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Quasilinear noncoercive parabolic bilateral variational inequalities in $L^p(0,\tau;D^{1,p}(\mathbb{R}^N))$

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Dedicated to Enzo Mitidieri in Honour of his 70th birthday

Abstract. *In this paper, we prove existence results for quasilinear parabolic bilateral variational inequalities of the form: Find $u \in K \subset X$ with $u(\cdot,0)=0$ satisfying*

$$0 \in u_t - \Delta_p u + aF(u) + \partial I_K(u) \quad \text{in } X^*$$

in the unbounded cylindrical domain $\mathbb{Q} = \mathbb{R}^N \times (0,\tau)$, where Δ_p is the p-Laplacian acting on $X = L^p(0,\tau;D^{1,p}(\mathbb{R}^N))$ with its dual space X^*, and with $D^{1,p}(\mathbb{R}^N)$ denoting the Beppo-Levi space (or homogeneous Sobolev space). The bilateral constraint is represented by the closed convex set $K \subset X$ given by

$$K = \{v \in X : \phi(x,t) \le v(x,t) \le \psi(x,t) \text{ for a.a. } (x,t) \in \mathbb{Q}\}$$

and I_K is the indicator function related to K with ∂I_K denoting its subdifferential in the sense of convex analysis. The main goal and the novelty of this paper is to prove existence and directedness results without assuming coercivity conditions on the operator $-\Delta_p + aF : X \to X^*$, and without supposing the existence of sub- and supersolutions. Moreover, additional difficulties we are faced with arise due to the lack of compact embedding of $D^{1,p}(\mathbb{R}^N)$ into Lebesgue spaces $L^\sigma(\mathbb{R}^N)$, and the fact that the domain K of ∂I_K has empty interior, which prevents us to use recent results on evolutionary variational inequality. Instead our approach is based on an appropriately designed penalty technique and the use of weighted Lebesgue spaces as well as pseudomontone operator theory.

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1. Introduction and Main Results

Let $\mathbb{Q} = \mathbb{R}^N \times (0, \tau)$ be the unbounded space-time cylindrical domain, and let $V = D^{1,p}(\mathbb{R}^N)$ be the homogeneous Sobolev space (also called Beppo-Levi space), which is the completion of $C_c^\infty(\mathbb{R}^N)$ (space of infinitely differentiable functions with compact support in $\mathbb{R}^N$) with respect to the norm

$$\|u\|_V = \Big(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \Big)^{\frac{1}{p}}.$$

For the range $1 < p < N$, the Beppo-Levi space V is a Banach space which is separable, reflexive and even uniformly convex, see [5, Theorem 12.2.3] and [2]. Due to the Gagliardo-Nirenberg-Sobolev inequality the Beppo-Levi space V is continuously embedded into $L^{p^*}(\mathbb{R}^N)$ with

$$p^* = \frac{Np}{N-p} \text{ denoting the critical Sobolev exponent.}$$

Thus V can be characterized as

$$V = \Big\{ v \in L^{p^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p < \infty \Big\}.$$

Clearly $V \subset W^{1,p}_{\text{loc}}(\mathbb{R}^N)$, where $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ stands for the local Sobolev space on $\mathbb{R}^N$. However, the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is a strict subspace of V, which can easily be seen by the following example.

Example 1.1. For $p = 2$ and $N = 3$ the function

$$u(x) = \big(1 + |x|^2\big)^{-\frac{1}{2}}$$

belongs to $V = D^{1,2}(\mathbb{R}^3)$, but u does not belong to $W^{1,2}(\mathbb{R}^3)$, because $u \notin L^2(\mathbb{R}^3)$, which is easily seen by the following elementary calculation making use of spherical coordinates:

$$\begin{aligned} \int_{\mathbb{R}^3} |u(x)|^2 dx &= \int_{\mathbb{R}^3} \frac{1}{1+|x|^2} dx = c \int_0^\infty \frac{r^2}{1+r^2} dr \\ &\geq c \int_1^\infty \frac{r^2}{1+r^2} dr \geq c \int_1^\infty \frac{r^2}{2r^2} dr = \infty, \end{aligned}$$

where c is some positive constant.

Let $X = L^p(0, \tau; V)$ be the Banach-valued Lebesgue space with its dual space $X^* = L^{p'}(0, \tau; V^*)$, where p' is the Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$, and assume throughout $2 \leq p < N$. In this paper we consider the following parabolic bilateral variational inequality: Find $u \in K \subset X$ with $u(\cdot, 0) = 0$ such that

$$0 \in u_t - \Delta_p u + aF(u) + \partial I_K(u) \quad \text{in } X^*, \tag{1}$$

where $\Delta_p u = |\nabla u|^{p-2}\nabla u$ is the p-Laplacian, and K denotes the closed convex subset of X representing the bilateral constraint given by

$$K = \{v \in X \,:\, \phi(x,t) \le v(x,t) \le \psi(x,t) \text{ for a.a. } (x,t) \in \mathbb{Q}\} \tag{2}$$

with ϕ, $\psi \in X$, and I_K is the indicator function related to K with ∂I_K denoting its subdifferential in the sense of convex analysis. The time derivative $u_t := \frac{\mathrm{d}u(\cdot,t)}{\mathrm{d}t}$ is understood as the distributional time-derivative of the Banach-valued function $u\colon (0,\tau) \to V$. By definition of the subdifferential ∂I_K, problem (1) is equivalent to the following parabolic variational inequality: Find $u \in K$ with $u(\cdot, 0) = 0$ such that

$$\langle u_t - \Delta_p u + aF(u), v - u\rangle \ge 0 \quad \text{for all } v \in K, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its dual X^*. The nonlinear lower order term F is the Nemytskij operator generated by the Carathéodory function $f\colon \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ through $F(u)(x,t) = f(x,t,u(x,t))$, and the measurable bounded function $a : \mathbb{Q} \to \mathbb{R}$ is supposed to decay at infinity like $|x|^{-(N+\alpha)}$ with some $\alpha > 0$. Let us introduce the function space Y as follows

$$Y = \{u \in X \,:\, u_t \in X^*\}\,.$$

Since the Beppo-Levi space is separable, reflexive and even uniformly convex, it follows that the Lebesgue space $X = L^p(0,\tau;V)$, and Y are separable, and uniformly convex, and thus reflexive Banach spaces as well (see e.g. Zeidler [12, Proposition 23.2, Proposition 23.7]) equipped with the norms

$$\|u\|_Y = \|u\|_X + \|u_t\|_{X^*},$$

where $\|\cdot\|_X$ and $\|u\|_{X^*}$ are defined by

$$\|u\|_X = \left(\int_0^\tau \|u(\cdot,t)\|_V^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \quad \|u\|_{X^*} = \left(\int_0^\tau \|u(\cdot,t)\|_{V^*}^{p'} \,\mathrm{d}t\right)^{\frac{1}{p'}}$$

with $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ being the norms in $V = D^{1,p}(\mathbb{R}^N)$ and V^*, respectively. We assume the following hypotheses on the data of the variational inequality (3).

(Ha) The function $a : \mathbb{Q} \to \mathbb{R}$ is measurable and satisfies the decay for some positive constants c_a, α

$$|a(x,t)| \le c_a w(x), \quad \text{for a.a. } (x,t) \in \mathbb{Q}, \text{ with } w(x) = \frac{1}{1+|x|^{N+\alpha}}. \tag{4}$$

(Hψ) The function $\psi\colon \mathbb{Q} \to \mathbb{R}$ of K is supposed to satisfy: $\psi \in Y$, $\psi(\cdot, 0) \geq 0$ in $\mathbb{R}^N$, and

$$\langle \psi_t - \Delta_p \psi, \varphi \rangle \geq 0 \quad \text{for all } \varphi \in X \text{ with } \varphi \geq 0.$$

(Hϕ) The function $\phi\colon \mathbb{Q} \to \mathbb{R}$ of K is supposed to satisfy: $\phi \in Y$, $\phi(\cdot, 0) \leq 0$ in $\mathbb{R}^N$, and

$$\langle \phi_t - \Delta_p \phi, \varphi \rangle \leq 0 \quad \text{for all } \varphi \in X \text{ with } \varphi \geq 0.$$

(Hf) $f\colon \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $(x, t) \mapsto f(x, t, s)$ is measurable in $\mathbb{Q}$ for all $s \in \mathbb{R}$, and $s \mapsto f(x, t, s)$ is continuous in $\mathbb{R}$ for a.a. $(x, t) \in \mathbb{Q}$. Further, f satisfies the following growth condition

$$|f(x, t, s)| \leq k(x, t) + c_f |s|^{p-1} \tag{5}$$

for a.a. $(x, t) \in \mathbb{Q}$ and for all $s \in \mathbb{R}$, where $c_f \geq 0$ and $k \in L^{p'}(\mathbb{Q}, w)$.

Remark 1.2. A few remarks regarding the hypotheses are in order.

(i) Hypotheses (Hψ) and (Hϕ) imply that $\phi \leq \psi$ in $\mathbb{Q}$. These conditions, however, do not imply that ψ is a supersolution and ϕ is a subsolution for the variational inequality (3).

(ii) The space $L^{p'}(\mathbb{Q}, w)$, which appears in (Hf), is the weighted Lebesgue space with weight w given by (4) that will be specified in Section 2.

(iii) Clearly the bilateral constraint $K \subset X$ has empty interior and satisfies the following lattice condition

$$K \vee K \subset K \quad \text{and} \quad K \wedge K \subset K, \tag{6}$$

where

$$K \vee K = \{v \vee w : w, v \in K\} \text{ with } v \vee w = \max\{v, w\},$$

$$K \wedge K = \{v \wedge w : w, v \in K\} \text{ with } v \wedge w = \min\{v, w\}.$$

This lattice property will be used in the proof of the directedness of the solution set of (3), see Theorem 1.5.

(iv) We are going to provide examples for ψ and ϕ that satisfy hypotheses (Hψ) and (Hϕ), and show that the operator $-\Delta_p + aF : X \to X^*$ is, in general, noncoercive.

Definition 1.3. *A function $u \in Y \cap K$ is called a solution of the bilateral parabolic variational inequality (3) if $u(\cdot, 0) = 0$ and the variational inequality*

$$\langle u_t, v - u \rangle + \int_{\mathbb{Q}} |\nabla u|^{p-2} \nabla u (\nabla v - \nabla u) \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{Q}} aF(u)(v - u) \,\mathrm{d}x\mathrm{d}t \geq 0$$

is satisfied for all $v \in K$.

The main results of this paper are the following.

THEOREM 1.4. *Let hypotheses (Ha), (Hψ), (Hϕ), and (Hf) be satisfied. Then the bilateral parabolic variational inequality (3) admits at least one solution* $u \in Y$.

THEOREM 1.5. *Let hypotheses of Theorem 1.4 be satisfied, and assume* $a(x,t) \geq 0$. *Then the solution set* $\mathcal{S}$ *of all solutions of (3) is a directed set with respect to the pointwise order of functions, that is,* $\mathcal{S}$ *is directed upward, which means for each pair* u_1, $u_2 \in \mathcal{S}$ *there is a* $u \in \mathcal{S}$ *such that* $u_i \leq u$, $i = 1, 2$, *as well as* $\mathcal{S}$ *is directed downward, which means for each pair* u_1, $u_2 \in \mathcal{S}$ *there is a* $v \in \mathcal{S}$ *such that* $u_i \geq v$, $i = 1, 2$.

Unlike in case of elliptic bilateral variational inequalities, in the treatment of its parabolic (evolutionary) counterpart considered here, an additional difficulty arises due to the subdifferential of the indicator function ∂I_K in (1). This is because no growth condition can be assumed on ∂I_K, and thus, in general there is no growth estimate of the time derivative u_t in the dual space X^* available, which would be needed for proving existence of solutions. Usually, this lack is compensated in the treatment of evolutionary variational inequalities by requiring that K admits a nonempty interior, that is, $\mathrm{int}(K) \neq \emptyset$, see e.g. [6, 14]. Namely, if $\mathrm{int}(K) \neq \emptyset$, then Rockafellar's theorem about the sums of maximal monotone operators may be applied, which allows one to study evolutionary variational inequalities by implementation of arguments and results for elliptic variational inequalities to evolutionary variational inequalities. Unfortunately, the interior of the constraint K we are dealing with is empty, i.e., $\mathrm{int}(K) = \emptyset$, and therefore a similar approach as for elliptic variational inequalities cannot be applied. Instead, we are going to deal with this difficulty by using an appropriately designed penalty technique, which also will enable us to handle the lack of coercivity of the operator $-\Delta_p + aF : X \to X^*$. A further difficulty arises due to the unboundedness of the space domain $\mathbb{R}^N$, and hence the lack of compact embedding $V \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N)$ which will be resolved by working in weighted Lebesgue spaces.

Existence results for general parabolic variational inequalities of the form:

$$u \in Y \cap K : 0 \in u_t + A(u) + aF(u) + \partial I_K \quad \text{in } X^*$$

as well as systems of such parabolic variational inequalities in bounded and unbounded cylindrical domains can be found as parts of the recent monograph [3], see also relevant references therein, and the following related papers [8, 9, 10, 11]. Even though in [3] the constraint K satisfies $\mathrm{int}(K) = \emptyset$, general existence results and a detailed study of the quality of the solution set has been obtained under either certain coercivity assumptions on the (possibly) multi-valued operator $A + aF : X \to 2^{X^*}$, or the existence of appropriately defined sub- and supersolutions, where A may be a general Leray-Lions operator.

The main goal and the novelty of this paper is to prove existence and directedness results without assuming coercivity conditions on the operator $-\Delta_p + aF : X \to X^*$, and without supposing the existence of sub- and supersolutions. Our approach is based on an appropriately designed penalty technique and the use of weighted Lebesgue spaces as well as pseudomontone operator theory.

The paper is organized as follows. In Section 2 we present main tools which are needed in the sequel and provide some examples. In Section 3 and Section 4 we prove our main results Theorem 1.4 and Theorem 1.5, respectively.

2. Preliminaries and Examples

Throughout this paper, we assume $2 \le p < N$, and hypotheses (Ha), (Hψ), (Hϕ), and (Hf). The following notations will be used: For any $\sigma \in (1, \infty)$, its Hölder conjugate is denoted by σ', i.e., $1/\sigma + 1/\sigma' = 1$, the $L^\sigma(\mathbb{R}^N)$-norm is denoted by $\|\cdot\|_\sigma$ and the $L^\sigma(\mathbb{Q})$-norm is denoted by $\|\cdot\|_{\mathbb{Q},\sigma}$. For normed linear spaces W and Z, $W \hookrightarrow Z$ denotes the continuous embedding, and $W \hookrightarrow\hookrightarrow Z$ stands for the compact embedding of W into Z. With the weight function w given by (4) we introduce the weighted Lebesgue spaces $L^q(\mathbb{R}^N, w)$ and $L^q(\mathbb{Q}, w)$ as follows

$$L^q(\mathbb{R}^N, w) = \Big\{u \in L^0(\mathbb{R}^N) : \int_{\mathbb{R}^N} w|u|^q \, dx < \infty\Big\}$$

with norm

$$\|u\|_{q,w} = \Big(\int_{\mathbb{R}^N} w|u|^q \, dx\Big)^{\frac{1}{q}},$$

and

$$L^q(\mathbb{Q}, w) = \Big\{u \in L^0(\mathbb{Q}) : \int_{\mathbb{Q}} w|u|^q \, dxdt = \int_0^\tau \Big(\int_{\mathbb{R}^N} w|u|^q \, dx\Big) dt < \infty\Big\}$$

with norm

$$\|u\|_{\mathbb{Q},q,w} = \Big(\int_{\mathbb{Q}} w|u|^q \, dxdt\Big)^{\frac{1}{q}},$$

where $L^0(\mathbb{R}^N)$ and $L^0(\mathbb{Q})$ denotes the space of real-valued measurable functions on $\mathbb{R}^N$ and $\mathbb{Q}$, respectively. The weighted Lebesgue spaces $L^q(\mathbb{R}^N, w)$ and $L^q(\mathbb{Q}, w)$ are separable and reflexive Banach spaces for $1 < q < \infty$.

LEMMA 2.1. *The weight function w given by (4) belongs to $L^r(\mathbb{R}^N)$ for $1 \le r \le \infty$.*

Proof. In fact, clearly $w \in L^\infty(\mathbb{R}^N)$, and using spherical coordinates we get for any $r \in [1, \infty)$

$$\begin{aligned}\int_{\mathbb{R}^N} w^r\,dx &= \int_{|x|<1} w^r\,dx + \int_{|x|\geq 1} w^r\,dx \\ &\leq |B(0,1)| + c\int_1^\infty \Big(\frac{1}{1+\varrho^{N+\alpha}}\Big)^r \varrho^{N-1}\,d\varrho \\ &\leq c + c\int_1^\infty \varrho^{-(N+\alpha)r+N-1}\,d\varrho < \infty,\end{aligned}$$

since $-(N+\alpha)r + N < 0$. Here, $|B(0,1)|$ denotes the Lebesgue measure of the unit ball $B(0,1)$ in $\mathbb{R}^N$. □

Corollary 2.2. *If $1 < q < r < \infty$, then $L^r(\mathbb{R}^N, w) \hookrightarrow L^q(\mathbb{R}^N, w)$, and $L^r(\mathbb{Q}, w) \hookrightarrow L^q(Q, w)$.*

Proof. We only show $L^r(\mathbb{R}^N, w) \hookrightarrow L^q(\mathbb{R}^N, w)$, since the proof for $L^r(\mathbb{Q}, w) \hookrightarrow L^q(Q, w)$ follows in the same way. Using Hölder inequality we get

$$\int_{\mathbb{R}^N} w|u|^q\,dx = \int_{\mathbb{R}^N} w^{\frac{q}{r}}|u|^q w^{1-\frac{q}{r}}\,dx \leq \Big(\int_{\mathbb{R}^N} w|u|^r\,dx\Big)^{\frac{q}{r}} \Big(\int_{\mathbb{R}^N} w\,dx\Big)^{\frac{r-q}{r}},$$

which yields

$$\|u\|_{q,w} \leq c\|u\|_{r,w}, \quad \text{where } c = \Big(\int_{\mathbb{R}^N} w\,dx\Big)^{\frac{r-q}{rq}}. \qquad \square$$

Lemma 2.3. $V \hookrightarrow L^q(\mathbb{R}^N, w)$ *for* $1 \leq q \leq p^*$.

Proof. Let $u \in V = D^{1,p}(\mathbb{R}^N)$, then $u \in L^{p^*}(\mathbb{R}^N)$, and thus with Lemma 2.1 we get for any $q \in [1, p^*)$

$$\int_{\mathbb{R}^N} w|u|^q\,dx \leq \|w\|_{\frac{p^*}{p^*-q}} \|u\|_{p^*}^q \leq c\|w\|_{\frac{p^*}{p^*-q}} \|u\|_V^q,$$

that is, $\|u\|_{q,w} \leq c\|w\|_{\frac{p^*}{p^*-q}}^{\frac{1}{q}} \|u\|_V$, and for $q = p^*$ we have

$$\|u\|_{p^*,w}^{p^*} = \int_{\mathbb{R}^N} w|u|^{p^*}\,dx \leq \|u\|_{p^*}^{p^*} \leq c\|u\|_V^{p^*},$$

which shows that $i_w : V \to L^q(\mathbb{R}^N, w)$ is linear and continuous for $q \in [1, p^*]$. □

For the following embedding result we refer to [3, Lemma 6.1].

LEMMA 2.4 ([3]). *The embedding $V \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ is compact for $1 \le q < p^*$, that is, the embedding operator $i_w : V \to L^q(\mathbb{R}^N, w)$ defined by $u \mapsto i_w u = u$ is linear and compact.*

From Lemma 2.4 it follows, in particular, $V \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N, w)$. Therefore, identifying the Hilbert space $L^2(\mathbb{R}^N, w)$ with its dual, we have the following evolution triple $(V, L^2(\mathbb{R}^N, w), V^*)$ with the embeddings

$$V \overset{i_w}{\hookrightarrow} L^2(\mathbb{R}^N, w) \overset{i_w^*}{\hookrightarrow} V^*$$

being dense and compact, where $i_w : V \to L^q(\mathbb{R}^N, w)$ is the embedding operator of V into $L^q(\mathbb{R}^N, w)$, and $i_w^* : L^{q'}(\mathbb{R}^N, w) \to V^*$ its adjoint operator defined by

$$v \in L^{q'}(\mathbb{R}^N, w) : \quad \langle i_w^* v, \varphi \rangle = \int_{\mathbb{R}^N} w v \varphi \, dx, \quad \forall\, \varphi \in V.$$

The spaces $X = L^p(0, \tau; V)$ and Y introduced in the preceding section along with the evolution triple $(V, L^2(\mathbb{R}^N, w), V^*)$ yield the following result.

LEMMA 2.5. *The following holds true:*

(i) Continuous embedding: $Y \hookrightarrow C([0, \tau]; L^2(\mathbb{R}^N, w))$;

(ii) If $u \in Y$, then the following integration by parts formula is valid

$$\int_0^\tau \langle u_t(\cdot, t), u(\cdot, t) \rangle \, \mathrm{d}t = \frac{1}{2} \left(\|u(\cdot, \tau)\|_{2,w}^2 - \|u(\cdot, 0)\|_{2,w}^2 \right);$$

(iii) If $u \in Y$, then it holds

$$\int_0^\tau \langle u_t(\cdot, t), u(\cdot, t)^+ \rangle \, \mathrm{d}t = \frac{1}{2} \left(\|u(\cdot, \tau)^+\|_{2,w}^2 - \|u(\cdot, 0)^+\|_{2,w}^2 \right),$$

where $s^+ = \max\{s, 0\}$.

Proof. (i) and (ii) are immediate consequences of Proposition 23.23 in Zeidler [12].

(iii) In a similar way as in the proof of Lemma 2.146 in Carl-Le-Motreanu [4] one obtains this formula by regularization and density arguments. □

LEMMA 2.6. *The following embeddings hold:*

$$X \hookrightarrow L^p(\mathbb{Q}, w), \quad Y \hookrightarrow\hookrightarrow L^p(\mathbb{Q}, w).$$

Proof. The continuous embedding $X \hookrightarrow L^p(\mathbb{Q}, w)$ is an immediate consequence of $V \hookrightarrow L^p(\mathbb{R}^N, w)$. Since V is even compactly embedded into $L^p(\mathbb{R}^N, w)$, that is, $V \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N, w)$ and $L^p(\mathbb{R}^N, w) \hookrightarrow L^{p'}(\mathbb{R}^N, w) \hookrightarrow V^*$ (note that $2 \le p < N$), we finally get $V \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N, w) \hookrightarrow V^*$. Hence, we may apply Lions-Aubin Theorem (see e.g. Carl-Le [3, Theorem 2.52]), which results in $Y \hookrightarrow\hookrightarrow L^p(0, \tau; L^p(\mathbb{R}^N, w)) = L^p(\mathbb{Q}, w)$. □

With the coefficient a satisfying (Ha) we define the operator $i_a^* : L^{p'}(\mathbb{Q}, w) \to X^*$ as follows:

$$\eta \in L^{p'}(\mathbb{Q}, w) : \quad \langle i_a^* \eta, \varphi \rangle = \int_{\mathbb{Q}} a\eta\varphi \, dxdt, \quad \forall \, \varphi \in X. \tag{7}$$

Lemma 2.7. *The operator $i_a^* : L^{p'}(\mathbb{Q}, w) \to X^*$ is linear and continuous. Analogously, $i_{|a|}^* : L^{p'}(\mathbb{Q}, w) \to X^*$ and $i_w^* : L^{p'}(\mathbb{Q}, w) \to X^*$ are linear and continuous, where $i_{|a|}^*$ and i_w^* are defined as in (7) with a replaced by $|a|$ and w, respectively.*

Proof. For any $\eta \in L^{p'}(\mathbb{Q}, w)$, using Hölder inequality and Lemma 2.6 we have the following estimate:

$$\begin{aligned} |\langle i_a^* \eta, \varphi \rangle| &\le \int_{\mathbb{Q}} |a| \, |\eta| \, |\varphi| \, dxdt \le c_a \int_{\mathbb{Q}} w \, |\eta| \, |\varphi| \, dxdt \\ &\le c_a \int_{\mathbb{Q}} w^{\frac{1}{p'}} \, |\eta| \, w^{\frac{1}{p}} |\varphi| \, dxdt \le c_a \|\eta\|_{\mathbb{Q},p',w} \|\varphi\|_{\mathbb{Q},p,w} \\ &\le c \, \|\eta\|_{\mathbb{Q},p',w} \|\varphi\|_X, \quad \forall \, \varphi \in X. \end{aligned}$$

As the linearity of i_a^* is obvious, the above estimate shows that i_a^* bounded. The proof for $i_{|a|}^*$ and i_w^* follows the same line, which completes the proof. □

Corollary 2.8. *The Nemytskij operator $F : L^p(\mathbb{Q}, w) \to L^{p'}(\mathbb{Q}, w)$ generated by f satisfying (Hf), and $aF : X \to X^*$ are continuous and bounded.*

Proof. In view of the continuity and growth condition on f, by standard arguments on Nemytskij operators it follows that $F : L^p(\mathbb{Q}, w) \to L^{p'}(\mathbb{Q}, w)$ is continuous and bounded. Lemma 2.6 and Lemma 2.7 imply that $aF : X \to X^*$ is continuous and bounded. □

By means of the functions ϕ and ψ of the bilateral constraint K we are introducing the operator P defined by

$$\langle P(u), \varphi \rangle = \int_{\mathbb{Q}} w \left([u - \psi)^+]^{p-1} - [(\phi - u)^+]^{p-1} \right) \varphi \, dxdt, \quad u, \varphi \in X, \tag{8}$$

Let $b : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ be given by

$$b(x, t, s) = [(s - \psi(x, t))^+]^{p-1} - [(\phi(x, t) - s)^+]^{p-1}, \tag{9}$$

which can equivalently be characterized by

$$b(x, t, s) = \begin{cases} (s - \psi(x, t))^{p-1} & \text{if} \quad s > \psi(x, t) \\ 0 & \text{if} \quad \phi(x, t) \le s \le \psi(x, t) \\ -(\phi(x, t) - s)^{p-1} & \text{if} \quad s < \phi(x, t). \end{cases}$$

One readily verifies that $b : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which satisfies the following growth condition

$$|b(x,t,s)| \le \beta(x,t) + c_b|s|^{p-1}, \quad \forall (x,t,s) \in \mathbb{Q} \times \mathbb{R},\ c_b > 0, \tag{10}$$

where $\beta(x,t) = c\big(|\psi(x,t)|^{p-1} + |\phi(x,t)|^{p-1}\big)$ with some positive constant c, and thus $\beta \in L^{p'}(\mathbb{Q},w)$, since $\phi,\ \psi \in X \hookrightarrow L^p(\mathbb{Q},w)$. Therefore, b fulfills qualitatively the same regularity and growth conditions like f in (Hf), and hence the Nemytskij operator B associated with b through $B(u)(x,t) = b(x,t,u(x,t))$ yields a continuous and bounded mapping from $L^p(\mathbb{Q},w)$ to $L^{p'}(\mathbb{Q},w)$. Moreover, $s \mapsto b(x,t,s)$ is monotone nondecreasing.

In view of (8) the operator P can be characterized as $P = wB$ or

$$P = i_w^* \circ B \circ i_w : X \to X^*, \tag{11}$$

with the embeddings $i_w : X \to L^p(\mathbb{Q},w)$, and $i_w^* : L^{p'}(\mathbb{Q},w) \to X^*$ (see Lemma 2.7), and thus $P : X \to X^*$ is bounded and continuous.

Lemma 2.9. *The operator $P = wB : X \to X^*$ defined by (8) (resp. (11)) is a penalty operator associated with K, that is, $P : X \to X^*$ is a bounded, hemicontinuous, and monotone operator, which satisfies*

$$P(u) = 0 \Longleftrightarrow u \in K.$$

Proof. $P : X \to X^*$ is bounded, and continuous, hence hemicontinuous, and also monotone due to the monotonicity of $s \mapsto b(x,t,s)$. Therefore, it only remains to show

$$P(u) = 0 \Longleftrightarrow u \in K.$$

If $u \in K$, then by the definition of the function b we have $b(x,t,u) = 0$, and thus $P(u) = 0$. To show the converse, let $P(u) = 0$, that is, $\langle P(u), \varphi\rangle = 0$ for all $\varphi \in X$. Using the special test function $\varphi = (u-\psi)^+ \in X$ we get

$$0 = \langle P(u), (u-\psi)^+\rangle = \int_{\mathbb{Q}} w[(u-\psi)^+]^p\, dxdt,$$

which implies $(u-\psi)^+ = 0$, i.e., $u \le \psi$ a.e. in $\mathbb{Q}$. Testing $\langle P(u), \varphi\rangle = 0$ with $\varphi = (\phi - u)^+ \in X$ yields

$$0 = \langle P(u), (\phi-u)^+\rangle = -\int_{\mathbb{Q}} w[(\phi-u)^+]^p\, dxdt,$$

which implies $(\phi-u)^+ = 0$, i.e., $\phi \le u$ a.e. in $\mathbb{Q}$. □

Lemma 2.10. *The penalty operator $P\colon X \to X^*$ fulfills the inequality*

$$\langle P(u), (u-\psi)^+ - (\phi-u)^+\rangle \ge d\,\|P(u)\|_{X^*}\big(\|(u-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u)^+\|_{\mathbb{Q},p,w}\big)$$

with some $d > 0$.

Proof. From (8) we get

$$\begin{aligned}\langle P(u),(u-\psi)^+-(\phi-u)^+\rangle\\ &= \int_{\mathbb{Q}} w\left([(u-\psi)^+]^{p-1}-[(\phi-u)^+]^{p-1}\right)\cdot\left((u-\psi)^+-(\phi-u)^+\right)\,dxdt\\ &= \int_{\mathbb{Q}} w\left([(u-\psi)^+]^{p}+[(\phi-u)^+]^{p}\right)\,dxdt,\end{aligned}$$

that is

$$\langle P(u),(u-\psi)^+-(\phi-u)^+\rangle = \|(u-\psi)^+\|^p_{\mathbb{Q},p,w}+\|(\phi-u)^+\|^p_{\mathbb{Q},p,w}. \tag{12}$$

Applying Hölder inequality and $X\hookrightarrow L^p(\mathbb{Q},w)$ we estimate

$$\begin{aligned}|\langle P(u),\varphi\rangle| &\le \int_{\mathbb{Q}} w\left([(u-\psi)^+]^{p-1}+[(\phi-u)^+]^{p-1}\right)|\varphi|\,dxdt\\ &\le \int_{\mathbb{Q}} w^{\frac{1}{p'}}\left([(u-\psi)^+]^{p-1}+[(\phi-u)^+]^{p-1}\right)w^{\frac{1}{p}}|\varphi|\,dxdt\\ &\le \left(\|(u-\psi)^+\|^{p-1}_{\mathbb{Q},p,w}+\|(\phi-u)^+\|^{p-1}_{\mathbb{Q},p,w}\right)\|\varphi\|_{\mathbb{Q},p,w}\\ &\le c\left(\|(u-\psi)^+\|^{p-1}_{\mathbb{Q},p,w}+\|(\phi-u)^+\|^{p-1}_{\mathbb{Q},p,w}\right)\|\varphi\|_X,\end{aligned}$$

where c is some positive constant, which yields

$$\|P(u)\|_{X^*}\le c\left(\|(u-\psi)^+\|^{p-1}_{\mathbb{Q},p,w}+\|(\phi-u)^+\|^{p-1}_{\mathbb{Q},p,w}\right). \tag{13}$$

Using the elementary inequality $r^p+s^p\ge\frac{1}{2}(r^{p-1}+s^{p-1})(r+s)$ for any real numbers $r\ge 0$, $s\ge 0$, from (12) and (13), we get for some positive constant d independent of u, ψ, ϕ

$$\langle P(u),(u-\psi)^+-(\phi-u)^+\rangle\ge d\,\|P(u)\|_{X^*}\left(\|(u-\psi)^+\|_{\mathbb{Q},p,w}+\|(\phi-u)^+\|_{\mathbb{Q},p,w}\right),$$

which completes the proof. □

Lemma 2.11. *For any $u\in Y$ with $u(\cdot,0)=0$ it holds*

$$\langle u_t-\Delta_p u,(u-\psi)^+-(\phi-u)^+\rangle\ge 0.$$

Proof. We use hypotheses (Hψ) and (Hϕ) and note that $u-\psi\in Y$ and $\phi-u\in Y$, as well as $(u-\psi)^+(x,0)=0$ and $(\phi-u)^+(x,0)=0$, which by applying the integration by parts formula (see Lemma 2.5) yields

$$\langle (u-\psi)_t,(u-\psi)^+\rangle=\frac{1}{2}\|(u-\psi)^+(\cdot,\tau)\|^2_{2,w}, \tag{14}$$

$$\langle (\phi-u)_t,(\phi-u)^+\rangle=\frac{1}{2}\|(\phi-u)^+(\cdot,\tau)\|^2_{2,w}. \tag{15}$$

With (14) we get by taking into account (Hψ) and the fact that $-\Delta_p : X \to X^*$ is a bounded, continuous and monotone operator the following estimate

$$\begin{aligned}&\langle u_t - \Delta_p u - (\psi_t - \Delta_p \psi), (u-\psi)^+\rangle \\ &\qquad = \langle (u-\psi)_t, (u-\psi)^+\rangle + \langle -\Delta_p u - (-\Delta_p \psi), (u-\psi)^+\rangle \ge 0,\end{aligned}$$

which results in

$$\langle u_t - \Delta_p u, (u-\psi)^+\rangle \ge \langle \psi_t - \Delta_p \psi, (u-\psi)^+\rangle \ge 0. \tag{16}$$

Similarly with (Hϕ) and (15) we get

$$\begin{aligned}&\langle u_t - \Delta_p u - (\phi_t - \Delta_p \phi), -(\phi-u)^+\rangle \\ &\qquad = \langle (u-\phi)_t, -(\phi-u)^+\rangle + \langle -\Delta_p u - (-\Delta_p \phi), -(\phi-u)^+\rangle \ge 0,\end{aligned}$$

which yields

$$\langle u_t - \Delta_p u, -(\phi-u)^+\rangle \ge \langle \phi_t - \Delta_p \phi, -(\phi-u)^+\rangle \ge 0, \tag{17}$$

and thus (16) and (17) complete the proof. □

We conclude this section with a few examples.

EXAMPLE 2.12. Let $p = 2$ and $N = 6$, which gives $p^* = 3$ and $p^{*\prime} = \frac{3}{2}$, and let ψ, $\phi : \mathbb{Q} \to \mathbb{R}$ with $\mathbb{Q} = \mathbb{R}^6 \times (0,\tau)$, be given by

$$\psi(x,t) = \left(2\tau - t + |x|^2\right)^{-2}, \quad \text{and} \quad \phi(x,t) = -\psi(x,t). \tag{18}$$

Let us show that hypotheses (Hψ) and (Hϕ) are fulfilled with $X = L^2(0,\tau;V)$, $X^* = L^2(0,\tau;V^*)$, and $V = D^{1,2}(\mathbb{R}^6)$. We get

$$\psi_t = \frac{\partial \psi}{\partial t} = 2\left(2\tau - t + |x|^2\right)^{-3}.$$

To verify that $\psi_t \in X^*$, we first show that $\frac{\partial \psi}{\partial t}(\cdot,t) \in L^{p^{*\prime}}(\mathbb{R}^6) \hookrightarrow V^*$.

$$\begin{aligned}\int_{\mathbb{R}^6} |\frac{\partial \psi}{\partial t}(x,t)|^{\frac{3}{2}}\,dx &= 2^{\frac{3}{2}} \int_{\mathbb{R}^6} \left(2\tau - t + |x|^2\right)^{\frac{-9}{2}}\,dx \\ &= 2^{\frac{3}{2}} \int_{B(0,1)} \left(2\tau - t + |x|^2\right)^{\frac{-9}{2}}\,dx \\ &\qquad + 2^{\frac{3}{2}} \int_{\mathbb{R}^6\setminus B(0,1)} \left(2\tau - t + |x|^2\right)^{\frac{-9}{2}}\,dx \\ &\le c\left(1 + \int_{\mathbb{R}^6\setminus B(0,1)} \left(2\tau - t + |x|^2\right)^{\frac{-9}{2}}\,dx\right) \\ &\le c\left(1 + \int_1^\infty \frac{\varrho^5}{\varrho^9}\,d\varrho\right) < \infty,\end{aligned}$$

which shows that $\frac{\partial\psi}{\partial t}(\cdot,t) \in L^{p^{*\prime}}(\mathbb{R}^6)$ for all $t \in [0,\tau]$. Since

$$t \mapsto \int_{\mathbb{R}^6} |\frac{\partial\psi}{\partial t}(x,t)|^{\frac{3}{2}}\,dx \text{ is continuous}$$

it follows that $\frac{\partial\psi}{\partial t} \in L^2(0,\tau;L^{p^{*\prime}}(\mathbb{R}^6)) \hookrightarrow X^*$. Let us prove that $\psi \in X$. Calculating $\frac{\partial\psi}{\partial x_i}$ we get

$$\frac{\partial\psi}{\partial x_i} = -4x_i\left(2\tau - t + |x|^2\right)^{-3},$$

that is

$$|\nabla\psi|^2 = 16|x|^2\left(2\tau - t + |x|^2\right)^{-6}$$

and thus by using again spherical coordinates

$$\int_{\mathbb{R}^6} |\nabla\psi(x,t)|^2\,dx = 16\int_{\mathbb{R}^6} \frac{|x|^2}{\left(2\tau - t + |x|^2\right)^6}\,dx \le c\left(1 + \int_1^\infty \frac{\varrho^2\varrho^5}{\varrho^{12}}\,d\varrho\right) < \infty.$$

Since $t \mapsto \int_{\mathbb{R}^6} |\nabla\psi(x,t)|^2\,dx$ is continuous on $[0,\tau]$, it follows that $\psi \in X$, and hence with $\frac{\partial\psi}{\partial t} \in X^*$ we have $\psi \in Y$.

Clearly $\psi(x,0) \ge 0$. Finally, calculating $\psi_t - \Delta\psi$ yields

$$\psi_t - \Delta\psi = 2\left(2\tau - t + |x|^2\right)^{-3} - [-24(2\tau - t)]\left(2\tau - t + |x|^2\right)^{-4} \ge 0,$$

which proves that (Hψ) is fulfilled. Apparently, $\phi(x,t) = -\psi(x,t)$ fulfills (Hϕ).

Example 2.13. Let $p = 2$ and $N = 6$, $K = \{v \in X : \phi \le v \le \psi\}$ with ϕ, ψ, and X as in Example 2.12, and $f : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ and $a : \mathbb{Q} \to \mathbb{R}$ be given by

$$f(x,t,s) = -c_f s + k(x,t), \quad a(x,t) = w(x) = \frac{1}{1 + |x|^{6+\alpha}}, \tag{19}$$

where $c_f > 0$, and $k \in L^2(\mathbb{Q}, w)$. Then a and f satisfy hypotheses (Ha) and (Hf), respectively. According to our main result, Theorem 1.4, the bilateral parabolic variational inequality: Find $u \in Y \cap K$ with $u(\cdot,0) = 0$ such that

$$\langle u_t - \Delta u + wF(u), v - u\rangle \ge 0, \quad \forall\, v \in K, \tag{20}$$

admits at least one solution. We note that the bilateral constraint K with ϕ and ψ given by (18) is unbounded in $X = L^2(0,\tau;V)$, where $V = D^{1,2}(\mathbb{R}^6)$, which is demonstrated as follows. Consider the sequence (u_n) defined by

$$u_n(x) = \begin{cases} \frac{|x|^n}{(2\tau+1)^2} & \text{if} \quad 0 \le |x| \le 1 \\ \frac{1}{(2\tau+|x|^2)^2} & \text{if} \quad |x| \ge 1, \end{cases}$$

which is independent of t. Clearly, $\phi \le u_n \le \psi$, that is $u_n \in K$. Calculating $|\nabla u_n|^2$ yields

$$|\nabla u_n(x)|^2 = \begin{cases} \frac{1}{(2\tau+1)^4} n^2 |x|^{2n-2} & \text{if} \quad 0 \le |x| \le 1 \\ 16 \frac{|x|^2}{(2\tau+|x|^2)^3} & \text{if} \quad |x| > 1. \end{cases}$$

One readily observes that $u_n \in X$, but the sequence (u_n) is unbounded in X, which is seen as follows:

$$\begin{aligned} \|u_n\|_X^2 &= \int_0^\tau \|u_n\|_V^2 \, dt = \int_0^\tau \left(\int_{\mathbb{R}^6} |\nabla u_n(x)|^2 \, dx \right) dt \\ &= \tau \int_{\mathbb{R}^6} |\nabla u_n(x)|^2 \, dx \ge \tau \int_{B(0,1)} |\nabla u_n(x)|^2 \, dx \\ &\ge \frac{\tau}{(2\tau+1)^4} n^2 \int_{B(0,1)} |x|^{2n-2} \, dx \\ &= c \frac{\tau}{(2\tau+1)^4} n^2 \int_0^1 \varrho^{2n+3} \, d\varrho = c \frac{\tau}{(2\tau+1)^4} \frac{n^2}{2n+4} \to \infty \text{ as } n \to \infty. \end{aligned}$$

Next let us verify that the operator $-\Delta + wF : X \to X^*$ fails to be coercive on $X = L^2(0,\tau;V)$ with f and w as in (19).

$$\begin{aligned} \langle -\Delta u + wF(u), u \rangle &= \int_{\mathbb{Q}} |\nabla u|^2 \, dxdt - c_f \int_{\mathbb{Q}} w|u|^2 \, dxdt + \int_{\mathbb{Q}} wku \, dxdt \\ &\le \|u\|_X^2 - c_f \|u\|_{\mathbb{Q},2,w}^2 + \|k\|_{\mathbb{Q},2,w} \|u\|_{\mathbb{Q},2,w}. \end{aligned}$$

Let $u_0 \neq 0$, $u_0 \in X$, using $u = \lambda u_0$ we obtain

$$\frac{\langle -\Delta u + wF(u), u \rangle}{\|u\|_X} \le \frac{\lambda}{\|u_0\|_X} \left(\|u_0\|_X^2 - c_f \|u_0\|_{\mathbb{Q},2,w}^2 \right) + \frac{\|k\|_{\mathbb{Q},2,w} \|u_0\|_{\mathbb{Q},2,w}}{\|u_0\|_X}.$$

If $c_f \ge \frac{\|u_0\|_X^2}{\|u_0\|_{\mathbb{Q},2,w}^2}$, then the right-hand side of the last inequality is bounded above for all $\lambda \ge 0$, which proves that $-\Delta + wF : X \to X^*$ is not coercive.

3. Proof of Theorem 1.4

Before proving Theorem 1.4, first we are going to reformulate the parabolic variational inequality (3). We recall that

$$V \hookrightarrow L^2(\mathbb{R}^N, w) \hookrightarrow V^*$$

forms an evolution triple, and thus the initial value $u(\cdot,0) = 0$ is well defined. Let $Lu := u_t$ be the time derivative operator with domain $D(L)$ given by

$$D(L) = \{u \in Y : u(\cdot,0) = 0\}.$$

Then by using Proposition 32.10] of Zeidler [13] we have the following result.

Lemma 3.1. *The operator $L\colon D(L) \to X^*$ is densely defined, closed and maximal monotone.*

Now we can reformulate the parabolic variational inequality (3) as follows:

$$u \in D(L) \cap K : \langle Lu - \Delta_p u + aF(u), v - u\rangle \geq 0 \quad \text{for all } v \in K. \tag{21}$$

The existence proof for (21) makes use of an abstract existence result for evolution equations of the form

$$u \in D(L) : Lu + T(u) = h \quad \text{in } X^* \tag{22}$$

with $h \in X^*$, where $T : X \to X^*$. To this end let $D(L)$ be equipped with its graph norm $\|u\|_{D(L)} = \|u\|_X + \|Lu\|_{X^*}$.

Definition 3.2. *$T\colon X \to X^*$ is called pseudomonotone with respect to the graph norm topology of $D(L)$ (for short: pseudomonotone w.r.t. $D(L)$), if for any sequence $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in X, $Lu_n \rightharpoonup Lu$ in X^* and*

$$\limsup_{n\to\infty}\langle T(u_n), u_n - u\rangle \leq 0$$

implies $T(u_n) \rightharpoonup T(u)$ and $\langle T(u_n), u_n\rangle \to \langle T(u), u\rangle$.

Since the space X is a separable, and uniformly convex Banach space, thus reflexive, the following surjectivity theorem plays an important role in the proof of our main result, see Berkovits-Mustonen [1] or Lions [7].

Theorem 3.3. *Let $T\colon X \to X^*$ be bounded, demicontinuous, and pseudomonotone w.r.t. $D(L)$. If T is coercive, that is,*

$$\frac{1}{\|u\|_X}\langle Tu, u\rangle \to \infty \quad \text{as } \|u\|_X \to \infty,$$

then $L + T\colon D(L) \to X^*$ is surjective, that is, $(L + T)(D(L)) = X^*$.

Proof of Theorem 1.4. The proof is based on a penalty approach and is carried out in three steps.

Step 1: Assumption (Hψ) implies that the function ψ is nonnegative, that is, $\psi \geq 0$, which is seen as follows. By (Hψ), $\psi \in Y$, and $\psi(\cdot, 0) \geq 0$, and satisfies

$$\langle \psi_t - \Delta_p\psi, \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X \text{ with } \varphi \geq 0.$$

Testing the inequality with $\varphi = \psi^- \in X$, where $s^- = \max\{-s, 0\}$, we get

$$\langle \psi_t, \psi^-\rangle + \int_{\mathbb{Q}} |\nabla\psi|^{p-2}\nabla\psi\nabla\psi^- \,\mathrm{d}x\mathrm{d}t \geq 0.$$

By means of the integration by parts formula we get

$$\langle \psi_t, \psi^- \rangle = \langle \psi_t, \psi^+ - \psi \rangle \leq 0,$$

which yields

$$\int_{\mathbb{Q}} |\nabla \psi|^{p-2} \nabla \psi \nabla \psi^- \,\mathrm{d}x\mathrm{d}t \geq 0.$$

Thus

$$-\int_{\mathbb{Q}} |\nabla \psi^-|^p \,\mathrm{d}x\mathrm{d}t \geq 0,$$

which results in $\nabla \psi^- = 0$, and therefore $\psi^- = 0$, that is, $\psi \geq 0$.

In a similar way one shows that $\phi \leq 0$, and thus $0 \in K$.

Step 2: Penalty equations

With the penalty operator $P : X \to X^*$ introduced in Section 2, we consider the penalty equation

$$u \in D(L) : \langle Lu - \Delta_p u + aF(u), \varphi \rangle + \langle \lambda P(u) + \frac{1}{\varepsilon} P(u), \varphi \rangle = 0,\ \forall \varphi \in X, \quad (23)$$

for any $\varepsilon > 0$, where $\lambda > 0$ will be chosen in such a way that the operator $-\Delta_p + aF + \lambda P : X \to X^*$ becomes coercive, which is seen as follows.

$$\begin{aligned}\langle -\Delta_p u + aF(u) + \lambda P(u), u \rangle \geq \|u\|_X^p - c_f c_a \|u\|_{\mathbb{Q},p,w}^p - c_a \|k\|_{\mathbb{Q},p',w} \|u\|_{\mathbb{Q},p,w} \\ + \lambda c_1 \|u\|_{\mathbb{Q},p,w}^p - \lambda c_2,\end{aligned}$$

where c_i are some positive constants, which implies that if $\lambda > \frac{c_f c_a}{c_1}$, then

$$\frac{1}{\|u\|_X} \langle -\Delta_p u + aF(u) + \lambda P(u), u \rangle \to \infty \quad \text{as } \|u\|_X \to \infty,$$

that is, $-\Delta_p + aF + \lambda P : X \to X^*$ is coercive for $\lambda > \frac{c_f c_a}{c_1}$. Since $P : X \to X^*$ is monotone (see Lemma 2.9) and $0 \in K$, that is $P(0) = 0$, we have $\langle P(u), u \rangle \geq 0$, and thus for any $\varepsilon > 0$ it follows that also

$$T = -\Delta_p + aF + \lambda P + \frac{1}{\varepsilon} P : X \to X^* \quad \text{is coercive.} \quad (24)$$

Recall that the operators $-\Delta_p$, and aF, are given by

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{Q}} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dxdt$$

$$\langle aF(u), \varphi \rangle = \int_{\mathbb{Q}} af(x,t,u) \varphi \, dxdt.$$

Clearly, $-\Delta_p : X \to X^*$ is bounded, continuous and strictly monotone, which implies that $-\Delta_p : X \to X^*$ is pseudomonotone in the usual sense (see e.g. Zeidler [13, Proposition 27.6 (a)]), and thus, in particular, pseudomonotone w.r.t. $D(L)$. By Corollary 2.8 the Nemytskij operator $F : L^p(\mathbb{Q}, w) \to L^{p'}(\mathbb{Q}, w)$ is continuous and bounded, and $aF : L^p(\mathbb{Q}, w) \to X^*$ is continuous and bounded as well. To show that $aF : X \to X^*$ is even pseudomonotone w.r.t. $D(L)$, let $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in X and $Lu_n \rightharpoonup Lu$ in X^*, that is, $u_n \rightharpoonup u$ in Y. Then due to Lemma 2.6 we know that $Y \hookrightarrow\hookrightarrow L^p(\mathbb{Q}, w)$, and hence $aF(u_n) \to aF(u)$ in X^*, which is already enough to ensure that $aF(u_n) \rightharpoonup aF(u)$ and $\langle aF(u_n), u_n\rangle \to \langle aF(u), u\rangle$, that is, $aF : X \to X^*$ is pseudomonotone w.r.t. $D(L)$. The penalty operator introduced in (8) is given by $P = wB$, where B denotes the Nemytskij operator generated by the function $b : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$, which qualitatively satisfies the same regularity and growth conditions (see (10)) as the function $f : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ in (Hf). Therefore, the penalty operator P possesses the same mapping properties as aF (note: $|a(x, t)| \le c_a w(x)$), hence $P : X \to X^*$ is continuous, bounded, and pseudomonotone w.r.t. $D(L)$. Since the sum of operators that are pseudomonotone w.r.t. $D(L)$ is again pseudomonotone w.r.t. $D(L)$, we finally get that $T : X \to X^*$ defined in (24) is bounded, continuous, coercive and pseudomonotone w.r.t. $D(L)$, which allows us to apply Theorem 3.3 that ensures the existence of a solution $u_\varepsilon \in D(L)$ of the penalty equation (23) for any $\varepsilon > 0$. Let (ε_n) with $\varepsilon_n \searrow 0$ and select an associated sequence of penalty solutions $(u_{\varepsilon_n}) := (u_n)$, that is,

$$u_n \in D(L) : \quad u_{nt} - \Delta_p u_n + aF(u_n) + \lambda P(u_n) + \frac{1}{\varepsilon_n} P(u_n) = 0 \quad \text{in } X^*. \tag{25}$$

Testing (25) with $\varphi = u_n$, and taking the monotonicity of the penalty operator P into account as well as $P(0) = 0$, we get

$$\langle -\Delta_p u_n + aF(u_n) + \lambda P(u_n), u_n\rangle = -\frac{1}{\varepsilon_n}\langle P(u_n), u_n\rangle - \langle u_{nt}, u_n\rangle \le 0.$$

Hence it follows

$$\frac{1}{\|u_n\|_X}\langle -\Delta_p u_n + aF(u_n) + \lambda P(u_n), u_n\rangle \le 0,$$

which in view of the coercivity of $-\Delta_p + aF + \lambda P$ implies that $(\|u_n\|_X)$ is bounded, and therefore, $(-\Delta_p u_n)$, $(aF(u_n))$, and $P(u_n)$ are bounded in X^*. Consider next the sequence $(\frac{1}{\varepsilon_n} P(u_n))$. By Lemma 2.10 we have

$$\begin{aligned}\langle P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+\rangle \\ \ge d\,\|P(u_n)\|_{X^*}\big(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w}\big).\end{aligned} \tag{26}$$

Testing the penalty equation (25) with $\varphi = (u_n - \psi)^+ - (\phi - u_n)^+$ we obtain

$$\langle u_{nt} - \Delta_p u_n, (u_n - \psi)^+ - (\phi - u_n)^+ \rangle \\ + \left\langle aF(u_n) + \lambda P(u_n) + \frac{1}{\varepsilon_n} P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle = 0. \quad (27)$$

Using Lemma 2.11 from (27) it follows

$$\left\langle \frac{1}{\varepsilon_n} P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle \leq - \left\langle aF(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle \\ - \left\langle \lambda P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle . \quad (28)$$

We estimate the first term on the right-hand side of (28) as follows:

$$\begin{aligned} &| \left\langle aF(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle | \\ &\quad \leq c_a \int_{\mathbb{Q}} w |F(u_n)| [(u_n - \psi)^+ + (\phi - u_n)^+] \, dxdt \\ &\quad \leq c_a \int_{\mathbb{Q}} w^{\frac{1}{p'}} |F(u_n)| \left(w^{\frac{1}{p}} (u_n - \psi)^+ + w^{\frac{1}{p}} (\phi - u_n)^+ \right) \, dxdt \\ &\quad \leq c_a \|F(u_n)\|_{\mathbb{Q},p',w} \left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w} \right) \\ &\quad \leq c \left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w} \right), \end{aligned}$$

for some positive constant c, since $(u_n) \subset X$ is bounded, which implies that (u_n) is bounded in $L^p(\mathbb{Q}, w)$ and thus $(F(u_n))$ is bounded in $L^{p'}(\mathbb{Q}, w)$. As $P = wB$, we get in the same way an analogous estimate for the second term on the right-hand side of (28), that is

$$| \left\langle \lambda P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle | \leq c \left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w} \right).$$

Hence from (28) it follows

$$\left\langle \frac{1}{\varepsilon_n} P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+ \right\rangle \leq c \left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w} \right),$$

which in view of Lemma 2.10 yields

$$\frac{d}{\varepsilon_n} \|P(u_n)\|_{X^*} \left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w} \right) \\ \leq c \left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w} \right)$$

and thus we obtain

$$\frac{1}{\varepsilon_n} \|P(u_n)\|_{X^*} \leq \frac{c}{d}, \quad \forall \, \varepsilon_n. \quad (29)$$

From the penalty equations (25) we have

$$u_{nt} = -\left(-\Delta_p u_n + aF(u_n) + \lambda P(u_n) + \frac{1}{\varepsilon_n} P(u_n)\right),$$

which in view of the boundedness of the sequence $(\frac{1}{\varepsilon_n} P(u_n))$ in X^* due to (29) along with the boundedness of the other terms on the right-hand side in X^* yields that (u_{nt}) is bounded in X^*, hence it follows that (u_n) is bounded in Y. Thus there exists a subsequence (again denoted by (u_n)) such that

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad u_{nt} \rightharpoonup u_t \quad \text{in } X^* \tag{30}$$

as $n \to \infty$ and $\varepsilon_n \searrow 0$. Since $D(L)$ is closed in Y and convex, it is weakly closed, and therefore $u \in D(L)$.

Step 3: The weak limit u in (30) is a solution of (21)

We are going to show that the weak limit u in (30) is in fact a solution of the parabolic bilateral variational inequality (21), resp. (3). To this end let us first show that $P(u) = 0$, and thus $u \in K$. From (29) it follows that $P(u_n) \to 0$ in X^*. Since the penalty operator $P\colon X \to X^*$ is monotone, we get $\langle P(v) - P(u_n), v - u_n\rangle \geq 0$ for all $v \in X$ and for all n, which by passing to the limit as $n \to \infty$ yields

$$\langle P(v), v - u\rangle \geq 0 \quad \text{for all } v \in X.$$

In particular, the last inequality holds for $v = u + \delta\varphi$ for any $\delta > 0$ and $\varphi \in X$, that is,

$$\langle P(u + \delta\varphi), \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X.$$

Passing to the limit as $\delta \searrow 0$ we get

$$\langle P(u), \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X,$$

which implies $P(u) = 0$, that is, $u \in K$.

Testing the penalty equation (25) with $\varphi = u_n - u$ and applying the inequality $\langle u_{nt} - u_t, u_n - u\rangle \geq 0$ one gets

$$\langle -\Delta_p u_n, u_n - u\rangle \leq -\langle u_t, u_n - u\rangle - \left\langle aF(u_n) + \left(\lambda + \frac{1}{\varepsilon_n}\right) P(u_n), u_n - u\right\rangle. \tag{31}$$

With (30) and the compact embedding $Y \hookrightarrow\hookrightarrow L^p(\mathbb{Q}, w)$ (see Lemma 2.6) it follows that $u_n \to u$ in $L^p(\mathbb{Q}, w)$, which yields by passing to the lim sup in (31) and taking into account $P : X \to X^*$ monotone and $P(u) = 0$ the following inequality

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle \leq 0. \tag{32}$$

Since $-\Delta_p : X \to X^*$ is a monotone operator, we obtain

$$0 \le \langle -\Delta_p u_n - (-\Delta_p u), u_n - u\rangle, \quad \forall n,$$

and in view of (30) and (32) we have

$$\limsup_{n\to\infty} \langle -\Delta_p u_n - (-\Delta_p u), u_n - u\rangle \le 0,$$

and thus

$$\lim_{n\to\infty} \int_{\mathbb{Q}} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\, dxdt = 0. \tag{33}$$

From Hölder's inequality it follows that

$$\begin{aligned}
&\int_{\mathbb{Q}} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\, dxdt \\
&\quad \ge \int_{\mathbb{Q}} (|\nabla u_n|^p + |\nabla u|^p)\, dxdt - \int_{\mathbb{Q}} \left(|\nabla u_n|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla u_n|\right) dxdt \\
&\quad \ge \|u_n\|_X^p + \|u\|_X^p - \|u_n\|_X^{p-1}\|u\|_X - \|u\|_X^{p-1}\|u_n\|_X \\
&\quad = (\|u_n\|_X^{p-1} - \|u\|_X^{p-1})(\|u_n\|_X - \|u\|_X) \ge 0,
\end{aligned}$$

which along with (33) yields

$$(\|u_n\|_X^{p-1} - \|u\|_X^{p-1})(\|u_n\|_X - \|u\|_X) \to 0,$$

and thus

$$\lim_{n\to\infty} \|u_n\|_X = \|u\|_X. \tag{34}$$

Since X is a uniformly convex Banach space, the weak convergence $u_n \rightharpoonup u$ in X together with (34) imply the strong convergence $u_n \to u$ in X. Now let $v \in K$ be arbitrarily given. Testing the penalty equation with $\varphi = v - u_n$ we get for all n (note: $P(v) = 0$)

$$\langle u_{nt} - \Delta_p u_n + aF(u_n), v - u_n\rangle = \left(\lambda + \frac{1}{\varepsilon_n}\right)\langle P(v) - P(u_n), v - u_n\rangle \ge 0,$$

which in view of $u_n \to u$ in X and $u_{nt} \rightharpoonup u_t$ and passing to the limit yields

$$\langle u_t - \Delta_p u + aF(u), v - u\rangle \ge 0 \quad \text{for all } v \in K,$$

that is, u is in fact a solution of the parabolic bilateral variational inequality (21), resp. (3), which completes the proof. □

Remark 3.4. Instead of the p-Laplacian in the parabolic bilateral variational inequality (3), we can deal likewise with a more general Leray-Lions operator A of the form

$$Au(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,t,\nabla u),$$

whose coefficient a_i satisfy the following conditions:

(A1) Each $a_i : \mathbb{Q}\times\mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x,t,\xi)$ is measurable in $(x,t) \in \mathbb{Q}$ for all $\xi \in \mathbb{R}^N$, and continuous in ξ for a.a. $(x,t) \in \mathbb{Q}$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^{p'}(\mathbb{Q})$ such that

$$|a_i(x,t,\xi)| \le k_0(x,t) + c_0\, |\xi|^{p-1}\,,$$

for a.a. $(x,t) \in \mathbb{Q}$ and for all $\xi \in \mathbb{R}^N$.

(A2) For a.a. $(x,t) \in \mathbb{Q}$ and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$, the following monotonicity holds:

$$\sum_{i=1}^{N} (a_i(x,t,\xi) - a_i(x,t,\xi'))(\xi_i - \xi_i') > 0.$$

(A3) There is a constant $\nu > 0$ such that for a.a. $(x,t) \in \mathbb{Q}$ and for all $\xi \in \mathbb{R}^N$, the inequality

$$\sum_{i=1}^{N} a_i(x,t,\xi)\xi_i \ge \nu|\xi|^p - k_1(x)$$

is satisfied for some function $k_1 \in L^1(\mathbb{Q})$.

4. Proof of Theorem 1.5

Let $u_k \in \mathcal{S}$, $k = 1, 2$, that is

$$u_k \in D(L) \cap K : \langle u_{kt} - \Delta_p u_k + aF(u_k), v - u_k\rangle \ge 0, \quad \forall v \in K. \tag{35}$$

First, we are focusing on the proof that $\mathcal{S}$ is directed upward, that is, to show the existence of $u \in \mathcal{S}$ such that $u \ge u_k$, $k = 1, 2$.

We define $u_0 = \max\{u_1, u_2\}$, and introduce the following truncated functions $f_k : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ of f

$$f_k(x,t,s) = \begin{cases} f(x,t,s) & \text{if} \quad s \ge u_k(x,t) \\ f(x,t,u_k(x,t) & \text{if} \quad s < u_k(x,t), \end{cases}$$

and denote by F_k the associated Nemytskij operators. By means of F_k and F we define G by

$$G(u) = -\sum_{j=1}^{2} |F_j(u) - F(u)|. \tag{36}$$

Thus F_k and G qualitatively have the same mapping properties as the Nemytskij operator F. Further, let $\hat{b} : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ be the following cut-off function

$$\hat{b}(x,t,s) = \begin{cases} 0 & \text{if} \quad s \geq u_0(x,t) \\ -(u_0(x,t) - s)^{p-1} & \text{if} \quad s < u_0(x,t), \end{cases}$$

and $\hat{B}$ its Nemytskij operator, which is readily seen to behave qualitatively like F. Now, we consider the auxiliary parabolic bilateral variational inequality: Find $u \in D(L) \cap K$ such that

$$\langle u_t - \Delta_p u + w\hat{B}(u) + aF(u) + aG(u), v - u \rangle \geq 0, \quad \forall v \in K. \tag{37}$$

Since the operator $w\hat{B} + aG : X \to X^*$ has the same mapping properties like F, we may apply Theorem 1.4, which ensures the existence of a solution $u \in D(L) \cap K$ of the auxiliary problem (37). The proof for $\mathcal{S}$ being upward directed is accomplished provided we can show that a solution u of the auxiliary problem (37) satisfies $u \geq u_k$, $k = 1, 2$, because then $u \geq u_0$, and hence it follows $G(u) = 0$ as well as $\hat{B}(u) = 0$, which shows $u \in \mathcal{S}$. To this end we recall that the bilateral constraint K has the lattice property (6), which allows us to use in (35) the test function

$$v = u_k \wedge u = u_k - (u_k - u)^+$$

and in (37) the test function

$$v = u_k \vee u = u + (u_k - u)^+$$

which yields

$$\langle u_{kt} - \Delta_p u_k + aF(u_k), -(u_k - u)^+ \rangle \geq 0 \tag{38}$$

$$\langle u_t - \Delta_p u + w\hat{B}(u) + aF(u) + aG(u), (u_k - u)^+ \rangle \geq 0. \tag{39}$$

Adding (38) and (39) results in

$$\begin{aligned} \langle w\hat{B}(u), (u_k - u)^+ \rangle \geq{}& \langle u_{kt} - u_t, (u_k - u)^+ \rangle + \langle -\Delta_p u_k - (-\Delta_p u), (u_k - u)^+ \rangle \\ &+ \langle aF(u_k) - aF(u) - aG(u), (u_k - u)^+ \rangle. \end{aligned} \tag{40}$$

Note that

$$\langle u_{kt} - u_t, (u_k - u)^+ \rangle \geq 0 \quad \text{and} \quad \langle -\Delta_p u_k - (-\Delta_p u), (u_k - u)^+ \rangle \geq 0.$$

As for $\langle aF(u_k) - aF(u) - aG(u), (u_k - u)^+\rangle$ we get by using the definition of f_k

$$
\begin{aligned}
&\langle aF(u_k) - aF(u) - aG(u), (u_k - u)^+\rangle \\
&= \int_{\mathbb{Q}} a(f(x,t,u_k) - f(x,t,u))(u_k - u)^+\,dxdt \\
&\qquad + \int_{\mathbb{Q}} a|f_1(x,t,u) - f(x,t,u)|(u_k - u)^+\,dxdt \\
&\qquad + \int_{\mathbb{Q}} a|f_2(x,t,u) - f(x,t,u)|(u_k - u)^+\,dxdt \\
&\ge \int_{\{u_k - u > 0\}} a\big[f(x,t,u_k) - f(x,t,u) + |f(x,t,u_k) - f(x,t,u)|\big](u_k - u)\,dxdt.
\end{aligned}
$$

Since $a\big[f(x,t,u_k) - f(x,t,u) + |f(x,t,u_k) - f(x,t,u)|\big](u_k - u) \ge 0$ on the set $\{u_k - u > 0\}$, we get $\langle aF(u_k) - aF(u) - aG(u), (u_k - u)^+\rangle \ge 0$, and thus from (40) we obtain

$$\langle w\hat{B}(u), (u_k - u)^+\rangle \ge 0,$$

which by definition of the function $\hat{b}$ yields

$$0 \le \int_{\mathbb{Q}} w\hat{b}(x,t,u)(u_k - u)^+\,dxdt = -\int_{\{u_k - u > 0\}} w(u_0(x,t) - u)^{p-1}(u_k - u)\,dxdt,$$

that is

$$
\begin{aligned}
0 &\ge \int_{\{u_k - u > 0\}} w(u_0(x,t) - u)^{p-1}(u_k - u)\,dxdt \\
&\ge \int_{\{u_k - u > 0\}} w(u_k - u)^p\,dxdt = \int_{\mathbb{Q}} w[(u_k - u)^+]^p\,dxdt \ge 0.
\end{aligned}
$$

Hence it follows $\|(u_k - u)^+\|_{\mathbb{Q},w,p} = 0$, which implies $u_k \le u$ for $k = 1, 2$, that is, $\mathcal{S}$ is upward directed. The proof for $\mathcal{S}$ being downward directed is done by obvious modifications and can be omitted. □

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