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Singular perturbations for diffusive competing species

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*"Dedicated to E. Mitidieri,
at the occasion of his 70th anniversary,
with steam and friendship."*

Abstract. *The first aim of this paper is to discuss some of the contents of Hutson et al. [15] versus the contents of a well known paper of Y. Lou, [20], as many experts are attributing, incorrectly, to Lou [20] some of the pioneering findings of Hutson et al. [15], published 11 years before. The second aim is contextualize the most pioneering results versus the most recent ones by the team of the author. Finally, some new multiplicity and uniqueness results are given for a symmetric diffusive competition model.*

Keywords: Singular perturbation, competing species, coexistence states, permanence, bi-stability, dominance.
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1. Introduction

The first aim of this paper is to discuss some of the contents of Hutson et al. [15] versus the contents of a well known paper of Y. Lou, [20], as many experts in the field have attributed, incorrectly, to Lou [20] some of the pioneering findings of Hutson et al. [15], published 11 years before. As the results of [15] have a large number of applications in many different fields, this paper tries to minimize as much as possible the damage already caused to [15] by the high number of incorrect attributions of Theorem 1.1 of Lou [20].

To reach this goal, the author will invoke a mathematical letter sent to K. J. Brown by V. Hutson in 1994 proposing two different problems, one of them already solved in Fraile et al. [7], which was left outside the bibliography of the influential monograph [1]. Incidentally, though V. Hutson sent to R. S. Cantrell and C. Cosner a copy of the manuscript of [15] by ordinary mail early 1995, and, actually, he maintained at that time a rather active mathematical collaboration with the authors of [1], published in 2003, Hutson et al. [15] was also left outside the bibliography of [1].

The second aim of this paper is to review some of the findings of Hutson et al. [15] at the light of the much sharper findings of Fernández-Rincón and López-Gómez [5, 6], using the methodology of Furter and López-Gómez [8, 9], adopted by W. M. Ni and his coworkers much later. This weights the relevance of the most pioneering findings of [15] versus the most recent ones.

Finally, we will deliver a number of new results concerning a symmetric competition system introduced in [5] to get a novel multiplicity result for sufficiently small diffusions. In particular, we will give some new uniqueness and exact multiplicity results.

The distribution of this paper is the following. In Section 2, we discuss the genesis of the underlying theory at the light of the mathematical correspondence crossed by V. Hutson with K. J. Brown and J. López-Gómez, at Heriot–Watt University in 1994-95. Section 3 reviews the findings and proofs of Hutson et al. [15]. Section 4 shows how Theorem 1.1 of Lou [20] was already known, at least, 11 years before, though, possibly, was attributed to Y. Lou by the incorrect quotations of W. M. Ni and his coworkers, as well as by the fact that [15] remained outside the bibliography of Cantrell and Cosner [1]. Section 5 recalls the (optimal) singular perturbation of Fernández-Rincón and López-Gómez [5]. Finally, Section 6 delivers some new uniqueness and multiplicity results for a symmetric competition model. These results measure how spatial heterogeneities enhance the multiplicity of coexistence states in competitive systems.

2. The genesis of the theory

On November 3rd, 1994, V. Hutson addressed a letter to K. J. Brown at Heriot–Watt University (Edinburgh, UK) asking him for any information concerning the diffusive competing species model

$$\begin{cases} \frac{\partial u}{\partial t} - \mu\Delta u = u(\alpha(x) - u - v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} - \mu\Delta v = v(\beta(x) - u - v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n}(x,t) = 0 = \frac{\partial v}{\partial n}(x,t) & \text{for all } (x,t) \in \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \geq 0, \;\; v(\cdot,0) = v_0 \geq 0, & \text{in } \Omega, \end{cases} \tag{1}$$

where n stands for the outward unit normal of a nice open bounded domain, Ω, of $\mathbb{R}^N$, $N \geq 1$, and α, β are two continuous functions such that $\alpha(x) > 0$, $\beta(x) > 0$ for all $x \in \Omega$. He was specially interested in the most intriguing case when the function $\alpha - \beta$ changes sign. Once proposed this problem, he added:

"It would be enough in the first place if $\Omega = (0,1)$, although one obviously would like to consider a more general Ω eventually.

Before we can even get off this ground, we need to know something about the following scalar problem

$$-\mu\frac{\partial^2 w}{\partial x^2} = w(\alpha(x) - w) \tag{2}$$

with zero Neumann conditions, $\Omega = (0,1)$, so that there is a unique solution w say. The speculation is that as $\mu \to 0$, $w \to \alpha$ uniformly on compact subsets of Ω —or perhaps even in more strong sense. There is a good reason to think this is true. For zero boundary data it is proved in [3]. The proof is really rather long, but I suspect it is easier for the Neumann problem, even easier for the ODE case above."

Then, V. Hutson went to express some of his personal suspicions about the behavior of the system for small μ. Actually, he had raised two problems in his letter: One for the scalar equation (2), and another one, much more sophisticated technically, for the diffusive competition system (1).

During the academic year 1994–1995 the author was on a sabbatical leave from Madrid at Heriot–Watt University supported by a Research Grant of the European Union assigned to J. M. Ball, and K. J. Brown had already got the preprints of Furter and López-Gómez [8] and Fraile et al. [7], which had been submitted for publication to the *Transactions of the American Mathematical Society* (TAMS) and the *Journal of Differential Equations* (JDE), respectively. In [8], submitted to J. Mallet-Paret, the dynamics of a prototype model for competing species, including the system of (1), under homogeneous Dirichlet boundary conditions and different diffusion coefficients (μ, ν), had been analyzed in depth in an heterogeneous environment. According to the abstract of the preprint [8], produced in 1993 and finally published in 1997 in [9] (where the reader might go whenever [8] is cited in this paper):

"When diffusion is switched off, at each point $x \in \Omega$ we have a pair of ODE's: *the kinetic*. If for some $x \in \Omega$ the kinetic has a unique stable coexistence state, we show that there exist $\hat{\mu} > 0$, $\hat{\nu} > 0$ such that, for every $(\mu,\nu) \in (0,\hat{\mu}) \times (0,\hat{\nu})$, the RD model is *persistent*, in the sense that it has a compact global attractor within the interior of the positive cone and has a stable coexistence state. The same result is true if there exist $x_u, x_v \in \Omega$ such that the semitrivial coexistence states $(u,0)$ and $(0,v)$ of the kinetic are globally asymptotically stable at $x = x_u$ and $x = x_v$, respectively. More generally, our main result shows that, for most kinetic patterns, stable coexistence of populations can be found for some range of the diffusion coefficients. Singular perturbation techniques, monotone schemes, fixed point index, global analysis of *persistence curves*, global continuation and singularity theory are some of the technical tools employed to get the previous results, among others."

In Fraile et al. [7], already submitted to J. K. Hale on October 26th 1994, the dynamics of the generalized boundary value problem of logistic type

$$\begin{cases} \partial_t u + \mathfrak{L}(x,D)u = m(x)u - a(x)f(x,u)u & \text{in } \Omega \times (0,\infty), \\ \mathfrak{B}(x,D)u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \gneq 0 & \text{in } \Omega, \end{cases} \tag{3}$$

had been fully characterized, where $\mathfrak{L}(x, D)$ is a second order uniformly elliptic operator, $\mathfrak{B}(x, D)$ is a general boundary operator of mixed type including Dirichlet, Neumann and Robin boundary conditions, $m, a \in \mathcal{C}^{\nu}(\bar{\Omega})$ for some $\nu \in (0,1)$, $a \gneq 0$, and the function $f : [0, +\infty) \times \bar{\Omega} \to \mathbb{R}$ is of class $\mathcal{C}^{\nu,1+\nu}$ and satisfies $f(x,0) \equiv 0$, $f(x,w) > 0$ and $\partial_w f(x,w) > 0$ for all $w \in \Omega$ and $x \in \Omega$. Moreover,

$$\lim_{w\to\infty} f(x,w) = \infty \quad \text{for all } \; x \in \Omega,$$

though these assumptions can be substantially relaxed by working in the setting of the L^p–theory. Precisely, denoting by $\sigma_1^{\Omega}[\mathfrak{L} - m, \mathfrak{B}]$ the principal eigenvalue of $(\mathfrak{L} - m, \mathfrak{B}, \Omega)$, Theorem 3.7 of [7] can be stated as follows.

Theorem 2.1. *The following assertions are true:*

(a) *If* $\sigma_1^{\Omega}[\mathfrak{L} - m, \mathfrak{B}] \geq 0$, *then the zero solution of* (3) *is globally asymptotically stable.*

(b) *If* $\sigma_1^{\Omega}[\mathfrak{L} - m, \mathfrak{B}] < 0$ *and there exists a positive steady-state* w_0 *of* (3), *then* w_0 *is globally asymptotically stable. In particular,* w_0 *is unique.*

(c) *If* $\sigma_1^{\Omega}[\mathfrak{L} - m, \mathfrak{B}] < 0$ *and* (3) *does not admit a positive steady-state, then*

$$\lim_{t\to\infty} \|u(\cdot, t, u_0)\|_{\mathcal{C}(\bar{\Omega})} = \infty \quad \textit{for each } \; u_0 \gneq 0.$$

Moreover, by Theorem 3.5 of Fraile et al. [7], the next result holds.

Theorem 2.2. *If* $a(x) > 0$ *for all* $x \in \bar{\Omega}$, *then, the problem* (3) *has some positive steady state solution if, and only if,* $\sigma^{\Omega}[\mathfrak{L} - m, \mathfrak{B}] < 0$.

By combining Theorems 2.1 and 2.2, the following result holds.

Corollary 2.3. *Suppose that* $a(x) > 0$ *for all* $x \in \bar{\Omega}$. *Then, the problem* (3) *has some positive steady state solution if, and only if,* $\sigma^{\Omega}[\mathfrak{L} - m, \mathfrak{B}] < 0$. *Moreover, if it exists, it is unique and globally asymptotically stable with respect to the positive solutions of* (3).

As, according to the offprint of Fraile et al. [7], its manuscript had been received by J. K. Hale on November 8th 1994, it is apparent that, when V. Hutson sent his letter to K. J. Brown, the manuscript of [7] was already traveling to Atlanta. Thus, since (3) is far more general than (2), the problem posed by V. Hutson concerning (2) had been already solved, independently, by J. M. Fraile et al. [7], except for the behavior of its positive steady-state solution as $\mu \downarrow 0$. The singular perturbation problem as $\mu \downarrow 0$ had been already solved, under homogeneous Dirichlet boundary conditions, in Furter and López-Gómez [8].

As K. J. Brown had already got copies of the preprints of [8] and [7], he provided with a copy of Hutson's letter to the author, who solved the problems proposed by V. Hutson in his interesting letter, except for the uniqueness of the coexistence state of (1) for sufficiently small μ. The uniqueness was a very challenging problem 30 years ago solved recently in its greatest generality, in a general heterogeneous context, by Fernández-Rincón and López-Gómez in [6]. As suspected by V. Hutson, most of the results for Neumann boundary conditions had simpler proofs than for Dirichlet boundary conditions, by the absence of boundary layers.

Short time later, V. Hutson visited Heriot–Watt University to gather K. J. Brown and the author. After some technical discussions, V. Hutson prepared the manuscript of [15] from a preliminary (rather complete) draft of the author. Although K. J. Brown participated in these discussions, he declined to sign the paper. Finally, V. Hutson shortened the original draft of the author, re-elaborated it, and submitted the final manuscript, on February 10th 1995, for the special volume *Dynamical Systems and Applications*, published by R. P. Agarwal as the fourth issue of *World Scientific Series on Applied Analysis.* By the limitation to 14 pages of the contributions to this special issue, the authors had to shorten substantially the detailed draft of J. López-Gómez. In his letter to the author, containing the final manuscript submitted for publication to R. P. Agarwal, V. Hutson knowledged to J. López-Gómez that the day before he had already posted to Ravi Agarwal the paper and that he planned to send copies to Chris Cosner, or Steve Cantrell jointly, Hal Smith and Alan Lazer. The author possesses and can exhibit the supporting documentation on request.

The joint paper of the author with J. E. Furter, which permitted him to solve the problems proposed by V. Hutson, was finally published, after 4 years, in the *Proceedings of the Royal Society of Edinburgh*, [9], in 1997. The final version had some minor changes with respect to [8] (one of them, very significative, in its title), but the contents were identical. Although [8] had been submitted to J. Mallet-Paret for the TAMS, a negative report of its reviewer provoked its rejection. The reviewer ended his (rather biased) report by arguing that

> "Because misuse of terminology is likely to further damage the already tarnished reputation of mathematicians among ecologists, I hope this paper will not be published in its current form. If the terminology is corrected then the paper should be publishable. The authors should note that the existence of equilibria and uniform persistence (or "Permanence" in the European terminology) are treated for more general heterogeneous ecological systems with diffusion in (Ghoreishi and Logan, Bull. Australian Math. Soc. **44** (1991), 79–94) and (Cantrell, Cosner, and Hutson, (Proc. Royal Soc. Edinburgh **123A** (1993), 533–559) respectively. "

The rejection was communicated to the author by J. Mallet-Paret in August 1995, though the paper had been submitted to publication in 1993. By whatever reason, the rejection letter, sent to J. E. Furter on September 1993, never reached him. It would be interesting to know how many ecologists have read some paper in the TAMS!

It is perplexing that Hutson et al. [15] and Fraile et al. [7] have not been included in the Bibliography of Cantrell and Cosner [1], published 7 years later, in 2003. According to the letter of V. Hutson to the author, R. S. Cantrell and C. Cosner should have had, at least, a copy of [15] since 1995. By the way, since Fraile et al. [7] incorporated *protection zones* in their general abstract setting, most of the findings of [7] are substantially sharper than those of [1] for the single equation.

3. The main findings of Hutson et al. [15]

On page 345 of Hutson et al. [15], the following linear eigenvalue problem is introduced

$$\begin{cases} \mu\Delta\varphi + h(x)\varphi = \lambda\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Adopting the notations of Theorem 2.1, the lowest principal eigenvalue of (4), denoted by $\lambda(\mu, h)$ in [15], is given by

$$\lambda(\mu, h) = -\sigma_1^\Omega[-\mu\Delta - h(x), \mathfrak{N}],$$

where $\mathfrak{N}$ stands for the Neumann operator on $\partial\Omega$. The next result, which is Lemma 2.1 of [15], establishes some important properties of $\lambda(\mu, h)$.

Lemma 3.1. *For fixed $h(x)$ the principal eigenvalue of problem* (4) *is a continuous, non-increasing function of μ, and is strictly decreasing if $h(x)$ is not a constant. Furthermore, the following hold:*

(i) *If $h < 0$ in Ω, then $\lambda(\mu, h) < 0$,*

(ii) *$\lambda(\mu, h) \uparrow \max_{\bar{\Omega}} h$ as $\mu \to 0$,*

(iii) *$\lambda(\mu, h) \downarrow \hat{h}$ as $\mu \to \infty$, where $\hat{h} := \frac{1}{|\Omega|}\int_\Omega h(x)\,dx$.*

Also, if $h_1(x) \geq h_2(x)$ for $x \in \Omega$, then $\lambda(\mu, h_1) \geq \lambda(\mu, h_2)$ with strict inequality if $h_1 \neq h_2$.

Although most of these properties, based on the variational characterization of the principal eigenvalue, go back to Furter and López-Gómez [8], Part (iii) was a novel result of Hutson et al. [15]. Thus, a detailed proof of it was delivered on page 346 of [15]. Next, based on Theorems 2.1, 2.2 and Corollary 2.3, Hutson et al. [15] described the dynamics of a single species with diffusion μ by stating the following result.

LEMMA 3.2. *Consider the initial value problem for the scalar equation*

$$\frac{\partial w}{\partial t} = \mu \Delta w + w(h - w). \tag{5}$$

If $\lambda(\mu, h) \leq 0$, then 0 is a global attractor for positive solutions. If $\lambda(\mu, h) > 0$, there is a unique, strictly positive steady-state solution, which is a global attractor for non-trivial positive solutions, the convergence in both cases being in $\| \cdot \|_\infty$.

Concerning this lemma, Hutson et al. [15] claimed that

"These results are fairly well known, [8], [2], although sometimes for zero boundary conditions, but the proofs hold with minor amendments for the zero Neumann boundary conditions assumed here."

Indeed, Lemma 3.2 is a direct consequence of Theorems 3.5 and 3.7 of Fraile et al. [7], because, thanks to Theorems 2.1, 2.2 and Corollary 2.3, the elliptic problem

$$\begin{cases} -\mu \Delta w = hw - w^2 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

has a positive solution if, and only if,

$$\lambda(\mu, h) = -\sigma_1^\Omega[-\mu\Delta - h(x), \mathfrak{N}] > 0.$$

As [7] had been already received by the JDE on November 8th 1994, it is difficult to explain why V. Hutson decided to send the readers to his joint paper with R. S. Cantrell and C. Cosner, [2], for a proof of Lemma 3.2, instead of sending them to Theorems 3.5 and 3.7 of Fraile et al [7], which was substantially far more general than [2], because, besides it allowed the existence of *protection zones* for the species, it was valid for general boundary conditions of mixed type. Astonishingly, Fraile et al. [7] was left outside the list of references of Hutson et al. [15]. Note that, according to his letter to K. J. Brown, the problem was open even in one spatial dimension three months before!

After denoting by $\theta_{[\mu,h]}$ the maximal non-negative solution of (6) and making precise that $\theta_{[\mu,h]} \gg 0$ if $\lambda(\mu, h) > 0$ and $\theta_{[\mu,h]} = 0$ if $\lambda(\mu, h) \leq 0$, the following result was stated as Lemma 2.4 of Hutson et al. [15].

LEMMA 3.3. *The following hold when μ is sufficiently large. If $\hat{h} < 0$, 0 is a global attractor for positive solutions. If $\hat{h} \geq 0$ and $h \neq 0$, then there is a unique globally-attracting, positive equilibrium $\theta_{[\mu,h]} > 0$ (for non-trivial positive solutions). Furthermore, with convergence in $\| \cdot \|_\infty$,*

$$\lim_{\mu \to \infty} \theta_{[\mu,h]} = \hat{h}. \tag{7}$$

The proof of this result given in [15] simply indicates that

"The statement concerning the existence of $\theta_{[\mu,h]} \gg 0$ follow from Lemma 3.1(iii) and Lemma 3.2. The limiting behavior (7) is proved in Theorem 4.1 of [2]."

It turns out that Theorem 4.1 of Cantrell, Cosner and Hutson [2] has nothing to do with the proof of (7), because it deals with a singular perturbation problem for Dirichlet boundary conditions. Namely, the theorem of A. J. De Santi [3] already cited by V. Hutson in his letter to K. J. Brown. Incidentally, in [2], the result of A. J. De Santi was given in Section 4, entitled *Conditions for permanence when diffusion rates are small*, which had been the leitmotif of Furter and López-Gómez [8], rejected by J. Mallet-Paret in September 1993.

Nevertheless, at the light of Lemma 3.1(iii), (7) is almost obvious, because the positive solution of (6) inherits the behavior of the principal eigenfunction of the linearized eigenvalue problem (4), even when dealing with large solutions of diffusive degenerate problems (see, e.g., the monograph [18]). Indeed, since the $\theta_{[\mu,h]}$'s are uniformly bounded above by $\|h\|_\infty$, $\lim_{\mu\to\infty} |\Delta\theta_{[\mu,h]}| = 0$ uniformly in $\bar{\Omega}$. Thus, by the compactness of $(-\Delta)^{-1}$, any sequence $\theta_{[\mu_n,h]}$, $n \geq 1$, with $\lim_{n\to\infty} \mu_n = \infty$, contains a subsequence, relabeled by n, such that $\lim_{n\to\infty} \theta_{[\mu_n,h]} = C$ for some constant $C \geq 0$, as these are the eigenfunctions of $-\Delta$ under Neumann boundary conditions. On the other hand, by definition,

$$\left(\mu\Delta + h - \theta_{[\mu,h]}\right)\theta_{[\mu,h]} = 0.$$

Thus, $\lambda(\mu, h - \theta_{[\mu,h]}) = 0$ for all $\mu > 0$. In particular, $\lambda(\mu_n, h - \theta_{[\mu_n,h]}) = 0$ for all $n \geq 1$ and hence, letting $n \to \infty$, Lemma 3.1(iii) implies that $\hat{h} - C = 0$. As this argument can be repeated along any subsequence, (7) holds.

Finally, the next result is Lemma 2.5 of Hutson et al. [15].

LEMMA 3.4. *Assume that $h(x) > 0$ for some $x \in \Omega$. Then, for small enough μ, there is a unique, globally-attracting, positive equilibrium $\theta_{[\mu,h]}$ (for non-trivial positive solutions), and with convergence in $\|\cdot\|_\infty$,*

$$\lim_{\mu\to 0} \theta_{[\mu,h]} = h^+. \tag{8}$$

The proof of this result in [15] reads as follows:

"From Lemma 3.1(ii), $\lambda(\mu, h) > 0$ for small μ, so the first assertion follows from Lemma 3.2. The limiting behavior (8) is established in [8, Th. 3.5] for Dirichlet boundary conditions, the convergence being uniform on compact subsets of Ω. However, a minor amendment of the proof for Neumann conditions also yields (8). "

Although very schematic, by the restriction to 14 pages of the length of [15], it describes exactly how to get (8) in many circumstances. Indeed, following *mutatis mutandis* the proof of [8, Th. 3.5] (see [9]), for any given $\varepsilon > 0$, we can consider a smooth function ψ in $\bar{\Omega}$, with $\frac{\partial\psi}{\partial n} = 0$ on $\partial\Omega$, such that

$$\tfrac{\varepsilon}{2} \leq h^+ + \tfrac{\varepsilon}{2} \leq \psi \leq h^+ + \varepsilon \qquad \text{in } \bar{\Omega},$$

which coincides, exactly, with (3.10) of [9]. For this choice, arguing as in the proof of Theorem 3.5 on p. 300 of [9], we find that

$$h\psi - \psi^2 = \psi(h-\psi) \le \psi(h - h^+ - \tfrac{\varepsilon}{2}) \le -\tfrac{\varepsilon}{2}\psi \le -\mu\Delta\psi$$

for sufficiently small μ, say $\mu \le \mu(\varepsilon)$. Thus, ψ is a supersolution of (6) for all $\mu \le \mu(\varepsilon)$. Hence, as (6) has arbitrarily small subsolutions, by the uniqueness of the positive solution, $\theta_{[\mu,h]} \le \psi$ for all $\mu \le \mu(\varepsilon)$. Therefore, $\theta_{[\mu,h]} \le h^+ + \varepsilon$ for every $\mu \le \mu(\varepsilon)$ and

$$\limsup_{\mu\to 0} \theta_{[\mu,h]} \le h^+ + \varepsilon \quad \text{for all } \ \varepsilon > 0.$$

Consequently, letting $\varepsilon \to 0$ yields

$$\limsup_{\mu\to 0} \theta_{[\mu,h]} \le h^+.$$

Finally, note that the unique solution of the associated Dirichlet problem

$$\begin{cases} -\mu\Delta w = w(h(x) - w) & \text{in } \ \Omega, \\ w = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{9}$$

denoted by w_μ, satisfies $w_\mu \le \theta_{[\mu,h]}$, because $\theta_{[\mu,h]}$ is a supersolution of (9). Thus, by Theorem 3.5 of [8] (or [9]), it becomes apparent that

$$\liminf_{\mu\to 0} \theta_{[\mu,h]} \ge \lim_{\mu\to 0} w_\mu = h^+ \quad \text{in } \ \Omega,$$

which ends the proof of (8), the convergence being uniform on compact subsets of Ω. Naturally, if $h^+ = 0$ on a neighborhood of $\partial\Omega$, then the convergence is uniform in $\bar{\Omega}$. The uniform convergence in the general case is a bit more delicate and the reader is sent to Fernández-Rincón and López-Gómez [5], where complete technical details are given for a general class of semilinear boundary value problems with boundary operators of non-classical mixed type.

In Section 3 of Hutson et al. [15], the dynamics of (1) was analyzed, where $\alpha, \beta : \bar{\Omega} \to \mathbb{R}$ are smooth and $\mu > 0$. It is not assumed that α, β are everywhere positive. The main results there, Theorems 3.2, 3.3 and 3.4, gave some necessary and sufficient conditions for permanence, or extinction, for small, or large, μ's. These results were very well known by the author, as he had already found many of them under Dirichlet boundary conditions in his previous papers with J. E. Furter [8, 9] and J. C. Sabina [19]. Finally, in Section 4 of Hutson et al. [15], the authors borrowed the monotone scheme introduced by López-Gómez and Sabina [19] (see (3.10) in [19]) to prove the main theorem of Hutson et al. [15], which can be stated as follows. This result corroborated to be true the suspicions of V. Hutson in his letter to K. J. Brown.

THEOREM 3.5. *Let* (u_μ, v_μ) *be a family of coexistence states of* (1). *Set*

$$u_0(x) := \begin{cases} 0, & \text{if } \alpha(x) \le 0, \ \text{ or } \ \beta(x) > \alpha(x) > 0, \\ \alpha(x), & \text{if } \ \alpha(x) > \beta(x) \ \text{ and } \ \alpha(x) > 0. \end{cases}$$

and

$$v_0(x) := \begin{cases} 0, & \text{if } \beta(x) \le 0, \ \text{ or } \ \alpha(x) > \beta(x) > 0, \\ \beta(x), & \text{if } \beta(x) > \alpha(x) \ \text{ and } \ \beta(x) > 0. \end{cases}$$

Then,

$$\lim_{\mu \to 0} (u_\mu, v_\mu) = (u_0, v_0)$$

uniformly on compact subsets of

$$\bar{\Omega} \setminus \{x \in \bar{\Omega} : \alpha(x) = \beta(x)\}.$$

Consequently, the individuals of the species u segregate, as $\mu \downarrow 0$, towards the region where $\alpha > \beta$, whereas the individuals of v segregate towards $\beta > \alpha$ as $\mu \downarrow 0$, as illustrated by Figure 1.

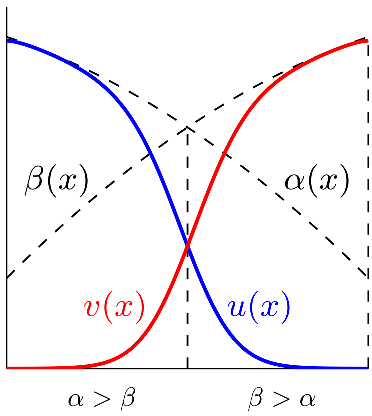

Figure 1: The behavior of (u, v) as $\mu \downarrow 0$

Figure 1 is a true solution computed by Tellini [22] through a finite difference scheme for the choices $\Omega = (0, 1)$,

$$\alpha(x) = 3 - (x + 0.5)^2, \qquad \beta(x) = 3 - (x + 1.5)^2, \qquad x \in (0, 1),$$

and $\mu = 0.003$.

Theorem 3.5 explains why spatial segregation facilitates coexistence, as sketched in Figure 6.1 on p. 309 of Cantrell and Cosner [1], though, as we have already mentioned, Hutson et al. [15], published in 1995, was left outside the Bibliography of [1], published in 2003.

4. A theorem attributed to Y. Lou

Almost 11 years later, in 2006, also Y. Lou considered the problem (6) and stated the following result (see Theorem 1.1 of Lou [20]):

THEOREM 4.1. *Suppose that $h(x)$ is non-constant, bounded and measurable, and $\int_\Omega h(x)\,dx > 0$. Then:*

(a) *For every $\mu > 0$, problem (6) has a unique positive solution θ_μ such that $\theta_\mu \in W^{2,p}(\Omega)$ for every $p \geq 1$.*

(b) *As $\mu \to 0+$, the solution $\theta_\mu \to h_+$ in $L^p(\Omega)$ for every $p \geq 1$, where $h_+(x) = \sup\{h(x), 0\}$; as $\mu \to \infty$, the solution $\theta_\mu \to \frac{1}{|\Omega|}\int_\Omega h(x)\,dx$ in $W^{2,p}(\Omega)$ for every $p \geq 1$.*

(c) *If $h(x)$ is Hölder continuous in $\bar{\Omega}$, then $\theta_\mu \in \mathcal{C}^2(\bar{\Omega})$. Moreover, $\theta_\mu \to h_+$ in $L^\infty(\Omega)$ as $\mu \to 0$, and $\theta_\mu \to \frac{1}{|\Omega|}\int_\Omega h(x)\,dx$ in $\mathcal{C}^2(\bar{\Omega})$ as $\mu \to \infty$.*

After stating Theorem 4.1, Y. Lou wrote:

"We refer the proofs of (a) and (c) to [1] and the references therein. Since we cannot locate the proof of the first part of (b) in the literature, we prove it in the beginning of Section 2."

Surprisingly, though Theorem 4.1 goes back to Hutson et al. [15], and [15] was included in the list of references of [20], Y. Lou preferred to send his readers to Cantrell and Cosner [1] for a proof of Parts (a) and (c).

Intriguingly, though Furter and López-Gómez [9] had been also added to the list of references of Lou [20], and a detailed proof of the first claim of Theorem 4.1 (b) for Dirichlet boundary conditions had been already delivered in [9], Y. Lou wrote that he could not find the proof of the first claim of Part (b) in the literature; even being folklore that the Dirichlet case is far more sophisticated than the Neumann one by the existence of boundary layers.

Actually, the first proof of Theorem 4.1 (b) for Dirichlet boundary conditions had been already given in the preprint [8], published in 1993. Thus, 13 years before than [20].

Since Lou [20] was published, many authors have attributed incorrectly Theorem 4.1 to Y. Lou, as well as some other closely related results. For example, the properties (b,c) established by Theorem 4.1 were stated by X. He and W. M. Ni in Lemma 2.3(i,a) of [10] with the next proof:

"The proofs of limiting behaviors of θ_μ as $\mu \to 0$ or ∞ are standard, see, e.g., Ni [21]."

Although He and Ni [10] could have attributed correctly Theorem 4.1, because Hutson et al. [15] has been added to the references of [10], they preferred, instead, to send the readers to Ni [21]. Once again He and Ni [11] repeat that:

"The proofs of limiting behaviors of θ_μ as μ goes to 0+ and ∞ are standard; see, e.g., Ni [21]."

It was not until [12, 13] that X. He and W. M. Ni decided to send the readers to Cantrell and Cosner [1] and Hutson et al. [15] for the limiting behaviors of θ_μ as $\mu \to 0^+$ or ∞, though Theorem 4.1 goes back to Hutson et al [15].

Not only Lemmas 3.3 and 3.4 are attributed incorrectly by W. M. Ni and his students and coworkers, but also Lemma 3.1. Indeed, concerning Proposition 2.2 on page 534 of [10], X. He and W. M. Ni claim that

"The following proposition collects some well known properties of $\mu_1(d, h)$ in connection with $\lambda_1(h)$. For a proof, see e.g. p. 95 in [1], or p. 69 in [21].

By obvious reasons, the most important part of Proposition 2.2 in [10] is Part (iii), which is the result established by Lemma 3.1, going back to Hutson et al. [15]. Surprisingly, Hutson et al. [15] had been added to the list of references of He and Ni [10]. Even at the end of the proof of Theorem 4.2 in [10], X. He and W. M. Ni do recognize that their result was a slight generalization of Theorem 3.5, going back to Hutson et al. [15]:

"The proof of (iii) uses the same arguments as in that of Theorem 4.1 of [15], as is therefore omitted here. (Note that the extra assumption that $d_1 = d_2$ in [15] is not needed in the proof). "

As far as Lemma 3.1 concerns, similar (incorrect) attributions are repeated in He and Ni [11], though in this occasion Hutson et al. [15] was left outside their bibliography.

Finally, it is perplexing that Hutson et al. [15] was left outside the list of references of Hutson, Mischaikow and Polácik [17], though Lemma 2.4 (b,c) of [17] is the natural periodic-parabolic counterpart of Lemma 3.1. Moreover, though Lemma 3.3 of Hutson, Lou and Mischaikow [16] is a natural extension of Theorem 3.5, with an obvious adaptation of the iterative scheme of López-Gómez and Sabina [19], no mention to this crucial fact was done in [16], but simply the authors commented that:

"We set $\bar{u}_0 = \tilde{u}$ and adopt a standard iteration method."

However, at the very end of the proof of Lemma 3.3 of [16], the authors stated, very carefully, Dini's convergence criterium from Rudin's book "Principles of Mathematical Analysis".

5. The sharpest singular perturbation theorem for competing species

In this section, we are going to consider the following generalized version of (1)

$$\begin{cases} \frac{\partial u}{\partial t} - \mu\Delta u = u(\alpha(x) - a(x)u - b(x)v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} - \nu\Delta v = v(\beta(x) - c(x)u - d(x)v) & \text{in } \Omega \times (0,\infty), \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega \text{ for all } t > 0, \\ u(\cdot,0) = u_0 \gneq 0, \;\; v(\cdot,0) = v_0 \gneq 0, & \text{in } \Omega, \end{cases} \tag{10}$$

together with its elliptic counterpart

$$\begin{cases} -\mu\Delta u = u(\alpha(x) - a(x)u - b(x)v) & \text{in } \Omega, \\ -\nu\Delta v = v(\beta(x) - c(x)u - d(x)v) & \text{in } \Omega, \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

whose non-negative solutions are the steady states of (10). In (10) and (11), μ and ν are positive constants and α, β, a, b, c, $d \in \mathcal{C}(\bar\Omega)$ satisfy

$$b(x) > 0 \text{ and } c(x) > 0 \text{ for all } x \in \Omega, \quad \min_{\bar\Omega} a > 0, \;\; \min_{\bar\Omega} d > 0. \tag{12}$$

As far as to the boundary operators $\mathcal{B}_i$, we assume that, for every $i \in \{1,2\}$, $\partial\Omega$ consists of finitely many (disjoint) components of class $\mathcal{C}^2$,

$$\Gamma^D_{i,j}, \;\; \Gamma^R_{i,\kappa}, \quad 1 \le j \le n_i^D, \;\; 1 \le \kappa \le n_j^R,$$

for some integers n_i^D, $n_j^R \ge 0$; some, or several of these components, might be empty. Then, for every $i \in \{1,2\}$, we set

$$\Gamma_i^D = \bigcup_{j=1}^{n_i^D} \Gamma^D_{i,j}, \qquad \Gamma_i^R = \bigcup_{j=1}^{n_i^R} \Gamma^R_{i,j},$$

and the boundary operator $\mathcal{B}_i$ is defined by

$$\mathcal{B}_i h := \begin{cases} \mathcal{D}_i h := h & \text{on } \Gamma_i^D, \\ \mathcal{R}_i h := \frac{\partial h}{\partial n} + \gamma_i h & \text{on } \Gamma_i^R, \end{cases}$$

for all $h \in \mathcal{C}^1(\bar\Omega)$, where $\gamma_i \in \mathcal{C}(\Gamma_i^R)$. Thus, for every $i \in \{1,2\}$, Γ_i^D and Γ_i^R are the portions of the edges of the inhabiting territory where the species u and v obeys a boundary condition of Dirichlet ($\mathcal{D}$) or Robin ($\mathcal{R}$) type, respectively. In particular, $\mathcal{B}_i = \mathcal{D}$ if $\Gamma_i^D = \partial\Omega$, and $\mathcal{B}_i = \mathcal{R}$ if $\Gamma_i^R = \partial\Omega$.

As in Furter and López-Gómez [8, 9], the dynamics of (10) for sufficiently small μ and ν depends on the dynamics of the associated *kinetic model*

$$\begin{cases} u'(t) = \alpha(x)u(t) - a(x)u^2(t) - b(x)u(t)v(t), \\ v'(t) = \beta(x)v(t) - c(x)u(t)v(t) - d(x)v^2(t), \end{cases} \tag{13}$$

where $x \in \bar{\Omega}$ is viewed as a parameter in (13). Indeed, according to the dynamics of (13), Ω can be partitioned into the following (disjoint) patches:

$$\begin{aligned}
\Omega_{\rm ext} &:= \{x \in \bar{\Omega} \; : \; \alpha(x),\, \beta(x) \leq 0\}, \\
\Omega_{\rm per} &:= \{x \in \bar{\Omega} \; : \; \alpha(x), \beta(x) > 0,\; \alpha(x)d(x) > \beta(x)b(x),\; \beta(x)a(x) > \alpha(x)c(x)\}, \\
\Omega_{\rm bi} &:= \{x \in \bar{\Omega} \; : \; \alpha(x), \beta(x) > 0,\; \alpha(x)d(x) < \beta(x)b(x),\; \beta(x)a(x) < \alpha(x)c(x)\}, \\
\Omega_{\rm do}^u &:= \{x \in \bar{\Omega} \; : \; \alpha(x) > 0,\; \alpha(x)d(x) > \beta(x)b(x),\; \beta(x)a(x) < \alpha(x)c(x)\}, \\
\Omega_{\rm do}^v &:= \{x \in \bar{\Omega} \; : \; \beta(x) > 0,\; \alpha(x)d(x) < \beta(x)b(x),\; \beta(x)a(x) > \alpha(x)c(x)\}, \\
\Omega_{\rm junk} &:= \bar{\Omega} \setminus (\Omega_{\rm ext} \cup \Omega_{\rm per} \cup \Omega_{\rm bi} \cup \Omega_{\rm do}^u \cup \Omega_{\rm do}^v)\, .
\end{aligned}$$

As suggested by their names:

- $\Omega_{\rm ext}$ consists of the set of $x \in \bar{\Omega}$ for which $(0, 0)$ is a global attractor for the positive solutions of (13).
- $\Omega_{\rm per}$ consists of the set of $x \in \bar{\Omega}$ for which both semi-trivial positive solutions, $\big(\frac{\alpha(x)}{a(x)}, 0\big)$ and $\big(0, \frac{\beta(x)}{d(x)}, 0\big)$, are linearly unstable. Thus, (13) is *permanent*.
- $\Omega_{\rm bi}$ consists of the set of $x \in \bar{\Omega}$ where $\big(\frac{\alpha(x)}{a(x)}, 0\big)$ and $\big(0, \frac{\beta(x)}{d(x)}\big)$ are linearly stable. Thus, there is *bi-stability* of the semi-trivial positive solutions, and (13) shows a genuine founder control competition.
- $\Omega_{\rm do}^u$ consists of the set of $x \in \bar{\Omega}$ where $\alpha(x) > 0$ and $\big(\frac{\alpha(x)}{a(x)}, 0\big)$ is linearly stable, while $\beta(x) = 0$ or $\beta(x) > 0$ and $\big(0, \frac{\beta(x)}{d(x)}\big)$ is linearly unstable.
- $\Omega_{\rm do}^v$ consists of the set of $x \in \bar{\Omega}$ where $\beta(x) > 0$ and $\big(0, \frac{\beta(x)}{d(x)}\big)$ is linearly stable, while $\alpha(x) = 0$ or $\alpha(x) > 0$ and $\big(\frac{\alpha(x)}{a(x)}, 0\big)$ is linearly unstable.

Finally, we are denoting by $\Omega_{\rm junk}$ the complement in $\bar{\Omega}$ of the union of the previous regions. It is well known that, for every $x \in \Omega_{\rm per}$, (13) has a unique coexistence state which is a global attractor for the component-wise positive solutions of (13), whereas, for every $x \in \Omega_{\rm bi}$, (13) has a unique unique coexistence state which is a saddle point, whose stable manifold, linking $(0, 0)$ to the coexistence state, divides the first quadrant, $u > 0$, $v > 0$, into two regions, each of them being the attraction source of one of the semi-trivial positive solutions.

For example, in the special case when $\mu = \nu$ and

$$a = b = c = d = 1, \qquad \mathcal{B}_1 = \mathcal{B}_2 = \tfrac{\partial}{\partial n} \;\text{ on }\; \partial\Omega, \tag{14}$$

the problem (10) reduces to (1). Suppose that $\alpha(x) > 0$ and $\beta(x) > 0$ for all $x \in \Omega$ and $\alpha - \beta$ changes sign in Ω. Then:

$$\begin{aligned}
\Omega_{\text{ext}} &:= \{x \in \bar{\Omega} \;:\; \alpha(x), \beta(x) \le 0\} = \varnothing, \\
\Omega_{\text{per}} &:= \{x \in \bar{\Omega} \;:\; \alpha(x), \beta(x) > 0,\ \alpha(x) > \beta(x),\ \beta(x) > \alpha(x)\} = \varnothing, \\
\Omega_{\text{bi}} &:= \{x \in \bar{\Omega} \;:\; \alpha(x), \beta(x) > 0,\ \alpha(x) < \beta(x),\ \beta(x) < \alpha(x)\} = \varnothing,
\end{aligned}$$

and

$$\Omega^u_{\text{do}} := \{x \in \bar{\Omega} \;:\; \alpha(x) > \beta(x)\}, \quad \Omega^v_{\text{do}} := \{x \in \bar{\Omega} \;:\; \beta(x) > \alpha(x)\}.$$

Thus,

$$\Omega_{\text{junk}} := \{x \in \bar{\Omega} \;:\; \alpha(x) = \beta(x)\}.$$

under these assumptions, Theorem 3.5 can be stated, equivalently, as follows.

THEOREM 5.1. *Suppose $\mu = \nu$ in (1), $\alpha(x) > 0$ and $\beta(x) > 0$ for all $x \in \Omega$, and $\alpha - \beta$ changes sign in Ω. Then, for every family of coexistence states of (1), (u_μ, v_μ), $\mu > 0$,*

$$\lim_{\mu \to 0} (u_\mu, v_\mu) = \begin{cases} \left(\frac{\alpha(x)}{a(x)}, 0\right) & \text{if } x \in \Omega^u_{\text{do}} \;\; (\alpha(x) > \beta(x)), \\ \left(0, \frac{\beta(x)}{d(x)}\right) & \text{if } x \in \Omega^v_{\text{do}} \;\; (\beta(x) > \alpha(x)). \end{cases}$$

More generally, under the general heterogeneous setting of this section, the following singular perturbation result of Fernández-Rincón and López-Gómez [4] holds.

THEOREM 5.2. *Let $(u_{\mu,\nu}, v_{\mu,\nu})$, $\mu, \nu \in (0, \varepsilon)$, be a family of coexistence states of (11) for some $\varepsilon > 0$. Then,*

$$\lim_{(\mu,\nu) \to (0,0)} (u_{\mu,\nu}, v_{\mu,\nu}) = \begin{cases} (0,0) & \text{if } x \in \Omega_{\text{ext}}, \\ \left(\frac{\alpha(x)d(x) - \beta(x)d(x)}{a(x)d(x) - b(x)c(x)}, \frac{\beta(x)a(x) - \alpha(x)c(x)}{a(x)d(x) - b(x)c(x)}\right) & \text{if } x \in \Omega_{\text{per}}, \\ \left(\frac{\alpha(x)}{a(x)}, 0\right) & \text{if } x \in \Omega^u_{\text{do}}, \\ \left(0, \frac{\beta(x)}{d(x)}\right) & \text{if } x \in \Omega^v_{\text{do}}, \end{cases}$$

uniformly on compact subsets of Ω.

The proof of Theorem 5.2 delivered in [4] also uses the monotone scheme of López-Gómez and Sabina [19] by adapting the proof of Theorem 3.5 in Hutson et al. [15]. Since, for every $x \in \Omega_{\rm per}$,

$$\left(\tfrac{\alpha(x)d(x)-\beta(x)d(x)}{a(x)d(x)-b(x)c(x)}, \tfrac{\beta(x)a(x)-\alpha(x)c(x)}{a(x)d(x)-b(x)c(x)}\right)$$

is the coexistence state of (13), Theorem 5.2 establishes that, on each of the patches of the partition of Ω induced by the dynamics of the pure kinetic model (13), the coexistence states of (11) approximate the corresponding global attractor of the kinetic model as μ and ν approximate zero.

Theorem 5.2 is a substantial generalization of Theorem 5.1 and of the singular perturbation result of Theorem 1.1 in Hutson, Lou and Mischaikow [16], which covers the very special case when $\Omega_{\rm par} = \bar{\Omega}$ under Neumann boundary conditions.

Theorem 5.2 is optimal in the sense that in the bi-stability region $\Omega_{\rm bi}$ the limiting behavior of the coexistence states $(u_{\mu,\nu}, v_{\mu,\nu})$ might not be uniquely determined, as it will become apparent in the next section.

6. The symmetric model

This section analyzes the following symmetric counterpart of (11)

$$\begin{cases} -\nu\Delta u = \lambda u - a(x)u^2 - b(x)uv & \text{in } \Omega, \\ -\nu\Delta v = \lambda v - a(x)v^2 - b(x)uv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $\lambda > 0$ is a constant and $a, b \in \mathcal{C}(\bar{\Omega})$ satisfy $a(x) > 0$ and $b(x) > 0$ for all $x \in \bar{\Omega}$, whose associated kinetic model is given by

$$\begin{cases} u'(t) = \lambda u(t) - a(x)u^2(t) - b(x)u(t)v(t), \\ v'(t) = \lambda v(t) - a(x)v^2(t) - b(x)u(t)v(t). \end{cases} \tag{16}$$

Naturally, (15) is of type (11) for the special choice $\mu = \nu$, $\alpha(x) = \beta(x) \equiv \lambda$, $a = d$, $b = c$, and $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{D}$, the Dirichlet operator on $\partial\Omega$. This model was introduced in Section 7 of [4] to give a multiplicity result when $\Omega_{\rm bi} \neq \varnothing$.

Since $\lambda > 0$, the semi-trivial positive solutions of (16) are $\left(\frac{\lambda}{a(x)}, 0\right)$ and $\left(0, \frac{\lambda}{a(x)}\right)$ for each $x \in \Omega$. Thus, since the nonlinearity of (16) is given by

$$f(u,v) := \begin{pmatrix} \lambda u - au^2 - buv \\ \lambda v - av^2 - buv \end{pmatrix}, \qquad (u,v) \in \mathbb{R}^2,$$

whose Jacobian matrix is

$$Df(u,v) = \begin{pmatrix} \lambda - 2au - bv & -bu \\ -bv & \lambda - 2av - bu \end{pmatrix},$$

the linearizations of f at the semi-trivial positive solutions are

$$Df(\tfrac{\lambda}{a},0)=\begin{pmatrix} -\lambda & -b\frac{\lambda}{a} \\ 0 & \lambda\left(1-\frac{b}{a}\right)\end{pmatrix}, \qquad Df(0,\tfrac{\lambda}{a})=\begin{pmatrix} \lambda\left(1-\frac{b}{a}\right) & 0 \\ -b\frac{\lambda}{a} & -\lambda\end{pmatrix}.$$

Thus, $\left(\frac{\lambda}{a},0\right)$ and $\left(0,\frac{\lambda}{a}\right)$ are linearly stable if, and only if, $1-\frac{b}{a}<0$. Hence,

$$\Omega_{\rm bi}=\{x\in\Omega\ :\ a(x)<b(x)\}.$$

Similarly, $\left(\frac{\lambda}{a},0\right)$ and $\left(0,\frac{\lambda}{a}\right)$ are linearly unstable if, and only if, $1-\frac{b}{a}>0$. So,

$$\Omega_{\rm per}=\{x\in\Omega\ :\ b(x)<a(x)\}.$$

In particular, $\Omega_{\rm per}\neq\varnothing$ and $\Omega_{\rm bi}\neq\varnothing$ if, and only if, $a(x)-b(x)$ changes sign in Ω. As in (15) the local character of each of the semi-trivial positive solutions is identical, it becomes apparent that

$$\Omega^u_{\rm do}=\Omega^v_{\rm do}=\varnothing.$$

The problem (15) has a coexistence state with $u=v=w$ if, and only if,

$$\begin{cases} -\nu\Delta w=\lambda w-(a(x)+b(x))w^2 & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega,\end{cases} \tag{17}$$

has a positive solution. It is folklore that (17) admits a positive solution, w, if, and only if, $\lambda>\nu\sigma_1$, where σ_1 stands for the principal eigenvalue of $-\Delta$ in Ω subject to Dirichlet boundary conditions on $\partial\Omega$. Thus, the positive solution exists if $\nu<\frac{\lambda}{\sigma_1}$. Let denote by w_ν the unique positive solution of (17) for these ν's. Then, according to Theorem 3.5 of Furter and López-Gómez [9],

$$\lim_{\nu\downarrow 0} w_\nu=\left(\tfrac{\lambda}{a+b}\right)_+=\tfrac{\lambda}{a+b}$$

uniformly in compact subsets of Ω. Therefore, $(u,v)=(w_\nu,w_\nu)$ provides us with a coexistence state of (15) such that

$$\lim_{\nu\downarrow 0}(w_\nu,w_\nu)=\left(\tfrac{\lambda}{a+b},\tfrac{\lambda}{a+b}\right) \tag{18}$$

uniformly on compact subsets of Ω. Actually, $\left(\frac{\lambda}{a+b},\frac{\lambda}{a+b}\right)$ is a coexistence state of (16) and

$$Df\left(\tfrac{\lambda}{a+b},\tfrac{\lambda}{a+b}\right)=-\tfrac{\lambda}{a+b}\begin{pmatrix} a & b\\ b & a\end{pmatrix},$$

whose spectrum consists of the eigenvalues

$$z_\pm=-\tfrac{\lambda}{a+b}(a\pm b).$$

Consequently, $\left(\frac{\lambda}{a+b},\frac{\lambda}{a+b}\right)$ is a saddle point in $\Omega_{\rm bi}$, and a stable node in $\Omega_{\rm per}$.

Subsequently, we will consider four different situations according to the relative sizes of $a(x)$ and $b(x)$.

6.1. Case 1: $a(x) - b(x)$ changes sign in Ω

In this case,

$$\Omega_{\mathrm{per}} \neq \varnothing \quad \text{and} \quad \Omega_{\mathrm{bi}} \neq \varnothing.$$

Thus, since $\left(\frac{\lambda}{a+b}, \frac{\lambda}{a+b}\right)$ is linearly unstable in Ω_{bi}, it follows from (18) and the *Principle of Parabolic Instability* of Fernández-Rincón and López-Gómez [4], that, for sufficiently small $\nu > 0$, the coexistence state (w_ν, w_ν) is linearly unstable with respect to the positive solutions of the associated parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u = \lambda u - a(x)u^2 - b(x)uv & \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} - \nu \Delta v = \lambda v - a(x)v^2 - b(x)uv & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 = v(x,t) & \text{for all } (x,t) \in \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 \gneq 0, \;\; v(\cdot,0) = v_0 \gneq 0, & \text{in } \Omega. \end{cases} \tag{19}$$

On the other hand, the semi-trivial positive solutions of (15) are given by $(\theta_\nu, 0)$ and $(0, \theta_\nu)$, where, for every $\lambda > \nu\sigma_1$, θ_ν stands for the unique positive solution of the boundary value problem

$$\begin{cases} -\nu \Delta \theta = \lambda \theta - a(x)\theta^2 & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

According to [9, Th. 3.5.],

$$\lim_{\nu \downarrow 0} \theta_\nu = \tfrac{\lambda}{a} \;\; \text{uniformly in compact subsets of } \Omega.$$

Moreover, $\left(\frac{\lambda}{a}, 0\right)$ and $\left(0, \frac{\lambda}{a}\right)$ provide us with the semi-trivial positive solutions of (16). As, for every $x \in \Omega_{\mathrm{per}}$, these solutions are linearly unstable, it follows from the Principle of Parabolic Instability of [4] that $(\theta_\nu, 0)$ and $(0, \theta_\nu)$ are linearly unstable for sufficiently small $\nu > 0$ as steady-state solutions of (19). Therefore, (19) is permanent and, according to Hess [14], (15) possess some stable coexistence state, (u_ν, v_ν). Since (w_ν, w_ν) is linearly unstable, necessarily $u_\nu \neq v_\nu$. By the symmetry of the problem, (v_ν, u_ν) provides us with a third coexistence state of (15). Consequently, for sufficiently small $\nu > 0$, (15) possesses, at least, three coexistence states if $a(x) - b(x)$ changes sign in Ω.

6.2. Case 2: $a = b$ in Ω

Suppose that $a = b$ in Ω. Equivalently,

$$\Omega_{\mathrm{per}} = \Omega_{\mathrm{bi}} = \varnothing.$$

Then, it is easily seen that, for every $s \in (0,1)$, the pair

$$(u, v) = (s\theta_\nu, (1-s)\theta_\nu) \tag{21}$$

provides us with a coexistence state of (15), where θ_ν is the unique positive solution of (20) for every $\nu < \frac{\lambda}{\sigma_1}$. Thus, there is a segment of coexistence states linking the two semi-trivial positive solutions $(\theta_\nu, 0)$ and $(0, \theta_\nu)$ as soon as $\lambda > \nu\sigma_1$. We claim that these are the unique coexistence states of (15) when $a = b$ in Ω. Indeed, since $u \gg 0$, $v \gg 0$, and

$$\begin{cases} (-\nu\Delta + au + av)u = \lambda u, \\ (-\nu\Delta + au + av)v = \lambda v, \end{cases}$$

it becomes apparent that

$$\lambda = \sigma_1^\Omega[-\nu\Delta + au + av, \mathcal{D}]$$

and, since the principal eigenvalue is algebraically simple, there exists $\gamma > 0$ such that

$$v = \gamma u.$$

Thus,

$$-\nu\Delta u = \lambda u - au^2 - auv = \lambda u - a(1+\gamma)u^2.$$

Consequently, by the uniqueness of the positive solution of (20), it is apparent that

$$u = \frac{1}{\gamma+1}\theta_\nu, \qquad v = \frac{\gamma}{\gamma+1}\theta_\nu.$$

So, setting $s = \frac{1}{\gamma+1} \in (0,1)$, we find that

$$(u, v) = (s\theta_\nu, (1-s)\theta_\nu).$$

Therefore, (21) provides us with the set of coexistence states of (15) when $a = b$.

6.3. Case 3: $b \lneq a$ (i.e., $b \leq a$ but $b \neq a$)

Then,

$$\Omega_{\rm bi} = \varnothing \quad \text{and} \quad \Omega_{\rm per} \neq \varnothing.$$

Note that $\Omega_{\rm per}$ might have an arbitrarily large number of components, with the Lebesgue measure $|\Omega_{\rm per}|$ arbitrarily small. In this case, the following result holds.

THEOREM 6.1. *Suppose that* $b \lneq a$. *Then, for every* $\nu < \frac{\lambda}{\sigma_1}$, $(u, v) = (w_\nu, w_\nu)$ *is the unique coexistence state of* (15), *where* w_ν *stands for the unique positive solution of* (17).

Proof. By the uniqueness of the positive solution of (17), it is apparent that (15) has a unique coexistence state (u,v) with $u=v$. Suppose that (15) admits a coexistence state (u,v) with $u\neq v$. Then, multiplying the u-equation by v, the v-equation by u, subtracting and rearranging terms, it is easily seen that

$$\nu(u\Delta v - v\Delta u) = (b-a)(u-v)uv.$$

Thus, integrating by parts in Ω yields

$$\int_\Omega (b-a)(u-v)uv = \nu\int_\Omega (u\Delta v - v\Delta u) = \nu\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) = 0.$$

Hence, since $u \gg 0$ and $v \gg 0$, it becomes apparent that $u-v$ changes sign in the region where $b<a$. On the other hand,

$$-\nu\Delta u - \lambda u + au^2 = -buv = -\nu\Delta v - \lambda v + av^2.$$

Thus,

$$[-\nu\Delta + a(u+v)](u-v) = \lambda(u-v).$$

Hence, since $u-v\neq 0$ changes of sign in Ω, λ must be an eigenvalue of the differential operator $-\nu\Delta + a(u+v)$ in Ω subject to Dirichlet boundary conditions on $\partial\Omega$, with associated eigenfunction $u-v$. Therefore, since $u-v$ changes sign in Ω,

$$\lambda \geq \sigma_2^\Omega[-\nu\Delta + a(u+v),\mathcal{D}] > \sigma_1^\Omega[-\nu\Delta + a(u+v),\mathcal{D}],$$

where σ_2^Ω stands for second eigenvalue of $-\nu\Delta + a(u+v)$ in Ω under Dirichlet boundary conditions. Since $b \lneq a$ and $v \gg 0$, it follows from the previous estimate that

$$\lambda > \sigma_1^\Omega[-\nu\Delta + au + bv,\mathcal{D}]. \tag{22}$$

However, as

$$(-\nu\Delta + au + bv)u = \lambda u,$$

by the uniqueness of the principal eigenvalue, we also find that

$$\lambda = \sigma_1^\Omega[-\nu\Delta + au + bv,\mathcal{D}],$$

which contradicts (22). Therefore, (15) cannot admit a coexistence state (u,v) with $u\neq v$. □

6.4. Case 4: $a \lneq b$ in Ω

Suppose that $a\leq b$ but $a\neq b$. Then,

$$\Omega_{\rm bi}\neq\varnothing \quad \text{and} \quad \Omega_{\rm per}=\varnothing.$$

In this case, the following uniqueness result, of a local nature, holds.

THEOREM 6.2. *Suppose that* $a \lneq b$. *Then, for every* $\nu < \frac{\lambda}{\sigma_1}$, *there exists* $\eta \equiv \eta(\nu) > 0$ *such that* $(u, v) = (w_\nu, w_\nu)$ *is the unique coexistence state of* (15) *if* $b - a \leq \eta$.

Proof. The proof will proceed by contradiction. Fixed $a(x)$, assume that there exists a sequence, $b_n(x)$, $n \geq 1$, of b's such that $a \lneq b_n$ for all $n \geq 1$,

$$\lim_{n\to\infty} \max_{\bar{\Omega}} (b_n - a) = 0,$$

and, for every $n \geq 1$, the system

$$\begin{cases} -\nu\Delta u = \lambda u - a(x)u^2 - b_n(x)uv & \text{in } \Omega, \\ -\nu\Delta v = \lambda v - a(x)v^2 - b_n(x)uv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

has a coexistence state (u_n, v_n) with $u_n \neq v_n$. Then, reasoning as in the proof of Theorem 6.1, it is apparent that $u_n - v_n$ changes sign in Ω for all $n \geq 1$. Moreover, for every $n \geq 1$,

$$[-\nu\Delta + a(u_n + v_n)](u_n - v_n) = \lambda(u_n - v_n)$$

and hence, since $u_n - v_n$ changes sign in Ω, we find that

$$\lambda \geq \sigma_2^\Omega[-\nu\Delta + a(u_n + v_n), \mathcal{D}] \quad \text{for all } n \geq 1. \tag{24}$$

Also, by the uniqueness of the principal eigenvalue, it follows from (23) that

$$\lambda = \sigma_1^\Omega[-\nu\Delta + au_n + b_n v_n, \mathcal{D}] \quad \text{for all } n \geq 1. \tag{25}$$

On the other hand, setting $a_L := \min_{\bar{\Omega}} a > 0$, it follows from (23) that

$$\max\{u_n, v_n\} \leq \tfrac{\lambda}{a_L} \quad \text{for all } n \geq 1.$$

Thus, by a rather standard compactness argument, there is a subsequence of (u_n, v_n), relabeled by $n \geq 1$, such that

$$\lim_{n\to\infty} (u_n, v_n) = (u, v) \geq (0, 0) \quad \text{in } \mathcal{C}^1(\bar{\Omega})$$

for some non-negative solution (u, v) of

$$\begin{cases} -\nu\Delta u = \lambda u - a(x)u^2 - a(x)uv & \text{in } \Omega, \\ -\nu\Delta v = \lambda v - a(x)v^2 - a(x)uv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Letting $n \to \infty$ in (24), from the continuous dependence of the spectrum with respect to the potentials, we find that

$$\lambda \geq \sigma_2^\Omega[-\nu\Delta + au + av, \mathcal{D}] > \sigma_1^\Omega[-\nu\Delta + au + av, \mathcal{D}].$$

Similarly, letting $n \to \infty$ in (25), it becomes apparent that

$$\lambda = \sigma_1^\Omega[-\nu\Delta + au + av, \mathcal{D}],$$

which is impossible. This contradiction ends the proof. □

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