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Pairs of positive solutions of a quasilinear elliptic Neumann problem driven by the mean curvature operator

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Dedicated to Enzo Mitidieri on the occasion of his 70th birthday

Abstract. *We establish the existence of multiple positive weak solutions of the quasilinear elliptic Neumann problem driven by the mean curvature operator*

$$\begin{cases} -\mathrm{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big) = \lambda\, w(x)\, |u|^{p-2}u & \text{in } \Omega, \\ -\nabla u\, \nu/\sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded regular domain in $\mathbb{R}^N$, with $N \geq 2$, $p \in (1, 1^*)$, w is a sign-changing weight function, and $\lambda > 0$ is a parameter. Our findings provide the existence, for sufficiently small λ, of two positive solutions, the smaller in $C^1(\overline{\Omega})$, the larger in $BV(\Omega)$, which respectively bifurcate from $(\lambda, u) = (0, 0)$ and from $(\lambda, u) = (0, +\infty)$. This way we extend to a genuine PDE setting some results obtained in [22, 23] for the corresponding one-dimensional problem.

Keywords: Quasilinear elliptic problem, mean curvature operator, Neumann boundary condition, classical solution, bounded variation solution, positive solution, existence, multiplicity, non-smooth critical point theory, minimisation, mountain pass theorem.
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1. Introduction and statements

In this paper we shall establish the existence of pairs of positive weak solutions of the model problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda\, w(x)\, |u|^{p-2}u & \text{in } \Omega, \\ -\dfrac{\nabla u\, \nu}{\sqrt{1+|\nabla u|^2}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

(h_1) *Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $C^{0,1}$ boundary $\partial\Omega$ and unit outer normal ν to $\partial\Omega$,*

(h_2) *$p \in (1, 1^*)$, with $1^* = \frac{N}{N-1}$, is a fixed exponent, $w \in L^{\frac{1^*}{1^*-p}}(\Omega)$ is a non-zero function, and $\lambda \in \mathbb{R}$ is a parameter.*

In a quite similar fashion, but at the expense of an additional, somewhat cumbersome, technical effort, we could also deal with more general differential equations involving quasilinear operators $-\mathrm{div} A(x, \nabla u)$ patterned on the mean curvature operator $-\frac{1}{N}\mathrm{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$, like in [4, 9, 24], and functions $f(x, u)$ modelled on the power function $|u|^{p-2}u$, like in [13, 22, 23, 25]. Yet, while the extensions of our results along these directions will sometimes be indicated, the corresponding proofs will always be omitted in order to avoid obscuring ideas and blurring arguments.

As it is natural for Problem (1), two notions of weak solutions shall be used here, bounded variation solutions and classical weak solutions, the precise definitions being given in Section 2. The introduction of bounded variation functions when investigating prescribed mean curvature problems has become rather customary since the early seventies [10] and reveals absolutely imperative in the present context. Indeed, we shall show in a moment that, already in the one-dimensional case and even for Hölder continuous weight functions w, if (1) admits a pair of positive solutions, then the larger one necessarily displays jump discontinuities for sufficiently small values of the parameter.

This work is mainly motivated by the results we obtained in some recent papers, in collaboration with J. López-Gómez [16–21] and with J. López-Gómez and S. Rivetti [22, 23], concerning the one-dimensional counterpart of (1),

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda\, w(x)\, |u|^{p-2}u & \text{in } (0,1), \\ u'(0) = 0,\ u'(1) = 0, \end{cases} \tag{2}$$

and variations or extensions thereof. A particular example of our findings is provided by the following statement. The notion of bounded variation solution is expressed by Definition 2.1.

THEOREM 1.1. *Assume*

(k_1) *$p \in (1, 2)$,*

(k_2) *$w \in L^\infty(0, 1)$ and there is $z \in (0, 1)$ such that $w(x) > 0$ a.e. in $(0, z)$ and $w(x) < 0$ a.e. in $(z, 1)$,*

(k_3) $\displaystyle\int_0^1 w(x)\,\mathrm{d}x < 0.$

Then, there exists $\lambda^* \in (0,+\infty)$ *such that, for all* $\lambda > \lambda^*$*, Problem* (2) *has no positive bounded variation solutions and, for all* $\lambda \in (0,\lambda^*)$*, has at least two positive decreasing bounded variation solutions* $u_\lambda^{(1)}, u_\lambda^{(2)}$*, with* $u_\lambda^{(1)} \in W^{2,\infty}(0,1)$ *and* $u_\lambda^{(2)} \in W^{2,\infty}_{\mathrm{loc}}[0,z) \cap W^{2,\infty}_{\mathrm{loc}}(z,1]$*, satisfying*

$$\|u_\lambda^{(1)}\|_{C^1} \to 0, \qquad \operatorname{ess\,inf} u_\lambda^{(2)} \to +\infty \quad \textit{as } \lambda \to 0.$$

Moreover, the following regularity criterion is valid: if

$$\int_0^z \left(\int_z^x w(t)\,\mathrm{d}t\right)^{-\frac{1}{2}} \mathrm{d}x = +\infty \quad \textit{or} \quad \int_z^1 \left(\int_z^x w(t)\,\mathrm{d}t\right)^{-\frac{1}{2}} \mathrm{d}x = +\infty \tag{3}$$

holds, then, for all $\lambda > 0$*, all positive solutions of* (2) *belong to* $W^{2,\infty}(0,1)$*, while if* (3) *fails, i.e.,*

$$\int_0^z \left(\int_z^x w(t)\,\mathrm{d}t\right)^{-\frac{1}{2}} \mathrm{d}x \in \mathbb{R} \quad \textit{and} \quad \int_z^1 \left(\int_z^x w(t)\,\mathrm{d}t\right)^{-\frac{1}{2}} \mathrm{d}x \in \mathbb{R} \tag{4}$$

hold, then, for all small $\lambda > 0$*,* $u_\lambda^{(2)}$ *exhibits a jump discontinuity at* z *and*

$$\lim_{\lambda \to 0} \left(u_\lambda^{(2)}(z^-) - u_\lambda^{(2)}(z^+)\right) = +\infty.$$

A few comments about Theorem 1.1 are in order.

1. The non-existence of positive solutions follows by adapting the arguments given in [22, Theorem 6.1, Remark 6.1], or [18, Theorem 2.1], whereas, their existence and multiplicity come from [23, Theorem 1.6], [22, Theorem 9.1]. The regularity properties has been established in [16, Proposition 3.6], [17, Theorem 6.1], [20, Theorems 3.1–3.3]. The two plots of Figure 1 illustrate admissible configurations for the set of the positive solutions of (2), displaying branches which bifurcate from $(\lambda, u) = (0, 0)$ and from $(\lambda, u) = (0, +\infty)$, as guaranteed by Theorem 1.1.

2. It has been proved in [23, Proposition 1.1] that the existence of a bounded variation solution u of (1.1), with $\operatorname{ess\,inf} u > 0$, entails that the essential support of w^+ must have positive measure and (k_3) must hold. Accordingly, under (k_2), Assumption (k_3) is necessary for the existence of solutions $u_\lambda^{(2)}$ satisfying $\operatorname{ess\,inf} u_\lambda^{(2)} \to +\infty$ as $\lambda \to 0$.

3. Assumption (3) has been introduced in [17] and provides a mild one sided condition at z guaranteeing the regularity of all positive solutions of (2).

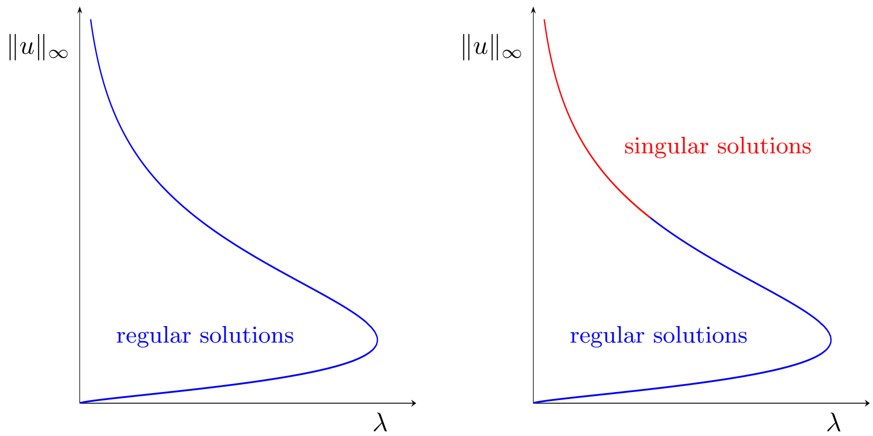

Figure 1: Admissible branches of positive solutions of (2) within the context of Theorem 1.1: on the left when (3) holds, on the right when (3) fails.

We stress that (3) does not imply the continuity of w at z. For instance, it holds if $w(z^-) = 0$ and $w'(z^-) \in \mathbb{R}$, or $w(z^+) = 0$ and $w'(z^+) \in \mathbb{R}$, even though the function w is discontinuous at z. Conversely, it is manifest that (4) is fulfilled by a large class of Hölder continuous functions w. Condition (3) is optimal, as witnessed by the fact that its failure, that is, the validity of (4), yields the existence of singular, discontinuous, solutions of (2). Consequently, conditions (3) and (4) produce a sharp local criterion characterising the regularity properties of the large positive solutions of (2) for all small λ. This way we also clarify and refine a result obtained in [11, Section 3C] (see also [5, Section 3.3]) where it is shown, through examples, that the membership of w to the Sobolev spaces $W^{1,q}(0,1)$ for any finite $q \geq 1$ is not enough to guarantee the $W^{1,1}$-regularity of the solutions. Figure 2 displays a picture of the profiles of the positive solutions of (2) in the frame of Theorem 1.1.

Our aim here is to provide a partial extension of Theorem 1.1 to a genuine PDE setting, as formulated by Theorem 1.2. Yet, we stress that, while we are able to establish for (1) existence and multiplicity conclusions quite similar to those of Theorem 1.1, extending to (1) the optimal regularity/singularity assertions, based on conditions such as (3) and (4), remains so far an extremely challenging unsolved problem. The notions of bounded variation solution and of classical weak solution used below are given by Definitions 2.1 and 2.9 respectively.

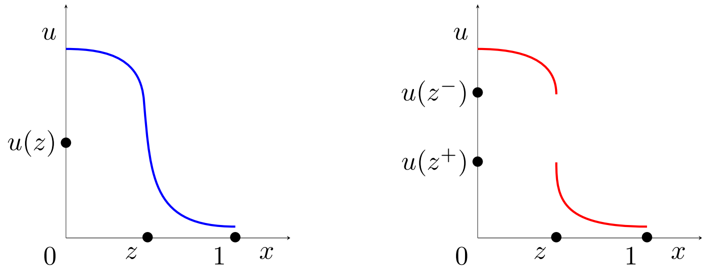

Figure 2: Profiles of the positive solutions of (2) within the context of Theorem 1.1: regular on the left, singular on the right.

THEOREM 1.2. *Assume* (h_1), (h_2),

(h_3) $\displaystyle\int_\Omega w\,\mathrm{d}x < 0$ *and the essential support of* w^+ *has non-empty interior.*

Then, there exists $\lambda^* \in (0, +\infty)$ *such that, for all* $\lambda \in (0, \lambda^*)$, *Problem* (1) *has at least two positive bounded variation solutions* $u_\lambda^{(1)}, u_\lambda^{(2)}$ *satisfying*

$$\|u_\lambda^{(1)}\|_{BV} \to 0, \qquad \|u_\lambda^{(2)}\|_{L^{1^*}} \to +\infty \qquad \textit{as } \lambda \to 0.$$

Suppose, in addition, that

(h_4) $\partial\Omega$ *is of class* $C^{1,\sigma}$ *for some* $\sigma \in (0, 1]$,

(h_5) $w \in L^\infty(\Omega)$.

Then, there exist $\lambda_* \in (0, \lambda^*)$ *and* $\tau \in (0, 1]$ *such that, for all* $\lambda \in (0, \lambda_*]$, $u_\lambda^{(1)}$ *can be chosen to be a classical weak solution satisfying*

$$u_\lambda^{(1)} \in C^{1,\tau}(\overline{\Omega}) \cap W^{2,q}_{\mathrm{loc}}(\Omega), \ \textit{for all finite } q \geq 1, \ \textit{and} \ \|u_\lambda^{(1)}\|_{C^{1,\tau}} \to 0 \ \textit{as } \lambda \to 0.$$

REMARK 1.3. Due to Assumptions (h_1)–(h_3), Problem (1) belongs to the class, widely investigated in the literature, of sub-superlinear indefinite elliptic problems. Specifically, according to the features of the mean curvature operator, the primitive $\frac{1}{p}|s|^p$ of the power function at the right hand side of the differential equation in (1) is “2-sublinear” at zero and “1-superlinear” at infinity in the essential support of w^+: this fact indeed determines the existence of multiple solutions.

REMARK 1.4. Under (h_1) and (h_2), it is very easy to verify that Assumption (h_4) is almost necessary for the existence of strictly positive classical weak solutions of (1). Indeed, if, for some $\lambda > 0$, $u \in C^1(\overline{\Omega})$ is a classical weak solution of (1) with $\min_\Omega u > 0$, then testing Equation (10) against $\phi = 1$ and $\phi = u^{1-p}$, respectively, yields

$$\int_\Omega w\, u^{p-1} \mathrm{d}x = 0 \qquad \text{and} \qquad (1-p) \int_\Omega \frac{u^{-p}|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\, \mathrm{d}x = \lambda \int_\Omega w\, \mathrm{d}x. \tag{5}$$

From the former identity in (5) it is apparent, as (h_2) requires $w \neq 0$, that

$$\operatorname*{ess\,inf}_\Omega w \cdot \operatorname*{ess\,sup}_\Omega w < 0,$$

so that, in particular, the essential support of w^+ must have positive measure. Obviously, the support of w^+ must have non-empty interior whenever w is continuous in Ω. On the other hand, the latter identity in (5) implies, as $p > 1$, that either

$$\int_\Omega w\, \mathrm{d}x < 0,$$

or u is a positive constant. This last eventuality is ruled out immediately, because otherwise Equation (10) would imply that $\int_\Omega w\, \phi\, \mathrm{d}x = 0$, for all $\phi \in C^1(\overline{\Omega})$, and thus $w = 0$, while $w \neq 0$.

As already observed above, it follows from [23, Proposition 1.1] that the same conclusion holds for the positive bounded variation solutions of (2).

REMARK 1.5. The proof of Theorem 1.2 is variational and combines minimisation and non-smooth critical point theory. Thanks to this features, further information are provided on the nature of $u_\lambda^{(1)}, u_\lambda^{(2)}$ as (sub-)critical points of the action functional $\mathcal{I}_\lambda$ associated with Problem (1) and defined by (7). Namely, we shall see that the smaller solution, $u_\lambda^{(1)}$, is a local minimiser of $\mathcal{I}_\lambda$, while the larger one, $u_\lambda^{(2)}$, is a (sub-)critical point of mountain pass type, these and other variational properties of $u_\lambda^{(1)}, u_\lambda^{(2)}$ being expressed by conditions (22), (37), (49), (53).

REMARK 1.6. The odd symmetry of Problem (1), which however is not necessary at all for establishing Theorem 1.2, obviously implies that $-u_\lambda^{(1)}$, $-u_\lambda^{(2)}$ constitute a pair of negative classical weak solutions of Problem (1). The fact of $u_\lambda^{(1)}, -u_\lambda^{(1)}$ being local minimisers predicts the existence of a third, sign-changing, classical weak solution. Actually, we can say more: from [28] we see that an infinite sequence of classical weak solutions tending to 0 exists for any given $\lambda > 0$. A second sequence of arbitrarily large bounded variation solutions, not necessarily classical, is expected to exist as well. These and other related topics will be investigated in a future paper.

REMARK 1.7. The power function $|s|^{p-2}s$ can be replaced in Theorem 1.2 by any continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying, for some $p \in (1, 1^*)$, $q \in (1, 2)$, $\vartheta > 1$,

$$\lim_{s\to 0} \frac{f(s)}{s^{q-1}} = 1, \qquad \lim_{s\to +\infty} \frac{f(s)}{s^{p-1}} = 1, \qquad \lim_{s\to +\infty} \frac{\vartheta \int_0^s f(t)\,\mathrm{d}t - f(s)s}{s} = 0.$$

Based on the proof of Theorem 1.2, the interested reader can provide the necessary details for establishing this statement.

REMARK 1.8. A counterpart of Theorem 1.2 for Problem (1) where the Neumann condition is replaced by a homogeneous Dirichlet condition can be established by combining [13, Theorem 2.1] and [29, Theorem 4.4]. In this setting (h_1)–(h_3) can still be assumed, although the condition $\int_\Omega w\,\mathrm{d}x < 0$ in (h_3) is not necessary anymore. As a result, the situation is substantially less delicate in the Dirichlet case and the corresponding proofs are somehow simpler.

The remainder of this paper is structured as follows: Section 2 provides all definitions, preparatory results, and complementary information relevant to the proof of Theorem 1.2, which is eventually delivered in Section 3.

Notations. Throughout, we write, for all $s, t \in \mathbb{R}$, $s \vee t = \max\{s, t\}$ and $s \wedge t = \min\{s, t\}$, as well as $s^+ = s \vee 0$ and $s^- = -(s \wedge 0)$; similar notations are also used for functions. We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure and by $\mathcal{H}^{N-1}$ the $N-1$-dimensional Hausdorff measure. We say that a function $v \in L^1(\Omega)$ is positive if $\operatorname{ess\,inf} v \geq 0$ and $\operatorname{ess\,sup} v > 0$, while we say that it is sign-changing if $\operatorname{ess\,inf} v \cdot \operatorname{ess\,sup} v < 0$.

2. Preliminaries

In order to introduce the notion of bounded variation solution we use in this paper a few preliminaries are needed.

Bounded variation functions and the area functional.
Let $BV(\Omega)$ be the space of bounded variation functions in Ω, i.e., functions $v \in L^1(\Omega)$ whose distributional gradient Dv is a finite vector valued Radon measure. When equipped with the norm

$$\|v\|_{BV} = |\overline{v}| + \|Dv\|, \tag{6}$$

where $\overline{v} = \frac{1}{|\Omega|}\int_\Omega v\,\mathrm{d}x$ and $\|Dv\| = |Dv|(\Omega) = \int_\Omega \mathrm{d}|Dv|$, $BV(\Omega)$ is a Banach space compactly imbedded into $L^q(\Omega)$ for all $q \in [1, 1^*)$, where $1^* = \frac{N}{N-1}$ if $N \geq 2$ and $1^* = \infty$ in $N = 1$, and continuously imbedded into $L^{1^*}(\Omega)$. Thanks to the Poincaré inequality,

$$\|v - \overline{v}\|_{L^q} \leq c_\Omega \|Dv\|,$$

with $q \in [1, 1^*]$ and $c_\Omega > 0$ a constant depending only on Ω, the norm defined by (6) is in fact equivalent to the standard one.

For any $v \in BV(\Omega)$, $Dv = D^a v \ + D^s v$ is the Lebesgue decomposition of the measure Dv in an absolutely continuous part $D^a v = \nabla v \, \mathcal{L}^n$, with ∇v the approximate differential of v, and a singular part $D^s v$ with respect to $\mathcal{L}^n$. $|Dv|$ stands for the total variation of the measure Dv. Moreover, $|Dv| = |Dv|^a + |Dv|^s = |D^a v| + |D^s v|$ is the Lebesgue decomposition of $|Dv|$ and $\frac{Dv}{|Dv|}$ is the density of Dv with respect to $|Dv|$.

Let us define the functional $\mathcal{J} : BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}(v) = \int_\Omega \sqrt{1 + |Dv|^2} - |\Omega|,$$

where

$$\int_\Omega \sqrt{1 + |Dv|^2} = \int_\Omega \sqrt{1 + |\nabla v|^2} \, \mathrm{d}x + \int_\Omega \mathrm{d}|Dv|^s$$

is the area functional in $BV(\Omega)$. The functional $\mathcal{J}$ vanishes at 0, it is convex and Lipschitz continuous in $BV(\Omega)$, and lower semicontinuous with respect to the L^1-convergence in $BV(\Omega)$.

For $\lambda > 0$, let us introduce the action functional $\mathcal{I}_\lambda : BV(\Omega) \to \mathbb{R}$ associated with Problem (1) by setting

$$\mathcal{I}_\lambda(v) = \mathcal{J}(v) - \lambda \int_\Omega \tfrac{1}{p} w \, |v|^p \, \mathrm{d}x. \tag{7}$$

It is apparent that $\mathcal{I}_\lambda$ is even and it is the sum of a convex and of a differentiable term; moreover, as $p \in (1, 1^*)$, it is lower semicontinuous with respect to the weak* convergence in $BV(\Omega)$, i.e., if $(v_n)_n$ is a sequence in $BV(\Omega)$ which converges in $L^1(\Omega)$ to $v \in BV(\Omega)$ and satisfies $\sup_n \|Dv_n\| < +\infty$, then

$$\liminf_{n \to +\infty} \mathcal{I}_\lambda(v_n) \geq \mathcal{I}_\lambda(v).$$

For further information on the subject of this paragraph we refer to [1, 11, 12].

Bounded variation solutions and their characterisation.
Two equivalent notions of bounded variation solutions are given below: the former is formulated through an Euler equation, like in [4], the latter through a variational inequality, like in [24].

Definition 2.1. *Assume* (h_1) *and* (h_2). *A function* $u \in BV(\Omega)$ *is a bounded variation solution of* (1) *if*

$$\int_\Omega \frac{\nabla u \, \nabla \phi}{\sqrt{1 + |\nabla u|^2}} \, \mathrm{d}x + \int_\Omega \frac{Du}{|Du|} \frac{D\phi}{|D\phi|} \mathrm{d}|D\phi|^s = \lambda \int_\Omega w \, |u|^{p-2} u \, \phi \, \mathrm{d}x \tag{8}$$

for all $\phi \in BV(\Omega)$ *such that* $|D\phi|^s$ *is absolutely continuous with respect to* $|Du|^s$.

Remark 2.2. It is apparent that Assumption (h_2) ensures that the integral on the right of (8) is finite. Further, whenever $D^s u = 0$, Definition 2.1 reduces to the notion of weak solution in $W^{1,1}(\Omega)$.

Remark 2.3. From the results in [4, Section 3] it follows that the conditions required by Definition 2.1 are satisfied by $u \in BV(\Omega)$ if and only if u minimises in $BV(\Omega)$ the convex functional

$$\mathcal{H}_u(v) = \mathcal{J}(v) - \lambda \int_\Omega w\,|u|^{p-2} u\, v\, \mathrm{d}x.$$

Consequently, the following alternative formulation of the notion of bounded variation solution can be given.

Definition 2.4. *Assume* (h_1) *and* (h_2). *A function* $u \in BV(\Omega)$ *is a bounded variation solution of* (1) *if*

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_\Omega w\,|u|^{p-2} u\,(v-u)\, \mathrm{d}x,$$

for all $v \in BV(\Omega)$.

Remark 2.5. According to Definition 2.4 u is a sub-critical point of the functional $\mathcal{I}_\lambda$, in the sense that

$$0 \in \partial\mathcal{J}(u) - \lambda \int_\Omega w\,|u|^{p-2} u\, \mathrm{d}x \quad \iff \quad \lambda \int_\Omega w\,|u|^{p-2} u\, \mathrm{d}x \in \partial\mathcal{J}(u),$$

where $\partial\mathcal{J}(u)$ denotes the sub-differential of $\mathcal{J}$ at u. It is also easy to see that any local minimiser of $\mathcal{I}_\lambda$ in $BV(\Omega)$ is a solution of (1).

The next result establishes some relevant properties enjoyed by the bounded variation solutions of (1). Following [2] we set

$$X(\Omega)_N = \{z \in L^\infty(\Omega, \mathbb{R}^N) \mid \operatorname{div} z \in L^N(\Omega)\}.$$

Then, by [2, Theorem 1.9] the integration by parts formula

$$\int_\Omega v \operatorname{div} z\, \mathrm{d}x = \int_{\partial\Omega} [z \cdot \nu]\, v\, \mathrm{d}\mathcal{H}^{N-1} - \int_\Omega \mathrm{d}(Dv, z) \tag{9}$$

holds for all $v \in BV(\Omega)$ and $z \in X(\Omega)_N$, where $[z \cdot \nu] \in L^\infty(\partial\Omega)$ is the weak trace on $\partial\Omega$ of the component of z along the outer normal ν to $\partial\Omega$, which exists by [2, Theorem 1.2], and (Dv, z) is the Radon measure defined in [2, Definition 1.4].

Proposition 2.6. *Let* u *be a bounded variation solution of* (1). *Then, the following holds:*

1. *the distributional divergence of the vector field* $\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in L^\infty(\Omega)$ *satisfies*
$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda\, w\, |u|^{p-2}u \qquad a.e.\ in\ \Omega,$$
accordingly, $\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in X(\Omega)_N$,

2. *the identity, between measures in* Ω,
$$\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, Du\right) = \sqrt{1+|Du|^2} - \sqrt{1-\frac{|\nabla u|^2}{1+|\nabla u|^2}}\,\mathcal{L}^N$$
holds,

3. *the weak trace* $\left[\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot \nu_{\partial\Omega}\right] \in L^\infty(\partial\Omega)$ *of the component on* $\partial\Omega$ *of the vector field* $\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in X(\Omega)_N$ *along to the unit outer normal* ν *exists and satisfies*
$$-\left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot \nu_{\partial\Omega}\right] = 0 \qquad \mathcal{H}^{N-1}\text{-}a.e.\ on\ \partial\Omega.$$

Proof. The first assertion follows by testing (8) against all $v \in C_0^\infty(\Omega)$ and recalling that
$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in L^\infty(\Omega) \qquad \text{and} \qquad w\,|u|^{p-2}u \in L^N(\Omega).$$

The second assertion follows from (8) and the definition of the measure $\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}, Du\right)$, as expressed by [2, Definition 1.4, Theorem 1.5]. Indeed, if we pick any $\phi \in C_0^\infty(\Omega)$ and we test (8) against $v = u\phi$, we get
$$\begin{aligned}\int_\Omega \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\phi\,\mathrm{d}x + \int_\Omega \frac{\nabla u \nabla\phi}{\sqrt{1+|\nabla u|^2}} u\,\mathrm{d}x + \int_\Omega \phi\,\mathrm{d}|D^s u| &= \int_\Omega w\,|u|^p\,\phi\,\mathrm{d}x\\ &= -\int_\Omega \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) u\,\phi\,\mathrm{d}x\\ &= \int_\Omega \phi\,\mathrm{d}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, Du\right) + \int_\Omega \frac{\nabla u\nabla\phi}{\sqrt{1+|\nabla u|^2}} u\,\mathrm{d}x.\end{aligned}$$

Hence, we infer that

$$\int_\Omega \phi \, \mathrm{d}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, Du\right) = \int_\Omega \phi \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \, \mathrm{d}x + \int_\Omega \phi \, \mathrm{d}|D^s u|$$

and then

$$\begin{aligned}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, Du\right) &= \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \, \mathcal{L}^N + |D^s u| \\ &= \sqrt{1+|\nabla u|^2} \, \mathcal{L}^N + |D^s u| - \frac{1}{\sqrt{1+|\nabla u|^2}} \, \mathcal{L}^N \\ &= \sqrt{1+|Du|^2} - \sqrt{1 - \frac{|\nabla u|^2}{1+|\nabla u|^2} \, \mathcal{L}^N}.\end{aligned}$$

The third assertion follows from the integration by parts formula in $BV(\Omega)$ expressed by (9). Indeed, by testing (8) against any $v \in W^{1,1}(\Omega)$, we find

$$\begin{aligned}\int_{\partial\Omega} \left[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu\right] v \, \mathrm{d}\mathcal{H}^{N-1} = &\int_\Omega \frac{\nabla u \nabla v}{\sqrt{1+|\nabla u|^2}} \, \mathrm{d}x \\ &+ \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) v \, \mathrm{d}x = 0.\end{aligned}$$

Thus, the proof is concluded. □

Remark 2.7. The third conclusion of Propositon 2.6 implies that all bounded variation solutions of (1) satisfy the homogeneous Neumann boundary condition in the weak sense explained above.

Remark 2.8. Proposition 2.6 shows that the notion of solution expressed by Definition 2.1, or equivalently by Definition 2.4, implies the validity of the Neumann counterpart of the definition of solution recently used, first, in [30] and, later, in [8] for the Dirichlet problem associated with the mean curvature equation.

Classical weak solutions.
The following definition of classical weak solution of (1) is quite standard [31], nevertheless we remind it for completeness.

Definition 2.9. *Assume* (h_1) *and* (h_2). *A function* $u \in C^1(\overline{\Omega})$ *is a classical weak solution of* (1) *if, for all* $\phi \in C^1(\overline{\Omega})$,

$$\int_\Omega \frac{\nabla u \nabla \phi}{\sqrt{1+|\nabla u|^2}} \, \mathrm{d}x = \lambda \int_\Omega w \, |u|^{p-2} u \, \phi \, \mathrm{d}x. \tag{10}$$

Remark 2.10. Combining the third conclusion of Propositon 2.6 with [3, Remark 1.8] entails that, if Ω has a C^1 boundary $\partial\Omega$, then any classical weak solution u of (1) satisfies the homogeneous Neumann boundary condition everywhere on $\partial\Omega$. Moreover, from [14, Theorem 2.1] it follows that, if $w \in L^\infty(\Omega)$, then $u \in W^{2,q}_{\mathrm{loc}}(\Omega)$ for all finite $q \geq 1$, and if in addition $\partial\Omega$ has a $C^{1,1}$ boundary, then $u \in W^{2,q}(\Omega)$ for all finite $q \geq 1$.

3. Proof of Theorem 1.2

The proof is divided into two parts, each subdivided into a number of steps.

Part 1. Existence of pairs of bounded variation solutions

In this part conditions (h_1)–(h_3) are assumed.

Step 1. Preliminary estimates.

We establish two estimates that are crucial in this proof.

First estimate: for all $v \in BV(\Omega)$,

$$\mathcal{J}(v) \geq \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Dv\|^2}}. \tag{11}$$

Indeed, fix $v \in BV(\Omega)$ and let $(v_n)_n$ be a sequence in $W^{1,1}(\Omega)$ such that

$$v_n \to v \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \mathcal{J}(v_n) \to \mathcal{J}(v), \quad \text{as } n \to +\infty.$$

By [4, Fact 3.1], we also have that

$$\|\nabla v_n\|_{L^1} \to \|Dv\|, \quad \text{as } n \to +\infty.$$

Hence, thanks to Jensen's inequality, we get

$$\begin{aligned} \mathcal{J}(v) = \lim_{n\to+\infty} \mathcal{J}(v_n) &= \lim_{n\to+\infty} \int_\Omega \sqrt{1 + |\nabla v_n|^2}\, \mathrm{d}x - |\Omega| \\ &\geq \lim_{n\to+\infty} |\Omega| \sqrt{1 + \left(\frac{\|\nabla v_n\|_{L^1}}{|\Omega|}\right)^2} - |\Omega| \\ &= |\Omega| \sqrt{1 + \left(\frac{\|Dv\|}{|\Omega|}\right)^2} - |\Omega| = \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Dv\|^2}}. \end{aligned}$$

Second estimate: there exist constants $c_1, c_2, c_3, c_4 > 0$, such that, for all $\lambda > 0$, $v \in BV(\Omega)$, and $\varepsilon > 0$,

$$\begin{aligned} \mathcal{I}_\lambda(v) \geq & \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Dv\|^2}} \\ & + \lambda \Big(- c_1 \|Dv\|^p - c_2\, \varepsilon |\overline{v}|^{2(p-1)} - c_3 \tfrac{1}{\varepsilon} \|Dv\|^2 + c_4 |\overline{v}|^p \Big), \end{aligned} \tag{12}$$

where $\overline{v} = \frac{1}{|\Omega|}\int_\Omega v\,\mathrm{d}x$. Indeed, by (11), we have that, for all $\lambda > 0$ and $v \in BV(\Omega)$,

$$\begin{aligned}\mathcal{I}_\lambda(\overline{v}+\widetilde{v}) &= \mathcal{J}(v) - \lambda\int_\Omega \tfrac{1}{p} w\,|\overline{v}+\widetilde{v}|^p\,\mathrm{d}x\\ &\geq \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2+\|Dv\|^2}} - \lambda\int_\Omega \tfrac{1}{p} w\left(|\overline{v}+\widetilde{v}|^p - |\overline{v}|^p\right)\mathrm{d}x\\ &\qquad - \lambda\tfrac{1}{p}|\overline{v}|^p\int_\Omega w\,\mathrm{d}x, \end{aligned} \tag{13}$$

where $\widetilde{v} = v - \overline{v}$. As $p \in (1,2)$, the inequality

$$|r+s|^p \leq |r|^p + |s|^p + p|r|^{p-1}|s|$$

holds for all $r, s \in \mathbb{R}$. Hence, by Hölder and Young inequalities, we get, for any given $\varepsilon > 0$,

$$\begin{aligned}\int_\Omega w\left(|\overline{v}+\widetilde{v}|^p - |\overline{v}|^p\right)\mathrm{d}x &\leq \int_\Omega |w|\,|\widetilde{v}|^p\,\mathrm{d}x + p|\overline{v}|^{p-1}\int_\Omega |w|\,|\widetilde{v}|\,\mathrm{d}x \\ &\leq \|w\|_{L^{\frac{1^*}{1^*-p}}}\||\widetilde{v}|^p\|_{L^{\frac{1^*}{p}}} + p|\overline{v}|^{p-1}\|w\|_{L^{\frac{1^*}{1^*-p}}}\|\widetilde{v}\|_{L^{\frac{1^*}{p}}}\\ &\leq \|w\|_{L^{\frac{1^*}{1^*-p}}}\|\widetilde{v}\|^p_{L^{1^*}} + \tfrac{1}{2}p\|w\|_{L^{\frac{1^*}{1^*-p}}}\varepsilon|\overline{v}|^{2(p-1)}\\ &\qquad + \tfrac{1}{2}p\|w\|_{L^{\frac{1^*}{1^*-p}}}\tfrac{1}{\varepsilon}\|\widetilde{v}\|^2_{L^{\frac{1^*}{p}}}\\ &\leq \|w\|_{L^{\frac{1^*}{1^*-p}}}\|\widetilde{v}\|^p_{L^{1^*}} + \tfrac{\varepsilon}{2}p\|w\|_{L^{\frac{1^*}{1^*-p}}}|\overline{v}|^{2(p-1)}\\ &\qquad + \tfrac{1}{2\,\varepsilon}p|\Omega|^{\frac{2(p-1)}{1^*}}\|w\|_{L^{\frac{1^*}{1^*-p}}}\|\widetilde{v}\|^2_{L^{1^*}}. \end{aligned} \tag{14}$$

Inserting (14) into (13) and using the Poincaré inequality, we can find constants $c_1, c_2, c_3, c_4 > 0$, independent of v, λ, ε, such that

$$\begin{aligned}\mathcal{I}_\lambda(v) &\geq \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2+\|Dv\|^2}} - \lambda\tfrac{1}{p}\|w\|_{L^{\frac{1^*}{1^*-p}}}\|\widetilde{v}\|^p_{L^{1^*}} - \lambda\tfrac{\varepsilon}{2}\|w\|_{L^{\frac{1^*}{1^*-p}}}|\overline{v}|^{2(p-1)}\\ &\qquad - \lambda\tfrac{1}{2\,\varepsilon}|\Omega|^{\frac{2(p-1)}{1^*}}\|w\|_{L^{\frac{1^*}{1^*-p}}}\|\widetilde{v}\|^2_{L^{1^*}} + \lambda\tfrac{1}{p}\,|\overline{v}|^p\left|\int_\Omega w\,\mathrm{d}x\right|\\ &\geq \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2+\|Dv\|^2}}\\ &\qquad + \lambda\Big(-c_1\|Dv\|^p - c_2\,\varepsilon|\overline{v}|^{2(p-1)} - c_3\tfrac{1}{\varepsilon}\|Dv\|^2 + c_4|\overline{v}|^p\Big). \end{aligned}$$

Step 2. Existence of solutions $u^{(1)}_\lambda$ *such that* $\mathcal{I}_\lambda(u^{(1)}_\lambda) < 0$ *and* $\lim_{\lambda\to 0}\|u^{(1)}_\lambda\|_{BV} = 0$.

The existence of a first solution $u_\lambda^{(1)}$, for all small $\lambda > 0$, is proved by minimising the functional $\mathcal{I}_\lambda$ over a ball in $BV(\Omega)$.

Let us denote by

$$\mathcal{B}_1 = \{v \in BV(\Omega) : \|Dv\| + |\overline{v}| < 1\}$$

the unit open ball in $BV(\Omega)$ and by

$$\mathcal{S}_1 = \{v \in BV(\Omega) : \|Dv\| + |\overline{v}| = 1\}$$

its boundary. Setting $c_0 = (|\Omega| + \sqrt{|\Omega|^2 + 1})^{-1}$, from (12) we infer that, for all $\lambda > 0$ and $v \in \mathcal{S}_1$,

$$\mathcal{I}_\lambda(v) \geq \Big(c_0 - \lambda c_3 \tfrac{1}{\varepsilon}\Big)\|Dv\|^2 + \lambda\big(- c_1\|Dv\|^p - c_2\varepsilon|\overline{v}|^{2(p-1)} + c_4|\overline{v}|^p\big). \tag{15}$$

Consider, for $s \in [0, 1]$, the function

$$\Big(c_0 - \lambda c_3 \tfrac{1}{\varepsilon}\Big)s^2 + \lambda\big(- c_1 s^p - c_2\varepsilon(1 - s)^{2(p-1)} + c_4(1 - s)^p\big).$$

Fix $\varepsilon > 0$ so small that $-c_2\varepsilon + c_4 > 0$. Hence, we can find $s_0 \in (0, 1)$ such that,

$$-c_1 s^p - c_2\varepsilon(1 - s)^{2(p-1)} + c_4(1 - s)^p > 0 \quad \text{in } [0, s_0].$$

Consequently, as $c_0 - \lambda \frac{c_3}{\varepsilon} > 0$ for all small $\lambda > 0$, we have that

$$\Big(c_0 - \lambda c_3 \tfrac{1}{\varepsilon}\Big)s^2 + \lambda\big(- c_1 s^p - c_2\varepsilon(1 - s)^{2(p-1)} + c_4(1 - s)^p\big) > 0 \quad \text{in } [0, s_0]. \tag{16}$$

Thus, possibly reducing $\lambda > 0$ further, we get

$$\Big(c_0 - \lambda c_3 \tfrac{1}{\varepsilon}\Big)s^2 + \lambda\big(- c_1 s^p - c_2\varepsilon(1 - s)^{2(p-1)} + c_4(1 - s)^p\big) > 0 \quad \text{in } [s_0, 1]. \tag{17}$$

From (16) and (17), we infer that, for all small $\lambda > 0$, there is $\delta_\lambda > 0$ such that

$$\Big(c_0 - \lambda c_3 \tfrac{1}{\varepsilon}\Big)s^2 + \lambda\big(- c_1 s^p - c_2\varepsilon(1 - s)^{2(p-1)} + c_4(1 - s)^p\big) \geq \delta_\lambda \quad \text{in } [0, 1].$$

Hence, taking any $v \in \mathcal{S}_1$ and setting $\|Dv\| = s$ and $|\overline{v}| = 1 - s$, from (15) we conclude that

$$\mathcal{I}_\lambda(v) \geq \delta_\lambda. \tag{18}$$

provided that $\lambda > 0$ is small enough.

Then, it is straightforward to show that there exists

$$\min_{\overline{\mathcal{B}}_1} \mathcal{I}_\lambda.$$

Indeed, Condition (18) implies that $\mathcal{I}_\lambda$ is bounded from below in $\overline{\mathcal{B}}_1$. Hence, let $(u_n)_n$ be a sequence minimising $\mathcal{I}_\lambda$ in $\overline{\mathcal{B}}_1$. By the weak* compactness in $BV(\Omega)$ of $\overline{\mathcal{B}}_1$, there is a subsequence of $(u_n)_n$, still denoted by $(u_n)_n$, which converges in $L^p(\Omega)$ to some $u \in \overline{\mathcal{B}}_1$. The lower semicontinuity of $\mathcal{I}_\lambda$, with respect to the weak* convergence in $BV(\Omega)$ implies that

$$\mathcal{I}_\lambda(u) \leq \liminf_{n\to+\infty} \mathcal{I}_\lambda(u_n) = \inf_{\overline{\mathcal{B}}_1} \mathcal{I}_\lambda,$$

that is,

$$\mathcal{I}_\lambda(u) = \min_{\overline{\mathcal{B}}_1} \mathcal{I}_\lambda.$$

Let us prove that

$$\mathcal{I}_\lambda(u) = \min_{\overline{\mathcal{B}}_1} \mathcal{I}_\lambda < 0. \tag{19}$$

By (h_3) there exist open balls B_1, B_2 in $\mathbb{R}^N$ such that $\overline{B}_1 \subset B_2$, $\overline{B}_2 \subset \Omega$ and $w(x) > 0$ a.e. in B_2. Let $\psi \in C^1(\overline{\Omega})$ be such that $0 \leq \psi(x) \leq 1$ in Ω, $\psi(x) = 1$ in B_1 and $\psi(x) = 0$ in $\Omega \setminus B_2$. Then, for all $t > 0$, we have that

$$\begin{aligned}\mathcal{I}_\lambda(t\psi) &= \int_{B_2} \frac{t^2|\nabla\psi|^2}{1+\sqrt{1+t^2|\nabla\psi|^2}}\,\mathrm{d}x - \lambda\int_{B_1} \tfrac{1}{p} w\, t^p\psi^p\,\mathrm{d}x - \lambda\int_{B_2\setminus B_1} \tfrac{1}{p} w\, t^p\psi^p\,\mathrm{d}x \\ &\leq t^p\Big(t^{2-p}\int_{B_2} |\nabla\psi|^2\,\mathrm{d}x - \lambda\int_{B_1} \tfrac{1}{p} w\,\mathrm{d}x\Big).\end{aligned} \tag{20}$$

As $p \in (1,2)$, letting $t \to 0$ yields

$$\mathcal{I}_\lambda(t\psi) < 0$$

for all small $t > 0$. Hence, (19) folllows.

From (18) and (19) we infer that $u \in \mathcal{B}_1$. Therefore, u is a local minimiser of $\mathcal{I}_\lambda$ in $BV(\Omega)$ and hence a solution of (1). Moreover, (19) implies that $u \neq 0$, because $\mathcal{I}_\lambda(0) = 0$.

Let show that there exists a positive minimiser of $\mathcal{I}_\lambda$ in $\mathcal{B}_1$. Indeed, by the even symmetry and the lattice property enjoyed by the area functional, we have that

$$\begin{aligned}\mathcal{J}(|u|) = \tfrac{1}{2}\big(\mathcal{J}(-|u|) + \mathcal{J}(|u|)\big) &= \tfrac{1}{2}\big(\mathcal{J}(u \wedge (-u)) + \mathcal{J}(u \vee (-u))\big) \\ &\leq \tfrac{1}{2}\big(\mathcal{J}(u) + \mathcal{J}(-u)\big) = \mathcal{J}(u),\end{aligned}$$

where $\mathcal{J}(|u|) < \mathcal{J}(u)$ may occur if u exhibits jump discontinuities when changing sign. Hence, we get

$$\mathcal{I}_\lambda(|u|) = \mathcal{J}(|u|) - \lambda\int_\Omega \tfrac{1}{p} w\,|u|^p\,\mathrm{d}x \leq \mathcal{J}(u) - \lambda\int_\Omega \tfrac{1}{p} w\,|u|^p\,\mathrm{d}x = \mathcal{I}_\lambda(u). \tag{21}$$

Thus, we conclude that $|u| \in \overline{\mathcal{B}_1}$ is a local minimiser of $\mathcal{I}_\lambda$ too. Hereafter, we denote by $u_\lambda^{(1)}$ any positive solution of (1) such that

$$\mathcal{I}_\lambda(u_\lambda^{(1)}) = \min_{\overline{\mathcal{B}}_1} \mathcal{I}_\lambda. \tag{22}$$

We finally prove that

$$\lim_{\lambda \to 0} \|u_\lambda^{(1)}\|_{BV} = 0. \tag{23}$$

Indeed, from (11) and $\mathcal{I}_\lambda(u_\lambda^{(1)}) < 0$, we infer that

$$\begin{aligned} \frac{\|Du_\lambda^{(1)}\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Du_\lambda^{(1)}\|^2}} &\le \mathcal{J}(u_\lambda^{(1)}) \\ &\le \lambda \int_\Omega \tfrac{1}{p}|w|\,|u|^p \,\mathrm{d}x \le \lambda \tfrac{1}{p}\|w\|_{L^{\frac{1^*}{1^*-p}}} \|u_\lambda^{(1)}\|^p_{L^{1^*}}. \end{aligned}$$

The continuous embedding of $BV(\Omega)$ into $L^{1^*}(\Omega)$ and the bound $\|u_\lambda^{(1)}\|_{BV} < 1$ yield the existence of a constant $c > 0$ such that

$$\|Du_\lambda^{(1)}\| \le c\sqrt{\lambda}.$$

Hence, we get

$$\lim_{\lambda \to 0} \|Du_\lambda^{(1)}\| = 0$$

and, by the Poincaré inequality,

$$\lim_{\lambda \to 0} \|\widetilde{u}_\lambda^{(1)}\|_{L^{1^*}} = 0. \tag{24}$$

Let us show that

$$\lim_{\lambda \to 0} \overline{u}_\lambda^{(1)} = 0.$$

Otherwise, we can find a sequence $(\lambda_n)_n$, with $\lambda_n \to 0^+$ as $n \to +\infty$, such that

$$\lim_{n \to +\infty} \overline{u}_{\lambda_n}^{(1)} = m > 0. \tag{25}$$

Testing (8), with $u = u_{\lambda_n}^{(1)}$, against $\phi = 1$ and dividing by $\lambda_n > 0$, we obtain

$$\int_\Omega w\,\big(\widetilde{u}_{\lambda_n}^{(1)} + \overline{u}_{\lambda_n}^{(1)}\big)^{p-1}\,\mathrm{d}x = 0.$$

Hence, possibly passing to a subsequence and using the dominated convergence theorem, by (24), (25) and (h_3), we get the contradiction

$$0 = \lim_{n \to +\infty} \int_\Omega w\,\big(\widetilde{u}_{\lambda_n}^{(1)} + \overline{u}_{\lambda_n}^{(1)}\big)^{p-1}\,\mathrm{d}x = \int_\Omega w\, m^{p-1}\,\mathrm{d}x < 0.$$

Thus, (23) is proved.

Step 3. Existence of solutions $u_\lambda^{(2)}$ *such that* $\mathcal{I}_\lambda(u_\lambda^{(2)}) > 0$ *and* $\lim_{\lambda\to 0} \|u_\lambda^{(2)}\|_{L^{1^*}} = +\infty$.

The existence of a second solution $u_\lambda^{(2)}$, for all small $\lambda > 0$, is proved by applying a non-smooth version of the mountain pass theorem to the functional $\mathcal{I}_\lambda$ in $BV(\Omega)$.

Fix any

$$\alpha \in \left(\tfrac{1}{p}, \tfrac{2}{2p-1}\right) \subset (0,1) \tag{26}$$

and, for all $\lambda > 0$, define

$$\mathcal{S}_\lambda = \{v \in BV(\Omega) : \|Dv\| + |\overline{v}| = \lambda^{-\alpha}\}.$$

Pick any $v \in \mathcal{S}_\lambda$. Setting $\varepsilon = 1$, estimate (12) reads

$$\begin{aligned}\mathcal{I}_\lambda(v) \geq{}& \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Dv\|^2}} \\ &+ \lambda\big(-c_1\|Dv\|^p - c_2|\overline{v}|^{2(p-1)} - c_3\|Dv\|^2 + c_4|\overline{v}|^p\big). \end{aligned}\tag{27}$$

As $\|Dv\| \leq \lambda^{-\alpha}$, we have that

$$\frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Dv\|^2}} \geq \frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \lambda^{-2\alpha}}} = \frac{\lambda^\alpha\|Dv\|^2}{\lambda^\alpha|\Omega| + \sqrt{\lambda^{2\alpha}|\Omega|^2 + 1}}$$

and hence, for all small $\lambda > 0$,

$$\frac{\|Dv\|^2}{|\Omega| + \sqrt{|\Omega|^2 + \|Dv\|^2}} \geq \tfrac{1}{2}\lambda^\alpha\|Dv\|^2. \tag{28}$$

Combining (27) with (28) yields

$$\mathcal{I}_\lambda(v) \geq \left(\tfrac{1}{2} - c_3\lambda^{1-\alpha}\right)\lambda^\alpha\|Dv\|^2 + \lambda\big(c_4|\overline{v}|^p - c_1\|Dv\|^p - c_2|\overline{v}|^{2(p-1)}\big)$$

and, as $\alpha < 1$, possibly reducing $\lambda > 0$,

$$\mathcal{I}_\lambda(v) \geq \tfrac{1}{4}\lambda^\alpha\|Dv\|^2 + \lambda\big(c_4|\overline{v}|^p - c_1\|Dv\|^p - c_2|\overline{v}|^{2(p-1)}\big). \tag{29}$$

Set $\beta = \frac{3}{4}\alpha$ and, if necessary, further reduce $\lambda > 0$ so that $\lambda^{-\beta} < \lambda^{-\alpha}$. Then, the following estimates hold:

(a) Let $\|Dv\| \leq \lambda^{-\beta}$. As $|\overline{v}| = \lambda^{-\alpha} - \|Dv\|$, from (29) we obtain

$$\begin{aligned}\mathcal{I}_\lambda(v) &\geq \tfrac{1}{4}\lambda^\alpha\|Dv\|^2 + c_4\lambda\big|\lambda^{-\alpha} - \|Dv\|\big|^p - c_1\lambda\|Dv\|^p - c_2\lambda\big|\lambda^{-\alpha} - \|Dv\|\big|^{2(p-1)} \\ &\geq c_4\lambda\big|\lambda^{-\alpha} - \lambda^{-\beta}\big|^p - c_1\lambda^{1-\beta p} - c_2\lambda^{1-2\alpha(p-1)} \\ &= \lambda^{1-\alpha p}\big(c_4\big|1 - \lambda^{\alpha-\beta}\big|^p - c_1\lambda^{(\alpha-\beta)p} - c_2\lambda^{\alpha(2-p)}\big).\end{aligned}$$

Since $\alpha - \beta = \frac{1}{4}\alpha > 0$ and $p < 2$, possibly taking $\lambda > 0$ smaller, we have that

$$\mathcal{I}_\lambda(v) \geq \tfrac{1}{2} c_4 \lambda^{1-\alpha p}, \tag{30}$$

where $1 - \alpha p < 0$, because $\alpha > \frac{1}{p}$.

(b) Let $\lambda^{-\beta} < \|Dv\| \leq \lambda^{-\alpha}$. From (29) we obtain

$$\begin{aligned}\mathcal{I}_\lambda(v) &\geq \tfrac{1}{4}\lambda^{\alpha}\|Dv\|^2 + c_4\lambda\big|\lambda^{-\alpha} - \|Dv\|\big|^p - c_1\lambda\|Dv\|^p - c_2\lambda\big|\lambda^{-\alpha} - \|Dv\|\big|^{2(p-1)} \\ &\geq \tfrac{1}{4}\lambda^{\alpha-2\beta} - c_1\lambda^{1-\alpha p} - c_2\lambda^{1-2\alpha(p-1)} \\ &= \lambda^{\alpha-2\beta}\big(\tfrac{1}{4} - c_1\lambda^{1-\alpha p-\alpha+2\beta} - c_2\lambda^{1-2\alpha(p-1)-\alpha+2\beta}\big).\end{aligned}$$

Since the condition $\alpha < \frac{2}{2p-1}$ yields

$$1 - \alpha p - \alpha + 2\beta = 1 - \alpha\big(p - \tfrac{1}{2}\big) > 0$$

and the conditions $p < 2$ and $\alpha > 0$ imply that

$$1 - 2\alpha(p-1) > 1 - \alpha p,$$

and, consequently,

$$1 - 2\alpha(p-1) - \alpha + 2\beta > 0,$$

we infer that, possibly for smaller $\lambda > 0$,

$$\mathcal{I}_\lambda(v) \geq \tfrac{1}{8}\lambda^{\alpha-2\beta}, \tag{31}$$

where $\alpha - 2\beta = -\frac{1}{2}\alpha < 0$.

Finally, combining (30) and (31) and observing that, due to $\alpha < \frac{2}{2p-1}$, one has

$$\alpha - 2\beta = -\tfrac{1}{2}\alpha < 1 - \alpha p < 0,$$

we conclude that, for all small $\lambda > 0$,

$$\inf_{v \in \mathcal{S}_\lambda} \mathcal{I}_\lambda(v) \geq \tfrac{1}{2} c_4 \lambda^{1-\alpha p} =: \eta_\lambda, \tag{32}$$

with

$$\lim_{\lambda \to 0} \eta_\lambda = +\infty. \tag{33}$$

Take $\lambda \in (0,1)$ so small that, according to Step 2, $u_\lambda^{(1)}$ exists and satisfies

$$\mathcal{I}_\lambda(u_\lambda^{(1)}) < 0 \qquad \text{and} \qquad \|u_\lambda^{(1)}\|_{BV} < 1 < \lambda^{-\alpha}. \tag{34}$$

Next, let $C \subset \Omega$ be a Caccioppoli set such that

$$\int_C w \,\mathrm{d}x > 0. \tag{35}$$

Denoting by $\chi_C \in BV(\Omega)$ the characteristic function of C and by $\mathrm{Per}_\Omega(C)$ the perimeter of C in Ω, we get, for all $s > 0$,

$$\mathcal{I}_\lambda(s\chi_C) = s\int_\Omega \mathrm{d}|D^s\chi_C| - \lambda \tfrac{1}{p} s^p \int_\Omega w\chi_C \,\mathrm{d}x = s\left(\mathrm{Per}_\Omega(C) - \lambda \tfrac{1}{p} s^{p-1}\int_C w\,\mathrm{d}x\right).$$

Hence, by using (35) and $p > 1$, we infer that, for all large $s > 0$,

$$\mathcal{I}_\lambda(s\chi_C) < 0.$$

Fix $s_\lambda > 0$ so large that

$$\mathcal{I}_\lambda(s_\lambda\chi_C) < 0 \qquad \text{and} \qquad \|s_\lambda\chi_C\|_{BV} > \lambda^{-\alpha}. \tag{36}$$

Then, set

$$e_0 = u_\lambda^{(1} \qquad \text{and} \qquad e_1 = s_\lambda\chi_C$$

and define

$$\Gamma = \big\{\gamma \in C^0([0,1], BV(\Omega)) : \gamma(0) = e_0, \gamma(1) = e_1\big\}.$$

Thanks to (32), (33), (36), (34), we have that, for all small $\lambda > 0$,

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \eta_\lambda > 0 > \max\big\{\mathcal{I}_\lambda(e_0), \mathcal{I}_\lambda(e_1)\big\}. \tag{37}$$

Fix any such $\lambda > 0$. Since e_0, e_1 are positive in Ω, we have that $|\gamma| \in \Gamma$, whenever $\gamma \in \Gamma$. Thus, by (21), we obtain, for all $\gamma \in \Gamma$,

$$\max_{t\in[0,1]} \mathcal{I}_\lambda(|\gamma(t)|) \leq \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)).$$

Hence, setting

$$\Gamma^+ = \big\{\gamma \in \Gamma : \ \operatorname*{ess\,inf}_\Omega \ \gamma(t) \geq 0 \text{ for all } t \in [0,1]\big\},$$

it follows that

$$\begin{aligned}\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) &\geq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(|\gamma(t)|) \\ &= \inf_{\gamma\in\Gamma^+}\max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) \geq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)),\end{aligned}$$

and then

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(|\gamma(t)|).$$

For each $k \geq 1$ pick $\gamma_k \in \Gamma^+$ such that

$$c_\lambda \leq \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma_k(t)) \leq c_\lambda + \tfrac{1}{k}.$$

The Ekeland's Variational Principle [7, Theorem 1.1] implies that there exists a sequence $(\gamma_k^*)_k$ in Γ such that

$$c_\lambda \leq \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma_k^*(t)) \leq c_\lambda + \tfrac{1}{k},$$

$$\max_{t\in[0,1]} \|\gamma_k^*(t) - \gamma_k(t)\|_{BV} \leq \tfrac{1}{\sqrt{k}}, \tag{38}$$

and, for all $\gamma \in \Gamma$ with $\gamma \neq \gamma_k^*$,

$$\max_{t\in[0,1]} \mathcal{I}(\gamma(t)) > \max_{t\in[0,1]} \mathcal{I}(\gamma_k^*(t)) - \tfrac{1}{\sqrt{k}} \max_{t\in[0,1]} \|\gamma_k^*(t) - \gamma_k(t)\|_{BV}.$$

From the proof of Lemma 3.7 in [27], as delivered in [26], we infer that sequences $(\varepsilon_k)_k$ in $\mathbb{R}$ and $(t_k)_k$ in $[0,1]$ exist such that

$$\lim_{k\to+\infty} \varepsilon_k = 0 \tag{39}$$

and, for all $k \geq 1$

$$c_\lambda - \tfrac{1}{k} \leq \mathcal{I}_\lambda(v_k) \leq \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma_k(t)) \leq c_\lambda + \tfrac{1}{k}, \quad \text{with } \ v_k = \gamma_k^*(t_k), \tag{40}$$

$$\min_{t\in[0,1]} \|v_k - \gamma_k(t)\|_{BV} \leq \tfrac{1}{\sqrt{k}},$$

$$\begin{aligned}\mathcal{J}(v) - \mathcal{J}(v_k) \geq \lambda \int_\Omega w\, |v_k|^{p-2} v_k (v - v_k)\, \mathrm{d}x \\ + \varepsilon_k \|v - v_k\|_{BV}, \quad \text{for all } \ v \in BV(\Omega).\end{aligned} \tag{41}$$

Let us prove that $(v_k)_k$ is bounded in $BV(\Omega)$. Take any $k \geq 1$. Multiplying (40) by p, we have that

$$p\mathcal{J}(v_k) \leq \lambda \int_\Omega w\, |v_k|^p \, \mathrm{d}x + p(c_\lambda + 1). \tag{42}$$

Testing (41) against $v = 2v_k$ and observing that

$$\mathcal{J}(2v_k) - \mathcal{J}(v_k) \leq \int_\Omega \sqrt{1 + |\nabla v_k|^2}\mathrm{d}x + \int_\Omega \mathrm{d}|Dv_k|^s = \mathcal{J}(v_k) + |\Omega|,$$

we obtain

$$-\mathcal{J}(v_k) - |\Omega| \le -\lambda \int_\Omega w\,|v_k|^p\,\mathrm{d}x - \varepsilon_k \|v_k\|_{BV}. \tag{43}$$

Summing up (42) and (43) yields

$$(p-1)\mathcal{J}(v_k) - |\Omega| \le p(c_\lambda + 1) - \varepsilon_k \|v_k\|_{BV}$$

and then, as $\mathcal{J}(v_k) \ge \|Dv_k\| - |\Omega|$,

$$(p-1)\|Dv_k\| \le p(c_\lambda + 1 + |\Omega|) + |\varepsilon_k|\|Dv_k\| + |\varepsilon_k||\overline{v}_k|.$$

Consequently, by using (39) and $p > 1$, we can find a constant $d_\lambda > 0$ such that, for all large k,

$$\|Dv_k\| \le d_\lambda(1 + |\varepsilon_k||\overline{v}_k|) \tag{44}$$

and hence, by the Poincaré inequality,

$$\|\widetilde{v}_k\|_{L^{1^*}} \le c_\Omega d_\lambda(1 + |\varepsilon_k||\overline{v}_k|). \tag{45}$$

Suppose by contradiction that there exists a subsequence of $(\overline{v}_k)_k$, still denoted by $(\overline{v}_k)_k$, such that

$$\lim_{k\to+\infty} |\overline{v}_k| = +\infty.$$

Without restriction we can assume that

$$\lim_{k\to+\infty} \overline{v}_k = +\infty \qquad \text{or} \qquad \lim_{k\to+\infty} \overline{v}_k = -\infty. \tag{46}$$

From (44) and (45), we infer that

$$\lim_{k\to+\infty} \frac{\|Dv_k\|}{|\overline{v}_k|} = 0 \qquad \text{and} \qquad \lim_{k\to+\infty} \frac{\|\widetilde{v}_k\|_{L^{1^*}}}{|\overline{v}_k|} = 0. \tag{47}$$

Let us set

$$z_k = \frac{v_k}{|\overline{v}_k|} = \frac{\widetilde{v}_k}{|\overline{v}_k|} + \frac{\overline{v}_k}{|\overline{v}_k|}.$$

From (47) it follows that $(z_k)_k$ converges in $L^{1^*}(\Omega)$ to 1 or to -1 according to the first or the second alternative of (46) holds and, hence, $(|z_k|^p)_k$ converges in $L^{\frac{1^*}{p}}(\Omega)$ to 1. Dividing (40) by $|\overline{v}_k|^p$ yields

$$\frac{c_\lambda + 1}{|\overline{v}_k|^p} \ge \frac{\mathcal{I}_\lambda(v_k)}{|\overline{v}_k|^p} \ge \frac{\|Dv_k\| - |\Omega|}{|\overline{v}_k|^p} - \lambda \int_\Omega \tfrac{1}{p} w\,|z_k|^p\,\mathrm{d}x. \tag{48}$$

Consequently, passing to the limit in (48) and using the condition (47) and the assumption $p > 1$, as well as the convergence of $(w|z_k|^p)_k$ in $L^1(\Omega)$ to w, we obtain

$$0 \ge -\lambda \tfrac{1}{p} \int_\Omega w\,\mathrm{d}x,$$

thus contradicting Assumption (h_3). Finally, as $(\overline{v}_k)_k$ must be bounded, by (44) we conclude that $(v_k)_k$ is bounded in $BV(\Omega)$.

The boundedness of $(v_k)_k$ in $BV(\Omega)$ implies the existence of a subsequence of $(v_k)_k$, still denoted by $(v_k)_k$, that converges weakly* in $BV(\Omega)$, strongly in $L^p(\Omega)$ and pointwise a.e. in Ω to some $u \in BV(\Omega)$. Consequently, from (41) we get, by the lower semicontinuity of $\mathcal{J}$,

$$\begin{aligned}\mathcal{J}(u) - \mathcal{J}(v) &\le \liminf_{k\to+\infty} \mathcal{J}(v_k) - \mathcal{J}(v)\\ &\le \lambda \lim_{k\to+\infty} \int_\Omega w\,|v_k|^{p-2} v_k\,(v - v_k)\,\mathrm{d}x + \lim_{k\to+\infty} \varepsilon_k \|v - v_k\|_{BV}\\ &= \lambda \int_\Omega w\,|u|^{p-2} u\,(v-u)\,\mathrm{d}x,\end{aligned}$$

for all $v \in BV(\Omega)$. Therefore, u is a solution of (1). Moreover, testing (41) against $v = u$, we deduce that

$$\begin{aligned}\mathcal{J}(u) &\ge \limsup_{k\to+\infty} \left(\mathcal{J}(v_k) + \lambda \int_\Omega w\,|v_k|^{p-2} v_k (u - v_k)\,\mathrm{d}x + \varepsilon_k \|u - v_k\|_{BV} \right)\\ &= \limsup_{k\to+\infty} \mathcal{J}(v_k)\end{aligned}$$

Hence, the lower semicontinuity of $\mathcal{J}$ yields

$$\lim_{k\to+\infty} \mathcal{J}(v_k) = \mathcal{J}(u)$$

and thus, by (40),

$$\lim_{k\to+\infty} \mathcal{I}_\lambda(v_k) = \mathcal{I}_\lambda(u) = c_\lambda.$$

Since $c_\lambda > 0$ and $\mathcal{I}_\lambda(0) = 0$, we conclude that $u \neq 0$.

Next, we show that u is positive. We know that there is a subsequence of $(v_k)_k$, still denoted by $(v_k)_k$ which converges to u strongly in $L^p(\Omega)$. As $v_k = \gamma_k^*(t_k)$, from (38) we infer that

$$\lim_{k\to+\infty} \gamma_k(t_k) = \lim_{k\to+\infty} \gamma_k^*(t_k) = u$$

in $L^p(\Omega)$. Consequently, possibly passing to a subsequence, $(\gamma_k(t_k))_k$ converges to u a.e. in Ω. Since $\operatorname{ess\,inf}_\Omega \gamma_k(t_k) \ge 0$, we conclude that $\operatorname{ess\,inf}_\Omega u \ge 0$ too. Consequently, u is a positive solution of (1). Hereafter, we denote by $u_\lambda^{(2)}$ any positive solution of (1) such that

$$\mathcal{I}_\lambda(u_\lambda^{(2)}) = c_\lambda. \tag{49}$$

Finally, we shall prove that

$$\lim_{\lambda\to 0} \|u_\lambda^{(2)}\|_{L^{1^*}} = +\infty. \tag{50}$$

Indeed, testing (8) against $\phi = u_\lambda^{(2)}$, we get

$$\int_\Omega \frac{\left|\nabla u_\lambda^{(2)}\right|^2}{\sqrt{1+\left|\nabla u_\lambda^{(2)}\right|^2}}\,\mathrm{d}x + \int_\Omega d\left|Du_\lambda^{(2)}\right|^s - \lambda \int_\Omega w\left|u_\lambda^{(2)}\right|^p \mathrm{d}x = 0. \tag{51}$$

Subtracting (51) from (49), we obtain

$$c_\lambda = \int_\Omega \left(\frac{\left|\nabla u_\lambda^{(2)}\right|^2}{1+\sqrt{1+\left|\nabla u_\lambda^{(2)}\right|^2}} - \frac{\left|\nabla u_\lambda^{(2)}\right|^2}{\sqrt{1+\left|\nabla u_\lambda^{(2)}\right|^2}} \right) \mathrm{d}x + \lambda\left(1-\tfrac{1}{p}\right) \int_\Omega w\left|u_\lambda^{(2)}\right|^p \mathrm{d}x$$

$$\leq \lambda \int_\Omega w\left|u_\lambda^{(2)}\right|^p \mathrm{d}x.$$

By (37) and (32), we infer that

$$\tfrac{1}{2} c_4 \lambda^{-\alpha p} = \eta_\lambda \lambda^{-1} \leq \int_\Omega w\left|u_\lambda^{(2)}\right|^p \mathrm{d}x \leq \|w^+\|_{L^{\frac{1^*}{1^*-p}}} \|u_\lambda^{(2)}\|_{L^{1^*}}^p$$

and hence

$$\liminf_{\lambda\to 0} \lambda^\alpha \|u_\lambda^{(2)}\|_{L^{1^*}} > 0, \tag{52}$$

where α satisfies (26). In particular, Condition (50) follows; however, with respect to (50), Condition (52) provides an explicit estimate, in terms of λ, of the blow up speed of $\|u_\lambda^{(2)}\|_{L^{1^*}}$ as $\lambda \to 0$.

This concludes the proof of the existence of a constant $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, Problem (1) has two positive bounded variation solutions $u_\lambda^{(1)}, u_\lambda^{(2)}$ satisfying

$$\mathcal{I}_\lambda(u_\lambda^{(1)}) < 0 < \mathcal{I}_\lambda(u_\lambda^{(2)}) \tag{53}$$

and

$$\lim_{\lambda\to 0} \|u_\lambda^{(1)}\|_{BV} = 0, \qquad \lim_{\lambda\to 0} \|u_\lambda^{(2)}\|_{L^{1^*}} = +\infty.$$

Part 2. Existence of classical weak solutions

In this part we assume, in addition to (h_1)–(h_3), Conditions (h_4) and (h_5).

Step 1. A modified problem.

First, we modify the mean curvature operator. Like in [13], we define the $C^{1,1}$ function $a : [0,+\infty) \to [0,+\infty)$ by

$$a(s) = \begin{cases} (1+s)^{-1/2} & \text{if } s \in [0,1), \\ \frac{\sqrt{2}}{16}(s-2)^2 + \frac{7\sqrt{2}}{16} & \text{if } s \in [1,2), \\ \frac{7\sqrt{2}}{16} & \text{if } s \in [2,+\infty) \end{cases}$$

and we set, for all $s \geq 0$,

$$A(s) = \int_0^s a(t)\,\mathrm{d}t.$$

We further define the C^1 vector field $\alpha : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\alpha(\eta) = \tfrac{1}{2}\nabla A(|\eta|^2) = a(|\eta|^2)\eta.$$

Elementary calculations show that, for all $s \geq 0$,

$$\tfrac{7\sqrt{2}}{16} \leq a(s) \leq 1, \tag{54}$$

$$-\tfrac{1}{2} \leq a'(s) \leq 0,$$

$$\min_{s\in\mathbb{R}} \big(a(s^2) + 2a'(s^2)s^2\big) = \tfrac{19\sqrt{2}}{80},$$

and

$$\tfrac{7\sqrt{2}}{16} s \leq A(s) \leq s. \tag{55}$$

Hence, we infer that, for all $\eta, \xi \in \mathbb{R}^N$,

$$\tfrac{19\sqrt{2}}{80}|\xi|^2 \leq \partial\alpha(\eta)\xi\cdot\xi \leq |\xi|^2$$

and

$$\big(\alpha(\eta) - \alpha(\xi)\big)\cdot(\eta-\xi) \geq \tfrac{19\sqrt{2}}{80}\,|\eta-\xi|^2.$$

As the function $\eta \mapsto \alpha(\eta) = \frac{1}{2}\nabla A(|\eta|^2)$ is strongly monotone, the function $\eta \mapsto A(|\eta|^2)$ is strictly convex.

Next, we modify the right hand side of the differential equation in (1). Let $\chi : [0,+\infty) \to [0,1]$ be the piecewise linear cut-off function such that

$$\chi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 2. \end{cases}$$

Let us define, for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$,

$$g(x,s) = w(x)\,\chi(|s|)\,|s|^{p-2}s + \operatorname{sgn}(s)(\chi(|s|) - 1). \tag{56}$$

Thanks to (h_5), there exists a constant $a_0 > 0$ such that, for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$

$$|g(x,s)| \leq a_0 \tag{57}$$

and, by construction,

$$g(x,-s) = -g(x,s). \tag{58}$$

We also set, for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$,

$$G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t.$$

By (56) and (57) there exists a constant $b_0 > 0$ such that, for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$,

$$G(x,s) \le -a_0|s| + b_0 \tag{59}$$

and, by (58),

$$G(x,-s) = G(x,s). \tag{60}$$

Finally, we introduce, for any given $\lambda > 0$, the following modification of Problem (1)

$$\begin{cases} -\mathrm{div}\big(a(|\nabla u|^2)\nabla u\big) = \lambda g(x,u) & \text{in } \Omega, \\ -\nabla u\,\nu/\sqrt{1+|\nabla u|^2} = 0 & \text{on } \partial\Omega. \end{cases} \tag{61}$$

A solution of (61) is a function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega a(|\nabla u|^2)\nabla u \cdot \nabla \phi\,\mathrm{d}x = \lambda \int_\Omega g(x,u)\phi\,\mathrm{d}x \tag{62}$$

for all $\phi \in H^1(\Omega)$. Further, for each $\lambda > 0$ we define $\mathcal{H}_\lambda : H^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{H}_\lambda(v) = \int_\Omega \tfrac{1}{2}A(|\nabla v|^2)\,\mathrm{d}x - \lambda \int_\Omega G(x,v)\,\mathrm{d}x.$$

Clearly, $\mathcal{H}_\lambda$ is even, of class C^1, and $u \in H^1(\Omega)$ is a solution of (61) if and only if it is a critical point of $\mathcal{H}_\lambda$. Notice that, for all $v \in C^1(\overline{\Omega})$ such that $\|v\|_{C^1}$ is small enough,

$$\begin{aligned} a(|\nabla v|^2)\nabla v &= \nabla v/\sqrt{1+|\nabla v|^2}, & g(\cdot,v) &= w\,|v|^{p-2}v, \\ \tfrac{1}{2}A(|\nabla v|^2) &= \sqrt{1+|\nabla v|^2} - 1, & G(\cdot,v) &= w\,|v|^p, \end{aligned} \tag{63}$$

and then, in particular,

$$\mathcal{H}_\lambda(v) = \mathcal{I}_\lambda(v). \tag{64}$$

Step 2. Existence of positive solutions of the modified problem.
As (55) and (59) imply that, for all $v \in H^1(\Omega)$,

$$\mathcal{H}_\lambda(v) \ge \tfrac{7\sqrt{2}}{32} \int_\Omega |\nabla v|^2\,\mathrm{d}x + \lambda a_0 \int_\Omega |v|\,\mathrm{d}x - \lambda b_0|\Omega|,$$

$\mathcal{H}_\lambda$ is coercive and bounded from below. Moreover, it is weakly lower semicontinuous, being the sum of two functionals, the former convex and continuous, and therefore weakly lower continuous, the latter weakly continuous [6, Chapter 1]. Consequently, $\mathcal{H}_\lambda$ admits a global minimiser in $H^1(\Omega)$, i.e., there exists $u \in H^1(\Omega)$ such that

$$\mathcal{H}_\lambda(u) = \inf_{H^1(\Omega)} \mathcal{H}_\lambda(v),$$

and u is a solutions of (62). Thanks to (64), adapting in an obvious fashion the calculations in (20) yields

$$\mathcal{H}_\lambda(u_\lambda) < 0 \tag{65}$$

and, as $\mathcal{H}_\lambda(0) = 0$, $u \neq 0$. Moreover, similarly as in Part 1-Step 2, we can see that $|u| \in H^1(\Omega)$ is a positive global minimiser of $\mathcal{H}_\lambda$. Indeed, by setting

$$\Omega^\pm = \{x \in \Omega : \pm u(x) > 0\}$$

and applying [32, Theorem 1.56], we obtain

$$\begin{aligned}\int_\Omega A(|\nabla|u||^2)\,\mathrm{d}x &= \int_{\Omega^+} A(|\nabla u^+|^2)\,\mathrm{d}x + \int_{\Omega^-} A(|\nabla u^-|^2)\,\mathrm{d}x \\ &= \int_{\Omega^+} A(|\nabla u^+|^2)\,\mathrm{d}x + \int_{\Omega^-} A(|-\nabla u^-|^2)\,\mathrm{d}x = \int_\Omega A(|\nabla u|^2)\,\mathrm{d}x\end{aligned}$$

and then, by (60),

$$\begin{aligned}\mathcal{H}_\lambda(|u|) &= \int_\Omega \tfrac{1}{2} A(|\nabla|u||^2)\,\mathrm{d}x - \lambda \int_\Omega G(x, |u|)\,\mathrm{d}x \\ &= \int_\Omega \tfrac{1}{2} A(|\nabla u|^2)\,\mathrm{d}x - \lambda \int_\Omega G(x, u)\,\mathrm{d}x = \mathcal{H}_\lambda(u).\end{aligned}$$

Hereafter, we denote by $u_\lambda^{(1)}$ any positive global minimiser of $\mathcal{H}_\lambda$.

Step 3. Existence of positive classical weak solutions of the original problem.

We start by proving that, for all $\lambda > 0$,

$$\|u_\lambda^{(1)}\|_\infty \leq 2. \tag{66}$$

Indeed, fix any $\lambda > 0$ and test (62) against $\phi = (u_\lambda^{(1)} - 2)^+ \in H^1(\Omega)$. By (54) and (56), we have that

$$\begin{aligned}0 \leq \tfrac{7\sqrt{2}}{16} \int_\Omega |\nabla(u_\lambda^{(1)} - 2)^+|^2\,\mathrm{d}x &\leq \int_\Omega a(|\nabla u_\lambda^{(1)}|^2)\,|\nabla(u_\lambda^{(1)} - 2)^+|^2\,\mathrm{d}x \\ &= \int_\Omega a(|\nabla u_\lambda^{(1)}|^2)\,\nabla u_\lambda^{(1)} \nabla(u_\lambda^{(1)} - 2)^+\,\mathrm{d}x \\ &= \int_\Omega g(x, u_\lambda^{(1)})(u_\lambda^{(1)} - 2)^+\,\mathrm{d}x \\ &= \int_\Omega -(u_\lambda^{(1)} - 2)^+\,\mathrm{d}x \leq 0.\end{aligned}$$

Hence, we get $(u_\lambda^{(1)} - 2)^+ = 0$, that is, $u_\lambda^{(1)}(x) \leq 2$ for a.a. $x \in \Omega$. Thus, (66) follows because $u_\lambda^{(1)}$ is positive.

Next, fix any $\hat{\lambda} > 0$. Thanks to (h_4), (66) and (57), the regularity theory for the quasilinear problem (61) (see, e.g., [15, Theorem 2]) yields the existence of constants $\rho \in (0,1]$ and $R > 0$ such that, for all $\lambda \in (0, \hat{\lambda})$, $u_\lambda^{(1)} \in C^{1,\rho}(\overline{\Omega})$ and

$$\|u_\lambda^{(1)}\|_{C^{1,\rho}} \leq R. \tag{67}$$

In addition, Remark 2.9 ensures that $u_\lambda^{(1)} \in W^{2,q}_{\mathrm{loc}}(\Omega)$, for all finite $q \geq 1$.

Fix any $\tau \in (0, \rho)$. We shall prove that

$$\lim_{\lambda \to 0} \|u_\lambda^{(1)}\|_{C^{1,\tau}} = 0. \tag{68}$$

Let $(\lambda_n)_n$ be a sequence in $(0, \hat{\lambda})$, with

$$\lim_{n \to +\infty} \lambda_n = 0, \tag{69}$$

and let $(u_n)_n = (u_{\lambda_n}^{(1)})_n$ be a corresponding sequence of positive minimisers of $\mathcal{H}_{\lambda_n}$. Estimate (67) and the Arzelà-Ascoli theorem imply the existence of a subsequence of $(u_n)_n$ converging in $C^{1,\tau}(\overline{\Omega})$ to some function $u \in C^{1,\tau}(\overline{\Omega})$. From (65), (55) and (59), we infer that

$$0 \leq \tfrac{7\sqrt{2}}{16} \int_\Omega |\nabla u_n|^2 \,\mathrm{d}x \leq \lambda_n \int_\Omega G(s, u_n) \,\mathrm{d}x \leq \lambda_n b_0 |\Omega|. \tag{70}$$

Consequently, we have that

$$\|\nabla u\|_{L^2} = \lim_{n \to +\infty} \|\nabla u_n\|_{L^2} = 0$$

and hence

$$\|\nabla u\|_\infty = \lim_{n \to +\infty} \|\nabla u_n\|_\infty = 0.$$

Thus, using estimate (66) too, u is constant, with $u \in [0, 2]$. Testing (61) against $\phi = 1$, dividing by $\lambda_n > 0$ and passing to the limit as $n \to +\infty$, we obtain, by the dominated convergence theorem,

$$0 = \lim_{n \to +\infty} \int_\Omega g(x, u_n) \,\mathrm{d}x = \int_\Omega g(x, u) \,\mathrm{d}x = \chi(u)\, u^{p-1} \int_\Omega w(x) \,\mathrm{d}x.$$

As the function $\chi(s) s^{p-1} > 0$ for all $s \in (0, 2)$, it follows that either $u = 0$ or $u = 2$. If $u = 2$, dividing (70) by $\lambda_n > 0$, we infer that

$$0 \leq \int_\Omega G(x, 2) \,\mathrm{d}x = \int_\Omega w \,\mathrm{d}x \int_0^2 \chi(s)\, s^{p-1} \mathrm{d}s.$$

This is impossible, because

$$\int_\Omega w\,\mathrm{d}x < 0 \qquad \text{and} \qquad \int_0^2 \chi(s)\, s^{p-1}\mathrm{d}s > 0.$$

Therefore, we have that $u = 0$. Since this occurs for any possible sequence $(\lambda_n)_n$ satisfying (69), we conclude that (68) holds true.

Finally, thanks to (63) and (68), we can find $\lambda_* \in (0, \lambda^*)$ such that, for all $\lambda \in (0, \lambda_*]$ the solutions $u_\lambda^{(1)}$ of (61) that we have just constructed are solutions of Problem (1) too.

This concludes the proof of Theorem 1.2.

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