x-y|^{N+ps}}\, dxdy \\
&\quad \le \frac{\hat{\gamma}}{(1-\tau)^p \rho^{sp}} \|(u-k)_\pm\|^p_{L^p(B_\rho(x_0))} \\
&\qquad + \frac{\hat{\gamma}}{(1-\tau)^{N+sp} \rho^{sp}} \|(u-k)_\pm\|_{L^1(B_\rho(x_0))}\, \mathit{Tail}((u-k)_\pm, x_0, \rho)^{p-1}
\end{aligned} \tag{34}$$

is satisfied. Finally,

$$DG_{s,p}(\hat{\gamma}, \Omega) := DG_{s,p}^{+}(\hat{\gamma}, \Omega) \cap DG_{s,p}^{-}(\hat{\gamma}, \Omega)\,.$$

Remark 3.2. Roughly speaking, what diversifies our definition of De Giorgi class $DG_{s,p}(\hat{\gamma}, \Omega)$ from the one of [7] is the fact that integral of the second term on the left hand side is taken in the unbounded set $\mathbb{R}^N$. However, this minor adjustment allows us to use a new method, adapted from [30], which is far simpler and more readable: in particular, it does not employ the usual clustering lemmas nor Krylov-Safonov covering arguments.

Remark 3.3. As in the local framework, if $u \in DG_{s,p}^{+}(\hat{\gamma}, \Omega)$ then $-u \in DG_{s,p}^{-}(\hat{\gamma}, \Omega)$. This is because again by taking $-k$ we get $(-u + k)_+ = (u - k)_-$.

As in the local framework, the first property that we aim to show, in order to construct a complete regularity theory, is

$$DG_{s,p}(\hat{\gamma}, \Omega) \subseteq L^\infty_{loc}(\Omega)\,.$$

To this aim, we recall a classical fractional embedding.

LEMMA 3.4 (Fractional embedding). *Let $\sigma \in (0,1)$. Then there exists a constant $\gamma > 1$, depending only on the data, such that for every $u \in W^{s,p}(B_\rho)$, compactly supported in $B_{(1-\sigma)\rho}$, there holds*

$$\left(\int_{B_\rho} |u|^{\frac{Np}{N-sp}} \, dx \right)^{\frac{N-sp}{N}} \leq \gamma \left\{ \iint_{B_\rho^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy + \frac{\sigma^{-(sp)}}{\rho^{ps}} \int_{B_\rho} |u(x)|^p \, dx \right\} . \tag{35}$$

Now, we can prove the local boundness.

THEOREM 3.5 (Local Boundness). *If $u \in DG^{\pm}_{s,p}(\hat{\gamma}, \Omega)$, then for all $B_\rho(x_0) \subset \Omega$, $\sigma \in (0,1)$ we have*

$$\sup_{B_{\sigma\rho}(x_0)} u_\pm \leq \gamma(\sigma) \left\{ \left(\fint_{B_\rho(x_0)} u_\pm^p \, dx \right)^{\frac{1}{p}} + Tail(u_\pm, x_0, \sigma\rho) \right\} \tag{36}$$

where $\gamma(\sigma) > 1$ depends only on the data $\{N, p, s\}$, and σ.

Proof. Let us define, for some $k \in \mathbb{R}$ to be determined later and $n \in \mathbb{N}$, the sequence of decreasing balls

$$\begin{cases} \rho_n = \rho(\sigma + \frac{(1-\sigma)}{2^n}) \\ \hat{\rho}_n = \frac{3\rho_n + \rho_{n+1}}{4} \\ \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2} \\ \bar{\rho}_n = \frac{\rho_n + 3\rho_{n+1}}{4} \end{cases} \quad \text{and} \quad \begin{cases} B_n = B_{\rho_n}(x_0) \\ \hat{B}_n = B_{\hat{\rho}_n}(x_0) \\ \tilde{B}_n = B_{\tilde{\rho}_n}(x_0) \\ \bar{B}_n = B_{\bar{\rho}_n}(x_0), \end{cases}$$

where $B_{n+1} \subset \bar{B}_n \subset \tilde{B}_n \subset \hat{B}_n \subset B_n$, and the sequence of increasing levels $k_n = k(1 - 2^{-n})$.

Let ξ_n be a cut-off function such that $\mathbf{1}_{B_{n+1}} \leq \xi_n \leq \mathbf{1}_{\bar{B}_n}$ and such that $|\nabla \xi_n| \leq \frac{\gamma 2^{n+1}}{(1-\sigma)\rho}$. Now we estimate (34), with positive truncation $(u - k_{n+1})_+$, between $\tilde{B}_n$ and $\hat{B}_n$.

To apply precisely Definition 3.1, first observe that

$$\tilde{\rho} = \hat{\rho} \left(\frac{2^{n+3}\sigma + 6(1-\sigma)}{2^{n+3}\sigma + 7(1-\sigma)} \right)$$

so that, with the notation of Definition 3.1,

$$1 - \tau \geq \frac{(1-\sigma)}{2^{n+3}} .$$

Using this remark, we get

$$\begin{aligned}
&[(u-k_{n+1})_+]^p_{W^{s,p}(\tilde{B}_n)} \le \frac{2^{n(N+ps)}\gamma}{\rho^{sp}(1-\sigma)^{N+ps}}\times \\
&\times\left\{\left(1+\frac{\gamma 2^{n(p-1)}}{k^{p-1}}\mathrm{Tail}((u-k_{n+1})_+,x_0,\hat{\rho}_n)^{p-1}\right)\|(u-k_n)_+\|^p_{L^p(\hat{B}_n)}\right\}, \quad (37)
\end{aligned}$$

where we have used the fact that $\hat{\rho}_n \ge \frac{\rho}{2^{n+3}}$ and

$$\|(u-k_{n+1})_+\|_{L^1(\hat{B}_n)} \le \frac{\gamma 2^{n(p-1)}}{k^{p-1}}\|(u-k_n)_+\|^p_{L^p(\hat{B}_n)}\,.$$

Now, we choose the level k such that

$$\begin{aligned}
k > \sigma^{-\frac{sp}{p-1}}\mathrm{Tail}(u_+,x_0,\sigma\rho) &\ge \left(\frac{\hat{\rho}_n}{\sigma\rho}\right)^{\frac{sp}{p-1}}\mathrm{Tail}(u_+,x_0,\sigma\rho) \\
&\ge \mathrm{Tail}((u-k_{n+1})_+,x_0,\hat{\rho}_n) \quad (38)
\end{aligned}$$

so from (37) we recover

$$[(u-k_{n+1})_+]^p_{W^{s,p}(\tilde{B}_n)} \le \frac{2^{n(N+ps)}\gamma}{\rho^{sp}(1-\sigma)^{N+ps}}\|(u-k_n)_+\|^p_{L^p(\hat{B}_n)}\,. \quad (39)$$

Now we consider the cut-off function ξ_n, use the Hölder inequality and the embedding (35) over the balls $\bar{B}_n$ and $\tilde{B}_n$,

$$\frac{1}{\tilde{\rho}_n-\bar{\rho}_n} \le \frac{\gamma\, 2^n}{(1-\sigma)\rho}\,,$$

to obtain

$$\begin{aligned}
&\int_{B_{n+1}}|(u-k_{n+1})_+|^p\,dx = \int_{B_{n+1}}|(u-k_{n+1})_+\xi_n|^p\,dx \\
&\le \left(\int_{B_{n+1}}|(u-k_{n+1})_+\xi_n|^{\frac{Np}{N-ps}}\,dx\right)^{\frac{N-ps}{N}}|B_{n+1}\cap\{u\ge k_{n+1}\}|^{\frac{ps}{N}} \\
&\le \gamma\Bigg\{\iint_{\tilde{B}_n^2}\frac{|(u-k_n)_+\xi_n(x)-(u-k_n)_+\xi_n(y)|^p}{|x-y|^{N+ps}}\,dxdy \\
&\quad+\frac{1}{\rho^{ps}}\frac{1}{[(1-\sigma)\rho]^{ps}}\int_{\tilde{B}_n}|(u(x)-k_{n+1})_+|^p\xi_n^p(x)\,dx\Bigg\}|B_n\cap\{u\ge k_{n+1}\}|^{\frac{ps}{N}}\,.
\end{aligned} \quad (40)$$

On the first term of the right hand side in (40) we estimate, adding and subtracting $(u(y)-k_n)_+\xi_n(x)$

$$\begin{aligned}
&|(u-k_n)_+\xi_n(x)-(u-k_n)_+\xi_n(y)|^p \\
&\le \gamma|(u(x)-k_n)_+-(u(y)-k_n)_+|^p\xi_n^p(x)+\gamma(u(y)-k_n)_+^p|\xi_n(x)-\xi_n(y)|^p\,.
\end{aligned}$$

So that it becomes, using the Lagrange theorem on ξ_n

$$
\begin{aligned}
&\iint_{\tilde{B}_n^2} \frac{|(u-k_n)_+\xi_n(x)-(u-k_n)_+\xi_n(y)|^p}{|x-y|^{N+ps}}\,dxdy \\
&\le \gamma\Bigg[\iint_{\tilde{B}_n^2} \frac{|(u(x)-k_n)_+-(u(y)-k_n)_+|^p \xi_n^p(x)}{|x-y|^{N+ps}}\,dxdy \\
&\qquad\qquad + \iint_{\tilde{B}_n^2} \frac{(u(y)-k_n)_+^p|\xi_n(x)-\xi_n(y)|^p}{|x-y|^{N+ps}}\,dxdy\Bigg] \\
&\le \gamma\Bigg[\iint_{\tilde{B}_n^2} \frac{|(u(x)-k_n)_+-(u(y)-k_n)_+|^p}{|x-y|^{N+ps}}\,dxdy \\
&\qquad\qquad + \frac{2^{np}}{\rho^p}\iint_{\tilde{B}_n^2} \frac{(u(y)-k_n)_+^p}{|x-y|^{N+(s-1)p}}\,dxdy\Bigg] \\
&\le \gamma\Bigg[[(u-k_n)_+]_{s,p}^p + \frac{2^{np}}{\rho^p}\int_{\tilde{B}_n}(u(y)-k_n)_+^p\,dy \sup_{y\in\tilde{B}_n}\int_{\tilde{B}_n}\frac{dx}{|x-y|^{N+(s-1)p}}\Bigg] \\
&\le \gamma\Bigg[[(u-k_n)_+]_{s,p}^p + \frac{2^{np}}{\rho^p}\left(\int_{\tilde{B}_n}(u(y)-k_n)_+^p\,dy\right)\frac{1}{\tilde{\rho}_n^{(s-1)p}}\Bigg] \\
&\le \gamma\Bigg[[(u-k_n)_+]_{s,p}^p + \frac{2^{np}}{\rho^{sp}}\|(u-k_n)_+\|_{L^p(\tilde{B}_n)}^p\Bigg]. \qquad (41)
\end{aligned}
$$

Combining (41) and (40) we get

$$
\begin{aligned}
\int_{B_{n+1}}|(u-k_{n+1})_+|^p\,dx &\le \gamma\Bigg\{\Bigg[[(u-k_n)_+]_{s,p}^p + \frac{2^{np}}{\rho^{sp}}\|(u-k_n)_+\|_{L^p(\tilde{B}_n)}^p\Bigg] \\
&\quad + \frac{2^{n(sp)}}{\rho^{sp}(1-\sigma)^{sp}}\int_{\tilde{B}_n}|(u(x)-k_{n+1})_+|^p\,dx\Bigg\}|B_n\cap\{u\ge k_{n+1}\}|^{\frac{ps}{N}} \\
\le \gamma\Bigg\{\Bigg[[(u-k_n)_+]_{s,p}^p &+ \frac{2^{np}}{\rho^{sp}}\|(u-k_n)_+\|_{L^p(B_n)}^p\Bigg] \\
&+ \frac{2^{n(sp)}}{\rho^{sp}(1-\sigma)^{sp}}\|(u-k_n)_+\|_{L^p(B_n)}^p\Bigg\}|B_n\cap\{u\ge k_{n+1}\}|^{\frac{ps}{N}},
\end{aligned}
$$

and using (39) we arrive at

$$
\begin{aligned}
&\|(u-k_{n+1})_+\|_{L^p(B_{n+1})}^p \\
&\le \left(\frac{\gamma\, 2^{n(N+ps)}}{(1-\sigma)^{N+sp}\rho^{ps}}\right)\left\{\|(u-k_n)_+\|_{L^p(B_n)}^p\right\}\left(\frac{\gamma 2^{np}}{k^p}\right)^{\frac{sp}{N}}(\|(u-k_n)_+\|_{L^p(B_n)}^p)^{\frac{sp}{N}} \\
&\le \frac{\gamma 2^{n(N+p)(1+\frac{sp}{N})}}{(1-\sigma)^{N+sp}\rho^{ps}k^{\frac{sp^2}{N}}}\|(u-k_n)_+\|_{L^p(B_n)}^{p(1+\frac{sp}{N})}, \qquad (42)
\end{aligned}
$$

having used

$$\|(u-k_n)_+\|^p_{L^p(B_n)} \geq \int_{B_n\cap\{u\geq k_{n+1}\}} (u-k_n)^p\, dx \geq \left(\frac{k}{2^{n+1}}\right)^p |B_n \cap \{u \geq k_{n+1}\}|\,.$$

Now we call $\beta = \frac{sp}{N} > 0$, $b = 2^{(N+p)(1+\beta)} > 1$ and

$$Y_n = \fint_{B_n} (u-k_n)^p_+\, dx = \frac{\|(u-k_n)_+\|^p_{L^p(B_n)}}{|B_n|}\,,$$

so that we get in (42), multiplying and dividing in the right hand side by $|B_n|$ and remembering that $2^N|B_{n+1}| \geq |B_n|$,

$$Y_{n+1} \leq \frac{\gamma b^n}{(1-\sigma)^{N+ps} k^{\frac{sp^2}{N}}} Y_n^{1+\beta}\,.$$

We now want to use the same argument of (8): to this end we choose k so that the condition

$$Y_0 = \fint_{B_\rho} u^p_+\, dx \leq \left(\frac{\gamma}{(1-\sigma)^{N+ps}}\right)^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}} k^p$$

is satisfied. This means that k has to be chosen so that

$$k > \left(\frac{\gamma}{(1-\sigma)^{N+ps}}\right)^{\frac{1}{p\beta}} b^{\frac{1}{p\beta^2}} \left(\fint_{B_\rho} u^p_+\, dx\right)^{\frac{1}{p}} =: \gamma(\sigma)\left(\fint_{B_\rho} u^p_+\, dx\right)^{\frac{1}{p}}.$$

Recalling (38) our final choice of k is

$$k := \gamma(\sigma)\left(\fint_{B_\rho} u^p_+\, dx\right)^{\frac{1}{p}} + \sigma^{-\frac{sp}{p-1}} \mathrm{Tail}(u_+, x_0, \sigma\rho)\,,$$

then, we have $Y_n \to 0$ as $n \to \infty$, which means $u_+ \leq k_\infty = k$ in $B_{\sigma\rho}$ or equivalently (36). $\square$

Remark 3.6. Constant $\gamma(\sigma)$ deteriorates as soon as $\sigma \downarrow 0$ or $\sigma \uparrow 1$.

To Continuity and Beyond

In this subsection we show the arrow

$$DG_{s,p}(\hat{\gamma}, \Omega) \hookrightarrow C^{0,\alpha}_{loc}(\Omega),$$

for some $\alpha \in (0,1)$, meaning with this that elements of $DG_{s,p}(\hat{\gamma},\Omega)$ admit an Hölder continuous representative. From now on we consider $u \in L^\infty_{\mathrm{loc}}(\Omega)$, $B_{l\rho}(x_0) \subset \Omega$, for some radius $\rho > 0$ and $l > 2$ to be defined. Let

$$\mu^+ \geq \sup_{B_{l\rho}(x_0)} u\,, \qquad \mu^- \leq \inf_{B_{l\rho}(x_0)} u\,, \qquad \omega \geq \mu^+ - \mu^-\,. \tag{43}$$

The choice of this constant l is of great importance in the fractional framework. In fact, all of our results present a dichotomy, in the sense that either the tail is big, or we can have our result. In the last part of the work we will choose l so that we can negate the possibility of a big tail and have a guarantee that the result holds.

We show here a measure-to-point property, analogous to Lemma 2.4, for the nonlocal case.

LEMMA 3.7 (Nonlocal Critical Mass Lemma). *Suppose that $u \in DG^+_{s,p}(\hat{\gamma},\Omega)$, and for any $a \in (0,1)$ there exists a constant $\nu \in (0,1)$, depending only on the data, such that*

$$\big|[u > \mu^+ - \epsilon\omega] \cap B_\rho(x_0)\big| \leq \nu |B_\rho| \qquad \textit{for some } \epsilon \in (0,1) \tag{44}$$

and

$$l^{-\frac{sp}{p-1}}\, \mathit{Tail}(u-\mu^+)_+, x_0, l\rho) \leq \epsilon\omega\,. \tag{45}$$

Then

$$u \leq \mu^+ - a\epsilon\omega \qquad \textit{a.e. in } B_{\rho/2}(x_0)\,. \tag{46}$$

On the other hand if $u \in DG^-_{s,p}(\hat{\gamma},\Omega)$, for any $a \in (0,1)$ there exists a constant ν, depending only on the data, such that if

$$\big|[u < \mu^- + \epsilon\omega] \cap B_\rho(x_0)\big| \leq \nu |B_\rho| \qquad \textit{for some } \epsilon \in (0,1)$$

and

$$l^{-\frac{sp}{p-1}}\, \mathit{Tail}(u-\mu^-)_-, x_0, l\rho) \leq \epsilon\omega\,.$$

Then

$$u \geq \mu^- + a\epsilon\omega \qquad \textit{a.e. in } B_{\rho/2}(x_0)\,. \tag{47}$$

Proof. We will prove (44)-(45)-(46), the other case being the same, thanks to remark 3.3. Define the sequence of decreasing balls as in Theorem 3.5 with $\sigma = 1/2$ and the sequence of increasing levels

$$k_n = \mu^+ - a\epsilon\omega - \frac{(1-a)\epsilon\omega}{2^n}\,.$$

For $n \in \mathbb{N}$ let ξ_n be a cut-off function such that $\mathbf{1}_{B_{n+1}} \leq \xi_n \leq \mathbf{1}_{\tilde{B}_n}$ and $|\nabla \xi_n| \leq \frac{2^n}{\rho}$. Finally we define the set $A_n = [u > k_n] \cap B_n$.

Now we estimate

$$\begin{aligned}\left[\frac{(1-a)\epsilon\omega}{2^{n+1}}\right]^p|A_{n+1}| &= (k_{n+1}-k_n)^p|A_{n+1}| \le \|(u-k_n)_+\xi_n\|^p_{L^p(\tilde B_n)}\\ &\le \|(u-k_n)_+\xi_n\|^p_{L^{p^*}(\tilde B_n)}|A_n|^{\frac{ps}{N}} \quad \text{by Hölder's inequality}\\ &\le |A_n|^{\frac{ps}{N}}\gamma\Bigg[\iint_{\tilde B_n^2}\frac{|(u-k_n)_+\xi_n(x)-(u-k_n)_+\xi_n(y)|^p}{|x-y|^{N+ps}}\,dxdy\\ &\qquad + \frac{2^{(N+sp)n}}{\rho^{sp}}\int_{\tilde B_n}(u-k_n)_+^p\xi_n^p(x)\,dx\Bigg] \qquad (48)\end{aligned}$$

where in the last passage, we used the fractional embedding (35). We now work exactly as in (41) to arrive, from (48), at

$$\begin{aligned}\left[\frac{(1-a)\epsilon\omega}{2^{n+1}}\right]^p|A_{n+1}| \le |A_n|^{\frac{ps}{N}}\gamma\Bigg[&[(u-k_n)_+]^p_{s,p}+\frac{2^{np}}{\rho^{sp}}\|(u-k_n)_+\|^p_{L^p(\tilde B_n)}\\ &+\frac{2^{(N+sp)n}}{\rho^{sp}}\int_{\tilde B_n}(u-k_n)^p\xi_n^p(x)\,dx\Bigg] \qquad (49)\end{aligned}$$

Now for the first term in the right hand side we consider (34) and estimate it as in (37), the only difference being that now $k_{n+1}-k_n=\frac{(1-a)\epsilon\omega}{2^{n+1}}$, to have

$$\begin{aligned}&[(u-k_n)_+]^p_{s,p} \qquad (50)\\ &\le \frac{2^n\gamma}{\rho^{sp}}\left\{\left(1+\frac{\gamma 2^{n(p-1)}}{[(1-a)\epsilon\omega]^{p-1}}\mathrm{Tail}((u-k_{n+1})_+,x_0,\hat\rho_n)^{p-1}\right)\|(u-k_n)_+\|^p_{L^p(\hat B_n)}\right\}.\end{aligned}$$

We estimate the tail, considering $\rho/2\le\hat\rho_n\le\rho$, $\forall\, n\in\mathbb{N}$, so we have

$$\begin{aligned}&\mathrm{Tail}((u-k_{n+1})_+,x_0,\hat\rho_n)^{p-1} = \hat\rho_n^{sp}\int_{\mathbb{R}^N\setminus\hat B_n}\frac{(u(x)-k_{n+1})_+^{p-1}}{|x-x_0|^{N+ps}}\,dx\\ &=\hat\rho_n^{sp}\int_{\mathbb{R}^N\setminus B_{l\rho}(x_0)}\frac{(u(x)-k_{n+1})_+^{p-1}}{|x-x_0|^{N+ps}}\,dx+\hat\rho_n^{sp}\int_{B_{l\rho}(x_0)\setminus\hat B_n}\frac{(u(x)-k_{n+1})_+^{p-1}}{|x-x_0|^{N+ps}}\,dx\\ &\le \hat\rho_n^{sp}c(p)\int_{\mathbb{R}^N\setminus B_{l\rho}(x_0)}\frac{(u(x)-\mu^+)_+^{p-1}}{|x-x_0|^{N+ps}}\,dx+\hat\rho_n^{sp}c(p)\int_{\mathbb{R}^N\setminus B_{l\rho}(x_0)}\frac{(\epsilon\omega)^{p-1}}{|x-x_0|^{N+ps}}\,dx\\ &\qquad+\hat\rho_n^{sp}\int_{B_{l\rho}(x_0)\setminus\hat B_n}\frac{(u(x)-k_{n+1})_+^{p-1}}{|x-x_0|^{N+ps}}\,dx\\ &\le \gamma l^{-sp}\mathrm{Tail}((u-\mu^+)_+,x_0,l\rho)^{p-1}+\gamma l^{-sp}(\epsilon\omega)^{p-1}+\gamma(\epsilon\omega)^{p-1}\le\gamma(\epsilon\omega)^{p-1}\end{aligned}$$

where we also used (45), that $(u-k_{n+1})_+\le\epsilon\omega$ in $B_{l\rho}(x_0)$, and $l>2$ so that γ is independent of l. Inserting this result in (50) delivers

$$[(u-k_n)_+]^p_{s,p}\le\left(\frac{\gamma 2^{np}}{\rho^{sp}(1-a)^{p-1}}\right)\|(u-k_n)_+\|^p_{L^p(\hat B_n)}$$

and again putting this in (49) provides

$$\left[\frac{(1-a)\epsilon\omega}{2^{n+1}}\right]^p |A_{n+1}| \le |A_n|^{\frac{ps}{N}} \ \frac{\gamma\ 2^{(N+sp)n}}{\rho^{sp}} \left(\frac{1}{(1-a)^{p-1}}\right) \|(u-k_n)_+\|^p_{L^p(\hat{B}_n)} .$$

Now, let us just consider that

$$\|(u-k_n)_+\|^p_{L^p(\tilde{B}_n)} \le (\epsilon\omega)^p |\{u > k_n\} \cap B_n| = (\epsilon\omega)^p |A_n|$$

to finally have

$$\left[\frac{(1-a)\epsilon\omega}{2^{n+1}}\right]^p |A_{n+1}| \le \frac{2^{n(N+sp)}\gamma\,(\epsilon\omega)^p}{\rho^{sp}(1-a)^{p-1}} |A_n|^{1+\frac{sp}{N}} .$$

Putting all together, dividing both sides for $|B_n|$, remembering that $2^n|B_{n+1}| \ge |B_n|$ and calling $b = 2^{N+p(s+1)}$ and $\beta = \frac{sp}{N}$, we arrive at

$$Y_{n+1} := \frac{|A_{n+1}|}{|B_{n+1}|} \le \frac{\gamma b^n}{(1-a)^{2p-1}} Y_n^{1+\beta}$$

and again we are in a recursive situation as in (8): this time, to estimate the first step Y_0, we recall that assumption (44) gives us

$$Y_0 = \frac{|[u > \mu^+ - \epsilon\omega] \cap B_\rho(x_0)|}{|B_\rho|} \le \nu := b^{-\frac{1}{\beta^2}} \left(\frac{\gamma}{(1-a)^{2p-1}}\right)^{-\frac{1}{\beta}}$$

observe that $\gamma > 1$ so that $\nu \in (0,1)$. Hence, $Y_n \to 0$ when $n \to \infty$ and (46) is proved. □

Remark 3.8. The previous lemma can be reformulated in a more general way for a nonnegative function $u \in DG^-_{s,p}(\hat{\gamma}, \Omega)$, in fact, in this case we can rescale u to $(u - \mu_-)$ and assume $\mu_- = 0$. Then, for any $k > 0$ we can always choose ϵ so that $k = \epsilon\omega$, finally take $h \in (0,1)$ so our statement take the form of:
if

$$|[u < k] \cap B_{h\rho}(x_0)| \le \nu |B_{h\rho}| \quad \text{and} \quad h^{\frac{sp}{p-1}} \operatorname{Tail}(u_-, x_0, \rho) \le k$$

then

$$u \ge ak \qquad \text{a.e. in } B_{h\rho/2}(x_0) \text{ for } a \in (0,1) .$$

Next, we prove a Shrinking Lemma,

Lemma 3.9 (Nonlocal Shrinking Lemma). *Let $u \in DG^+_{s,p}(\hat{\gamma}, \Omega)$ and let us assume that for some $\epsilon \in (0, 1/2)$ it holds*

$$\big|[u \le \mu^+ - \epsilon\omega] \cap B_\rho(x_0)\big| \ge \alpha |B_\rho| \qquad \textit{for some } \alpha \in (0,1) .$$

Then, there exist $\gamma(l)$, depending only on the data and l, such that, for any $\sigma \in (0, 1/2)$ chosen, either

$$l^{-\frac{sp}{p-1}}\, Tail((u-\mu^+)_+, x_0, l\rho) \le \sigma\epsilon\omega\,,$$

or

$$\big|[u \ge \mu^+ - \sigma\epsilon\omega] \cap B_\rho(x_0)\big| \le \frac{\gamma(l)\sigma^{p-1}}{\alpha}|B_\rho|\,.$$

On the other hand if $u \in DG^-_{s,p}(\hat\gamma, \Omega)$ let us assume that for some $\epsilon \in (0, 1/2)$ it holds

$$\big|[u \ge \mu^- + \epsilon\omega] \cap B_\rho(x_0)\big| \ge \alpha|B_\rho| \qquad \textit{for some } \alpha \in (0,1)\,. \tag{51}$$

Then, there exist $\gamma(l)$, depending only on the data and l, such that, for any $\sigma \in (0, 1/2)$ chosen, either

$$l^{-\frac{sp}{p-1}}\, Tail((u-\mu^-)_-, x_0, l\rho) \le \sigma\epsilon\omega\,, \tag{52}$$

or

$$\big|[u \le \mu^- + \sigma\epsilon\omega] \cap B_\rho(x_0)\big| \le \frac{\gamma(l)\sigma^{p-1}}{\alpha}|B_\rho|\,.$$

Proof. This time, we show the proof for $u \in \mathrm{DG}^-_{s,p}(\hat\gamma, \Omega)$, the other case can be done again using remark 3.3. We use (34) over balls $B_\rho(x_0) \subset B_{2\rho}(x_0)$ for $(u-k)_-$ with $k = \mu^- + \sigma\epsilon\omega$. Observing that in $B_{2\rho} \subset B_{l\rho}(x_0)$ we have $(u-k)_- \le \sigma\epsilon\omega$ and we get

$$\begin{aligned}
&\iint_{B^2_\rho(x_0)} \frac{(u(x)-k)_-(u(y)-k)_+^{p-1}}{|x-y|^{N+ps}}\,dxdy \\
&\qquad \le \frac{\gamma}{\rho^{ps}}\Big[(\sigma\epsilon\omega)^p|B_{2\rho}| + (\sigma\epsilon\omega)|B_{2\rho}|\mathrm{Tail}((u-k)_-, x_0, 2\rho)^{p-1}\Big] \\
&\qquad\qquad \le \frac{\gamma}{\rho^{ps}}(\sigma\epsilon\omega)^p|B_{2\rho}|
\end{aligned} \tag{53}$$

where we enforced the tail condition (52) to compute

$$\begin{aligned}
&\mathrm{Tail}((u-k)_-, x_0, 2\rho)^{p-1} \\
&\le \gamma\rho^{ps}\int_{\mathbb{R}^N\setminus B_{2\rho}(x_0)} \frac{(u-\mu_-)_-^{p-1}}{|x-x_0|^{N+ps}}\,dx + \gamma\rho^{ps}\int_{\mathbb{R}^N\setminus B_{2\rho}(x_0)} \frac{(\sigma\epsilon\omega)^{p-1}}{|x-x_0|^{N+ps}}\,dx \\
&\le \gamma\rho^{ps}\int_{\mathbb{R}^N\setminus B_{l\rho}(x_0)} \frac{(u-\mu_-)_-^{p-1}}{|x-x_0|^{N+ps}}\,dx \\
&\quad + \gamma\rho^{ps}\int_{B_{l\rho}(x_0)\setminus B_{2\rho}(x_0)} \frac{(u-\mu_-)_-^{p-1}}{|x-x_0|^{N+ps}}\,dx + \gamma(\sigma\epsilon\omega)^{p-1} \\
&\le \gamma l^{-ps}\mathrm{Tail}((u-\mu_-)_-, x_0, l\rho)^{p-1} + \gamma(\sigma\epsilon\omega)^{p-1} \le \gamma(\sigma\epsilon\omega)^{p-1}\,.
\end{aligned}$$

Now, on (53), we estimate the left hand side from below as

$$
\begin{aligned}
&\iint_{B_\rho^2(x_0)} \frac{(u(x)-k)_-(u(y)-k)_+^{p-1}}{|x-y|^{N+ps}}\,dxdy \\
&\geq \left[\int_{B_\rho(x_0)\cap[u(x)\leq\mu^-+\sigma\epsilon\omega/2]} (k-u(x))\left(\int_{B_\rho(x_0)\cap\{u(y)\geq\mu^-+\epsilon\omega\}} \frac{(u(y)-k)^{p-1}}{|x-y|^{N+ps}}\,dy\right)dx\right] \\
&\geq \left(\frac{\sigma\epsilon\omega}{2}\right)\big|B_\rho(x_0)\cap[u(x)\leq\mu^-+\sigma\epsilon\omega/2]\big| \\
&\qquad \cdot \sup_{x\in\mathcal{B}_\rho(x_0)} \int_{B_\rho(x_0)\cap\{u(y)\geq\mu^-+\epsilon\omega\}} \frac{((1-\sigma)\epsilon\omega)^{p-1}}{|x-y|^{N+ps}}\,dy \\
&\geq \left(\frac{\sigma\epsilon\omega}{2}\right)\big|B_\rho(x_0)\cap[u(x)\leq\mu^-+\sigma\epsilon\omega/2]\big| \\
&\qquad \cdot \frac{((1-\sigma)\epsilon\omega)^{p-1}}{(2\rho)^{N+ps}}\big|B_\rho(x_0)\cap[u(y)\geq\mu^-+\epsilon\omega]\big| \\
&\geq \left(\frac{\sigma\epsilon\omega}{2}\right)\big|B_\rho(x_0)\cap[u(x)\leq\mu^-+\sigma\epsilon\omega/2]\big| \\
&\qquad \cdot \frac{((1-\sigma)\epsilon\omega)^{p-1}}{(2\rho)^{N+ps}}\big|B_\rho(x_0)\cap[u(y)\geq\mu^-+\epsilon\omega]\big| \\
&\geq \frac{1}{\gamma}\frac{\alpha\sigma(\epsilon\omega)^p}{\rho^{sp}}(1-\sigma)^{p-1}\big|B_\rho(x_0)\cap[u(x)\leq\mu^-+\sigma\epsilon\omega/2]\big| \qquad (54)
\end{aligned}
$$

where we used (51) and the fact that for $x,\, y \in B_\rho(x_0)$ $|x-y|\leq 2\rho$ at the third inequality. Using (54) in (53) gives us (as $1-\sigma\geq\frac{1}{2}$)

$$\big|B_\rho(x_0)\cap[u(x)\leq\mu^-+\sigma\epsilon\omega/2]\big| \leq \frac{\gamma\sigma^p}{\alpha}|B_\rho|\,,$$

which is the thesis. □

Remark 3.10. Let number ν be determined, by the only data $\{N,p,s,\hat{\gamma}\}$, from Lemma 3.7. Now, if we choose

$$\sigma = \left(\frac{\alpha\nu}{2^{p-1}\gamma}\right)^{\frac{1}{p-1}} \leq \left(\frac{1}{2^{p-1}}\right)^{\frac{1}{p-1}} \leq \frac{1}{2}$$

then the assumption in (44) is verified. Hence Lemma 3.7 implies that either (45) is valid or the reduction

$$\sup_{B_{\rho/2}(x_0)} u \leq \mu^- + a\sigma\epsilon\omega$$

holds true. A similar reasoning applies with (51) and (52).

Remark 3.11. Similarly as in Theorem 2.6, it is possible to prove that $u \in DG^{\pm}_{s,p}(\hat{\gamma},\Omega)$ has a lower(upper) semi-continuous representative (see for instance Thm 9 in [22], or more in general [36]). Being the reasoning similar to the one Theorem 2.6, we prefer to show directly that elements of $DG^{\pm}_{sp}(\hat{\gamma},\Omega)$ have an Hölder continuous representative. A proof of this result can be found in [7]; see also [4], [21] (here $p=2$), [30] for a different proof.

Theorem 3.12 (Oscillation estimates). *Let $u \in DG_{s,p}(\hat{\gamma},\Omega)$. There exist $l>2$ and $\delta \in (0,1)$ depending only on the data $\{N,p,s\}$ such that, if $B_\rho(x_0) \subset B_{l\rho}(x_0) \subset B_{\tilde{R}}(x_0) \subset \Omega$, then*

$$\operatorname*{osc}_{B_{\rho_i}(x_0)} u \le \omega\delta^i \qquad \rho_i = \rho(2l)^{-i}l \qquad \forall\, i \in \mathbb{N}\,,$$

where

$$\omega = 2\|u\|_{L^\infty(B_{\tilde{R}}(x_0))} + \mathit{Tail}(u,x_0,\tilde{R}).$$

Proof. We can consider $x_0=0$ without loss of generality. We are first going to prove that

$$\operatorname*{osc}_{B_{\rho/2}} u \le \delta\omega, \qquad \text{for some } \delta \in (0,1). \tag{55}$$

Let us recall (43) and assume $\mu^+ - \mu^- \ge \frac{1}{2}\omega$; as indeed otherwise (55) holds true with $\delta = 1/2$.

Thanks to our assumption, we have the following dichotomy

$$|[u < \mu^+ - \frac{\omega}{4}] \cap B_\rho| \ge \frac{1}{2}|B_\rho|\,, \tag{56}$$

or

$$|[u > \mu^- + \frac{\omega}{4}] \cap B_\rho| \ge \frac{1}{2}|B_\rho|\,. \tag{57}$$

If (56) holds true, then using both the Shrinking Lemma 3.9 and the Critical Mass Lemma 3.7, as in Remark 3.10 with $a=1/2$, taking $\sigma = \big(\frac{\nu}{4^{p-1}\gamma}\big)^{\frac{1}{p-1}}$, that depends only on the data, we have that either

$$l^{-\frac{ps}{p-1}}\operatorname{Tail}((u-\mu^+)_+,0,l\rho) > \sigma\frac{\omega}{4}\,, \tag{58}$$

or

$$\sup_{B_{\rho/2}} u \le \mu^+ - \frac{\sigma\omega}{8}\,. \tag{59}$$

We see that from (59) follows (55), in fact

$$\operatorname*{osc}_{B_{\rho/2}} u \le \sup_{B_{\rho/2}} -\mu^- \le \mu^+ - \frac{\sigma\omega}{8} - \mu^- \le \left(1-\frac{\sigma}{8}\right)\omega \le \delta\omega\,,$$

for $\delta = \max\{\frac{1}{4}, 1-\frac{\sigma}{8}\}$.

The trick now is to give a condition on l such that (58) does not happen. Indeed

$$\begin{aligned}
&\mathrm{Tail}((u-\mu^+)_+,0,l\rho)^{p-1}\\
&\le (l\rho)^{ps}\left[\int_{\mathbb{R}^N\setminus B_{l\rho}}\frac{|u|^{p-1}}{|x|^{N+ps}}\,dx+\int_{\mathbb{R}^N\setminus B_{l\rho}}\frac{\omega^{p-1}}{|x|^{N+ps}}\,dx\right]\\
&\le (l\rho)^{ps}\left[\int_{\mathbb{R}^N\setminus B_{\tilde R}}\frac{|u|^{p-1}}{|x|^{N+ps}}\,dx+\int_{B_{\tilde R}\setminus B_{l\rho}}\frac{|u|^{p-1}}{|x|^{N+ps}}\,dx+\gamma\,\omega^{p-1}(l\rho)^{-sp}\right]\\
&\le (l\rho)^{ps}\left[\tilde R^{-ps}\mathrm{Tail}(u,0,\tilde R)^{p-1}+\gamma\,\omega^{p-1}(l\rho)^{-ps}\right]\\
&\le \left(\frac{l\rho}{\tilde R}\right)^{ps}\omega^{p-1}+\gamma\omega^{p-1}\le\gamma\omega^{p-1}\,.
\end{aligned}$$

So we just need to take l so that

$$l^{-\frac{ps}{p-1}}\gamma\omega\le\frac{\sigma\omega}{4}\iff l>\left(\frac{4\gamma}{\sigma}\right)^{\frac{p-1}{ps}}\,. \tag{60}$$

If, on the other hand, (57) is valid, then we can repeat the procedure by using Lemma 3.7, in particular (47), and, taking again l so that (60) holds true, observe that in this case $\mathrm{Tail}((u-\mu^-)_-,0,l\rho)^{p-1}\le\gamma\omega^{p-1}$ by a similar argument, we obtain

$$\inf_{B_{\rho/2}}u\ge\mu^-+\frac{\sigma\omega}{8}$$

so that

$$\operatorname*{osc}_{B_{\rho/2}}u\le\mu^+-\inf_{B_{\rho/2}}u\le\mu^+-\mu^--\frac{\sigma\omega}{8}\le\delta\omega\,.$$

In this way, (55) is proven.

Now, we want to iterate this result. We denote by

$$\rho_i=\frac{\rho}{l^i}\,,\qquad B_i=B_{\rho_i}\,,\qquad \mu_i^+=\sup_{B_i}u\,,\qquad \mu_i^-=\inf_{B_i}u\,,$$

and we want to prove that

$$\operatorname*{osc}_{B_i}u\le\delta^i\omega=:\omega_i\qquad\forall\,i\in\mathbb{N},, \tag{61}$$

by induction. The case $i=1$ being done in (55), suppose this is true for all $n\in\{1,\dots,i\}$: we prove it for $i+1$. We assume again $\mu_i^+-\mu_i^-\ge\omega_i/2$, because otherwise (61) is valid, as

$$\operatorname*{osc}_{B_{i+1}}u\le\operatorname*{osc}_{B_i}u\le\frac{\delta^i}{2}\omega=:\frac{\omega_i}{2}\le\delta\omega_i=:\omega_{i+1}\,.$$

We have again the following dichotomy

$$|[u<\mu_i^+-\frac{\omega_i}{4}]\cap B_i|\geq\frac{1}{2}|B_i|\,, \tag{62}$$

or

$$|[u>\mu_i^-+\frac{\omega_i}{4}]\cap B_i|\geq\frac{1}{2}|B_i|\,. \tag{63}$$

Suppose (62) is true, then again using the Shrinking Lemma 3.9 and the Critical Mass lemma 3.7, now on balls $B_i:=B_{\rho_i}(x_0)\subset\Omega$ we either allow

$$l^{-\frac{sp}{p-1}}\operatorname{Tail}((u-\mu_i^+)_+,0,\rho_{i+1})>\sigma\frac{\omega_i}{4}\,, \tag{64}$$

or

$$\sup_{B_{i+1}}u\leq\sup_{B_{\rho/(2^{i+1}l^{i-1})}}u\leq\mu_i^+-\frac{a\sigma\omega_i}{4}\,. \tag{65}$$

From (65) follows (61) because

$$\operatorname*{osc}_{B_{i+1}}u\leq\sup_{B_{i+1}}u-\mu_i^-\leq\mu_i^+-\frac{a\sigma\omega_i}{4}-\mu_i^-\leq\frac{(1-\sigma a)\omega_i}{4}=:\omega_{i+1}\,.$$

To make sure (64) does not happen consider that

$$\begin{aligned}&\operatorname{Tail}((u-\mu_i^+)_+,0,l\rho_i)^{p-1}\\&\quad\leq(l\rho_i)^{sp}\left[\int_{\mathbb{R}^N\setminus B_{l\rho}}\frac{(u-\mu_i^+)^{p-1}}{|x|^{N+ps}}\,dx+\sum_{j=1}^{i}\int_{B_{j-1}\setminus B_j}\frac{(u-\mu_i^+)^{p-1}}{|x|^{N+ps}}\,dx\right].\end{aligned}$$

The first term is estimated exactly as the previous case, while for the second term we use the inductive hypothesis (61) to deduce

$$(u-\mu_i^+)_+\leq\mu_{j-1}^+-\mu_i^+\leq\mu_{j-1}^+-\mu_{j-1}^-\leq\omega_{j-1}\;,\quad\text{for }x\in B_{j-1}\backslash B_j$$

so that

$$\int_{B_{j-1}\setminus B_j}\frac{(u-\mu_i^+)^{p-1}}{|x|^{N+ps}}\,dx\leq\gamma\frac{\omega_{j-1}^{p-1}}{\rho_j^{sp}}\,.$$

Combining these estimates gives us

$$\begin{aligned}\operatorname{Tail}((u-\mu_i^+)_+,0,l\rho_i)^{p-1}&\leq(l\rho_i)^{ps}\gamma\frac{\omega_i^{p-1}}{(l\rho)^{ps}}+\gamma(l\rho_i)^{ps}\sum_{j=1}^{i}\frac{\omega_{j-1}^{p-1}}{\rho_j^{sp}}\\&=\gamma\,\omega_i^{p-1}\frac{\delta^{i(1-p)}}{l^{i(ps)}}+\gamma\omega_i^{p-1}\sum_{j=1}^{i}\frac{\delta^{(i+1-j)(1-p)}}{l^{(i-j)ps}}\\&\leq\gamma\omega^{p-1}\sum_{j=0}^{i}\frac{\delta^{(i-j)(1-p)}}{l^{(i-j)ps}}\,.\end{aligned}$$

The summation of this last inequality is bounded if we choose l so that

$$\frac{\delta^{1-p}}{l^{ps}} \leq \frac{1}{2} \Longleftrightarrow l \geq \left(\frac{2}{\delta^{p-1}}\right)^{\frac{1}{ps}}.$$

Setting

$$l = \left(\frac{4\gamma}{\sigma}\right)^{\frac{p-1}{ps}} + \left(\frac{2}{\delta^{p-1}}\right)^{\frac{1}{ps}}$$

satisfies (60) again.

The case when (63) is true is similar. This finally proves (61). □

THEOREM 3.13 (Hölder continuity). *Let $u \in DG_{s,p}(\hat{\gamma}, \Omega)$. Then, u has a Hölder continuous representative. Moreover, for $x_0 \in \Omega$, $\tilde{R} > 0$, such that $B_{\tilde{R}}(x_0) \subset \Omega$ there exists a constant γ, depending only on the data, such that*

$$\operatorname*{osc}_{B_\rho(x_0)} u := \sup_{B_\rho(x_0)} u - \inf_{B_\rho(x_0)} u \leq \gamma\omega \left(\frac{\rho l}{\tilde{R}}\right)^{\alpha} \tag{66}$$

where $\omega = 2\|u\|_{L^\infty(B_{\tilde{R}}(x_0))} + Tail(u, x_0, \tilde{R})$ and $\rho \in (0, \tilde{R}/l)$, for some $l > 2$.

Proof. Consider again $x_0 = O$. Fix a ball $B_{\tilde{R}} \subset \Omega$, and l by the conditions in Theorem 3.12. Then by Theorem 3.12 for the ball $B_R := B_{\tilde{R}/l} \subset B_{\tilde{R}} \subset \Omega$ we have

$$\operatorname*{osc}_{B_{R/(2l)^i}} u \leq \omega\delta^i$$

for $\delta \in (0,1)$, which corresponds to (23) for suitable parameters. The procedure can be repeated now using the general principle offered by (23)-(24). □

REMARK 3.14. Observe that up until now we have used (34) without the need for the second integral of the second term in the LHS to be over $\mathbb{R}^N$. In the following section, we will use the definition in its full strength.

Fractional Harnack-type Inequalities

We prove now that elements of the De Giorgi classes satisfy a nonlocal version of the Harnack inequality. Our idea is reminiscent of the method of Moser and Trudinger, for we combine the estimates of the supremum with a weak-Harnack inequality (see [14], for boundary-value problems). We begin with the latter, following the steps of [29] .

PROPOSITION 3.15 (Weak Harnack inequality). *Let $u \in DG^{-}_{s,p}(\hat{\gamma}, \Omega)$ be non-negative in $B_\rho(x_0) \subset \Omega$. Then , there exists a constant $\eta \in (0,1)$, depending only on the data, such that, $\forall\, \sigma \in (0, \frac{1}{2})$, we have*

$$\begin{aligned}&\inf_{B_{\sigma\rho/2(x_0)}} u + (\sigma)^{\frac{sp}{p-1}}\, Tail(u_-, x_0, \rho) \\ &\qquad\qquad \geq \eta \Bigg\{ \bigg(\fint_{B_{\sigma\rho}(x_0)} u^{p-1}(x)\, dx \bigg)^{\frac{1}{p-1}} + Tail(u_+, x_0, \sigma\rho) \Bigg\}. \quad (67)\end{aligned}$$

In order to prove this Proposition, we first require some preliminary lemmata. These are, in fact, measure shrinking lemmata, really similar to Lemma 3.9, in that it involves the RHS of (34). It is here that the difference of our definition with [7] plays its role: see the proof of Lemma 3.16. The first lemma measures he level set of u with its local average.

LEMMA 3.16. *Let $u \in DG^{-}_{s,p}(\hat{\gamma}, \Omega))$ be non-negative in $B_R(x_0) \subset \Omega$. Let $k > 0$, $\sigma \in (0, 1/2)$ and $\rho > 0$ such that $B_\rho(x_0) \subset \Omega$. Then there exists a constant $\gamma > 1$, depending only on the data, such that either*

$$(\sigma)^{\frac{sp}{p-1}}\, Tail(u_-, x_0, \rho) > k$$

or

$$|[u \leq k] \cap B_{\sigma\rho}(x_0)| \leq \frac{\gamma k^{p-1}}{avg(u^{p-1})_{B_{\sigma\rho}(x_0)}} |B_{\sigma\rho}|$$

where $avg(u^{p-1})_{B_{\sigma\rho}(x_0)}$ is the integral average on $B_{\sigma\rho(x_0)}$.

Proof. Using the definition (34) on $B_{\sigma\rho}(x_0)$ and $B_{2\sigma\rho}(x_0)$ and operating as in (53), with the fact that now $(u-k)_- \leq k$ and the hypothesis on the tail, we have

$$\int_{B_{\sigma\rho}(x_0)} \int_{B_{\sigma\rho}(x_0)} \frac{(u(x)-k)_-(u(y)-k)_+^{p-1}}{|x-y|^{N+ps}}\, dydx \leq \frac{\gamma\, k^p}{(\sigma)^{N+2sp}\rho^{sp}} |B_{\sigma\rho}|\,. \quad (68)$$

We bound (68) from below: let us consider the fact that $u_+^{p-1} \leq \gamma(u-k)_+^{p-1} + \gamma k^{p-1}$ and hence

$$\begin{aligned}&\iint_{B^2_{\sigma\rho}(x_0)} \frac{(u(x)-k)_-(u(y)-k)_+^{p-1}}{|x-y|^{N+ps}}\, dxdy \\ &\geq \iint_{B^2_{\sigma\rho}(x_0)} \frac{(u(x)-k)_-[\frac{1}{\gamma}(u_+)^{p-1} - k^{p-1}]}{(2\sigma\rho)^{N+ps}}\, dxdy \\ &\geq \frac{\gamma^{-1}}{(\sigma\rho)^{ps}} \bigg(\int_{B_{\sigma\rho}(x_0)} (u(x)-k)_-\, dx \bigg) \bigg(\fint_{B_{\sigma\rho}(x_0)} u(y)^{p-1}\, dy \bigg) - \frac{\gamma\, k^p}{(\sigma\rho)^{ps}} |B_{\sigma\rho}|\,.\end{aligned}$$

So now from (68) we get

$$\frac{\gamma}{(\sigma\rho)^{ps}}\left(\int_{B_{\sigma\rho}(x_0)}(u(x)-k)_-\,dx\right)(\mathrm{avg}(u^{p-1})_{B_{\sigma\rho}(x_0)}) \le \frac{\gamma\,k^p}{(\sigma\rho)^{ps}}|B_{\sigma\rho}|\,. \tag{69}$$

Now, just observe that

$$\begin{aligned}
&\int_{B_{\sigma\rho}(x_0)}(u(x)-k)_-\,dx = \int_{B_{\sigma\rho}(x_0)\cap[u\le k]} k-u(x)\,dx\\
&\ge \int_{B_{\sigma\rho}(x_0)\cap[u\le\frac12 k]} k-u(x)\,dx \ge \int_{B_{\sigma\rho}(x_0)\cap[u\le\frac12 k]} k-\frac{k}{2}\,dx\\
&\ge \left|B_{\sigma\rho}(x_0)\cap[u\le\frac{1}{2}k]\right|\frac{k}{2}\,,
\end{aligned}$$

so (69) becomes

$$\left|B_{\sigma\rho}(x_0)\cap[u\le\frac{1}{2}k]\right| \le \frac{\gamma\,k^{p-1}}{\mathrm{avg}(u^{p-1})_{B_{\sigma\rho}(x_0)}}|B_{\sigma\rho}|\,. \qquad \square$$

Next lemma is similar to the previous one, however the measure of the set is estimated by a nonlocal integral.

LEMMA 3.17. *Let $u\in DG^-_{s,p}(\hat\gamma,\Omega))$ be non-negative in $B_R(x_0)\subset\Omega$. Let $k>0$, $\sigma\in(0,1/2)$ and $\rho>0$ such that $B_\rho(x_0)\subset\Omega$. Then, there exists a constant $\gamma>1$, depending only on the data, such that either*

$$(\sigma)^{\frac{sp}{p-1}}\,\mathit{Tail}(u_-,x_0,\rho)>k\,,$$

or

$$|\{u\le k\}\cap B_{\sigma\rho}(x_0)| \le \frac{\gamma k^{p-1}}{\mathit{Tail}(u_+,x_0,\sigma\rho)}|B_{\sigma\rho}|\,.$$

Proof. Again, the proof starts the same as Lemma 3.16, so that we recover again (68). The difference lies in the estimate from below. Again we employ that $u_+^{p-1}\le\gamma(u-k)_+^{p-1}+\gamma k^{p-1}$, but we also consider that when $|y-x_0|\ge\sigma\rho$ and $|x-x_0|\le\sigma\rho$, one has

$$|x-y|\le|x-x_0|+|y-x_0|\le\sigma\rho+|y-x_0|\le 2|y-x_0|$$

so that we get

$$
\begin{aligned}
&\int_{B_{\sigma\rho}(x_0)}\int_{\mathbb{R}^N}\frac{(u(x)-k)_-(u(y)-k)_+^{p-1}}{|x-y|^{N+ps}}\,dxdy\\
&\geq \int_{B_{\sigma\rho}(x_0)}\int_{\mathbb{R}^N\setminus B_{\sigma\rho}}\frac{(u(x)-k)_-(u(y)-k)_+^{p-1}}{|x-y|^{N+ps}}\,dxdy\\
&\geq \int_{B_{\sigma\rho}(x_0)}\int_{\mathbb{R}^N\setminus B_{\sigma\rho}}\frac{(u(x)-k)_-[\frac{1}{\gamma}(u_+)^{p-1}-k^{p-1}]}{(2|y-x_0|)^{N+ps}}\,dxdy\\
&\geq \frac{\gamma}{(\sigma\rho)^{ps}}[\mathrm{Tail}(u_+,x_0,\sigma\rho)^{p-1}]\int_{B_{\sigma\rho}(x_0)}(u(x)-k)_-\,dx\\
&\qquad -\gamma k^p\left(\int_{\mathbb{R}^N\setminus B_{\sigma\rho}}\frac{dy}{|y-x_0|^{N+ps}}\right)|B_{\sigma\rho}|\\
&\geq \frac{\gamma}{(\sigma\rho)^{ps}}[\mathrm{Tail}(u_+,x_0,\sigma\rho)^{p-1}]\int_{B_{\sigma\rho}(x_0)}(u(x)-k)_-\,dx-\frac{\gamma k^p}{(\sigma\rho)^{ps}}|B_{\sigma\rho}|\,.
\end{aligned}
$$

This gives us

$$
[\mathrm{Tail}(u_+,x_0,\sigma\rho)^{p-1}]\int_{B_{\sigma\rho}(x_0)}(u(x)-k)_-\,dx\leq \gamma k^p|B_{\sigma\rho}|\,,
$$

the right hand side is estimated as in 3.16 so that we arrive at

$$
\left|B_{\sigma\rho}(x_0)\cap\{u\leq\frac{1}{2}k\}\right|\leq\frac{\gamma\,k^{p-1}}{[\mathrm{Tail}(u_+,x_0,\sigma\rho)^{p-1}]}|B_{\sigma\rho}|\,. \qquad \square
$$

Proof of proposition 3.15. Consider ν be the constant fixed in the Critical Mass lemma and choose $a=1/2$, in the formulation of Remark 3.8. Let us choose k_1 and k_2 as

$$
k_1:=\left(\frac{(\mathrm{avg}(u^{p-1})_{B_{\sigma\rho}(x_0)})\nu}{2\gamma_1}\right)^{\frac{1}{p-1}}\quad\text{and}\quad k_2:=\left(\frac{[\mathrm{Tail}(u_+,x_0,\sigma\rho)^{p-1}]\nu}{2\gamma_2}\right)^{\frac{1}{p-1}}.
$$

Then, taking γ_1 and γ_2 as determined in Lemma 3.16 and Lemma 3.17 respectively, we observe that k_1 and k_2 satisfy

$$
\frac{\gamma_1 k_1^{p-1}}{(\mathrm{avg}(u^{p-1})_{B_{\sigma\rho}(x_0)})}\leq\nu\quad\text{and}\quad\frac{\gamma_2 k_2^{p-1}}{[\mathrm{Tail}(u_+,x_0,\sigma\rho)^{p-1}]}\leq\nu\,.
$$

So, by the critical mass lemma, we have that in both cases either

$$
\sigma^{\frac{sp}{p-1}}\mathrm{Tail}(u_-,x_0,\rho)>k_i\qquad i=1,2 \tag{70}
$$

or

$$u \geq \frac{k_i}{2} \qquad \text{in } B_{\sigma\rho/2}(x_0). \tag{71}$$

Summing on (70)-(71) for $i = 1, 2$

$$\inf_{B_{\sigma\rho/2(x_0)}} u + \sigma^{\frac{sp}{p-1}} \mathrm{Tail}(u_-, x_0, \rho) \geq \frac{k_1 + k_2}{2},$$

$$\inf_{B_{\sigma\rho/2(x_0)}} u + \sigma^{\frac{sp}{p-1}} \mathrm{Tail}(u_-, x_0, \rho) \geq \eta \left[\left(\fint_{B_{\sigma\rho}(x_0)} u_+^{p-1}\, dx \right)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+, x_0, \sigma\rho) \right].$$

Having chosen η the biggest between the two constant $\frac{\nu}{4\gamma_1}$ and $\frac{\nu}{4\gamma_2}$ that depend only on the data. □

With the aid of Lemmata 3.16-3.17, we can prove the following nonlocal version of the Harnack inequality. A first proof, for solutions to equations, can be found in [14].

Theorem 3.18 (Harnack inequality). *Let $u \in DG_{s,p}(\hat{\gamma}, \Omega)$ be non-negative in $B_R(x_0) \subset \Omega$. Then, there exists a constant $\gamma > 1$, depending only on the data, such that, for all $\rho \leq \frac{R}{2}$, we have*

$$\inf_{B_{\rho/2}(x_0)} u + \left(\frac{\rho}{R} \right)^{\frac{ps}{p-1}} \mathit{Tail}(u_-, x_0, R) \geq \frac{1}{\gamma} \sup_{B_{\rho/2}(x_0)} u\,. \tag{72}$$

Proof. We employ the estimate (36), for the restricted range $\sigma \in (1/3, 2/3)$, so that γ depends only on the data.

$$\begin{aligned}
\sup_{B_{\sigma\rho}(x_0)} u &\leq \gamma \left\{ \left(\fint_{B_\rho(x_0)} u^p(x)\, dx \right)^{\frac{1}{p}} + \mathrm{Tail}(u_+, x_0, B_{\sigma\rho}) \right\} \\
&\leq \gamma \left\{ \left(\fint_{B_\rho(x_0)} u^p(x)\, dx \right)^{\frac{1}{p}} + (\sigma\rho)^{\frac{ps}{p-1}} \left[\int_{B_\rho(x_0) \setminus B_{\sigma\rho}(x_0)} \frac{u^{p-1}(x)}{|x - x_0|^{N+ps}}\, dx \right. \right. \\
&\qquad \left. \left. + \int_{\mathbb{R}^N \setminus B_\rho(x_0)} \frac{u_+^{p-1}(x)}{|x - x_0|^{N+ps}}\, dx \right]^{\frac{1}{p-1}} \right\} \\
&\leq \gamma \left\{ \left(\fint_{B_\rho(x_0)} u^p(x)\, dx \right)^{\frac{1}{p}} \right. \\
&\qquad \left. + \left[\frac{1}{(\sigma\rho)^N} \int_{B_\rho(x_0) \setminus B_{\sigma\rho}(x_0)} u^{p-1}(x)\, dx + \mathrm{Tail}(u_+, x_0, \rho)^{p-1} \right]^{\frac{1}{p-1}} \right\} \\
&\leq \gamma \left\{ \left(\fint_{B_\rho(x_0)} u^p(x)\, dx \right)^{\frac{1}{p}} + \left[\frac{1}{\sigma^N} \fint_{B_\rho(x_0)} u^{p-1}(x)\, dx + \mathrm{Tail}(u_+, x_0, \rho)^{p-1} \right]^{\frac{1}{p-1}} \right\}
\end{aligned}$$

$$
\begin{aligned}
&\le \gamma\Big\{\Big(\fint_{B_\rho(x_0)} u^p(x)\,dx\Big)^{\frac{1}{p}} + \Big(\frac{1}{\sigma^N}\fint_{B_\rho(x_0)} u^{p-1}(x)\,dx\Big)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+,x_0,\rho)\Big\}\\
&\le \gamma\Big\{\Big(\sup_{B_\rho(x_0)} u\Big)^{\frac{1}{p}}\Big(\fint_{B_\rho(x_0)} u^{p-1}(x)\,dx\Big)^{\frac{1}{p}}\\
&\qquad + \Big(\frac{1}{\sigma^N}\fint_{B_\rho(x_0)} u^{p-1}(x)\,dx\Big)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+,x_0,\rho)\Big\}, \qquad (73)
\end{aligned}
$$

where we have used that $u \in L^\infty_{\mathrm{loc}}(\Omega)$, $|x-x_0| \ge (\sigma\rho)^{N+ps}$ in $B_\rho\backslash B_{\sigma\rho}$ and that $(a+b)^q \le \gamma(a^q+b^q)$ for a, b, $q>0$. Now, we define the following increasing sequence of balls

$$
\begin{cases}
\rho_0 = \tau\rho\\
\rho_n = \rho\big(\tau + \sum_{j=1}^n \frac{(1-\tau)}{2^j}\big)\\
\rho_\infty = \rho
\end{cases}
\qquad \text{and} \qquad
\begin{cases}
B_0 = B_{\rho_0}(x_0) = B_{\tau\rho}(x_0)\\
B_n = B_{\rho_n}(x_0)\\
B_\infty = B_{\rho_\infty}(x_0) = B_\rho(x_0).
\end{cases}
$$

If we denote by $S_n = \sup_{B_n} u$, apply the inequality (73) to the radius ρ_n and ρ_{n+1}, along with the Young inequality, we obtain

$$
\begin{aligned}
S_n &\le \gamma\epsilon S_{n+1} + \gamma\Big\{\frac{\gamma}{\epsilon^{p-1}}\Big(\fint_{B_{n+1}} u^{p-1}(x)\,dx\Big)^{\frac{1}{p-1}}\\
&\qquad + \Big(\frac{1}{(\tau^N}\fint_{B_{n+1}} u^{p-1}(x)\,dx\Big)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+,x_0,\rho_{n+1})\Big\}\\
&\le \gamma\epsilon S_{n+1} + \gamma\Big\{\frac{\gamma}{\epsilon^{p-1}\tau^N}\Big(\fint_{B_{n+1}} u^{p-1}(x)\,dx\Big)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+,x_0,\rho_{n+1})\Big\}. \qquad (74)
\end{aligned}
$$

For the tail of u, we use this estimate:

$$
\begin{aligned}
&\mathrm{Tail}(u_+,x_0,\rho_{n+1})\\
&= \Big(\rho_{n+1}^{ps}\int_{\mathbb{R}^N\backslash B_{n+1}} \frac{u_+^{p-1}}{|x-x_0|^{N+ps}}\,dx\Big)^{\frac{1}{p-1}}\\
&= \Big(\rho_{n+1}^{ps}\int_{\mathbb{R}^N\backslash B_\infty} \frac{u_+^{p-1}}{|x-x_0|^{N+ps}}\,dx + \rho_{n+1}^{ps}\int_{B_\infty\backslash B_{n+1}} \frac{u^{p-1}}{|x-x_0|^{N+ps}}\,dx\Big)^{\frac{1}{p-1}}\\
&\le \Big(\rho^{ps}\int_{\mathbb{R}^N\backslash B_\infty} \frac{u_+^{p-1}}{|x-x_0|^{N+ps}}\,dx + \frac{1}{\rho_{n+1}^N}\int_{B_\infty\backslash B_{n+1}} u^{p-1}\,dx\Big)^{\frac{1}{p-1}}\\
&\le \gamma\mathrm{Tail}(u_+,x_0,\rho) + \frac{\gamma}{\tau^N}\Big(\fint_{B_\infty} u^{p-1}\,dx\Big)^{\frac{1}{p-1}}.
\end{aligned}
$$

Hence, inequality (74) becomes

$$S_n \leq \gamma\epsilon S_{n+1} + \gamma\left\{\frac{\gamma}{\epsilon^{p-1}\tau^N}\left(\fint_{B_\infty} u^{p-1}(x)\,dx\right)^{\frac{1}{p-1}} + \gamma \mathrm{Tail}(u_+, x_0, \rho)\right\}.$$

Choosing ϵ, $\tau = \frac{1}{2}$ and iterating backwards from n to 0, delivers

$$S_0 \leq \gamma\left(\frac{1}{2}\right)^n S_n + \gamma\left(\sum_{j=1}^{n} 2^{-j}\right)\left\{\left(\fint_{B_\infty} u^{p-1}(x)\,dx\right)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+, x_0, \rho)\right\}.$$

Now we just let $n \to \infty$ and obtain

$$\sup_{B_{\rho/2}(x_0)} u \leq \gamma\left\{\left(\fint_{B_\rho(x_0)} u^{p-1}(x)\,dx\right)^{\frac{1}{p-1}} + \mathrm{Tail}(u_+, x_0, \rho)\right\}.$$

Finally, we just use (67) to have

$$\sup_{B_{\rho/2}(x_0)} u \leq \frac{\gamma}{\eta}\left\{\inf_{B_{\rho/2(x_0)}} u + \left(\frac{\rho}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-, x_0, R)\right\}$$

which is our thesis. □

Remark 3.19. We recovered Harnack's estimate (72) and the oscillation estimates (66) independently. However, The Harnack's estimate can be used to derive the oscillation estimates and vice versa.

3.1. How big is $DG_{s,p}(\hat{\gamma}, \Omega)$?

Also in this case De Giorgi classes $DG_{s,p}(\hat{\gamma}, \Omega)$ encompass weak solutions to a group of partial differential equations modeled after the fractional p-Laplacian and minima of suitable functionals of the Calculus of Variations. We are going to give just a quick glance at this topic: our presentation here is far from being complete. In order to give a "nonlocal" formulation, the introduction of the following space is necessary:

$$\mathbb{W}^{s,p}(\Omega) = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ meas.},\ u_{|\Omega} \in L^p(\Omega), \right.$$
$$\left. (x,y) \to \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \in L^1(\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2)\right\}.$$

Let us denote $\mathcal{C}_\Omega = \mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2$ for ease of notation.

Definition 3.20. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. A function $u \in \mathbb{W}^{s,p}(\Omega)$ is said to be a sub (super) minimizer of the functional*

$$\mathcal{E}(u;\Omega) = \iint_{\mathcal{C}_\Omega} |u(x)-u(y)|^p K(x,y)\,dxdy,$$

in Ω, where the kernel $K : \mathbb{R}^{2N} \to [0,+\infty)$ is a measurable function satisfying for some $\Lambda > 0$

$$\frac{1}{\Lambda|x-y|^{N+ps}} \le K(x,y) = K(y,x) \le \frac{\Lambda}{|x-y|^{N+ps}}\,, \tag{75}$$

if $\mathcal{E}(u;\Omega) \le \mathcal{E}(u+\varphi;\Omega)$, for any non-positive (non-negative) measurable function $\varphi : \mathbb{R}^N \to \mathbb{R}$ supported inside Ω.

With a little bit of extra care in the manipulation of the second term at the left-hand side of (34), the membership of minima of $\mathcal{E}$ to the class $DG_{s,p}(\hat\gamma,\Omega)$ can be ensured.

Lemma 3.21 (See [7], Proposition 7.5). *Let $u \in \mathbb{W}^{s,p}(\Omega)$ be a sub(super) minimizer for $\mathcal{E}(u;\Omega)$. Then, there exists a constant $\hat\gamma > 0$ depending only on $\{N,s,p,\Lambda\}$ such that $u \in DG^{\pm}_{s,p}(\hat\gamma,\Omega)$.*

Similarly, (*nonlocal*) weak solutions to fractional p-Laplacian equations are defined. Here, the definition is *local*, because the testing procedure allows functions compactly supported in Ω, but the solutions must be defined in the whole $\mathbb{R}^N$.

Definition 3.22. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. A function $u \in \mathbb{W}^{s,p}(\Omega)$ is a weak sub (super)-solution to equation $\mathcal{L}(u) = 0$ locally weakly in Ω, if for any nonnegative (non-positive) function $\varphi \in W^{s,p}(\mathbb{R}^N)$ such that $supp(\varphi) \subset \Omega$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,dxdy \le 0\,. \tag{76}$$

Again, the membership to $DG^{\pm}_{s,p}$ of weak solutions to $\mathcal{L}(\cdot) = 0$ in Ω can be ensured.

Lemma 3.23 (See [7], Proposition 8.5, [30] Corollary 3.2). *Let $u \in \mathbb{W}^{s,p}(\Omega)$ be a weak sub(super) solution to $\mathcal{L}(u) = 0$ locally weakly in Ω. Then, there exists a constant $\hat\gamma > 0$ depending only on $\{N,s,p,\Lambda\}$ such that $u \in DG^{\pm}_{s,p}(\hat\gamma,\Omega)$.*

Proof. Let u be a weak sub-solution to $\mathcal{L}(u) = 0$ and let us suppose as usual that $x_0 = O$. We consider two radii $0 < r < R$ and a cut off function $\xi \in C_0^\infty(B_{\frac{R+r}{2}})$, such that $0 \le \xi \le 1$, $\xi = 1$ in B_r, $|\nabla\xi| \le 4/(R-r)$. Here we observe that by

Lagrange's Theorem and the convexity of B_R, the increment of ξ between two points $x, y \in \mathcal{B}_R$ can be evaluated by

$$|\xi(x) - \xi(y)|^p \leq \left(\sup_{z \in B_R} |\nabla \xi(z)|^p\right)|x-y|^p \leq \gamma\left(\frac{|x-y|}{R}\right)^p.$$

Now, let us fix a level $k \in \mathbb{R}$, write $w_+ = (u-k)_+$ and $A^+(k,\rho) = [u > k] \cap B_\rho$. By testing (76) with the function $\varphi := \xi^p w_+$ and using the symmetry of the integrand we obtain

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi^p w_+(x) - \xi^p w_+(y))K(x,y)\,dxdy \\
&= \int_{B_R}\int_{B_R} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi^p w_+(x) - \xi^p w_+(y))K(x,y)\,dxdy \\
&\quad + \iint_{\mathcal{C}_{B_R}\setminus B_R^2} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi^p w_+(x) - \xi^p w_+(y))K(x,y)\,dxdy \\
&= 2\int_{B_R}\int_{B_R\cap[u(x)>u(y)]} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi^p w_+(x) - \xi^p w_+(y))K(x,y)\,dxdy \\
&\quad + 2\int_{B_R}\int_{\mathbb{R}^N\setminus B_R} |u(x)-u(y)|^{p-2}(u(x)-u(y))\xi^p w_+(x)K(x,y)\,dxdy \\
&= I_1 + I_2\,,
\end{aligned} \tag{77}$$

First, we consider the contributions from I_1. If $x, y \notin A^+(k,R)$ then $w_+(x) = w_+(y) = 0$. If otherwise $x \in A^+(k,R)$ and $y \in B_R \setminus A^+(k,R)$ then

$$\begin{aligned}
&|u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi^p w_+(x) - \xi^p w_+(y)) \\
&= \xi^p(x)|w_+(x) + w_-(y)|^{p-1} w_+(x) \\
&\geq \gamma^{-1}\xi^p(x)\, w_+^p(x) + \gamma^{-1}\,\xi^p(x) w_-^{p-1}(y) w_+(x)\,.
\end{aligned}$$

The case where the role of x and y is switched, which is $y \in A^+(k,R)$ and $x \in B_R \setminus A^+(k,R)$, cannot happen since we are restricting the evaluation on the set $[u(x) > u(y)]$.

Finally, when $x, y \in A^+(k,R)$, we can reformulate the expression as

$$\begin{aligned}
&|u(x)-u(y)|^{p-2}(u(x)-u(y))(\xi^p w_+(x) - \xi^p w_+(y)) \\
&\qquad = (w_+(x) - w_+(y))^{p-1}(w_+\xi^p(x) - w_+\xi^p(y))\,.
\end{aligned} \tag{78}$$

Here we have to consider the two cases $\xi(x) \geq \xi(y)$ and $\xi(x) \leq \xi(y)$ separately. Assuming the former, we easily estimate (78) by

$$(w_+(x) - w_+(y))^{p-1}(w_+\xi^p(x) - w_+\xi^p(y)) \geq \xi^p(x)|w_+(x) - w_+(y)|^p \,. \qquad (79)$$

Vice versa if $\xi(y) > \xi(x)$ we write (78) by summing and subtracting $w_+(x)\xi^p(y)$ we obtain

$$\begin{aligned}&(w_+(x) - w_+(y))^{p-1}(w_+\xi^p(x) - w_+\xi^p(y)) \\ &\quad = |w_+(x) - w_+(y)|^p\xi^p(y) + (w_+(x) - w_+(y))^{p-1}w_+(x)(\xi^p(x) - \xi^p(y)) \\ &\quad = A + B \,.\end{aligned}$$

The use the Young's inequality

$$a^p - b^p \leq \epsilon a^p + \frac{\gamma}{\epsilon^{p-1}}(a-b)^p \qquad \text{for } a \geq b \geq 0,\ \epsilon > 0\,,$$

with $a = \xi(y)$, $b = \xi(x)$ and $\epsilon = (w_+(x) - w_+(y))/2w_+(x)$, allows us to arrive again at a similar expression

$$\begin{aligned}&|w_+(x) - w_+(y)|^p\xi^p(y) - (w_+(x) - w_+(y))^{p-1}w_+(x)(\xi^p(y) - \xi^p(x)) \\ &\geq \frac{1}{2}\xi^p(y)|w_+(x) - w_+(y)|^p - \gamma^{-1}\max\{w_+(x)^p, w_+(y)^p\}|\xi(x) - \xi(y)|^p \,. \qquad (80)\end{aligned}$$

Hence, since both expressions in the estimates (80) and (79) are larger than

$$\frac{1}{2}\max\{\xi^p(x), \xi^p(y)\}|w_+(x) - w_+(y)|^p - \gamma^{-1}\max\{w_+(x)^p, w_+(y)^p\}|\xi(x) - \xi(y)|^p,$$

we estimate the piece of I_1 taken on $A^+(k,R) \times A^+(k,R)$ with this quantity, irrespectively of the values of ξ. By virtue of these inequalities and (75) we estimate I_1 with

$$\begin{aligned}\gamma I_1 \geqslant &\iint_{A^+(k,R)\times B_R\setminus A^+(k,R)} \Big[\xi^p\, w_+^p(x) + \xi^p(x) w_-^{p-1}(y) w_+(x)\Big] \frac{dxdy}{|x-y|^{N+ps}} \\ &+ \iint_{A^+(k,R)\times A^+(k,R)} \max\{\xi^p(x), \xi^p(y)\}|w_+(x) - w_+(y)|^p \frac{dxdy}{|x-y|^{N+ps}} \\ &- \iint_{A^+(k,R)\times A^+(k,R)} \max\{w_+^p(x), w_+^p(y)\}|\xi(x) - \xi(y)|^p \frac{dxdy}{|x-y|^{N+ps}} \\ \geq &\iint_{B_R\times B_R} |w_+(x) - w_+(y)|^p \min\{\xi(x)^p, \xi(y)^p\}\, \frac{dxdy}{|x-y|^{N+ps}} \\ &+ \int_{B_r} w_+(x)\bigg(\int_{B_R} \frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\, dy\bigg)\, dx \\ &- \iint_{B_R\times B_R} \max\{w_+^p(x), w_+^p(y)\} \frac{|\xi(x) - \xi(y)|^p}{|x-y|^{N+ps}}\, dxdy \,, \qquad (81)\end{aligned}$$

where the first integral of the right-hand side has been reconstructed in the whole set from the sum of the first and the third term of the left-hand side. The properties of ξ allow to estimate from below the positive integrals on the right-hand side on the smaller set $B_r \times B_r$ and from above the negative one,

$$
\begin{aligned}
\iint_{B_R\times B_R\cap[u(x)>u(y)]} & \max\{w_+^p(x), w_+^p(y)\}\frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+ps}}\,dxdy \\
&\le \int_{B_R} w_+^p(x)\int_{B_R}\frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+ps}}\,dy\,dx \\
&\le \frac{\gamma}{(R-r)^p}\int_{B_R} w_+^p(x)\int_{B_R}\frac{dy}{|x-y|^{N+p(s-1)}}\,dx \\
&= \gamma\,\frac{R^{(1-s)p}}{(R-r)^p}\|w_+\|^p_{L^p(B_R)}\,.
\end{aligned}
$$

In conclusion, (81) becomes

$$
\begin{aligned}
\gamma I_1 \ge [w_+]^p_{W^{s,p}(B_r)} + \int_{B_r} w_+(x)\int_{B_R}\frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\,dy\,dx& \\
- \frac{R^{(1-s)p}}{(R-r)^p}\|w_+\|^p_{L^p(B_R)}\,.& \qquad (82)
\end{aligned}
$$

Now we consider the contribution from I_2 in (77). It is here that the energy estimates diversify from the ones of [7]. We divide the cases

$$
\begin{aligned}
\gamma I_2 &= \iint_{B_R\times\{\mathbb{R}^N\setminus B_R\}} |u(x)-u(y)|^{p-2}(u(x)-u(y))\xi^p w_+(x)K(x,y)\,dxdy \\
&\ge \int_{B_r} w_+(x)\left[\int_{[\mathbb{R}^N\setminus B_R]\cap[u(x)\ge u(y)]}\frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy\right]dx \\
&\quad - \int_{B_r} w_+(x)\left[\int_{[\mathbb{R}^N\setminus B_R]\cap[u(y)>u(x)]}\frac{(u(y)-u(x))^{p-1}}{|x-y|^{N+ps}}\,dy\right]dx\,.
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
\int_{B_r} & w_+(x)\left[\int_{[\mathbb{R}^N\setminus B_R]\cap[u(x)\ge u(y)]}\frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy\right]dx \\
&= \int_{B_r} w_+(x)\left[\int_{\mathbb{R}^N\setminus B_R}\frac{(u(x)-u(y))_+^{p-1}}{|x-y|^{N+ps}}\,dy\right]dx \\
&\ge \gamma^{-1}\int_{B_r} w_+(x)\left[\int_{\mathbb{R}^N\setminus B_R}\frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\,dy\right]dx\,,
\end{aligned}
$$

since we are necessarily considering $x \in A^+(k,R)$. On the other hand, for the same reason,

$$
\begin{aligned}
&\int_{B_r} w_+(x)\left[\int_{[\mathbb{R}^N\setminus B_R]\cap[u(y)>u(x)]} \frac{(u(y)-u(x))^{p-1}}{|x-y|^{N+ps}}\,dy\right]dx \\
&\le \int_{B_r} w_+(x)\left[\int_{\mathbb{R}^N\setminus B_R} \frac{w_+^{p-1}(y)}{|x-y|^{N+ps}}\,dy\right]dx \\
&\le \gamma\left(\frac{R}{R-r}\right)^{N+ps}\int_{B_r} w_+(x)\left[\int_{\mathbb{R}^N\setminus B_R} \frac{w_+^{p-1}(y)}{|y|^{N+ps}}\,dy\right]dx\,,
\end{aligned}
$$

using that for $x \in B_r$ and $y \in \mathbb{R}^N\setminus B_R$ we have $\frac{|y|}{|x-y|} \le \frac{R}{R-r}$. So, on both hands we have the estimate

$$
\begin{aligned}
\gamma I_2 \ge \int_{B_r} w_+(x)&\left[\int_{\mathbb{R}^N\setminus B_R} \frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\,dy\right]dx \\
&- \left(\frac{R}{R-r}\right)^{N+ps}\int_{B_r} w_+(x)\left[\int_{\mathbb{R}^N\setminus B_R} \frac{w_+^{p-1}(y)}{|y|^{N+ps}}\,dy\right]dx\,.
\end{aligned}
$$

In conclusion, this estimate and (82) lead inequality (77) to

$$
\begin{aligned}
&\hat\gamma\frac{R^{(1-s)p}}{(R-r)^p}\|w_+\|^p_{L^p(B_R)} + \hat\gamma\left(\frac{R}{R-r}\right)^{N+ps}\int_{B_r} w_+(x)\,dx\left[\int_{\mathbb{R}^N\setminus B_R} \frac{w_+^{p-1}(y)}{|y|^{N+ps}}\,dy\right] \\
&\ge [w_+]^p_{W^{s,p}(B_r)} + \int_{B_r} w_+(x)\int_{B_R} \frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\,dydx \\
&\qquad + \int_{B_r} w_+(x)\left[\int_{\mathbb{R}^N\setminus B_R} \frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\,dy\right]dx \\
&= [w_+]^p_{W^{s,p}(B_r)} + \int_{B_r} w_+(x)\int_{\mathbb{R}^N} \frac{w_-^{p-1}(y)}{|x-y|^{N+ps}}\,dydx\,.
\end{aligned}
$$

Estimates (34) are proven once we take $r = \tau\rho$ and $R = \rho$, for some $\tau \in (0,1)$ and $\rho > 0$, and we rewrite the second term on the left-hand side in terms of the tail. Remark (3.3) finishes the job in the case of super-solutions. □

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