Rend. Istit. Mat. Univ. Trieste
Vol. 57 (2025), Art. No. 6, 14 pages
DOI: 10.13137/2464-8728/37094

Blow-up of a modified ODEs system arising from the Galerkin approximation of some Navier-Stokes equations

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Dedicated to Enzo Mitidieri, in occasion of his 70th birthday

Abstract. *For the third order Galerkin approximation of the Navier-Stokes equations under Navier boundary conditions in a cube we prove global existence and qualitative behaviour of the solution. By modifying properly the signs of the resulting ODEs system and using the test function technique developed by Mitidieri-Pohožaev we prove, instead, finite time blow-up.*

Keywords: Blow-up, test function method, Navier-Stokes equations.
MS Classification 2020: 34A34, 34C11.

1. Introduction

In a series of remarkable papers [9, 10, 11, 12], Enzo Mitidieri & Stanislav Pohožaev introduced a unified approach to nonexistence of solutions for a class of nonlinear differential inequalities, including systems of ODEs, see [1, 2, 3, 4, 5, 7, 8] for some applications. The main idea, is a clever use of suitable test functions with compact support which, after some scaling, shows blow-up in finite time or space for the considered differential inequality. Quite surprisingly, this technique was not applied to analyse the possible blow-up for the Navier-Stokes equations. It is our purpose to give a first contribution in this direction.

In a recent paper [6], we considered the evolution Navier-Stokes equations in the cube $\Omega = (0,\pi)^3$ under Navier boundary conditions with no friction, namely

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } \Omega\times\mathbb{R}_+\,,\\ \nabla\cdot u = 0 & \text{in } \Omega\times\mathbb{R}_+\,,\\ u_1 = \partial_x u_2 = \partial_x u_3 = 0 & \text{on } \{0,\pi\}\times(0,\pi)\times(0,\pi)\times\mathbb{R}_+\,,\\ u_2 = \partial_y u_1 = \partial_y u_3 = 0 & \text{on } (0,\pi)\times\{0,\pi\}\times(0,\pi)\times\mathbb{R}_+\,,\\ u_3 = \partial_z u_1 = \partial_z u_2 = 0 & \text{on } (0,\pi)\times(0,\pi)\times\{0,\pi\}\times\mathbb{R}_+\,, \end{cases} \tag{1}$$

complemented with an initial condition $u(x,y,z,0) = u_0(x,y,z)$ in Ω. For (1) we were able to write exactly the Galerkin approximation by means of a nonlinear system of autonomous ODEs with explicit coefficients. The third order Galerkin approximation furnishes the system

$$\begin{cases} \dot{A}_1(t) + \lambda_1 A_1(t) = K_2 A_1(t) A_2(t) - K_3 A_1(t) A_3(t) \\ \dot{A}_2(t) + \lambda_2 A_2(t) = -K_2 A_1(t)^2 \\ \dot{A}_3(t) + \lambda_3 A_3(t) = K_3 A_1(t)^2 \\ A_1(0) = a_1, \quad A_2(0) = a_2, \quad A_3(0) = a_3, \end{cases} \tag{2}$$

for some initial data a_i $(i = 1, 2, 3)$, see [6] for the computational details. In (2), the A_i's are the Fourier components of the Galerkin approximation while, for some positive integers m, n, p,

$$\lambda_1 = m^2 + n^2 + p^2\,, \qquad \lambda_2 = 4(m^2 + p^2)\,, \qquad \lambda_3 = 4(m^2 + n^2)\,, \tag{3}$$

are the related eigenvalues of the Stokes operator and

$$K_2 = \frac{mn^2p}{(n^2 + p^2)\sqrt{\pi^3(m^2 + p^2)}}\,, \qquad K_3 = \frac{mnp^2}{(n^2 + p^2)\sqrt{\pi^3(m^2 + n^2)}}\,. \tag{4}$$

Therefore, all the coefficients appearing in (2) are explicit and positive. Dropping the constraints (3)-(4) makes (2) a more general system where all the coefficients (including the initial conditions) play some role. In particular, in (2) we see that K_2 and K_3 appear with a precise combination of signs. It is our purpose to discuss how changing this combination of signs gives different responses for solutions to (2), either global existence or finite time blow-up. Our proofs combine standard tools from the Navier-Stokes equations with the technique developed by Mitidieri-Pohožaev.

2. No blow-up in the approximation of the Navier-Stokes equations

Local existence and uniqueness of a solution to (2) is guaranteed by the Cauchy Theorem. If the solutions are exactly those coming from the third order Galerkin approximation of (1), i.e. (2) with (3)-(4), they can be extended to the whole interval $[0, \infty)$ because the (weak) solution to (1) satisfies $u \in L^\infty(\mathbb{R}_+, L^2(\Omega))$, see [6]. We state a slightly more general result for (2) in the following

PROPOSITION 2.1. *For any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\setminus\{0\}$ and $K_2, K_3 \in \mathbb{R}$ the unique solution to (2) is global-in-time, regardless of the initial values a_1, a_2, a_3. Moreover, if $\lambda_1, \lambda_2, \lambda_3 > 0$ all the A_i converge exponentially to zero as $t \to \infty$.*

Proof. By multiplying $(2)_i$ by A_i $(i = 1, 2, 3)$ and by adding the three resulting equations, we find

$$\frac{1}{2}\frac{d}{dt}\Big[A_1(t)^2 + A_2(t)^2 + A_3(t)^2\Big] = -\lambda_1 A_1(t)^2 - \lambda_2 A_2(t)^2 - \lambda_3 A_3(t)^2 \le \Lambda\Big[A_1(t)^2 + A_2(t)^2 + A_3(t)^2\Big]$$

where $\Lambda := \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} > 0$. Then

$$A_1(t)^2 + A_2(t)^2 + A_3(t)^2 \le (a_1^2 + a_2^2 + a_3^2)e^{2\Lambda t} \qquad \forall t \ge 0 \tag{5}$$

so that the solution is global-in-time. If $\lambda_1, \lambda_2, \lambda_3 > 0$ the previous estimate can be refined

$$\frac{1}{2}\frac{d}{dt}\Big[A_1(t)^2 + A_2(t)^2 + A_3(t)^2\Big] \le -\overline{\Lambda}\Big[A_1(t)^2 + A_2(t)^2 + A_3(t)^2\Big]$$

where $\overline{\Lambda} := \min \lambda_1, \lambda_2, \lambda_3 > 0$, and the exponential decay follows plugging $-\overline{\Lambda}$ into (5) instead of Λ. □

The proof of Proposition 2.1 is obtained by testing the system (2) with the solution itself, as custumary for the Navier-Stokes equations in order to prove boundedness of the energy.

In the previous result we do not consider the case $\lambda_1 = \lambda_2 = \lambda_3 = 0$, because it corresponds to the Euler equations that we study in Section 4. In the next statement we emphasize the role of the initial conditions in the behaviour of the solution to (2).

PROPOSITION 2.2. *Assume that* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$ *and* $a_1, a_2, a_3, K_2, K_3 \in \mathbb{R}$.

If $a_1 = 0$, *then the unique solution to* (2) *is* $A_1(t) \equiv 0$, $A_i(t) = a_i e^{-\lambda_i t}$ $(i = 2, 3)$.

If $a_1 \neq 0$, *then* $A_1(t)$ *has the same sign as* a_1 *for all* $t \ge 0$ *and:*

– *if* $a_2 \neq 0$ *and* $a_2 K_2 \le 0$, *then* $A_2(t)$ *has the same sign as* a_2 *for all* $t \ge 0$.

– *if* $a_3 \neq 0$ *and* $a_3 K_3 \ge 0$, *then* $A_3(t)$ *has the same sign as* a_3 *for all* $t \ge 0$.

– *if* $a_2 K_2 > 0$ *(resp.* $a_3 K_3 < 0$*), then* $A_2(t)$ *(resp.* $A_3(t)$*) changes sign at most once.*

– *if* $a_2 = 0$ *then* $A_2(t)$ *has the sign opposite to* K_2 *for all* $t > 0$.

– *if* $a_3 = 0$ *then* $A_3(t)$ *has the same sign as* K_3 *for all* $t > 0$.

Proof. In fact, (2) has "almost explicit" solutions:

$$A_1(t) = a_1 e^{-\lambda_1 t + K_2 \int_0^t A_2(\tau)d\tau - K_3 \int_0^t A_3(\tau)d\tau},$$

$$A_2(t) = \left\{a_2 - K_2 \int_0^t A_1(\tau)^2 \, e^{\lambda_2 \tau} \, d\tau\right\} e^{-\lambda_2 t},$$

$$A_3(t) = \left\{a_3 + K_3 \int_0^t A_1(\tau)^2 \, e^{\lambda_3 \tau} \, d\tau\right\} e^{-\lambda_3 t}.$$

The statements then follow by replacing the values of a_i $(i = 1, 2, 3)$ and K_2, K_3 in these formulas. □

3. Appearance of blow-up by changing the signs of the coefficients

For some $a_1, a_2, a_3 \in \mathbb{R}$, we consider here the following initial value problem for the nonlinear system

$$\begin{cases} \dot{A}_1(t) + \lambda_1 A_1(t) = K_2 A_1(t) A_2(t) + K_3 A_1(t) A_3(t) \\ \dot{A}_2(t) + \lambda_2 A_2(t) = K_2 A_1(t)^2 \\ \dot{A}_3(t) + \lambda_3 A_3(t) = K_3 A_1(t)^2 & (t > 0) \\ A_1(0) = a_1, \quad A_2(0) = a_2, \quad A_3(0) = a_3, \end{cases} \tag{6}$$

which has been obtained from (2) by changing two signs prior to the coefficients K_2 in $(2)_2$ and K_3 in $(2)_3$. As in Proposition 2.2: if $a_1 = 0$, then $A_1(t) \equiv 0$, $A_i(t) = a_i e^{-\lambda_i t}$ $(i = 2, 3)$ and the solution to (6) is global in time.

In the next statement we prove that, for small initial data, the solution to (6) is global in time while, for large data, the Mitidieri-Pohožaev method enables us to prove blow-up for (6).

THEOREM 3.1. *Let $\lambda_1, \lambda_2, \lambda_3 > 0$ and $K_2, K_3 > 0$.*

(i) There exists $K > 0$ such that if

$$\sum_{i=1}^{3} a_i^2 < K$$

then the local solution to (6) is global-in-time and $A_i(t) \to 0$ as $t \to \infty$ at exponential rate.

(ii) For any $T > 0$ there exist initial data $a_1 > 0$, $a_2, a_3 \geq 0$, such that the (local) solution to (6) blows up in some finite time $T^* \leq T$.

REMARK 3.2. We assume positive signs for $\lambda_1, \lambda_2, \lambda_3$ and K_2, K_3 because these are the signs of the ODEs system coming from the third order Galerkin approximation of (1), see (3)-(4). The proof of Theorem 3.1 still works also for different combinations of signs.

Proof. For the global existence statement, we consider the unique (local) solution (A_1, A_2, A_3) to (6) and we define the function

$$E(t) := \frac{1}{2} \sum_{i=1}^{3} A_i(t)^2 \,.$$

By differentiating and by using (6), we find that

$$\dot{E}(t) = \sum_{i=1}^{3} \dot{A}_i(t)A_i(t) = -\sum_{i=1}^{3} \lambda_i A_i(t)^2 + 2A_1(t)^2\big(K_2A_2(t) + K_3A_3(t)\big)$$
$$\leq -C_1E(t) + C_2E(t)^{3/2}\,,$$

where $C_1 = 2\min_i \lambda_i > 0$ and $C_2 = 4\sqrt{2}(K_2 + K_3) > 0$. This shows that $t \mapsto E(t)$ is strictly decreasing whenever $E(t) < C_1^2/C_2^2$. In particular, if

$$E(0) = \frac{1}{2}\sum_{i=1}^{3} a_i^2 < \frac{C_1^2}{C_2^2} =: \frac{K}{2}\,,$$

then $t \mapsto E(t)$ is strictly decreasing in $\mathbb{R}_+$, proving global existence. The vanishing limit for $E(t)$ and the exponential decay are then obvious consequences.

For the blow-up statement, as in the proof of Proposition 2.2, since $a_1 > 0$ and $a_2, a_3 \geq 0$, by (6) we infer that

$$A_1(t), A_2(t), A_3(t) > 0 \qquad \forall t > 0.$$

Throughout the proof we also need the simple inequality

$$\log x \leq \sqrt{x} \qquad \forall x > 0\,. \tag{7}$$

Let $0 < T < T_1$ and consider $\varphi \in C^3(\mathbb{R}_+)$ such that

$$\varphi(t) = \begin{cases} 1 & 0 \leq t \leq T, \\ 0 & t \geq T_1, \end{cases} \qquad \begin{cases} \dot{\varphi}(T) = 0,\ \varphi(T_1) = \dot{\varphi}(T_1) = \ddot{\varphi}(T_1) = 0, \\ \varphi \text{ is non-increasing in } \mathbb{R}_+\ (-\dot{\varphi} = |\dot{\varphi}|). \end{cases} \tag{8}$$

Let also

$$\Phi(t) = -\int_t^{T_1} \varphi(s)ds \implies \dot{\Phi} = \varphi\,,\ \Phi(T_1) = 0\,,\ \Phi(t) \leq 0 \text{ on } [0, T_1]. \tag{9}$$

For the moment, T and T_1 do not have specific values, they will be assigned in the proof, for which we proceed in four steps: in the first two steps we introduce constants $\varepsilon, \eta > 0$ that will also be specified in the course. In the third step we obtain the crucial bound that, after applying the scaling method by Mitidieri-Pohožaev [9], enables us to reach the conclusion in the fourth step.

Step 1. Divide $(6)_1$ by $A_1(t) > 0$, then test it with φ to obtain

$$\int_0^{T_1} \big[K_2A_2(t) + K_3A_3(t)\big]\varphi(t)dt = \lambda_1\int_0^{T_1} \varphi(t)dt + \int_0^{T_1} \frac{\dot{A}_1(t)}{A_1(t)}\varphi(t)dt$$
$$\text{(by parts)} \ = \lambda_1\int_0^{T_1} \varphi(t)dt - \int_0^{T_1} \log[A_1(t)]\dot{\varphi}(t)dt - \log a_1. \tag{10}$$

By $(6)_2$, we find

$$
\begin{aligned}
-\int_0^{T_1} & \log[A_1(t)]\dot\varphi(t)dt \\
&= \frac{1}{2}\int_0^{T_1}\log[A_1(t)^2]|\dot\varphi(t)|dt = \frac{1}{2}\int_0^{T_1}\log\tfrac{\dot A_2(t)+\lambda_2 A_2(t)}{K_2}\,|\dot\varphi(t)|dt \\
\text{by (7)}\ &\le \frac{1}{2\sqrt{K_2}}\int_0^{T_1}\sqrt{\dot A_2(t)+\lambda_2 A_2(t)}\,|\dot\varphi(t)|dt \\
\text{see (9)}\ &= \frac{1}{2\sqrt{K_2}}\int_0^{T_1}\sqrt{\big(\dot A_2(t)+\lambda_2 A_2(t)\big)|\Phi(t)|}\,\frac{|\dot\varphi(t)|}{|\Phi(t)|^{\frac{1}{2}}}dt \\
\text{Hölder}\ &\le \frac{1}{2\sqrt{K_2}}\bigg(\int_0^{T_1}\big(\dot A_2(t)+\lambda_2 A_2(t)\big)|\Phi(t)|dt\bigg)^{\frac{1}{2}}\bigg(\int_0^{T_1}\frac{|\dot\varphi(t)|^2}{|\Phi(t)|}dt\bigg)^{\frac{1}{2}} \\
\text{Young}\ &\le \frac{\varepsilon}{4\sqrt{K_2}}\int_0^{T_1}\big(\dot A_2(t)+\lambda_2 A_2(t)\big)|\Phi(t)|dt \\
&\qquad + \frac{1}{4\varepsilon\sqrt{K_2}}\int_0^{T_1}\frac{|\dot\varphi(t)|^2}{|\Phi(t)|}dt. \qquad (11)
\end{aligned}
$$

Note that the very last integral in (11) converges because $\ddot\varphi(T_1) = 0$, see (8)-(9).

Moreover, since by (9) we also have (recall $a_2 \ge 0$ and $\Phi(0) < 0$)

$$
\int_0^{T_1}\dot A_2(t)|\Phi(t)|dt = -\int_0^{T_1}\dot A_2(t)\Phi(t)dt \qquad (12)
$$

$$
= \int_0^{T_1}A_2(t)\varphi(t)dt + a_2\Phi(0) \le \int_0^{T_1}A_2(t)\varphi(t)dt; \qquad (13)
$$

from (11) we deduce

$$
\begin{aligned}
-\int_0^{T_1} & \log[A_1(t)]\dot\varphi(t)dt \\
&\le \frac{\varepsilon}{4\sqrt{K_2}}\int_0^{T_1}\big(A_2(t)\varphi(t)+\lambda_2 A_2(t)|\Phi(t)|\big)dt + \frac{1}{4\varepsilon\sqrt{K_2}}\int_0^{T_1}\frac{|\dot\varphi(t)|^2}{|\Phi(t)|}dt.
\end{aligned}
$$

By $(6)_3$, we derive a similar inequality involving $A_3(t)$. Then, from (10) we infer that

$$
\begin{aligned}
\int_0^{T_1} & \big(K_2A_2(t)+K_3A_3(t)\big)\varphi(t)dt \\
&\le \frac{\varepsilon}{8}\int_0^{T_1}\left[\frac{A_2(t)}{\sqrt{K_2}}+\frac{A_3(t)}{\sqrt{K_3}}\right]\varphi(t)dt + \frac{\varepsilon}{8}\int_0^{T_1}\left[\frac{\lambda_2 A_2(t)}{\sqrt{K_2}}+\frac{\lambda_3 A_3(t)}{\sqrt{K_3}}\right]|\Phi(t)|dt \\
&\qquad + \frac{1}{8\varepsilon}\,\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1}\frac{|\dot\varphi(t)|^2}{|\Phi(t)|}dt + \lambda_1\int_0^{T_1}\varphi(t)dt - \log a_1. \qquad (14)
\end{aligned}
$$

Step 2. We apply the same strategy as for (10), dividing $(6)_1$ by $A_1(t) > 0$ but testing with Φ in (9), instead of φ:

$$\int_0^{T_1} \big(K_2A_2(t) + K_3A_3(t)\big)\Phi(t)dt = \lambda_1 \int_0^{T_1} \Phi(t)dt - \int_0^{T_1} \log[A_1(t)]\varphi(t)dt - \Phi(0)\log a_1. \quad (15)$$

Changing the signs and recalling that $\Phi(t) \le 0$ for all $t \in [0, T_1]$, we obtain

$$\int_0^{T_1} \big(K_2A_2(t) + K_3A_3(t)\big)|\Phi(t)|dt = \lambda_1 \int_0^{T_1} |\Phi(t)|dt + \int_0^{T_1} \log[A_1(t)]\varphi(t)dt + \Phi(0)\log a_1. \quad (16)$$

Thanks to $(6)_2$-$(6)_3$, we find

$$\int_0^{T_1} \log[A_1(t)]\,\varphi(t)dt = \frac{1}{2}\int_0^{T_1} \log[A_1(t)^2]\,\varphi(t)dt$$

$$= \frac{1}{4}\int_0^{T_1} \log\tfrac{\dot A_2(t)+\lambda_2A_2(t)}{K_2}\,\varphi(t)dt + \frac{1}{4}\int_0^{T_1} \log\tfrac{\dot A_3(t)+\lambda_3A_3(t)}{K_3}\,\varphi(t)dt,$$

by (7):

$$\le \frac{1}{4\sqrt{K_2}}\int_0^{T_1} \sqrt{\dot A_2(t)+\lambda_2A_2(t)}\,\varphi(t)dt + \frac{1}{4\sqrt{K_3}}\int_0^{T_1} \sqrt{\dot A_3(t)+\lambda_3A_3(t)}\,\varphi(t)dt,$$

by Hölder and Young inequalities:

$$\le \frac{\eta}{8\sqrt{K_2}}\int_0^{T_1} \big(\dot A_2(t)+\lambda_2A_2(t)\big)|\Phi(t)|dt + \frac{\eta}{8\sqrt{K_3}}\int_0^{T_1} \big(\dot A_3(t)+\lambda_3A_3(t)\big)|\Phi(t)|dt + \frac{1}{8\eta}\,\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1} \frac{\varphi(t)^2}{|\Phi(t)|}dt,$$

by parts:

$$= \frac{\eta}{8}\int_0^{T_1} \left(\frac{A_2(t)}{\sqrt{K_2}} + \frac{A_3(t)}{\sqrt{K_3}}\right)\varphi(t)dt + \frac{\eta}{8}\int_0^{T_1} \left(\frac{\lambda_2A_2(t)}{\sqrt{K_2}} + \frac{\lambda_3A_3(t)}{\sqrt{K_3}}\right)|\Phi(t)|dt + \frac{\eta}{8}\Phi(0)\left(\frac{a_2}{\sqrt{K_2}} + \frac{a_3}{\sqrt{K_3}}\right) + \frac{1}{8\eta}\,\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1} \frac{\varphi(t)^2}{|\Phi(t)|}dt,$$

since $a_2, a_3 \ge 0,\ \Phi(0) < 0$

$$\le \frac{\eta}{8}\int_0^{T_1} \left(\frac{A_2(t)}{\sqrt{K_2}} + \frac{A_3(t)}{\sqrt{K_3}}\right)\varphi(t)dt + \frac{\eta}{8}\int_0^{T_1} \left(\frac{\lambda_2A_2(t)}{\sqrt{K_2}} + \frac{\lambda_3A_3(t)}{\sqrt{K_3}}\right)|\Phi(t)|dt + \frac{1}{8\eta}\,\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1} \frac{\varphi(t)^2}{|\Phi(t)|}dt.$$

Therefore, from (16) we deduce the inequality

$$\left(K_2 - \frac{\eta\lambda_2}{8\sqrt{K_2}}\right)\int_0^{T_1} A_2(t)|\Phi(t)|dt + \left(K_3 - \frac{\eta\lambda_3}{8\sqrt{K_3}}\right)\int_0^{T_1} A_3(t)|\Phi(t)|dt$$
$$\leq \frac{\eta}{8}\int_0^{T_1}\left(\frac{A_2(t)}{\sqrt{K_2}} + \frac{A_3(t)}{\sqrt{K_3}}\right)\varphi(t)dt + \frac{1}{8\eta}\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1}\frac{\varphi(t)^2}{|\Phi(t)|}dt$$
$$+ \lambda_1\int_0^{T_1}|\Phi(t)|dt - |\Phi(0)|\log a_1. \quad (17)$$

Taking

$$0 < \eta < 8\min\left\{\frac{K_2^{\frac{3}{2}}}{\lambda_2}, \frac{K_3^{\frac{3}{2}}}{\lambda_3}\right\}, \qquad c := \min\left\{K_2 - \frac{\eta\lambda_2}{8\sqrt{K_2}}\,,\, K_3 - \frac{\eta\lambda_3}{8\sqrt{K_3}}\right\} > 0,$$

we then obtain

$$\int_0^{T_1}\big(A_2(t)+A_3(t)\big)|\Phi(t)|dt$$
$$\leq \frac{1}{8c}\left[\eta\int_0^{T_1}\left(\frac{A_2(t)}{\sqrt{K_2}} + \frac{A_3(t)}{\sqrt{K_3}}\right)\varphi(t)dt + 8\lambda_1\int_0^{T_1}|\Phi(t)|dt\right.$$
$$\left. + \frac{\sqrt{K_2}+\sqrt{K_3}}{\eta\sqrt{K_2K_3}}\int_0^{T_1}\frac{\varphi(t)^2}{|\Phi(t)|}dt - 8|\Phi(0)|\log a_1\right] \quad (18)$$

Step 3. Let $C := \max\{\frac{\lambda_2}{\sqrt{K_2}}, \frac{\lambda_3}{\sqrt{K_3}}\}$. By combining (14) with (18), we obtain

$$\left[K_2 - \frac{\varepsilon}{8\sqrt{K_2}}\left(1+\frac{C\eta}{8c}\right)\right]\int_0^{T_1} A_2(t)\varphi(t)dt$$
$$+ \left[K_3 - \frac{\varepsilon}{8\sqrt{K_3}}\left(1+\frac{C\eta}{8c}\right)\right]\int_0^{T_1} A_3(t)\varphi(t)dt$$
$$\leq \frac{1}{8\varepsilon}\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1}\frac{\dot\varphi(t)^2}{|\Phi(t)|}dt + \frac{\varepsilon}{\eta}\frac{C}{64c}\frac{\sqrt{K_2}+\sqrt{K_3}}{\sqrt{K_2K_3}}\int_0^{T_1}\frac{\varphi(t)^2}{|\Phi(t)|}dt$$
$$+ \lambda_1\int_0^{T_1}\varphi(t)dt + \varepsilon\frac{C\lambda_1}{8c}\int_0^{T_1}|\Phi(t)|dt - \left(1+\varepsilon|\Phi(0)|\frac{C}{8c}\right)\log a_1. \quad (19)$$

Take

$$0 < \varepsilon < \frac{64c}{8c+C\eta}\min\{K_2^{\frac{3}{2}}, K_3^{\frac{3}{2}}\},$$
$$\zeta = \min\left\{K_2 - \frac{\varepsilon}{8\sqrt{K_2}}\left(1+\frac{C\eta}{8c}\right),\, K_3 - \frac{\varepsilon}{8\sqrt{K_3}}\left(1+\frac{C\eta}{8c}\right)\right\} > 0,$$

so that (19) becomes

$$\begin{aligned}\int_0^{T_1} \big(A_2(t)+A_3(t)\big)\varphi(t)dt \\ \leq \frac{1}{\zeta}\Bigg\{ \frac{\sqrt{K_2}+\sqrt{K_3}}{8\sqrt{K_2K_3}}\left[\frac{1}{\varepsilon}\int_0^{T_1}\frac{\dot\varphi(t)^2}{|\Phi(t)|}dt + \frac{C\varepsilon}{8c\eta}\int_0^{T_1}\frac{\varphi(t)^2}{|\Phi(t)|}dt\right] \\ + \lambda_1\int_0^{T_1}\varphi(t)dt + \frac{C\varepsilon\lambda_1}{8c}\int_0^{T_1}|\Phi(t)|dt - \log a_1\Bigg\}. \quad (20)\end{aligned}$$

Step 4. So far, $0 < T < T_1$ and the test functions φ, Φ in (8)-(9), were arbitrary. We now fix $\varphi_0 \in C^3(\mathbb{R}_+)$ such that

$$\varphi_0(t) = \begin{cases} 1 & 0 \leq t \leq 1, \\ 0 & t \geq 2, \end{cases} \qquad \begin{cases} \dot\varphi_0(1)=0, \ \varphi_0(2)=\dot\varphi_0(2)=\ddot\varphi_0(2)=0, \\ \varphi_0 \text{ is non-increasing in } \mathbb{R}_+ \ (-\dot\varphi_0 = |\dot\varphi_0|). \end{cases} \quad (21)$$

Take also $\Phi_0(t) = -\int_t^2 \varphi_0(s)ds$. Then, for a given $T > 0$ as in the statement of Theorem 3.1, let

$$\varphi(t) = \varphi_0\left(\frac{t}{T}\right), \qquad \Phi(t) = -\int_t^{2T}\varphi(s)ds = T\Phi_0\left(\frac{t}{T}\right).$$

Therefore,

$$\begin{aligned}\int_0^{2T}\varphi(t)dt &= T\int_0^2 \varphi_0(\tau)\,d\tau := T\alpha, \\ \int_0^{2T}|\Phi(t)|dt &= T^2\int_0^2 |\Phi_0(\tau)|\,d\tau := T^2\beta, \\ \int_0^{2T}\frac{\varphi(t)^2}{|\Phi(t)|}dt &= \int_0^2 \frac{\varphi_0(\tau)^2}{|\Phi_0(\tau)|}d\tau := \gamma, \\ \int_0^{2T}\frac{\dot\varphi(t)^2}{|\Phi(t)|}dt &= \frac{1}{T^2}\int_1^2\frac{\dot\varphi_0(\tau)^2}{|\Phi_0(\tau)|}d\tau := \frac{\delta}{T^2},\end{aligned} \quad (22)$$

for some $\alpha, \beta, \gamma, \delta > 0$; by (21) all the integrals in (22) are convergent.

We now determine $a_1 > 0$ for which the solution to (6) does not remain bounded in $(0,2T)$. Formally, from (20) we get

$$
\begin{aligned}
0 &< \int_0^{2T} \big(A_2(t)+A_3(t)\big)\varphi(t)dt \\
&\le \frac{1}{\zeta}\left\{\frac{\sqrt{K_2}+\sqrt{K_3}}{8\sqrt{K_2K_3}}\left[\frac{1}{\varepsilon}\frac{\delta}{T^2}+\frac{C\varepsilon\gamma}{8c\eta}\right]+\lambda_1\alpha T+\frac{C\varepsilon\lambda_1\beta}{8c}T^2-\log a_1\right\} \\
&= \frac{1}{\zeta T^2}\left[\frac{C\varepsilon\lambda_1\beta}{8c}T^4+\lambda_1\alpha T^3+\left(\frac{\sqrt{K_2}+\sqrt{K_3}}{8\sqrt{K_2K_3}}\frac{C\varepsilon\gamma}{8c\eta}-\log a_1\right)T^2\right. \qquad (23)\\
&\qquad\qquad \left.+\frac{\sqrt{K_2}+\sqrt{K_3}}{8\sqrt{K_2K_3}}\frac{\delta}{\varepsilon}\right] \\
&=: \frac{1}{\zeta T^2}\left[\mathbb{A}T^4+\mathbb{B}T^3-\mathbb{C}T^2+\mathbb{D}\right] \; ;
\end{aligned}
$$

notice that $\mathbb{A},\mathbb{B},\mathbb{D}>0$ and, also $\mathbb{C}>0$, provided that $a_1>1$ is sufficiently large. It is clear that if $\mathbb{C}>0$ is sufficiently large (again, $a_1>1$ is sufficiently large), then the upper bound in (23) is strictly negative, which gives a a contradiction and shows that the solution to (6) cannot be extended to $(0,2T)$ if a_1 is sufficiently large. □

REMARK 3.3. In the above proof of Theorem 3.1 we reached the conclusion by simply using the magnitude of $a_1>0$. The proof can also be obtained similarly, by using the magnitude of a_2 or a_3 that disappeared because of inequality (12).

4. Approximations of the Euler equations

If we take $\lambda_1=\lambda_2=\lambda_3=0$, system (2) no longer approximates the velocity within the Navier-Stokes equations $(1)_{1,2}$ but, instead, the velocity within the Euler equations for inviscid fluids in the cube Ω

$$u_t+(u\cdot\nabla)u+\nabla p=0\,,\qquad \nabla\cdot u=0\quad \text{in }\Omega\times\mathbb{R}_+\,.$$

Then, (2) becomes

$$
\begin{cases}
\dot{A}_1(t)=K_2A_1(t)A_2(t)-K_3A_1(t)A_3(t)\\
\dot{A}_2(t)=-K_2A_1(t)^2\\
\dot{A}_3(t)=K_3A_1(t)^2\\
A_1(0)=a_1,\quad A_2(0)=a_2,\quad A_3(0)=a_3,
\end{cases}
\qquad (24)
$$

and we have

PROPOSITION 4.1. *For any $K_2,K_3,a_1,a_2,a_3\in\mathbb{R}$, the solution to (24) is global.*

Proof. After multiplying $(24)_i$ by $A_i(t)$ (for $i = 1, 2, 3$) and adding the resulting equations, we obtain

$$\frac{d}{dt}\Big[A_1(t)^2 + A_2(t)^2 + A_3(t)^2\Big] = 0\,.$$

This implies that $A_1(t)^2 + A_2(t)^2 + A_3(t)^2 \equiv a_1^2 + a_2^2 + a_3^2$ and, hence, that the solution to (24) does not blow up. □

Similarly to Proposition 2.1, Proposition 4.1 ensures global existence regardless of the initial data and of the signs of K_2 and K_3 (we take $K_2, K_3 \neq 0$, otherwise the problem becomes trivial). On the other hand, the modified Euler version of (6) reads

$$\begin{cases} \dot{A}_1(t) = K_2 A_1(t) A_2(t) + K_3 A_1(t) A_3(t) \\ \dot{A}_2(t) = K_2 A_1(t)^2 \\ \dot{A}_3(t) = K_3 A_1(t)^2 \\ A_1(0) = a_1, \quad A_2(0) = a_2, \quad A_3(0) = a_3, \end{cases} \tag{25}$$

and, arguing as in the proof of Proposition 2.1, we obtain

$$\frac{d}{dt}\Big[A_1(t)^2 + A_2(t)^2 + A_3(t)^2\Big] = 4A_1(t)^2\Big[K_2 A_2(t) + K_3 A_3(t)\Big]\,, \tag{26}$$

which does not allow to exclude finite time blow-up for (25) whatever the signs of K_2 and K_3 are. Moreover, by multiplying $(25)_i$ by K_i ($i = 2, 3$) and by adding the two resulting equations, we find

$$\frac{d}{dt}\Big[K_2 A_2(t) + K_3 A_3(t)\Big] = A_1(t)^2\big[K_2^2 + K_3^2\big] \geq 0\,.$$

Hence, unless we know that $t \mapsto K_2 A_2(t) + K_3 A_3(t)$ remains negative, (26) is more likely to suggest blow-up for (25), either in finite or infinite time. One then wonders whether this depends on the initial data a_1, a_2, a_3. The answer is affirmative:

► if $a_1 = 0$, then $A_i(t) \equiv a_i$ ($i = 1, 2, 3$) and the solution to (25) is global;

► if $a_1 = 1/\sqrt{K_2^2 + K_3^2}$, $a_2 = K_2/(K_2^2 + K_3^2)$, $a_3 = K_3/(K_2^2 + K_3^2)$, then the solution to (25) is

$$A_1(t) = \frac{1}{\sqrt{K_2^2 + K_3^2}}\,\frac{1}{1-t}\,,$$

$$A_2(t) = \frac{K_2}{K_2^2 + K_3^2}\,\frac{1}{1-t}\,, \quad A_3(t) = \frac{K_3}{K_2^2 + K_3^2}\,\frac{1}{1-t}\,,$$

and each component blows up as $t \to 1$, where $A_1(t)$ is always positive, while $A_i(t)$ has the same sign as K_i ($i = 2, 3$). Note that this solution can be determined explicitly through a precise choice of $a_2, a_3 > 0$. If, instead, we take $a_2 = a_3 = 0$ we obtain both a surprising analogy and a surprising difference between (24) coming from the Euler equations and the modified system (25). The solution to (24) is

$$A_1(t) = \frac{a_1}{\cosh\left(|a_1|\sqrt{K_2^2+K_3^2}\,t\right)}\,,$$
$$A_2(t) = \frac{-|a_1|K_2}{\sqrt{K_2^2+K_3^2}}\tanh\left(|a_1|\sqrt{K_2^2+K_3^2}\,t\right)\,,\quad A_3(t) = -\frac{K_3}{K_2}A_2(t)$$

while the solution to (25) is

$$A_1(t) = \frac{a_1}{\cos\left(|a_1|\sqrt{K_2^2+K_3^2}\,t\right)}\,,$$
$$A_2(t) = \frac{|a_1|K_2}{\sqrt{K_2^2+K_3^2}}\tan\left(|a_1|\sqrt{K_2^2+K_3^2}\,t\right)\,,\quad A_3(t) = \frac{K_3}{K_2}A_2(t).$$

The analogy is that the latter are obtained from the former by replacing hyperbolic with trigonometric functions (up to a sign). The difference is that the former are global solutions, while the latter blow up at $T^* = \pi/(2|a_1|\sqrt{K_2^2+K_3^2})$. Note that the blow-up time decreases as $|a_1|$ increases, as in Theorem 3.1.

To visualize these results, in Figures 1-2 we plot the solutions to (24) and (25) when $K_2 = 1$ and

$$a_1 = 1, \qquad a_2 = a_3 = 0, \tag{27}$$

for different choices of the remaining parameters.

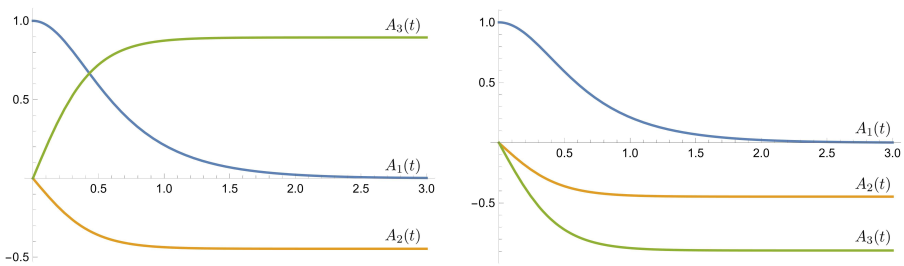
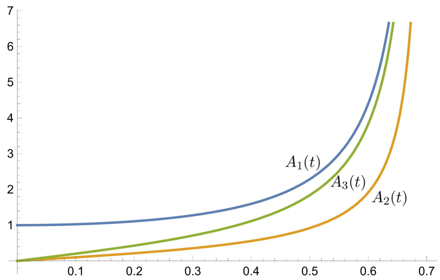


Figure 1: Solution of (24)+(27): $K_3 = 2$ (left) and $K_3 = -2$ (right).



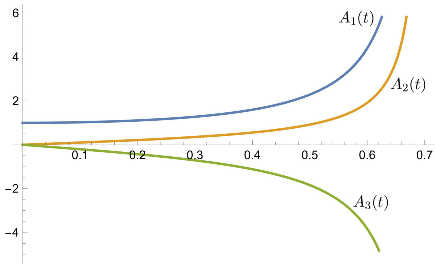


Figure 2: Solution of (25)+(27), $T^* = \frac{\pi}{2\sqrt{5}}$: $K_3 = 2$ (left) and $K_3 = -2$ (right).

Acknowledgements

The authors are grateful to an anonymous referee for suggesting how to improve Theorem 3.1. This work is part of the PRIN project 2022 "Partial differential equations and related geometric-functional inequalities", financially supported by the EU, in the framework of the "Next Generation EU initiative". The authors are also partially supported by INdAM.

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Received September 6, 2024
Revised November 22, 2024
Accepted November 23, 2024