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A note on the inverse maximum principle on Carnot groups

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Dedicated to Enzo with esteem and friendship

Abstract. *Let $\Delta_{\mathbb{G}}$ be a sublaplacian on a Carnot group, and let μ be a local measure on the open set $\Omega \subset \mathbb{G}$. If $u \in L^1_{loc}(\Omega)$ is such that*

$$-\Delta_{\mathbb{G}} u = \mu, \ u \geq 0 \quad \text{on } \Omega,$$

then $\mu_c \geq 0$, where μ_c is the concentrated component of μ with respect to the $\mathbb{G}$-capacity. This extends to the Carnot group setting a result contained in [9].

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1. Introduction

In [9] Dupaigne and Ponce prove the following *inverse maximum principle*.

Theorem 1.1 ([9]). *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, μ be a Radon measure and let $u \in L^1_{loc}(\Omega)$ be such that*

$$-\Delta u = \mu, \ u \geq 0 \quad \text{on } \Omega.$$

Then $\mu_c \geq 0$, where μ_c is the concentrated component of μ with respect to the capacity.

This interesting result is used in [9] to study the singularities of positive supersolutions in elliptic PDEs, and in [3] Brezis and Ponce exploits it to extend the Kato's inequality when Δu is a measure. For further applications of this result see also [2, 19, 20, 22, 23].

In this note, we extend the above inverse maximum principle to the Carnot group setting. That is, denoting by $\Delta_{\mathbb{G}}$ a sublaplacian (see Section 2), we have the following result.

Theorem 1.2 (Inverse maximum principle). *Let $\Omega \subset \mathbb{G}$ be open, μ be a local Radon measure and let $u \in L^1_{loc}(\Omega)$ be such that*

$$-\Delta_{\mathbb{G}} u = \mu, \ u \geq 0 \quad on\ \Omega.$$

Then $\mu_c \geq 0$, where μ_c is the concentrated component of μ with respect to the $\mathbb{G}$-capacity.

Besides the Carnot group framework, our results improve the ones in [9] slightly also in the Euclidean setting, since we deal with distributions μ which are locally Radon measures (see Definition 2.2), but may fail to be global Radon measures. Moreover, we deal also with the case of unbounded sets Ω.

Notice that in Theorem 1.2 the decomposition of the measure $\mu = -\Delta_{\mathbb{G}} u$ with respect to the $\mathbb{G}$-capacity is in some sense natural, since we decompose with respect to an object linked to the operator involved. Another classical decomposition of the measure can be done with respect to the Lebesgue measure λ: $\mu = \mu_s + \mu_a$, where $\mu_a \ll \lambda$ and μ_s singular. In this case the inverse maximum principle fails. Indeed, in dimension $N = 1$ the function $u(x) = |x|$ is nonnegative and solves

$$-u'' = \mu, \ u \geq 0 \quad \text{on } \mathbb{R},$$

where $\mu = -2\delta_0$. Clearly $\mu_s = \mu < 0$.

As a simple application of the main theorem, we show that there exists no solution of

$$-\Delta_{\mathbb{G}} u \geq \delta_{x_0} \quad \text{on } \Omega \tag{1}$$

which is unbounded from above.

Corollary 1.3 (Liouville-type result). *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$. Let $\Omega \subset \mathbb{G}$ be open, $x_0 \in \Omega$, and let $u \in L^1_{loc}(\Omega)$ be a solution of* (1)*. Then u^+ is unbounded in every neighborhood of x_0.*

The paper is organized as follows: in Section 2 we recall some notions about Carnot groups, and make clear some facts about capacities in this setting; moreover we deal with decompositions of local measures, which have an interest also in the Euclidean setting. In Section 3 we focus on the proof of the main theorem, by studying preliminarily some $W^{1,2}_{\mathbb{G}}$-regularity for distributional equations and the behaviour of such equations under truncation operators.

2. Preliminaires on Carnot groups

In this section we recall some facts concerning Carnot groups; for more information and proofs we refer the interested reader to [1, 10].

A *Carnot group* of dimension $N \geq 2$ is a connected, simply connected, nilpotent Lie group $\mathbb{G}$, with $\dim(\mathbb{G}) = N$ and with graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ such that $[V_1, V_i] = V_{i+1}$ for $i = 1 \ldots r-1$ and $[V_1, V_r] = 0$. A Carnot group $\mathbb{G}$ of dimension N can be identified, up to an isomorphism, with the structure of *homogeneous Carnot group* $(\mathbb{R}^N, \circ, \delta_\lambda)$ defined as follows. We identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with a smooth Lie group law $\circ$, and we consider $\mathbb{R}^N$ split in r subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$; we will write $\xi = (\xi^{(1)}, \ldots, \xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$ for each $\xi \in \mathbb{G}$. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N, \circ)$ is $\mathcal{G}$. For $i = 1, \ldots, n_1$ let Y_i be the unique vector field in $\mathcal{G}$ that coincides with $\partial/\partial\xi_i^{(1)}$ at the origin: we require that the Lie algebra generated by $Y_1, \ldots, Y_{n_1}$ is the whole $\mathcal{G}$. We shall moreover assume that there exists a family of Lie group automorphisms $(\delta_\lambda)_{\lambda>0}$, called *dilations*, of the form

$$\delta_\lambda(\xi) = (\lambda\xi^{(1)}, \lambda^2\xi^{(2)}, \ldots, \lambda^r\xi^{(r)}).$$

With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a homogeneous Carnot group; due to the identification, we will refer to this simply as Carnot group.

We denote by $Q := \sum_{i=1}^r i n_i$ the *homogeneous dimension* of $\mathbb{G}$. We recall that the Lebesgue measure is the bi-invariant Haar measure related to $\mathbb{G}$, and that for any measurable set $E \subset \mathbb{R}^N$, we have $|\delta_\lambda(E)| = \lambda^Q |E|$.

Write $l := n_1$. The *canonical sub-laplacian* on $\mathbb{G}$ is the second order differential operator $\mathcal{L} := \sum_{i=1}^l Y_i^2$. Let $X_1, \ldots, X_l$ be a basis of *span*$\{Y_1, \ldots, Y_l\}$, the second order differential operator

$$\Delta_{\mathbb{G}} := \sum_{i=1}^{l} X_i^2 \tag{2}$$

is called a *sub-laplacian* on $\mathbb{G}$. In what follows we fix the vector fields $X_1, \ldots, X_l$, a whatever basis of *span*$\{Y_1, \ldots, Y_l\}$, and hence we drop the dependence on the choice of the basis. Since $X_1, \ldots, X_l$ generate the whole $\mathbb{G}$, any sub-Laplacian $\Delta_{\mathbb{G}}$ satisfies the Hörmander's hypoellipticity condition. Moreover, the vector fields $X_1, \ldots, X_l$ are homogeneous of degree 1 with respect to δ_λ, thus $\Delta_{\mathbb{G}}$ is homogeneous of degree 2, and in addition $(\Delta_{\mathbb{G}}\phi)(y \circ z) = \Delta_{\mathbb{G}}\left(\phi(y \circ \cdot)\right)(z)$ for each $\phi \in C^2(\mathbb{G})$ and $y, z \in \mathbb{G}$.

In this setting we also use the symbol X_i^* for the formal adjoint of X_i and introduce the $\mathbb{G}$-gradient and the $\mathbb{G}$-divergence as

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_l)^T, \quad \mathrm{div}_{\mathbb{G}} := -\nabla_{\mathbb{G}}^*,$$

where $\nabla_{\mathbb{G}}^*$ is the formal adjoint of $\nabla_{\mathbb{G}}$.

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm* on $\mathbb{G}$, if $S(\xi) = 0 \iff \xi = 0$ and it is homogeneous of degree 1 with respect to δ_λ (i.e. $S(\delta_\lambda(\xi)) = \lambda S(\xi)$). We say that a homogeneous norm is *symmetric* if $S(\xi^{-1}) = S(\xi)$. In what follows we fix a homogeneous norm S which is symmetric and C^∞ away from 0.

Given $x, y \in \mathbb{G}$ we define the *S-homogeneous distance*[1] as

$$d_S(x,y) := S(x \circ y^{-1}).$$

Notice that d_S is right invariant, that is, for any $x, y, z \in \mathbb{G}$, $d_S(x \circ z, y \circ z) = d_S(x,y)$. For every $x \in \mathbb{G}$ and every $r > 0$, the set

$$B_S(x,r) := \big\{ y \in \mathbb{G},\ d_S(x,y) < r \big\}$$

is called the *S-ball* with *center at* x and *radius* r; notice that, by the symmetry of S, we have $y \in B_S(0,r) \iff y^{-1} \in B_S(0,r)$. In order to avoid cumbersome notations we shall omit the norm S in the above symbols. Finally for any $\Omega \subset \mathbb{G}$ and $\varepsilon > 0$ we define

$$\Omega_\varepsilon := \big\{ x \in \Omega,\ B(x,\varepsilon) \subset \Omega \big\}. \tag{3}$$

Example 2.1. Apart from the Euclidean case $\mathbb{R}^N$, a classical example of Carnot group is given by the *Heisenberg group* $\mathbb{H}^n := \mathbb{R}^{2n} \times \mathbb{R}$, with law $(x,y,t) \circ (x',y',t') := (x+x', y+y', t+t'+2(xy'+yx'))$, dilation $\delta_\lambda(x,y,t) := (\lambda x, \lambda y, \lambda^2 t)$ and homogeneous dimension $Q = 2n+2$. The *Kohn Laplacian* is given by $\Delta_{\mathbb{H}^n} := \sum_{i=1}^n (X_i^2 + Y_i^2)$, where $X_i := \partial_{x_i} + 2y_i\partial_t$ and $Y_i := \partial_{y_i} - 2x_i\partial_t$. Here a possible norm is given by $S(x,y,t) := \big(|x|^4 + |y|^4 + |t|^2\big)^{\frac{1}{4}}$.

2.1. Radon measures

In what follows we denote by $\mathcal{D}(\Omega) \equiv C_0^\infty(\Omega)$ the set of test functions on an open set $\Omega \subset \mathbb{G}$, and by $\mathcal{D}'(\Omega)$ the distributions on Ω.

Definition 2.2. *Let $\Omega \subseteq \mathbb{G}$ be an open set and let $\mathfrak{B}(\Omega)$ be the σ-algebra of Borelian sets on Ω. We say that $\mu : \mathfrak{B}(\Omega) \to [-\infty, +\infty]$ is a* Radon measure *if it is countably additive, $\mu(\emptyset) = 0$, μ takes at most one of the values $\pm\infty$, and it is locally finite, i.e. for each $K \subset \Omega$ compact we have that $|\mu(K)| < \infty$. We will denote the set of Radon measures by $M(\Omega)$, and observe that $M(\Omega) \subset \mathcal{D}'(\Omega)$.*

We say that $\mu \in \mathcal{D}'(\Omega)$ is a local Radon measures *– and we write $\mu \in M_{loc}(\Omega)$ – if for every $\Omega' \Subset \Omega$ we have $\mu_{|\Omega'} \in M(\Omega')$. Notice that $M_{loc}(\Omega)$ is the dual space of $C_0(\Omega)$.*

An example of $\mu \in M_{loc}(\Omega) \setminus M(\Omega)$ is, for instance, $\mu = \sum_n (-1)^n n \delta_{x_n}$ with $x_n \in \Omega$, $x_n \to \partial\Omega$.

[1] We mention that this may not be a distance in the usual sense, see [1, Section 5.1].

By the Jordan decomposition theorem [6, Corollary 4.1.6] for any $\mu \in M(\Omega)$ there exist two nonnegative measures $\mu^+, \mu^- \in M(\Omega)$ such that at least one of the two is finite and

$$\mu = \mu^+ - \mu^-.$$

The two measures are mutually singular (i.e. μ^+ and μ^- concentrated on two disjoint sets) and the decomposition is unique. Moreover we recall the *variation of* μ

$$|\mu| := \mu^+ + \mu^-.$$

We notice that the decomposition can be extended also to a $\mu \in M_{loc}(\Omega)$, by defining $\mu^\pm(\psi) := (\mu^\pm)_{|\Omega'}(\psi_{|\Omega'})$ for any $\psi \in \mathcal{D}(\Omega)$ and whatever $\Omega' \Subset \Omega$ such that $\psi_{|\Omega'} \in C_0^\infty(\Omega')$, thanks to [17, Theorem 6.22]. This moreover allows to define $|\mu| \in M(\Omega)$ (see also [5, Definition 2.18] for an equivalent definition).

Remark 2.3. In our setting every Radon measure is *σ-finite*, while every nonnegative Radon measure is also *(inner) regular*. In particular, for any $\mu \in M(\Omega)$, from the regularity of $\mu^\pm$ we obtain

$$\mu(K) \leq 0 \text{ for each compact set } K \subseteq E \implies \mu(E) \leq 0 \tag{4}$$

for any $E \subset \Omega$ Borel set. Indeed, if for each $K \subseteq E$ compact we have $\mu(K) \leq 0$, i.e. $\mu^+(K) \leq \mu^-(K)$, then by inner regularity we obtain $\mu^+(E) \leq \mu^-(E)$, that is (since one among μ^+ and μ^- is finite) $\mu(E) \leq 0$.

2.2. Mollifiers

On a Carnot group there is a good notion of *mollifier*. Let $m \in C_0^\infty(\mathbb{G})$, $m \geq 0$ be given such that

$$\operatorname{supp}(m) \subset B(0,1) \text{ and } \int_{\mathbb{G}} m \, dy = 1.$$

For any $\eta > 0$, we set

$$m_\eta(x) := \eta^{-Q} m(\delta_{1/\eta}(x)) \quad \text{for } x \in \mathbb{G};$$

notice that $\operatorname{supp}(m_\eta) \subset B(0,\eta)$. The family $(m_\eta)_{\eta>0}$ will be called a *family of mollifiers*. From now on, we fix a family of mollifiers which is *symmetric* (i.e. $m_\eta(x^{-1}) = m_\eta(x)$).

Let $\Omega \subset \mathbb{G}$ be an open set and let $\mu \in M_{loc}(\Omega)$. We define the *convolution* of μ and m_η for $x \in \Omega_\eta$ as

$$\mu_\eta(x) := (\mu \star_{\mathbb{G}} m_\eta)(x) := \int_\Omega m_\eta(x \circ y^{-1}) d\mu(y) = \int_{B(x,\eta)} m_\eta(x \circ y^{-1}) d\mu(y),$$

and call μ_η a *mollification of* μ; notice that we exploited that $\mathrm{supp}(m_\eta(x \circ \cdot^{-1})) \subset B(x,\eta) \subset \Omega$ and the definition of Ω_η by (3). Observe moreover that $\mu_\eta \in C^\infty(\Omega_\eta)$ and if $\mu \geq 0$, then $\mu_\eta \geq 0$ for each $\eta > 0$.

In particular, if $u \in L^1_{loc}(\Omega)$, the mollified function of u is given by

$$u_\eta(x) = \int_{B(x,\eta)} u(y) m_\eta(x \circ y^{-1}) dy = \int_{B(0,\eta)} u(z^{-1} \circ x) m_\eta(z) dz \quad \text{for } x \in \Omega_\eta.$$

Notice that $\|u_\eta\|_\infty \leq \|u\|_\infty$. It is easy that check that if $u \in L^1_{loc}(\Omega)$, then (see [1, Remark 5.3.8] for details) $u_\eta \to u$ as $\eta \to 0$ in $L^1_{loc}(\Omega)$ in the sense that, if $\Omega' \Subset \Omega$, then for η small we have $\Omega' \subset \Omega_\eta$ and $u_\eta \to u$ in $L^1(\Omega')$; if $u \in C_0(\Omega)$, then the convergence is also uniform.

Now, let $\mu \in M_{loc}(\Omega)$. Then we have that $\mu_\eta \to \mu$ as distributions, that is for any $\psi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega \psi \mu_\eta \to \int_\Omega \psi d\mu \quad \text{as } \eta \to 0;$$

here we mean that, if $\eta \ll 1$ is such that $\mathrm{supp}(\psi) \subset \Omega_\eta$, then we can extend $\psi\mu_\eta$ to the whole Ω and give sense to the integrals. We justify the limit: by using Fubini-Tonelli theorem and by symmetry of the functions m_η we get

$$\begin{aligned}
&\int_\Omega \psi(x)\mu_\eta(x)dx = \\
&\quad = \int_\Omega \psi(x) \left(\int_\Omega m_\eta(x \circ y^{-1}) d\mu(y) \right) dx = \int_\Omega \left(\int_\Omega \psi(x) m_\eta(x \circ y^{-1}) dx \right) d\mu(y) \\
&\quad = \int_\Omega \left(\int_\Omega \psi(x) m_\eta(y \circ x^{-1}) dx \right) d\mu(y) = \int_\Omega \psi_\eta(y) d\mu(y) \to \int_\Omega \psi(y) d\mu(y).
\end{aligned}$$

LEMMA 2.4. *Let $\mu \in M_{loc}(\Omega)$, and $u \in L^1_{loc}(\Omega)$ be such that $\Delta_{\mathbb{G}} u = \mu$ in $\mathcal{D}'(\Omega)$. We have $\Delta_{\mathbb{G}} u_\eta = \mu_\eta$ on Ω_η.*

Proof. Indeed, fix $\psi \in \mathcal{D}(\Omega_\eta)$. By using Fubini-Tonelli theorem we obtain

$$\begin{aligned}
\int_{\Omega_\eta} u_\eta(x) \Delta_{\mathbb{G}} \psi(x) dx &= \int_{\Omega_\eta} \Delta_{\mathbb{G}} \psi(x) \left(\int_{B(0,\eta)} u(y^{-1} \circ x) m_\eta(y) dy \right) dx = \\
&= \int_{B(0,\eta)} m_\eta(y) \left(\int_{\Omega_\eta} u(y^{-1} \circ x) \Delta_{\mathbb{G}} \psi(x) dx \right) dy \\
&= \int_{B(0,\eta)} m_\eta(y) \left(\int_{y^{-1} \circ \Omega_\eta} u(z) (\Delta_{\mathbb{G}} \psi)(y \circ z) dz \right) dy \\
&= \int_{B(0,\eta)} m_\eta(y) \left(\int_{y^{-1} \circ \Omega_\eta} u(z) \Delta_{\mathbb{G}} \left(\psi(y \circ \cdot) \right)(z) dz \right) dy.
\end{aligned}$$

Now, we observe that $y^{-1} \circ \Omega_\eta \subset \Omega$ for any $y \in B(0,\eta)$. Indeed, let $a \in \Omega_\eta$ that is $B_\eta(a) \subset \Omega$. Since $d(y^{-1} \circ a, a) = d(y^{-1}, 0) = S(y) < \eta$, we have that $y^{-1} \circ a \in B_\eta(a) \subset \Omega$. Further, we notice that $\psi(y \circ \cdot) \in \mathcal{D}(y^{-1} \circ \Omega_\eta)$ and we extend ψ to zero elsewhere in $\mathbb{G}$. Thus, since $\Delta_{\mathbb{G}} u = \mu$ in distributional sense, we have

$$\begin{aligned}
&\int_{B(0,\eta)} m_\eta(y) \left(\int_{y^{-1}\circ\Omega_\eta} u(z) \Delta_{\mathbb{G}} \left(\psi(y \circ \cdot) \right)(z) dz \right) dy = \\
&\quad = \int_{B(0,\eta)} m_\eta(y) \left(\int_{y^{-1}\circ\Omega_\eta} \psi(y \circ z) d\mu(z) \right) dy \\
&\quad = \int_{B(0,\eta)} m_\eta(y) \left(\int_{\Omega} \psi(y \circ z) d\mu(z) \right) dy \\
&\quad = \int_{\mathbb{G}} m_\eta(y) \left(\int_{\Omega} \psi(y \circ z) d\mu(z) \right) dy = \int_{\Omega} \left(\int_{\mathbb{G}} m_\eta(y) \psi(y \circ z) dy \right) d\mu(z) \\
&\quad = \int_{\Omega} \left(\int_{\mathbb{G}} m_\eta(x \circ z^{-1}) \psi(x) dx \right) d\mu(z) = \int_{\Omega} \left(\int_{\Omega_\eta} m_\eta(x \circ z^{-1}) \psi(x) dx \right) d\mu(z) \\
&\quad = \int_{\Omega_\eta} \left(\int_{\Omega} m_\eta(x \circ z^{-1}) d\mu(z) \right) \psi(x) dx = \int_{\Omega_\eta} \mu_\eta(x) \psi(x) dx,
\end{aligned}$$

that is the claim. □

2.3. Sobolev spaces

For a function $u \in L^1_{loc}(\Omega)$ we define as usual the vector field in the distributional sense, that is if $\nabla_{\mathbb{G}} u \in L^1_{loc}(\Omega, \mathbb{R}^l)$ and for each test function $\psi \in \mathcal{D}(\Omega)$ it holds that

$$\int_\Omega \psi \nabla_{\mathbb{G}} u \, dx = \int_\Omega u \nabla^*_{\mathbb{G}} \psi \, dx.$$

Moreover, let $1 \le p < \infty$. The Sobolev space

$$W^{1,p}_{\mathbb{G}}(\Omega) := \left\{ u \in L^p(\Omega) \mid |\nabla_{\mathbb{G}} u| \in L^p(\Omega) \right\}$$

endowed with

$$\|u\|_{W^{1,p}_{\mathbb{G}}(\Omega)} := \left(\int_\Omega \left(|u|^p + |\nabla_{\mathbb{G}} u|^p \right) dx \right)^{1/p}$$

is a Banach space [13]; if clear from the context, we will drop the dependence of the norm from Ω. We further define

$$W^{1,p}_{\mathbb{G},loc}(\Omega) := \left\{ u \in L^1_{loc}(\Omega) \mid u \in W^{1,p}_{\mathbb{G}}(\Omega') \text{ for each } \Omega' \Subset \Omega \right\}.$$

The following Meyers-Serrin type result holds, see [13, Theorem A.2] (see also [11, Theorem 1.2.3], [14, Theorem 1.7]).

THEOREM 2.5 (Meyers-Serrin type [11, 13, 14]). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \le p < \infty$, then the space $C^\infty(\Omega) \cap W^{1,p}_{\mathbb{G}}(\Omega)$ is dense in $W^{1,p}_{\mathbb{G}}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}_{\mathbb{G}}}$, i.e.*

$$W^{1,p}_{\mathbb{G}}(\Omega) = \overline{C^\infty(\Omega) \cap W^{1,p}_{\mathbb{G}}(\Omega)}^{\|\cdot\|_{W^{1,p}_{\mathbb{G}}}}. \tag{5}$$

We further recall that $\nabla_{\mathbb{G}}$ satisfies the usual Leibniz and Chain rules, and in particular [13, Lemma 3.5]

$$\nabla_{\mathbb{G}}|u| = \mathrm{sign}(u)\nabla_{\mathbb{G}}u, \quad \nabla_{\mathbb{G}}u^+ = \mathrm{sign}^+(u)\nabla_{\mathbb{G}}u.$$

Finally, we recall the following compact embedding result, see [13, Theorem 1.28 and pages 1085 and 1093].

THEOREM 2.6 (Sobolev embedding [13]). *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q, and let $\Omega \subset \mathbb{G}$ be an open set with $C^{1,1}$ boundary. Then the embedding $W^{1,p}_{\mathbb{G}}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for any $p \in [1,\infty)$ and $q \in [1, \frac{Qp}{Q-p})$. In particular, $W^{1,p}_{\mathbb{G}}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$.*

2.4. Capacity

In what follows we introduce and discuss the *Sobolev $\mathbb{G}$-capacity* of a set $E \subset \mathbb{G}$, defined as

$$\mathrm{Cap}_{\mathbb{G}}(E) := \inf_{\varphi \in S(E)} \|\varphi\|^2_{W^{1,2}_{\mathbb{G}}(\mathbb{R}^N)} \tag{6}$$

where $S(E)$ is given by

$$S(E) := \Big\{\varphi \in W^{1,2}_{\mathbb{G}}(\mathbb{R}^N),\ \varphi \ge 1 \text{ a.e. in some neighborhood of } E\Big\}.$$

The definition (6) is the Carnot group counterpart of the definition given in [15, Section 2.35] (see also [21, Appendix A], [18, page 5], [25] and [16] for an alternative definition).

We start with some useful observations. In the above definition of Sobolev $\mathbb{G}$-capacity, we can replace the set $S(E)$ with one of the following sets.

1. By considering $\varphi \mapsto |\varphi|$ and $\varphi \mapsto \min\{1, \varphi\}$ is it possible to restrict $S(E)$ to

$$S'(E) := \Big\{\varphi \in W^{1,2}_{\mathbb{G}}(\mathbb{R}^N), 0 \le \varphi \le 1,$$
$$\varphi = 1 \text{ a.e. in some open neighborhood of } E\Big\}.$$

2. Exploiting the Meyers-Serrin result (Theorem 2.5), Leibniz rule and arguing as in the Euclidean case [15, Lemma 2.36], when $E = K$ is compact we can restrict to regular functions

$$S''(K) := \Big\{\varphi \in C_0^\infty(\mathbb{R}^N),\, 0 \le \varphi \le 1,\\ \varphi = 1 \text{ in some open neighborhood of } K\Big\}.$$

For the sake of completeness, we mention also the following cases whose details are left to the interested readers.

3. When $E = K$ is compact, we can enlarge $S''(K)$ to functions who focus on K, i.e.

$$S'''(K) := \left\{\varphi \in C_0^\infty(\mathbb{R}^N),\ \varphi = 1 \text{ in } K\right\}.$$

Indeed if $(\varphi_n) \subset S'''(K)$ is a minimizing sequence for (6) in $S'''(K)$, then $(1 + \frac{1}{n})\varphi_n \in S(K)$, is still a minimizing sequence for (6).

4. If we consider $E \subset \Omega \subset \mathbb{G}$, Ω open, under suitable assumptions on $\mathbb{G}$ and Ω which allow the use of an extension operator (e.g. $\partial\Omega \in C^2$ and $\mathbb{G}$ is a two-steps group [24], see also [14]), we can minimize (6) on

$$S''''(E) := \Big\{\varphi \in W^{1,2}_{\mathbb{G}}(\Omega),\ \varphi \ge 1 \text{ a.e. in some neighborhood of } E\Big\},$$

where now we consider the infimum of the $W^{1,2}_{\mathbb{G}}(\Omega)$-norm, and obtain a notion of capacity which is equivalent to $\mathrm{Cap}_{\mathbb{G}}(E)$, in the sense that the sets of zero capacity coincide.

The following result can be proved following the Euclidean case [15, Chapter 2], by making use of the Chain Rule (see in particular [15, Theorem 2.37]). We recall that an *outer measure* is a positive set-function which is 0 on the empty set, increasing monotone with respect of the set inclusion and countable subadditive.

THEOREM 2.7. *The Sobolev $\mathbb{G}$-capacity is an outer measure.*

We finally notice that even if the value of the capacity of a set depends on the choice of the vector fields $X_1, \ldots, X_l$, the fact that a set has a zero capacity is independent of the choice of such a basis of $span\{Y_1, \ldots, Y_l\}$.

2.5. Decomposition

In what follows we describe the decomposition of a measure with respect to the $\mathbb{G}$-capacity.

Definition 2.8. *Let $\mu \in M_{loc}(\Omega)$. We say that μ is* $\mathbb{G}$-diffuse *if, for each $\Omega' \Subset \Omega$, $\mu_{|\Omega'}$ is diffuse with respect to the Sobolev capacity* $\mathrm{Cap}_{\mathbb{G}}$*, i.e. if for each measurable $A \subseteq \Omega'$ we have*

$$\mathrm{Cap}_{\mathbb{G}}(A) = 0 \implies \mu(A) = 0.$$

We say that μ is $\mathbb{G}$-concentrated *if, for each $\Omega' \Subset \Omega$, $\mu_{|\Omega'}$ is concentrated with respect to* $\mathrm{Cap}_{\mathbb{G}}$*, i.e. if there exists a measurable $N \subseteq \Omega'$ with $\mathrm{Cap}_{\mathbb{G}}(N) = 0$ such that for each measurable $A \subseteq \Omega'$ we have*

$$\mu(A) = \mu(A \cap N).$$

Remark 2.9. We observe that, if $\mu \in M(\Omega)$, then the previous definitions collapse to the standard ones. Indeed, if μ is diffuse and $A \subset \Omega$ is measurable with $\mathrm{Cap}_{\mathbb{G}}(A) = 0$, then by writing $A = \bigcup_k (A \cap \Omega_k)$ with $\Omega_k \Subset \Omega$ disjoint and invading Ω, we obtain $\mathrm{Cap}_{\mathbb{G}}(A \cap \Omega_k) = 0$ and thus $\mu(A) = \sum_k \mu(A \cap \Omega_k) = 0$. Moreover, if μ is concentrated, then for each k there exists N_k with $\mathrm{Cap}_{\mathbb{G}}(N_k) = 0$ where $\mu_{|\Omega_k}$ concentrate, thus set $N := \bigcup_k N_k$ we have $\mathrm{Cap}_{\mathbb{G}}(N) = 0$ and, for each $A \subset \Omega$ measurable, $\mu(A) = \sum_k \mu(A \cap \Omega_k) = \sum_k \mu(A \cap N_k) = \mu(A \cap N)$.

The above definitions naturally extends to a generic outer measure Φ. Now we state the decomposition of measures with respect to outer measures, so in particular with respect to the capacity.

Theorem 2.10. *Let $(X, \mathfrak{M})$ be a measurable space and let Φ be an outer measure on the σ-algebra $\mathfrak{M}$. Then for each σ-finite Radon measure μ on $(X, \mathfrak{M})$ there exists a unique pair (μ_0, μ_1) of measures on $(X, \mathfrak{M})$ such that*

1. *$\mu = \mu_0 + \mu_1$;*
2. *μ_0 is diffuse with respect to Φ, i.e. $\mu_0(A) = 0$ if $A \in \mathfrak{M}$ is such that $\Phi(A) = 0$.*
3. *μ_1 is concentrated on a measurable set N such stat $\Phi(N) = 0$, i.e. $\mu_1(A) = \mu_1(A \cap N)$ for any $A \in \mathfrak{M}$.*

Moreover if μ is nonnegative, so are μ_0 and μ_1.

Remark 2.11. We point out that the set function Φ could be defined on the whole space $\mathfrak{P}(X)$ (as in the case of $\mathrm{Cap}_{\mathbb{G}}$), but it is required to satisfy the properties which characterize outer measures only on the σ-algebra $\mathfrak{M}$.

Theorem 2.10 can be applied to the capacity $\Phi = \mathrm{Cap}_{\mathbb{G}}$ and general $\mu \in M(\Omega)$; we will write μ_d for the *diffuse part* and μ_c for the *concentrated part*, omitting the dependence on $\mathbb{G}$. A sketch of the proof can be found in [12, Lemma 2.1]: anyway, this result can be obtained by following the proof of the more famous Lebesgue decomposition theorem, where Φ is a (nonnegative) measure; see for instance [6, Theorem 4.3.2].

REMARK 2.12. Since we will deal with local arguments, we must highlight how decomposition works with restrictions. Indeed, let $\mu \in M(\Omega)$, and $\mu = \mu_d + \mu_c$ its decomposition with respect to $\mathrm{Cap}_{\mathbb{G}}$; if $\Omega' \subset \Omega$ is an open set, we also have the decomposition with respect to $\mathrm{Cap}_{\mathbb{G}}$ of $\mu_{|\Omega'}$. By using the uniqueness of the decomposition one can easily prove that

$$(\mu_d)_{|\Omega'} = (\mu_{|\Omega'})_d \text{ and } (\mu_c)_{|\Omega'} = (\mu_{|\Omega'})_c.$$

By the previous remark we can generalize Theorem 2.10 to local measures.

THEOREM 2.13. *Let $\mu \in M_{loc}(\Omega)$. Then there exists a unique pair (μ_d, μ_c) in $M_{loc}(\Omega)$ such that*

1. $\mu = \mu_d + \mu_c$;
2. μ_d *is diffuse with respect to* $\mathrm{Cap}_{\mathbb{G}}$;
3. μ_c *is concentrated with respect to* $\mathrm{Cap}_{\mathbb{G}}$.

Moreover if $\mu \in M(\Omega)$ so are μ_d and μ_c; if μ is nonnegative, so are μ_d and μ_c.

Proof. Let $\mu \in M_{loc}(\Omega)$. Consider $\psi \in \mathcal{D}(\Omega)$ and set a whatever $\Omega' \Subset \Omega$ such that $\psi_{|\Omega'} \in \mathcal{D}(\Omega')$; define $\mu_d(\psi) := (\mu_d)_{|\Omega'}(\psi_{|\Omega'})$ and $\mu_c(\psi) := (\mu_c)_{|\Omega'}(\psi_{|\Omega'})$. By Remark 2.12 we have that μ_d and μ_c are well defined (i.e. their definition does not depend on the choice of Ω') and moreover they belong to $M_{loc}(\Omega)$. The verification of the diffusivity, concentration and uniqueness is straightforward. □

REMARK 2.14. Let $\Lambda \subset \mathbb{G}$ be open. For every $K \subset \Lambda$ compact, we can define the *variational* (or *Dirichlet*, or *Newtonian*) $\mathbb{G}$*-capacity*, with respect to the *condenser* Λ as (see [15, page 27], [21, Section 12.1], [18, Section 4] and [4, 9, 26])

$$\mathrm{cap}_{\mathbb{G}}(K,\Lambda) := \inf\left\{\int_\Lambda |\nabla_{\mathbb{G}}\varphi|^2,\ \varphi \in C_0^\infty(\Lambda),\ \varphi \geq 1 \text{ on an open neighb. of } K\right\}.$$

The definition is then extended to open subsets $U \subset \Lambda$ as

$$\mathrm{cap}_{\mathbb{G}}(U,\Lambda) := \sup_{K \subset U \text{ compact}} \mathrm{cap}_{\mathbb{G}}(K,\Lambda),$$

and to general subsets of $E \subset \Lambda$ as

$$\mathrm{cap}_{\mathbb{G}}(E,\Lambda) := \inf_{E \supset U \text{ open}} \mathrm{cap}_{\mathbb{G}}(U,\Lambda).$$

Observed that a Poincaré inequality holds in Carnot groups (see e.g. [13]), and that $\nabla_{\mathbb{G}}$ is 1-homogeneous with respect to dilations δ_λ, one can follow the

Euclidean case [15, Chapter 2] to show that $\mathrm{cap}_{\mathbb{G}}$ is well defined, it is an outer measure (actually a Choquet capacity, as well as $\mathrm{Cap}_{\mathbb{G}}$), and that

$$\begin{aligned}&\mathrm{cap}_{\mathbb{G}}(E,\Lambda)=0 \text{ for some bounded } \Lambda\supset E\\ &\qquad\iff \mathrm{cap}_{\mathbb{G}}(E,\Lambda)=0 \text{ for each } \Lambda\supset E \iff \mathrm{Cap}_{\mathbb{G}}(E)=0;\end{aligned}$$

see [15, Theorem 2.2, Lemma 2.9, Theorem 2.38]. In particular, if we fix Λ bounded, then

$$\mathrm{Cap}_{\mathbb{G}}(E)=0 \iff \mathrm{cap}_{\mathbb{G}}(E,\Lambda)=0. \tag{7}$$

In particular, being $\mathrm{cap}_{\mathbb{G}}$ an outer measure, we can decompose a local Radon measure with respect to $\mathrm{cap}_{\mathbb{G}}$; if moreover Λ is bounded, thanks to (7) this decomposition is equivalent to the one given by $\mathrm{Cap}_{\mathbb{G}}$. As a consequence, if Ω is bounded, by choosing a whatever bounded $\Lambda\supset\Omega$ (possibly Ω itself), our main Theorem 1.2 can be rephrased in terms of the variational capacity $\mathrm{cap}_{\mathbb{G}}(\cdot,\Lambda)$ (see, indeed, [9]).

3. Main result

In what follows $\Omega\subset\mathbb{G}$ is an open set.

3.1. A regularity result

We discuss now the $W^{1,2}_{\mathbb{G},loc}$-regularity of distributional solutions.

LEMMA 3.1. *Let $u\in L^\infty_{loc}(\Omega)$, be such that*

$$-\Delta_{\mathbb{G}}u=\mu \quad \text{in } \mathcal{D}'(\Omega) \tag{8}$$

where $\mu\in M_{loc}(\Omega)$. Then $u\in W^{1,2}_{\mathbb{G},loc}(\Omega)$.

In the Euclidean setting, if $u\in L^1_{loc}(\Omega)$ solves (8), then we have $u\in W^{1,p}_{loc}(\Omega)$ for any $1\le p<\frac{N}{N-1}$ (see e.g. [8]), and in particular $u\in L^q_{loc}(\Omega)$ for every $q<\frac{N}{N-2}$; this summability is in general optimal, as one can easily see by considering the fundamental solution in $\mathbb{R}^N$, $N\ge 3$, that is $u(x)=C_N|x|^{2-N}$. In particular, u does not generally belong to $W^{1,2}_{loc}(\Omega)$. Lemma 3.1 assures this is the case provided u is locally bounded, even in a Carnot framework.

We mention that, in the Euclidean setting, this result is obtained in [15, Theorem 7.25] in the case of superharmonic functions (that is, μ is a positive measure), and we emphasize that here we do not make any assumption on the sign of the measure μ.

The following question arises: what is the smallest $q>1$ (actually, $q\ge\frac{Q}{Q-2}$) guaranteeing that, if $u\in L^q_{loc}(\Omega)$ solves (8), then $u\in W^{1,2}_{\mathbb{G},loc}(\Omega)$? Here we do not investigate the optimality of $q=\infty$ among $q>1$ to gain $W^{1,2}_{\mathbb{G},loc}$-regularity.

Proof of Lemma 3.1. Let B be a ball such that $\bar{B} \subset \Omega$; we shall prove that $u \in W^{1,2}_{\mathbb{G}}(B)$. Let $\phi \in C_0^\infty(\Omega)$ be such that $\phi = 1$ on B.

We begin by assuming that u is of class C^∞. Let $M > 0$ be such that $|u| + 1 \le M$ on $\omega := \mathrm{supp}(\phi)$. Testing (8) with $\frac{\phi^2}{M+u} \in \mathcal{D}(\Omega)$, and applying Leibniz rule, we obtain

$$\begin{aligned}\int_\omega \mu \frac{\phi^2}{M+u} &= \int_\omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \Big(\frac{\phi^2}{M+u} \Big) \\ &= - \int_\omega |\nabla_{\mathbb{G}} u|^2 \frac{\phi^2}{(M+u)^2} + 2 \int_\omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \phi \ \frac{\phi}{M+u}\end{aligned}$$

that is

$$\int_\omega |\nabla_{\mathbb{G}} u|^2 \frac{\phi^2}{(M+u)^2} = \int_\omega (-\mu) \frac{\phi^2}{M+u} + 2 \int_\omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \phi \ \frac{\phi}{M+u}. \tag{9}$$

Next by Young inequality we get

$$\begin{aligned}2 \int_\omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \phi \ \frac{\phi}{M+u} &\le 2 \int_\omega |\nabla_{\mathbb{G}} u| \frac{\phi}{M+u} |\nabla_{\mathbb{G}} \phi| \\ &\le \frac{1}{2} \int_\omega |\nabla_{\mathbb{G}} u|^2 \frac{\phi^2}{(M+u)^2} + 2 \int_\omega |\nabla_{\mathbb{G}} \phi|^2\end{aligned}$$

and since $\frac{1}{M+u} \le 1$ we obtain

$$\int_\omega (-\mu) \frac{\phi^2}{M+u} \le \int_\omega |\mu| \frac{\phi^2}{M+u} \le \int_\omega |\mu| \phi^2.$$

Plugging these inequalities in (9) we deduce

$$\frac{1}{2} \int_\omega |\nabla_{\mathbb{G}} u|^2 \frac{\phi^2}{(M+u)^2} \le \int_\omega |\mu| \phi^2 + 2 \int_\omega |\nabla_{\mathbb{G}} \phi|^2.$$

Now, taking into account that $\frac{1}{M+u} \ge \frac{1}{2M-1}$ we have the estimate

$$\frac{1}{2} \int_B |\nabla_{\mathbb{G}} u|^2 \le \frac{1}{2} \int_\omega |\nabla_{\mathbb{G}} u|^2 \phi^2 \le (2M-1)^2 \left(\int_\omega |\mu| \phi^2 + 2 \int_\omega |\nabla_{\mathbb{G}} \phi|^2 \right). \tag{10}$$

Next step is to show that a similar inequality holds for a general locally bounded distributional solution of (8). Let u be as in the claim of Lemma. Let $M > 0$ be such that $|u| + 1 \le M$ a.e. on a neighborhood of $\omega := \mathrm{supp}(\phi)$. Let $u_\eta := u \star_{\mathbb{G}} m_\eta$ and $\mu_\eta := \mu \star_{\mathbb{G}} m_\eta$ be the mollified functions of u and the measure μ respectively (see Section 2.2). We have that

$$-\Delta_{\mathbb{G}} u_\eta = \mu_\eta \text{ on } \Omega_\eta, \quad u_\eta \to u \ \textit{in} \ L^1_{loc}(\Omega), \quad \mu_\eta \to \mu \ \textit{in} \ \mathcal{D}'(\Omega), \quad |u_\eta| + 1 \le M.$$

The estimate (10) for u_η reads as

$$\frac{1}{2}\int_B |\nabla_{\mathbb{G}} u_\eta|^2 \leq (2M-1)^2 \left(\int_\omega |\mu_\eta|\phi^2 + 2\int_\omega |\nabla_{\mathbb{G}}\phi|^2\right).$$

Notice that by linearity we have that

$$\mu_\eta := \mu \star_{\mathbb{G}} m_\eta = \mu^+ \star_{\mathbb{G}} m_\eta - \mu^- \star_{\mathbb{G}} m_\eta =: \mu_\eta^+ - \mu_\eta^-$$

and hence

$$|\mu_\eta| \leq \mu_\eta^+ + \mu_\eta^-.$$

Observing that for $\eta \to 0$

$$\int |\mu_\eta|\phi^2 \leq \int \mu_\eta^+\phi^2 + \int \mu_\eta^-\phi^2 \to \int \phi^2 d\mu^+ + \int \phi^2 d\mu^-,$$

we deduce the bound

$$\int |\mu_\eta|\phi^2 \leq K$$

for η small and K independent of η. This implies that $\int_B |\nabla_{\mathbb{G}} u_\eta|^2$ is bounded for η small. Summing up: $(u_\eta)_\eta$ is a family of functions in $W^{1,2}_{\mathbb{G}}(B)$ such that $u_\eta \to u$ in $L^2(B)$ and it is bounded in $W^{1,2}_{\mathbb{G}}(B)$. By compactness (see Theorem 2.6), we get that, up to a subsequence, for any $i = 1 \ldots l$ there exists $g_i \in L^2(B)$ such that $X_i u_\eta \rightharpoonup g_i$, that is for any test function ψ we have

$$\int X_i u_\eta \psi \to \int g_i \psi \quad \text{as } \eta \to 0.$$

On the other hand since

$$\int X_i u_\eta \psi = \int u_\eta X_i^* \psi \to \int u X_i^* \psi \quad \text{as } \eta \to 0,$$

we deduce that $X_i u = g_i \in L^2(B)$ for any $i = 1 \ldots l$, hence $u \in W^{1,2}_{\mathbb{G}}(B)$. □

3.2. Kato's inequality and truncations

In order to prove the next results we need the following Kato's inequality, see [7].

THEOREM 3.2 (Kato's inequality [7]). *Let $v, f \in L^1_{loc}(\Omega)$ be such that*

$$\Delta_{\mathbb{G}} v \geq f \quad \textit{in } \mathcal{D}'(\Omega),$$

then

$$\Delta_{\mathbb{G}} v^+ \geq \operatorname{sign}(v)\, f \quad \textit{in } \mathcal{D}'(\Omega). \tag{11}$$

In what follows, we shall denote $u|^k := \min(u,k)$ for any $k \in \mathbb{R}$ and by χ_A the indicator function of a set $A \subset \mathbb{G}$. As a consequence of Lemma 3.1 and Theorem 3.2, we obtain the following result.

Corollary 3.3. *Let $u \in L^1_{loc}(\Omega)$ be such that $u \geq 0$ a.e. in Ω and*

$$-\Delta_{\mathbb{G}} u \geq h \quad \text{in } \mathcal{D}'(\Omega),$$

where $h \in L^1_{loc}(\Omega)$. Then, for each $k > 0$, we have $u|^k \in W^{1,2}_{\mathbb{G},loc}(\Omega)$ and

$$-\Delta_{\mathbb{G}} u|^k \geq h\chi_{[u<k]} \quad \text{in } \mathcal{D}'(\Omega). \tag{12}$$

Proof. The function $v := k - u$ satisfies the inequality

$$\Delta_{\mathbb{G}} v \geq h \quad \text{in } \mathcal{D}'(\Omega).$$

An application of (11) yields

$$\Delta_{\mathbb{G}} v^+ \geq \operatorname{sign}(v)\, h \quad \text{in } \mathcal{D}'(\Omega)$$

which in turns implies (12) since

$$v^+ = (k-u)^+ = k - u|^k \quad \text{and} \quad \operatorname{sign}(v) = \chi_{[v>0]} = \chi_{[u<k]}.$$

Finally, the fact that $u|^k \in W^{1,2}_{\mathbb{G},loc}(\Omega)$ is an application of Lemma 3.1, being $u|^k$ bounded. □

Another interesting consequence of Lemma 3.1 is the following result.

Corollary 3.4. *Let $u \in L^1_{loc}(\Omega)$ be such that $u \geq 0$ a.e. in Ω and*

$$\Delta_{\mathbb{G}} u = \mu \quad \text{in } \mathcal{D}'(\Omega),$$

where $\mu \in M_{loc}(\Omega)$ (resp. $M(\Omega)$). Then, for each $k > 0$, we have $u|^k \in W^{1,2}_{\mathbb{G},loc}(\Omega)$, $\Delta_{\mathbb{G}} u|^k \in M_{loc}(\Omega)$ (resp. $M(\Omega)$) and

$$\Delta_{\mathbb{G}} u|^k \leq (\Delta_{\mathbb{G}} u)^+ = \mu^+ \quad \text{in } \mathcal{D}'(\Omega). \tag{13}$$

Proof. We use the same notation of the proof of Lemma 3.1. Let $u_\eta := m_\eta \star_{\mathbb{G}} u$, we know that $u_\eta \to u$ in $L^1_{loc}(\Omega)$ as $\eta \to 0$ and we can assume that $u_\eta \to u$ a.e. in Ω as $\eta \to 0$. The function u_η satisfies the equation

$$\Delta_{\mathbb{G}} u_\eta = \mu_\eta = m_\eta \star \mu^+ - m_\eta \star \mu^- =: \mu_\eta^+ - \mu_\eta^- \quad \text{on } \Omega_\eta.$$

Arguing as in the proof of Corollary 3.3, by Kato's inequality we obtain

$$-\Delta_{\mathbb{G}} u_\eta|^k \geq -\chi_{[u_\eta<k]}\mu_\eta \quad \text{on } \Omega_\eta,$$

that is, for any nonnegative $\psi \in \mathcal{D}(\Omega_\eta)$,

$$\int_{\Omega_\eta} u_\eta|^k \Delta_{\mathbb{G}} \psi \le \int_{\Omega_\eta} \psi \chi_{[u_\eta<k]}(\mu_\eta^+ - \mu_\eta^-) \le \int_{\Omega_\eta} \psi \chi_{[u_\eta<k]} \mu_\eta^+ \le \int_{\Omega_\eta} \psi \mu_\eta^+. \quad (14)$$

For any $\psi \in \mathcal{D}(\Omega)$, by taking η sufficiently small we have $\psi \in \mathcal{D}(\Omega_\eta)$ and thus (14) still holds. Next by using Lebesgue dominated convergence (thanks to the boundedness of the functions involved) it results that $u_\eta|^k \to u|^k$ in $L^1_{loc}(\Omega)$. Hence by letting $\eta \to 0$ in (14) we obtain relation (13).

Next, from (13) we notice that $\mu^+ - \Delta_{\mathbb{G}} u|^k$ is a nonnegative distribution, thus it is a nonnegative Radon measure. This implies that $\Delta_{\mathbb{G}} u|^k \in M_{loc}(\Omega)$ (or $M(\Omega)$, if μ is so). Applying Lemma 3.1 to the bounded function $u|^k$, we conclude that $u|^k \in W^{1,2}_{\mathbb{G},loc}(\Omega)$. □

3.3. Proof of Theorem 1.2

Before presenting the main theorem, we state an approximation lemma.

LEMMA 3.5. *Let Ω be open, $K \subset \Omega$ be with zero $\mathbb{G}$-capacity, and $\psi \in C_0^\infty(\Omega)$ with $\psi \ge 0$. Then there exists $\psi_n \in C_0^\infty(\Omega \setminus K)$ such that $0 \le \psi_n \le \psi$ and $\psi_n \to \psi$ in $W^{1,2}_{\mathbb{G}}(\Omega)$.*

Proof. By definition of $\mathbb{G}$-capacity there exists $\varphi_n \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi_n = 1$ on a neighborhood of K and $\|\varphi_n\|_{W^{1,2}_{\mathbb{G}}(\mathbb{R}^N)} \to 0$. We can assume $0 \le \varphi_n \le 1$. We set $\psi_n := (1 - \varphi_n)\psi$, which satisfies $0 \le \psi_n \le \psi$ and $\psi_n \in C_0^\infty(\Omega \setminus K)$. Moreover

$$\begin{aligned} \|\psi_n - \psi\|_{W^{1,2}_{\mathbb{G}}(\Omega)} &= \|\varphi_n \psi\|_{W^{1,2}_{\mathbb{G}}(\Omega)} \\ &\le \|\varphi_n\|_2 \|\psi\|_2 + \|\nabla_{\mathbb{G}} \varphi_n\|_2 \|\psi\|_2 + \|\varphi_n\|_2 \|\nabla_{\mathbb{G}} \psi\|_2 \to 0. \end{aligned}$$ □

We are ready now to prove the main theorem. We highlight that the information given by Lemma 3.5 that is actually exploited in the proof of Theorem 1.2 is only $\nabla_{\mathbb{G}} \psi_n \to \nabla_{\mathbb{G}} \psi$ in $L^2(\Omega)$.

Proof of Theorem 1.2. For the sake of notation we set $\nu := \Delta_{\mathbb{G}} u = -\mu$. We will prove that $\nu_c \le 0$. Let $E \subset \Omega$ be a set of zero $\mathbb{G}$-capacity such that $|\nu_c|(\Omega \backslash E) = 0$. Let $K \subset E$ be a compact set. Clearly $\mathrm{cap}_{\mathbb{G}}(K, \Omega) = 0$.

Applying Corollary 3.4 to $\Omega \backslash K$, we have $u|^k \in W^{1,2}_{\mathbb{G},loc}(\Omega \setminus K)$ for each $k > 0$ and

$$\Delta_{\mathbb{G}} u|^k \le \nu^+ \quad \text{in } \mathcal{D}'(\Omega \backslash K). \quad (15)$$

Given $\psi \in \mathcal{D}(\Omega), \psi \ge 0$ in Ω, by Lemma 3.5 there exists $(\psi_n)_n \subset \mathcal{D}(\Omega \backslash K)$ such that $0 \le \psi_n \le \psi$ in Ω and $\psi_n \to \psi$ in $W^{1,2}(\Omega)$. Then we have

$$\int_\Omega \psi_n d\nu^+ \le \int_{\Omega \backslash K} \psi d\nu^+ \quad \text{for any } n \ge 1, \quad (16)$$

and

$$\int_\Omega \nabla_{\mathbb{G}} u|^k \cdot \nabla_{\mathbb{G}} \psi_n \overset{n\to\infty}{\longrightarrow} \int_\Omega \nabla_{\mathbb{G}} u|^k \cdot \nabla_{\mathbb{G}} \psi. \tag{17}$$

From relations (15), (16) and (17), we obtain

$$-\int_\Omega \nabla_{\mathbb{G}} u|^k \cdot \nabla_{\mathbb{G}} \psi \le \int_{\Omega\setminus K} \psi d\nu^+$$

and thus

$$\int_\Omega u|^k \Delta_{\mathbb{G}} \psi \le \int_{\Omega\setminus K} \psi d\nu^+.$$

As $k \to \infty$, exploiting that $u|^k \to u$ pointwise, that $\Delta_{\mathbb{G}}\psi$ has compact support and that $u \in L^1_{loc}(\Omega)$, by dominated convergence theorem we get

$$\int_\Omega u \Delta_{\mathbb{G}} \psi \le \int_{\Omega\setminus K} \psi d\nu^+. \tag{18}$$

Since (18) holds for any nonnegative $\psi \in \mathcal{D}(\Omega)$, we have

$$\nu = \Delta_{\mathbb{G}} u \le \chi_{\Omega\setminus K}\nu^+ \quad \text{in } \mathcal{D}'(\Omega).$$

Thus, $\nu_c|_K = \nu|_K \le 0$ in Ω.

First we assume that $\nu \in M(\Omega)$. Clearly $\nu_c \in M(\Omega)$. Since $K \subset \Omega$ is an arbitrary compact subset of E, then applying the inner regularity of Radon measures (see Remark 2.3) to ν_c, we conclude that $\nu_c(E) \le 0$, that is the claim.

Finally, let $\nu \in M_{loc}(\Omega)$, and fix $\Omega' \Subset \Omega$. Arguing as before we obtain $\nu_c|_{\Omega'} \le 0$ in the sense of measures, thus [21, Proposition 6.12] in the sense of distributions $\mathcal{D}'(\Omega')$; being Ω' arbitrary, we obtain $\nu_c \le 0$ in $\mathcal{D}'(\Omega)$. In particular, since ν_c has a sign, we have also [17, Theorem 6.22] $\nu_c \in M(\Omega)$. □

Proof of Corollary 1.3. Fixed a bounded neighborhood $U \subset \Omega$ of x_0, so that $-\Delta u \ge \delta_{x_0}$ on U as well, we first observe that $-\Delta_{\mathbb{G}} u = \delta_{x_0} + \lambda$ for some nonnegative $\lambda \in M(U)$. Then, we notice that $u - k$ is a solution of the same equation for each $k \in \mathbb{R}$. By showing that $(u-k)^+ \ne 0$ we have the claim. To show this, we apply Theorem 1.2. We only need to check that $\mathrm{Cap}_{\mathbb{G}}(\{x_0\}) = 0$. Being U bounded, by Remark 2.14 it is sufficient to check that $\mathrm{cap}_{\mathbb{G}}\big(\{x_0\}, B(x_0,R)\big) = 0$ for some $R > 0$. Noticed that we deal with the case $p = 2 \in (1, Q)$, such a result is ensured by [4, Theorem 2.2] and monotonicity of the capacity. □

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