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Nonlocal Schrödinger-Poisson systems in $\mathbb{R}^N$: the fractional Sobolev limiting case

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Dedicated to Enzo Mitidieri on the occasion of his 70th birthday

Abstract. *We study the existence of positive solutions for nonlocal systems in gradient form and set in the whole $\mathbb{R}^N$. A quasilinear fractional Schrödinger equation, where the leading operator is the $\frac{N}{s}$-fractional Laplacian, is coupled with a higher-order and possibly fractional Poisson equation. For both operators the dimension $N \geq 2$ corresponds to the limiting case of the Sobolev embedding, hence we consider nonlinearities with exponential growth. Since standard variational tools cannot be applied due to the sign-changing logarithmic Riesz kernel of the Poisson equation, we employ a variational approximating procedure for an auxiliary Choquard equation, where the Riesz kernel is uniformly approximated by polynomial kernels. Qualitative properties of solutions such as symmetry, regularity and decay are also established. Our results extend and complete the analysis carried out in the planar case in [13].*

Keywords: Schrödinger-Poisson system, Choquard equation, p-fractional Laplacian, exponential growth, variational methods, limiting fractional Sobolev embeddings.
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1. Introduction

We investigate existence of solutions for the nonlocal Schrödinger-Poisson system in the whole space given by

$$\begin{cases} (-\Delta)^s_{\frac{N}{s}} u + V(x)|u|^{\frac{N}{s}-2}u = \phi\, f(u) \\ (-\Delta)^{\frac{N}{2}} \phi = F(u) \end{cases} \quad \text{in} \quad \mathbb{R}^N, \tag{SP$_s$}$$

where $s \in (0,1)$, V is a positive and bounded potential, $f : \mathbb{R} \to \mathbb{R}$ is a positive nonlinearity, and F its primitive vanishing at zero. The nonlocal operator in the first Schrödinger equation is the $s-$fractional $p-$Laplacian with $p = \frac{N}{s}$, while the operator in the second Poisson equation is the $\frac{N}{2}$-power of the Laplacian, and therefore possibly of higher-order and fractional, depending on the dimension $N \geq 2$. As we are going to see, the functional setting is critical with respect to the Sobolev embeddings, and this on the one hand yields to consider nonlinearities with exponential growth. On the other hand, the Riesz kernel of the Poisson equation is logarithmic, hence sign-changing and unbounded from below and above. We are interested in understanding the interplay between these features, together with the nonlocal and higher-order character of the operators involved.

Schrödinger-Poisson systems emerge in several fields of Physics, such as in Hartree models for crystals, astrophysics, electromagnetism, and quantum mechanics. We refer the interested reader to [5, 33] for the background.

In the local semilinear case, when $N \geq 3$, namely

$$\begin{cases} -\Delta u + V(x)u = \phi\, f(u) \\ -\Delta \phi = F(u) \end{cases} \quad \text{in } \mathbb{R}^N, \tag{SP}$$

a well-known strategy to find solutions is to solve the Poisson equation by means of the positive polynomial Riesz kernel and substitute $\phi = \phi(u) := C_N|x|^{2-N} * F(u)$ into the first equation of the system. In this way, one obtains a Schrödinger equation with a convolution nonlinear term, the so-called Choquard type equation, which has the advantage to exhibit a variational structure and hence can be solved in the natural space $H^1(\mathbb{R}^N)$, see e.g. [37] and references therein. In the case $N = 2$ this technique cannot be directly employed: indeed, the Riesz kernel of the Laplace operator is $\frac{1}{2\pi} \log \frac{1}{|\cdot|}$, and the nonlocal term in the Choquard equation

$$-\Delta u + V(x)u = \frac{1}{2\pi}\left(\log \frac{1}{|\cdot|} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^2 \tag{Ch}$$

is no longer well-defined in the natural Hilbert space $H^1(\mathbb{R}^2)$. Let us also point out a more theoretical as well as delicate aspect: if from one side it is natural to consider the convolution with the Riesz kernel as solution of the Poisson equation, on the other side this choice is somehow arbitrary as one cannot distinguish solutions which differ by additive constants in some reasonable function space setting; see the discussion carried out in [4, 9, 13].

Following a first approach of Stubbe in [40], Cingolani and Weth managed to bypass this problem for $f(u) = u$ (strictly speaking the Choquard case) in [18] by restricting to the constrained space $\{u \in H^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \log(1 + |x|)u^2 \,\mathrm{d}x < +\infty\}$, in which the functional associated to the Choquard equation turns out to be well-defined; see also further extensions in [19, 26]. A different approach, suitable for tuning the phenomenon of logarithmic growth of the Riesz kernel and the maximal exponential growth for the nonlinearity, was proposed by

Cassani and Tarsi in [14]: in a log-weighted space built on $H^1(\mathbb{R}^2)$ they proved a Pohožaev-Trudinger inequality, which yields the well-posedness of the functional associated to (Ch) for subcritical or critical nonlinearities f in the sense of Trudinger-Moser. This approach has been then extended in [9] to quasilinear Schrödinger-Poisson systems in $\mathbb{R}^N$, namely (SP_s) with $s = 1$, and in [11] for Schrödinger-Poisson systems with potential and weight functions decaying to 0 at infinity. We also refer to [1, 2] for related results on Choquard equations with exponential nonlinearities and polynomial kernels, and to [3, 7, 8, 24] for exponential nonlinearities and logarithmic kernels different from (SP_s).

In all the aforementioned works the function space has to be adapted in order to have the energy functional well-defined. Recently, this difficulty was overcome in [34], see also [12] and the extension to the zero-mass case in [39], by means of an approximating procedure. The idea is to replace the logarithmic kernel with a more suitable power-like kernel, by observing that

$$\log\frac{1}{t} = \lim_{\mu\to 0^+} \frac{t^{-\mu}-1}{\mu}\ . \tag{1}$$

Hence, one is lead to find critical points of auxiliary approximating equations with power-like convolution terms, and then retrieve a solution of the original logarithmic Choquard equation by means of a limit procedure. The advantage is that one can deal with the approximating functionals in the standard space $H^1(\mathbb{R}^2)$ by means of the Hardy-Littlewood-Sobolev inequality. The price to pay is the need of estimates for the family of critical points, which have to be uniform with respect to the parameter μ.

Short-long range interactions in the physics model yield the appearance of nonlocal operators which, from the functional analysis point of view, turn out to be well defined within fractional Sobolev-Slobodeckij spaces $W^{s,p}(\mathbb{R}^N)$, see for instance [15] for more references on this topic. As in the Sobolev local case, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-sp}}(\mathbb{R}^N)$ is continuous provided $N > sp$. In the borderline case $N = ps$, still one has that the maximal degree of summability of a function with membership in $W^{s,N/s}(\mathbb{R}^N)$ is of exponential type, as obtained by Parini and Ruf [38] for bounded domains and by Zhang [43] for the whole space, see Theorem A below. This entices into investigating the system (SP_s), where f has the maximal exponential growth at infinity. To the best of our knowledge, the only results for p-fractional Choquard equations in $\mathbb{R}^N$ with exponential nonlinearities, which however do not derive from (SP_s), are obtained in [6, 17, 20, 41, 42]. The study of (SP_s) for the planar case $N = 2$ has been carried out in [13]. Since it seems difficult to obtain in this setting some log-weighted Pohožaev-Trudinger inequality in the spirit of [14], here we follow the approximating method of [12, 34]. However, the effort is not only technical because of the higher dimension, as we have to deal with the basic and still unknown fact, whether the energy level reached by Moser sequences (usually equivalent to instantons or Aubin-Talenti functions in the power case) matches the sharp exponent in the fractional Trudinger-Moser inequality of [38,43], and this prevents to exploit concentrating sequences to gain variational compactness. The restriction to the planar case somehow simplifies technicalities and moreover, in this case the Poisson equation is semilinear and of second-order. In [13], we also proved that from a weak solution of the Choquard equation we can retrieve a distributional solution of the system and this is somehow neglected in the literature. See also [10, 21, 35] for related problems and non existence results.

In this paper we aim at completing the analysis carried out in [13] by considering the general system (SP_s) in any dimension $N \geq 2$. Note that now the Poisson equation is of higher-order and nonlocal when N is odd. Before giving the precise statement of our results, let us recall some basic facts and make precise the definition of solutions we are going to use.

The $(s, \frac{N}{s})-$fractional Laplace operator pointwisely acts as

$$(-\Delta)^s_{\frac{N}{s}} u(x) := 2\, PV \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))}{|x-y|^{2N}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where PV stands for the Cauchy Principal Value. This is well-defined for all $x \in \mathbb{R}^N$ for functions in $C^{1,1}_{loc}(\mathbb{R}^N)$ which enjoy suitable integrability conditions at infinity, see [16, Lemma 5.2]. One can also consider such operator in a weak form in the corresponding fractional Sobolev-Slobodeckij space

$$W^{s,\frac{N}{s}}(\mathbb{R}^N) := \left\{ u \in L^{\frac{N}{s}}(\mathbb{R}^N) : [u]_{s,\frac{N}{s}} < +\infty \right\},$$

where $[u]_{s,\frac{N}{s}}$ denotes the Gagliardo seminorm

$$[u]_{s,\frac{N}{s}} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{s}{N}},$$

which is a uniformly convex Banach space with norm

$$\|u\| := \left(\|u\|^{\frac{N}{s}}_{\frac{N}{s}} + [u]^{\frac{N}{s}}_{s,\frac{N}{s}} \right)^{\frac{s}{N}}.$$

Note that, if the potential $V : \mathbb{R} \to \mathbb{R}$ is measurable and satisfies

(V) There exist constants $\underline{V_0}, \overline{V_0} > 0$ such that

$$0 < \underline{V_0} \leq V(x) \leq \overline{V_0} < +\infty \quad \text{for all } \; x \in \mathbb{R}^N,$$

then

$$\|u\|_V := \left(\|V(\cdot)^{\frac{s}{N}} u\|^{\frac{N}{s}}_{\frac{N}{s}} + [u]^{\frac{N}{s}}_{s,\frac{N}{s}} \right)^{\frac{s}{N}}$$

is an equivalent norm on $W^{s,\frac{N}{s}}(\mathbb{R}^N)$. In this functional setting we can define a weak notion of solution for (SP$_s$). For $\gamma > 0$ the weighted Lebesgue space $L_\gamma(\mathbb{R}^N)$ is defined as

$$L_\gamma(\mathbb{R}^N) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2\gamma}}\,\mathrm{d}x < +\infty \right\}.$$

As usual, we denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing functions and by $\mathcal{S}'$ the dual space of tempered distributions.

Definition 1.1. *For $\mathfrak{f} \in \mathcal{S}'(\mathbb{R}^N)$ we say that a function $\phi \in L_1(\mathbb{R}^N)$ is a solution of the linear Poisson equation $(-\Delta)^{\frac{N}{2}}\phi = \mathfrak{f}$ in $\mathbb{R}^N$ if*

$$\int_{\mathbb{R}^N} \phi(-\Delta)^{\frac{N}{2}}\varphi = \langle \mathfrak{f}, \varphi\rangle \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^N).$$

Definition 1.2 (Solution of (SP$_s$)). *We say that (u,ϕ) is a weak solution of the Schrödinger-Poisson system* (SP$_s$) *if*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{\frac{N}{s}-2}u\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N}\phi f(u)\varphi\,\mathrm{d}x$$

for all $\varphi \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$, and ϕ solves $(-\Delta)^{\frac{N}{2}}\phi = F(u)$ in $\mathbb{R}^N$ in the sense of Definition 1.1.

In order to find a solution of (SP$_s$) in the sense of Definition 1.2, we consider the logarithmic Choquard equation

$$(-\Delta)^s_{\frac{N}{s}} u + V(x)|u|^{\frac{N}{s}-2}u = C_N\left(\log\frac{1}{|\cdot|} * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \tag{Ch$_s$}$$

where the Riesz kernel of the Poisson equation in (SP$_s$)

$$I_N(x) := C_N \log\frac{1}{|x|} \qquad \text{with} \quad C_N^{-1} := 2^{N-1}\pi^{\frac{N}{2}}\Gamma\left(\tfrac{N}{2}\right), \tag{2}$$

has been substituted in the Schrödinger equation in (SP$_s$).

Definition 1.3 (Solution of (Ch$_s$)). *We say that $u \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$ is a weak solution of* (Ch$_s$) *if*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{\frac{N}{s}-2}u\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\log\frac{1}{|x-y|}F(u(y))\,\mathrm{d}y\right) f(u(x))\varphi(x)\,\mathrm{d}x \tag{3}$$

for all $\varphi \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$.

A key ingredient in our assumptions on f will be the following fractional version of the Trudinger-Moser inequality obtained in [38, 43]. Let

$$\Phi_{N,s}(t) := \mathrm{e}^t - \sum_{j=0}^{j_{\frac{N}{s}}-2} \frac{t^j}{j!}\,,$$

for $t \geq 0$, where $j_{\frac{N}{s}} := \min\{j \in \mathbb{N} :\, j \geq \frac{N}{s}\}$.

THEOREM A ([43, Theorem 1.3]). *Let* $s \in (0,1)$*, then for all* $\alpha > 0$ *one has*

$$\int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{\frac{N}{N-s}})\,\mathrm{d}x < +\infty\,. \tag{4}$$

Moreover, for all $\lambda > 0$*, setting*

$$\alpha_* := \sup\left\{\alpha : \sup_{u\in W^{s,\frac{N}{s}}(\mathbb{R}^N),\,[u]_{s,\frac{N}{s}}+\lambda\|u\|_{\frac{N}{s}}\leq 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{\frac{N}{N-s}})\,\mathrm{d}x < +\infty\right\},$$

one has $\alpha_* \leq \alpha^*_{s,N}$ *, where*

$$\alpha^*_{s,N} := N\left(\frac{2(N\omega_N)^2\Gamma(1+\frac{N}{s})}{N!}\sum_{k=0}^{+\infty}\frac{(N-1+k)!}{k!(N+2k)^{N/s}}\right)^{\frac{s}{N-s}}.$$

As remarked in [38, 43], the result of Theorem A is not sharp in the sense of Moser, since $\alpha^*_{s,N}$, which is the level reached by the concentrating fractional version of Moser sequences, is just an upper bound for the sharp exponent α_*. Obtaining the precise value of α_* is still a challenging open problem. This missing information is responsible for some extra issues we have to deal with in establishing fine estimates for the mountain-pass level, see the discussion which introduces Lemma 3.5 below.

Assumptions. According to Theorem A, the maximal growth for f is exponential. We will entail this in the following set of assumptions:

(f_1) $f \in C^1(\mathbb{R})$, $f \geq 0$, $f(t) = 0$ for $t \leq 0$, and $f(t) = o(t^{\frac{N}{s}-1})$ as $t \to 0^+$;

(f_2) there exist constants $b_1, b_2 > 0$ such that for any $t > 0$,

$$0 < f(t) \leq b_1 + b_2\Phi_{N,s}(\alpha_*|t|^{\frac{N}{N-s}})\,;$$

(f_3) there exists $\tau \in \left(\left(1-\frac{2}{N}\right)s, s\right)$ such that

$$1-s+\tau \leq \frac{F(t)f'(t)}{f^2(t)} < 1+\mu_N(s,\tau) \quad \text{for any } \ t>0,$$

where $\mu_N(s,\tau)$ is explicitly given in (45);

(f_4) $\lim\limits_{t\to+\infty} \frac{F(t)f'(t)}{f^2(t)} = 1$ or equivalently $\lim\limits_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\frac{F(t)}{f(t)} = 0$;

(f_5) there exists $\beta_0 > 1$ depending on s such that

$$f(t)F(t) \geq \beta\, t^{\frac{N}{s}} \qquad \text{for all } \ t > T_N(s)$$

for some $\beta > \beta_0$. The values of β_0 and $T_N(s)$ are explicitly given in Lemma 3.6.

The set of assumptions on f coincides for $N=2$ with the one considered in [13], to which we refer for further details and comments, see also Section 2. Let us just point out that our nonlinearities can have critical or subcritical growth in the sense of Theorem A, but at least exponential, as prescribed by (f_3)-(f_4). Finally, (f_5) is a condition in the interval $(T_N(s),+\infty)$, which requires a sufficiently fast growth for middle-range values of t. Indeed, at infinity, it is automatically satisfied.

Main results. We are now in a position to state our main results. First, we find a positive weak solution in the sense of Definition 1.3 to the Choquard equation (Ch_s).

THEOREM 1.4 (Existence for (Ch_s)). *Suppose V is radially symmetric and satisfies (V), and that (f_1)-(f_5) are fulfilled. Then, (Ch_s) possesses a positive radially symmetric weak solution $u \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$ such that*

$$\left| \int_{\mathbb{R}^N} \left(\log\frac{1}{|\cdot|} * F(u) \right) F(u)\,\mathrm{d}x \right| < +\infty\,. \tag{5}$$

Next, inspired by some ideas in [9], we rigorously establish the fact that from a solution of (Ch_s) one obtains a solution of (SP_s); in doing this we also investigate regularity and decaying properties of the obtained solutions.

THEOREM 1.5 ($(\mathrm{Ch}_s)\Longrightarrow(\mathrm{SP}_s)$). *Assume (V) and (f_1)-(f_2), let $u \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$ be a positive weak solution of the Choquard equation (Ch_s) in the sense of Definition 1.3 and define $\phi_u := I_N * F(u)$. Then, $(u,\phi_u) \in W^{s,\frac{N}{s}}(\mathbb{R}^N) \times L_\gamma(\mathbb{R}^N)$ for all $\gamma > 0$ is a solution of the Schrödinger-Poisson system (SP_s) in the sense of Definition 1.2. Moreover $u \in L^\infty(\mathbb{R}^N) \cap C^{0,\nu}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\nu \in (0,1)$, and*

$$\phi_u(x) = -C_N\|F(u)\|_1 \log|x| + o(1) \qquad \text{as } \ |x| \to +\infty\,. \tag{6}$$

Let us stress the fact that we do not prove that the two problems are equivalent, namely that they have the same set of solutions. Finding a proper functional setting in which this holds is still open, even for the local case (SP)–(Ch), see [9, Section 2]. However, combining Theorems 1.4 and 1.5 one directly deduces the existence results for (SP_s), which we next state for the sake of completeness.

Corollary 1.6 (Existence for (SP_s)). *Suppose V is radially symmetric and satisfies (V), and that (f_1)-(f_5) are fulfilled, then* (SP_s) *possesses a solution* $(u, \phi) \in W^{s,\frac{N}{s}}(\mathbb{R}^N) \times L_\gamma(\mathbb{R}^N)$ *for all* $\gamma > 0$ *such that:*

i) $u \in L^\infty(\mathbb{R}^N) \cap C^{0,\nu}_{\mathrm{loc}}(\mathbb{R}^N)$ *for some* $\nu \in (0,1)$*, is positive, radially symmetric and* (5) *holds;*

ii) $\phi = \phi_u := I_N * F(u)$ *and the asymptotic behaviour* (6) *is satisfied.*

Finally, in the special case of a constant potential, and assuming some additional regularity of the solutions in order to be able to deal with the pointwise definition of the operator $(-\Delta)^s_{\frac{N}{s}}$, it is possible to show that solutions are radially symmetric, by means of the moving planes technique and exploiting the connection between (Ch_s) and (SP_s) established in Theorem 1.5. Our result extends to system (SP_s) the corresponding results for $\left(s, \frac{N}{s}\right)$-fractional Schrödinger equations obtained in [16].

Theorem 1.7 (Symmetry for (Ch_s)). *Suppose (f_1)-(f_2) are satisfied, and let* $u \in C^{1,1}_{loc}(\mathbb{R}^N) \cap W^{s,\frac{N}{s}}(\mathbb{R}^N)$ *be a positive solution of* (Ch_s) *with* $V(x) = V_0 > 0$. *Then u is radially symmetric around the origin and monotone decreasing.*

Overview. After recalling some preliminary results in the next section, in Section 3 we prove the existence result for (Ch_s), namely Theorem 1.4, by means of a variational approach together with a uniform asymptotic approximation technique. The quite delicate relationship between (Ch_s) and (SP_s), which yields also existence for (SP_s), is investigated in Section 4. The moving planes argument used in the proof of Theorem 1.7 follows exactly the one employed in [13] for the planar case and thus we omit here the details.

Notation. For $R > 0$ and $x_0 \in \mathbb{R}^N$ we denote by $B_R(x_0)$ the ball of radius R and center x_0. Given a set $\Omega \subset \mathbb{R}^N$, we denote $\Omega^c := \mathbb{R}^N \setminus \Omega$, and its characteristic function by χ_Ω. The space of the infinitely differentiable functions which are compactly supported is denoted by $C^\infty_0(\mathbb{R}^N)$. The norm of the Lebesgue spaces $L^p(\mathbb{R}^N)$ with $p \in [1, +\infty]$ is denoted by $\|\cdot\|_p$. The spaces $C^{0,\nu}(\mathbb{R}^N)$ for $\nu \in (0,1)$ are usual spaces of Hölder continuous functions. For $q > 0$ we define $q! := q(q-1)\cdots(q-\lfloor q \rfloor)$, where $\lfloor q \rfloor$ denotes the largest integer strictly less than q; if $q > 1$ its conjugate Hölder exponent is $q' := \frac{q}{q-1}$. The symbol

$\lesssim$ indicates that an inequality holds up to a multiplicative constant depending only on the structural constants and not on the functions involved. Finally, $o_n(1)$ denotes a vanishing real sequence as $n \to +\infty$. Hereafter, the letter C will be used to denote positive constants which are independent of relevant quantities and whose value may change from line to line.

2. Preliminaries

In this section we summarize some consequences of our assumptions, we recall some basic estimates as well as two well-known results, Lions' compact embedding and the Hardy-Littlewood-Sobolev inequality, which will be used throughout the whole manuscript.

REMARK 2.1. From our set of assumptions can be derived the following:

(i) From (f_1) and (f_2) it is easy to infer that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$f(t) \leq \varepsilon t^{\frac{N}{s}-1} + C_\varepsilon t^{\frac{N}{s}-1} \Phi_{N,s}(\alpha_* |t|^{\frac{N}{N-s}}) \quad \text{for all } t > 0\,,$$

as well as

$$F(t) \leq \varepsilon t^{\frac{N}{s}} + C_\varepsilon t^{\frac{N}{s}} \Phi_{N,s}(\alpha_* |t|^{\frac{N}{N-s}}) \quad \text{for all } t > 0\,;$$

(ii) Assumption (f_3) implies

$$F(t) \leq (s-\tau) t f(t) \quad \text{for any } t \geq 0\,;$$

(iii) From (f_4) one has that for any $\varepsilon > 0$ there exists M_ε such that for $t \geq M_\varepsilon$

$$F(t) \leq \varepsilon t f(t)\,, \tag{7}$$

and that there exist $t_0, M_0 > 0$ such that

$$F(t) \leq M_0 f(t) \quad \text{for any } t \geq t_0\,. \tag{8}$$

For the proof of (i)-(iii), we refer to [13, Remark 2.3] and [12, p.2 (1.3)].

LEMMA 2.2. *Let $\mu \in (0,1]$. Then, the following elementary inequality holds*

$$\frac{t^{-\mu}-1}{\mu} \geq \log \frac{1}{t} \qquad \textit{for all } t \in (0,1]\,.$$

Moreover, for all $\nu > \mu$ there exists $C_\nu > 0$ such that

$$\frac{t^{-\mu}-1}{\mu} \leq C_\nu t^{-\nu} \qquad \textit{for all } t > 0\,.$$

LEMMA 2.3 (Lemma 2.3, [30]). *Let $\alpha > 0$ and $r > 1$. Then for any $\beta > r$ there exists a constant $C_\beta > 0$ such that*

$$\left(\Phi_{N,s}\left(\alpha|t|^{\frac{N}{N-s}}\right)\right)^r \le C_\beta \Phi_{N,s}\left(\alpha\beta|t|^{\frac{N}{N-s}}\right) \qquad \textit{for all } t > 0.$$

We denote by $W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ the subspace of radial functions in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$.

LEMMA 2.4. *(see [32]) Let $s \in (0,1]$. Then $W^{s,\frac{N}{s}}_{rad}(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ for any $\frac{N}{s} < q < \infty$.*

LEMMA 2.5. *(Hardy-Littlewood-Sobolev inequality, [31, Theorem 4.3]) Let $N \ge 1$, $\mu \in (0, N)$, and $q, r > 1$ with $\frac{1}{q} + \frac{\mu}{N} + \frac{1}{r} = 2$. There exists a constant $C = C(N, \mu, q, r)$ such that for all $f \in L^q(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * f\right) h \, \mathrm{d}x \le C\|f\|_q \|h\|_r \,.$$

3. Existence results for (Ch_s) by asymptotic approximation: Proof of Theorem 1.4

3.1. The approximating method

As mentioned in the Introduction, the energy functional formally associated to (Ch_s)

$$J(u) := \frac{s}{N}\|u\|_V^{\frac{N}{s}} - \frac{C_N}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \log\frac{1}{|x-y|} F(u(y))\,\mathrm{d}y\right) F(u(x))\,\mathrm{d}x$$

is not well-defined on the natural Sobolev-Slobodeckij space $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ because of the presence of the logarithmic convolution term. This prevents the use of standard variational tools. Following [13], we will make use of the approximating method, based on the simple convergence (1), which has been developed in the local case in [12, 34]. We set

$$G_\mu(x) := \frac{|x|^{-\mu} - 1}{\mu}, \qquad \mu \in (0, 1]\,, \quad x \in \mathbb{R}^N,$$

and consider the approximating problem

$$(-\Delta)^s_{\frac{N}{s}} u + V(x)|u|^{\frac{N}{s}-2}u = C_N\left(G_\mu(\cdot) * F(u)\right) f(u) \quad \text{in } \mathbb{R}^N, \tag{9}$$

whose related energy functional is given by

$$\begin{aligned}J_\mu(u) :&= \frac{s}{N}\|u\|_V^{\frac{N}{s}} - \frac{C_N}{2}\int_{\mathbb{R}^N}(G_\mu(\cdot) * F(u))F(u)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N}V(x)|u|^{\frac{N}{s}}\,\mathrm{d}x \\ &\quad + \frac{C_N}{2\mu}\left[\int_{\mathbb{R}^N}F(u)\,\mathrm{d}x\right]^2 - \frac{C_N}{2\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^\mu}F(u(x))F(u(y))\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The advantage of this approach is that the power-type singularity in G_μ can be handled by means of the Hardy-Littlewood-Sobolev inequality (Lemma 2.5), and it is thus possible to prove under conditions (V) and (f_0)-(f_2) that J_μ is well-defined and C^1 on $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ with Fréchet derivative

$$\begin{aligned}J'_\mu(u)[\varphi] &= \int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \\ &\quad + \int_{\mathbb{R}^N}V(x)|u|^{\frac{N}{s}-2}u\varphi\,\mathrm{d}x - C_N\int_{\mathbb{R}^N}(G_\mu(\cdot) * F(u))f(u)\varphi\,\mathrm{d}x\,. \qquad (10)\end{aligned}$$

We aim at proving first that, for all μ small, the functional J_μ possesses a critical point u_μ of mountain pass type. Being able to estimate the norm in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ of such critical points uniformly with respect to μ, thanks to a careful estimate on the mountain pass level, we will then pass to the limit as $\mu \to 0$ in order to show the existence of a critical point for the original functional J, which a posteriori will satisfy (5).

Since the potential V is radially symmetric, we will work in the radial subspace $W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$, where we can exploit the compactness given by Lemma 2.4.

Let us start showing that for all $\mu \in (0,1]$ the approximating functional J_μ enjoys a mountain pass geometry.

LEMMA 3.1. *Let $\mu \in (0,1]$ and assume (f_1)-(f_3). Then, there exist constants $\rho, \eta > 0$ and $e \in W^{s,\frac{N}{s}}_{rad}(\mathbb{R}^N)$ such that:*

(i) *$\|e\| > \rho$ and $J_\mu(e) < 0$;*

(ii) *$J_\mu|_{S_\rho} \geq \eta > 0$, where $S_\rho = \{u \in W^{s,\frac{N}{s}}_{rad}(\mathbb{R}^N) \mid \|u\| = \rho\}$.*

Proof. (i) Take $e_0 \in W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ with values in $[0,1]$, $\operatorname{supp} e_0 \subset B_{\frac{1}{4}}(0)$, and such that $e_0 \equiv 1$ in $B_{\frac{1}{8}}(0)$. For $t > 0$ set

$$\Psi(t) := \frac{1}{2}\left(\int_{\mathbb{R}^N}F(te_0)\,\mathrm{d}x\right)^2. \qquad (11)$$

By Remark 2.1 we have

$$\Psi'(t) = \frac{1}{t}\left(\int_{\mathbb{R}^N} F(te_0)\right)\left(\int_{\mathbb{R}^N} f(te_0)te_0\right) \geq \frac{2\Psi(t)}{t(s-\tau)}\,,$$

hence integrating on $[1,t]$ one gets $\Psi(t) \geq \Psi(1)t^{\frac{2}{s-\tau}}$. Inserting this in the functional J_μ and using Lemma 2.2 we infer

$$\begin{aligned}
J_\mu(te_0) &\leq \frac{s}{N}t^{\frac{N}{s}}\|e_0\|_V^{\frac{N}{s}} - \frac{C_N}{2}\int_{\{|x-y|\leq\frac{1}{2}\}} \log\frac{1}{|x-y|}F(te_0(y))F(te_0(x))\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{s}{N}t^{\frac{N}{s}}\|e_0\|_V^{\frac{N}{s}} - \frac{C_N\log 2}{2}\left(\int_{\mathbb{R}^N} F(te_0)\right)^2 \\
&\leq \frac{s}{N}t^{\frac{N}{s}}\|e_0\|_V^{\frac{N}{s}} - \frac{C_N\log 2}{2}\left(\int_{\mathbb{R}^N} F(e_0)\right)^2 t^{\frac{2}{s-\tau}}.
\end{aligned}$$

Since $\tau > \left(1-\frac{2}{N}\right)s$ by (f_3), one deduces that $J_\mu(te_0) \to -\infty$ as $t \to +\infty$. Hence one can take $e := te_0$ for a sufficiently large t.

(ii) By Lemma 2.2 with $\nu = 1$ we have

$$J_\mu(u) \geq \frac{s}{N}\|u\|_V^{\frac{N}{s}} - \frac{C_1C_N}{2}\int_{\{|x-y|<1\}} \frac{F(u(x))F(u(y))}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Applying the Hardy-Littlewood-Sobolev inequality with $\mu = 1$ and $q = r = \frac{2N}{2N-1}$ we infer

$$\begin{aligned}
\int_{\{|x-y|<1\}} \frac{F(u(x))F(u(y))}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \lesssim \Bigg[\int_{\mathbb{R}^N} |u|^{\frac{2N^2}{s(2N-1)}} + \left(\int_{\mathbb{R}^N} |u|^{\frac{2N^2\theta'}{s(2N-1)}}\right)^{\frac{1}{\theta'}} \\
\cdot\left(\int_{\mathbb{R}^N} \Phi_{N,s}\left(\alpha_* r\|u\|^{\frac{N}{N-s}}\left(\frac{u}{\|u\|}\right)^{\frac{N}{N-s}}\right)\right)^{\frac{1}{\theta}}\Bigg]^{\frac{2N-1}{N}}
\end{aligned}$$

with $r > \frac{2N\theta}{2N-1}$ by Lemma 2.3. Since $\theta > 1$ is arbitrary, it is sufficient to require $\rho < \left(\frac{2N-1}{2N}\right)^{\frac{N-s}{N}}$ to be able to use on S_ρ the Moser-Trudinger inequality provided by Theorem A and uniformly bound the last term. To control the first two, instead, we use the continuous embedding given by Lions' Lemma 2.4. We end up with

$$J_\mu(u) \gtrsim \|u\|^{\frac{N}{s}} - \|u\|^{\frac{2N}{s}} \geq \eta > 0$$

for all $u \in S_\rho$, up to a smaller ρ, as desired. □

The geometry given by Lemma 3.1 ensures the existence of a PS-sequence at the mountain pass level

$$c_\mu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\mu(\gamma(t)),$$

where

$$\Gamma := \left\{\gamma \in C\big([0,1], W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)\big) \mid \gamma(0)=0, \gamma(1)=e\right\}$$

for any fixed $\mu \in (0,1)$, see e.g. [27]. In other words, there exists a sequence $(u_n^\mu)_n \subset W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$J_\mu(u_n^\mu) \to c_\mu \quad \text{and} \quad J'_\mu(u_n^\mu) \to 0 \quad \text{in} \quad \Big(W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)\Big)' \tag{12}$$

as $n \to +\infty$. For the sake of a lighter notation, hereafter we set $u_n := u_n^\mu$.

Remark 3.2. Observe that from the proof of Lemma 3.1 there exist two constants $\underline{c}, \overline{c} > 0$ independent of μ such that $\underline{c} < c_\mu < \overline{c}$.

Next, we show that PS-sequences at level c_μ are bounded.

Lemma 3.3. *Assume that (f_1)–(f_4) hold. Let $(u_n)_n \subset W^{s,\frac{N}{s}}(\mathbb{R}^N)$ be a PS-sequence of J_μ at level c_μ, then $(u_n)_n$ is bounded in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ and*

$$\left|\int_{\mathbb{R}^N} [G_\mu(x) * F(u_n)]F(u_n)\,\mathrm{d}x\right| < C\,, \quad \left|\int_{\mathbb{R}^N} [G_\mu(x) * F(u_n)]f(u_n)u_n\,\mathrm{d}x\right| < C\,. \tag{13}$$

Proof. Define

$$v_n := \begin{cases} \frac{F(u_n)}{f(u_n)}, & u_n > 0\,, \\ (s-\tau)u_n, & u_n \le 0\,. \end{cases}$$

By Remark 2.1 one has $|v_n| \lesssim |u_n|$ in $\mathbb{R}^N$, hence $v_n \in L^{\frac{N}{s}}(\mathbb{R}^N)$. Following the computations in the proof of [13, Lemma 5.6], one gets $[v_n]_{s,\frac{N}{s}} \lesssim [u_n]_{s,\frac{N}{s}}$, as well as

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{\frac{N}{s}-2}\,(u_n(x)-u_n(y))\,(v_n(x)-v_n(y))}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y \le (s-\tau)[u_n]^{\frac{N}{s}}_{s,\frac{N}{s}}\,. \tag{14}$$

In particular $v_n \in W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ and we may use it as a test function for $J'_\mu(u_n)$. In light of (14), combining the two relations in (12), by the fact that in both expressions the term $\int_{\mathbb{R}^N}[G_\mu(x)*F(u_n)]F(u_n)\,\mathrm{d}x$ appears due to the choice of the test function v_n, one obtains

$$(s-\tau)[u_n]_{s,\frac{N}{s}}^{\frac{N}{s}}+(s-\tau)\int_{\{u_n<0\}}V(x)|u_n|^{\frac{N}{s}} \\ +\int_{\{u_n\geq 0\}}V(x)|u_n|^{\frac{N}{s}-2}u_n\frac{F(u_n)}{f(u_n)}+2c_\mu-\frac{2N}{s}\|u\|_V^{\frac{N}{s}}\geq o_n(1)\,.$$

Using again Remark 2.1 to control the third term, the above expression reduces to

$$\left(\tau-\left(1-\frac{2}{N}\right)s\right)\|u_n\|_V^{\frac{N}{s}}\leq 2c_\mu+o_n(1)\,, \tag{15}$$

hence a uniform bound in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$. The bounds in (13) then follow combining (15) and (12) with $\varphi=v_n$. □

Remark 3.4. Arguing as in [13, Remark 5.7], combining the uniform bound given by Lemma 3.3 and (12), from now on we may assume that PS-sequences at level c_μ are nonnegative.

In order to have a more precise bound on the norm of $(u_n)_n$, in light of (15), we are lead to a careful estimate of the mountain pass level c_μ. Usually, in presence of nonlinearities with exponential critical growth, this is accomplished combining a growth assumption in the spirit of de Figueiredo-Miyagaki-Ruf [25] with fine estimates by means of a Moser sequence, which detects the critical Moser level in the Trudinger inequality. However, in the fractional setting this way is not feasible, since it is still not known whether the fractional analogue of the Moser sequence proposed in [38] detects the critical Moser value of the fractional Trudinger inequality, see Theorem A. A possible alternative way, often used in the literature, is to prescribe a strong control of f near zero and with a large constant, the upper bound of which is often not explicit, see e.g. [1, 3, 6, 7, 11, 17]. We propose instead an approach which combines the choice of a simple fixed test function, which allows explicit computations on its seminorm, and our assumption (f_5), in which the growth of $F(\cdot)f(\cdot)$ is bounded from below away from zero by a suitable power multiplied by a constant, on which we have an explicit lower bound. Since it is automatically satisfied by the exponential growth of f at infinity due to (f_3)-(f_4), (f_5) is a sort of middle-range assumption, which in most of the cases it is easily verifiable.

First, let us introduce our test function and compute its norm. Let $R>0$ and $\bar{w}\in C(\mathbb{R}^N)$ be such that

$$\bar{w}(x):=\begin{cases}1 & \text{for } |x|\leq\frac{R}{2}\,,\\ 2-\frac{2}{R}|x| & \text{for } |x|\in(\frac{R}{2},R)\,,\\ 0 & \text{for } |x|\geq R\,.\end{cases} \tag{16}$$

LEMMA 3.5. *For all $R \in (0,1]$ and $s \in (0,1)$ we have $\bar{w} \in W^{s,\frac{N}{s}}_{rad}(\mathbb{R}^N)$ and*

$$\|\bar{w}\|_V^{\frac{N}{s}} \leq \mathfrak{J}_N(s,R) + \mathfrak{K}_N(s),$$

where the constants $\mathfrak{J}_N(s,R)$ and $\mathfrak{K}_N(s)$ are given explicitly in (18) *and* (27), *respectively.*

Proof. First, by (V) one has

$$\int_{\mathbb{R}^N} V(x)|\bar{w}|^{\frac{N}{s}}\,\mathrm{d}x \leq \overline{V_0}\left(|B_{\frac{R}{2}}(0)| + N\omega_{N-1}\int_{\frac{R}{2}}^{R}\left|2-\frac{2}{R}\rho\right|^{\frac{N}{s}}\rho^{N-1}\,\mathrm{d}\rho\right). \quad (17)$$

By the change of variable $y = R-\rho$, we estimate

$$\begin{aligned}\int_{\frac{R}{2}}^{R}\left|2-\frac{2}{R}\rho\right|^{\frac{N}{s}}\rho^{N-1}\,\mathrm{d}\rho &= \left(\frac{2}{R}\right)^{\frac{N}{s}}\int_0^{\frac{R}{2}} y^{\frac{N}{s}}(R-y)^{N-1}\,\mathrm{d}y \\ &\leq \left(\frac{2}{R}\right)^{\frac{N}{s}}\int_0^{\frac{R}{2}} y^{\frac{N}{s}}(R-y)\,\mathrm{d}y = \frac{s(N+3s)R^2}{4(N+s)(N+2s)}\,.\end{aligned}$$

Combining this with (17), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} V(x)|\bar{w}|^{\frac{N}{s}}\,\mathrm{d}x \\ &\qquad \leq \overline{V_0}\left(\omega_N\left(\frac{R}{2}\right)^N + \omega_{N-1}\frac{Ns(N+3s)R^2}{4(N+s)(N+2s)}\right) =: \mathfrak{J}_N(s,R)\,. \quad (18)\end{aligned}$$

Since $\bar{w}$ is radially symmetric, according to [38, Proposition 4.3], we have

$$\begin{aligned}[\bar{w}]_{s,\frac{N}{s}}^{\frac{N}{s}} &= (N\omega_N)^2\int_0^{+\infty}\int_0^{+\infty}|\bar{w}(r)-\bar{w}(t)|^{\frac{N}{s}}r^{N-1}t^{N-1}\frac{r^2+t^2}{|r^2-t^2|^{N+1}}\,\mathrm{d}r\,\mathrm{d}t \\ &= 2(N\omega_N)^2\left(\int_0^{\frac{R}{2}}\int_{\frac{R}{2}}^{R} + \int_0^{\frac{R}{2}}\int_R^{+\infty} + \int_{\frac{R}{2}}^{R}\int_R^{+\infty}\right) + (N\omega_N)^2\int_{\frac{R}{2}}^{R}\int_{\frac{R}{2}}^{R} \\ &=: 2(N\omega_N)^2\left(I_1+I_2+I_3\right) + (N\omega_N)^2 I_4\end{aligned} \quad (19)$$

and let us next estimate I_i, $i=1,\dots,4$, separately. Recalling that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{N}\frac{r^N}{(t^2-r^2)^N}\right) = r^{N-1}\frac{r^2+t^2}{|t^2-r^2|^{N+1}}\,, \quad (20)$$

we have

$$\begin{aligned} I_1 &= \int_{\frac{R}{2}}^{R} \left|\frac{2}{R}t-1\right|^{\frac{N}{s}} t^{N-1}\left(\int_0^{\frac{R}{2}} r^{N-1}\frac{r^2+t^2}{|t^2-r^2|^{N+1}}\,\mathrm{d}r\right)\mathrm{d}t \\ &= \frac{1}{N}\left(\frac{R}{2}\right)^{N-\frac{N}{s}} \int_{\frac{R}{2}}^{R}\left|t-\frac{R}{2}\right|^{\frac{N}{s}} \frac{t^{N-1}}{\left(t-\frac{R}{2}\right)^N\left(t+\frac{R}{2}\right)^N}\,\mathrm{d}t \\ &\le \frac{R^{N-1}}{N}\left(\frac{2}{R}\right)^{\frac{N}{s}}\left[\frac{\left(t-\frac{R}{2}\right)^{\frac{N}{s}-N+1}}{\frac{N}{s}-N+1}\right]_{t=\frac{R}{2}}^{t=R} = \frac{1}{N2^{N-1}\left(\frac{N}{s}-N+1\right)}\,. \end{aligned} \tag{21}$$

Concerning the second term we get similarly

$$I_2 = \frac{R^N}{N}\int_0^{\frac{R}{2}}\frac{r^{N-1}}{(R^2-r^2)^N}\,\mathrm{d}r \le \frac{1}{N}\left(\frac{R}{2}\right)^{N-1}\int_0^{\frac{R}{2}}\frac{\mathrm{d}r}{(R-r)^N} = \frac{2^{N-1}-1}{2^{N-1}(N-1)N}\,. \tag{22}$$

The third term can be estimated as

$$\begin{aligned} I_3 &= \int_{\frac{R}{2}}^{R}\left|\frac{2}{R}r-1\right|^{\frac{N}{s}} r^{N-1}\left(\int_R^{+\infty} t^{N-1}\frac{r^2+t^2}{(t^2-r^2)^{N+1}}\,\mathrm{d}t\right)\mathrm{d}r \\ &= \frac{1}{N}\left(\frac{2}{R}\right)^{\frac{N}{s}} R^N \int_{\frac{R}{2}}^{R}(r-R)^{\frac{N}{s}-N}\frac{r^{N-1}}{(R+r)^N}\,\mathrm{d}r \\ &\le \frac{2^{\frac{N}{s}}}{N}R^{N-1-\frac{N}{s}}\int_0^{\frac{R}{2}} y^{\frac{N}{s}-N}\,\mathrm{d}y = \frac{2^{N-1}}{N\left(\frac{N}{s}-N+1\right)}\,. \end{aligned} \tag{23}$$

Finally,

$$\begin{aligned} I_4 &= \left(\frac{2}{R}\right)^{\frac{N}{s}}\int_{\frac{R}{2}}^{R} r^{N-1}\left(\int_{\frac{R}{2}}^{R}|r-t|^{\frac{N}{s}}t^{N-1}\frac{r^2+t^2}{|t^2-r^2|^{N+1}}\,\mathrm{d}t\right)\mathrm{d}r \\ &= \left(\frac{2}{R}\right)^{\frac{N}{s}}\int_{\frac{R}{2}}^{R} r^{N-1}\left[|r-t|^{\frac{N}{s}}\cdot\frac{1}{N}\frac{t^N}{|t^2-r^2|^N}\right]_{t=\frac{R}{2}}^{t=R}\mathrm{d}r \\ &\quad + \left(\frac{2}{R}\right)^{\frac{N}{s}}\int_{\frac{R}{2}}^{R}\frac{r^{N-1}}{s}\int_{\frac{R}{2}}^{r}\left[(r-t)^{\frac{N}{s}-1}\cdot\frac{t^N}{(r^2-t^2)^N}\right]\mathrm{d}t\,\mathrm{d}r \\ &\quad - \left(\frac{2}{R}\right)^{\frac{N}{s}}\int_{\frac{R}{2}}^{R}\frac{r^{N-1}}{s}\int_{r}^{R}\left[(t-r)^{\frac{N}{s}-1}\cdot\frac{t^N}{(t^2-r^2)^N}\right]\mathrm{d}t\,\mathrm{d}r \\ &=: A_1 + A_2 - A_3 \le A_1 + A_2\,, \end{aligned} \tag{24}$$

since $A_3 \geq 0$. Moreover,

$$\begin{aligned} A_1 &\leq \left(\frac{2}{R}\right)^{\frac{N}{s}} \int_{\frac{R}{2}}^{R} \frac{r^{N-1}}{N} (R-r)^{\frac{N}{s}-N} \frac{R^N}{(R+r)^N} \, \mathrm{d}r \\ &\leq \left(\frac{2}{R}\right)^{\frac{N}{s}} R^{N-1} \int_{\frac{R}{2}}^{R} (R-r)^{\frac{N}{s}-N} \, \mathrm{d}r = \frac{1}{N2^{N-1}\left(\frac{N}{s}-N+1\right)} \end{aligned} \tag{25}$$

and

$$\begin{aligned} A_2 &\leq \left(\frac{2}{R}\right)^{\frac{N}{s}} \frac{R^{N-1}}{s} \int_{\frac{R}{2}}^{R} \int_{\frac{R}{2}}^{r} (r-t)^{\frac{N}{s}-1-N} \, \mathrm{d}t \, \mathrm{d}r \\ &\leq \left(\frac{2}{R}\right)^{\frac{N}{s}} \frac{R^{N-1}}{s} \int_{\frac{R}{2}}^{R} \left(r - \frac{R}{2}\right)^{\frac{N}{s}-N} \mathrm{d}r = \frac{2^{N-1}}{s\left(\frac{N}{s}-N+1\right)} \, . \end{aligned} \tag{26}$$

Eventually, from (19)-(26) we get

$$[\bar{w}]^{\frac{N}{s}}_{s,\frac{N}{s}} \leq \frac{N\omega_N^2}{\frac{N}{s}-N+1} \left(\frac{3}{2^N} + \frac{(2^N-2)\left(\frac{N}{s}-N+1\right)}{2^N(N-1)} + 2^N \left(1+\frac{N}{2s}\right)\right) =: \mathfrak{K}_N(s) \, . \tag{27}$$

□

Thanks to this information on $\bar{w}$ in $W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ we are now able to estimate the mountain pass level via (f_5).

Lemma 3.6. *Under (f_1)-(f_5) one has $c_\mu < \frac{s}{2N}$ for all $\mu \in (0,1]$.*

Proof. Let $\bar{w} \in W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ defined in (16) and $w := \bar{w}\|\bar{w}\|_V^{-1}$. Then there exists $T > 0$ such that $J_\mu(Tw) = \max_{t\geq 0} J_\mu(tw)$ and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} J_\mu(tw) = J'_\mu(Tw)w = 0$. Suppose by contradiction that $J_\mu(Tw) \geq \frac{s}{2N}$, then

$$\frac{s}{N} T^{\frac{N}{s}} \geq \frac{s}{2N} + \frac{C_N}{2} \int_{\mathbb{R}^N} (G_\mu(\cdot) * F(Tw)) \, F(Tw) \, . \tag{28}$$

Requiring $R \leq \frac{1}{3}$, then $G_\mu(x-y) \geq \log \frac{1}{|x-y|} \geq \log \frac{3}{2} > 0$, namely the convolution term is positive. We therefore obtain from (28) that

$$T \geq \left(\frac{1}{2}\right)^{\frac{s}{N}} . \tag{29}$$

Exploiting the fact that $J'_\mu(Tw)[Tw]=0$, we get

$$\begin{aligned}
T^{\frac{N}{s}} &= C_N \int_{\mathbb{R}^N} (G_\mu(\cdot) * F(Tw))\, f(Tw) Tw \,\mathrm{d}x \\
&\geq \int_{B_{\frac{R}{2}}(0)} \int_{B_{\frac{R}{2}}(0)} \log \frac{1}{|x-y|} F(Tw(y)) f(Tw(x)) Tw(x) \,\mathrm{d}x \,\mathrm{d}y \\
&\geq \log 3 \int_{B_{\frac{R}{2}}(0)} F(Tw) \,\mathrm{d}y \int_{B_{\frac{R}{2}}(0)} f(Tw) Tw \,\mathrm{d}x \\
&\geq \log 3 \left(\int_{B_{\frac{R}{2}}(0)} \sqrt{F(Tw) f(Tw) Tw} \,\mathrm{d}x \right)^2 .
\end{aligned}$$

Since $\bar{w} \equiv 1$ in $B_{\frac{R}{2}}(0)$, then

$$Tw = T\|\bar{w}\|_V^{-1} \geq \left(2^{\frac{s}{N}} (\mathfrak{I}_N(s,R) + \mathfrak{K}_N(s))\right)^{-\frac{s}{N}}$$

having used (29) and Lemma 3.5. Defining thus

$$T_N(s,R) := \left(2^{\frac{s}{N}} (\mathfrak{I}_N(s,R) + \mathfrak{K}_N(s))\right)^{-\frac{s}{N}}, \tag{30}$$

we may apply (f_5) and get

$$T^{\frac{N}{s}} \geq C_N \log 3 |B_{\frac{R}{2}}(0)|^2 \beta (Tw)^{\frac{N}{s}+1},$$

which yields, using again (28),

$$\beta \leq \beta 2^{\frac{N}{s}} T \leq \left(\frac{2}{R}\right)^N \frac{2^{\frac{N}{s}} \|\bar{w}\|_V^{\frac{N}{s}+1}}{\omega_N^2 C_N \log 3}.$$

Fixing now $R = \frac{1}{3}$ and defining $\mathfrak{I}_N(s) := \mathfrak{I}_N(s, \frac{1}{3})$ and $T_N(s) := T_N(s, \frac{1}{3})$, from Lemma 3.5 one gets

$$\beta \leq \frac{2^{\frac{N}{s}+N} 3^N}{\omega_N^2 C_N \log 3} \left(\mathfrak{I}_N(s) + \mathfrak{K}_N(s)\right)^{\frac{s}{N}+1} =: \beta_0, \tag{31}$$

which finally contradicts the lower bound on β in (f_5). This proves that $J_\mu(Tw) < \frac{s}{2N}$ and the conclusion follows from the definition of c_μ. □

Combining (15) and Lemma 3.6 we immediately obtain an estimate on the norm of $(u_n)_n$ *which is uniform with respect to* μ:

$$\|u_n\|_V^{\frac{N}{s}} \leq \frac{\frac{s}{N} + o_n(1)}{\tau - \left(1 - \frac{2}{N}\right) s} < \frac{s}{\tau - \left(1 - \frac{2}{N}\right) s}. \tag{32}$$

In order to get a critical point for the functional J_μ, we need first an integrability result for $F(u_n)$ in Lebesgue spaces.

LEMMA 3.7. *Suppose* (f_1)*–*(f_5) *hold and let* $(u_n)_n$ *be a PS-sequence of* J_μ *at level* c_μ. *Then there exists* $C > 0$ *independent of* n *and* μ *such that*

$$\int_{\mathbb{R}^N} f(u_n)u_n \,\mathrm{d}x \le C \quad \text{and} \quad \int_{\mathbb{R}^N} F(u_n)^\kappa \,\mathrm{d}x \le C \tag{33}$$

for any $\kappa \in \left[1, \frac{1}{\gamma_N(s,\tau)}\right)$, *where* $\gamma_N(s,\tau) \in (0,1)$ *is a constant depending just on* N, s, *and* τ.

Proof. Let us introduce the following auxiliary function

$$H_N(t) := t - \frac{N}{2s}\frac{F(t)}{f(t)} \quad \text{for } t \ge 0\,,$$

and define $v_n := H_N(u_n)$. Similarly to the proof of Lemma 3.3, one has $v_n \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$. We first *claim* that there exists $\gamma_N(s,\tau) \in (0,1)$ depending just on N, s, and τ, such that

$$\|v_n\|_V^{\frac{N}{s}} \le \gamma_N(s,\tau) < 1 \tag{34}$$

for n large enough. If so, then by (f_4) for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\frac{F(t)}{f(t)} \le \varepsilon(t - t_\varepsilon) + \frac{F(t_\varepsilon)}{f(t_\varepsilon)} \quad \text{for all } t \ge t_\varepsilon\,.$$

Hence,

$$v_n = H_N(u_n) = \left(1 - \frac{\varepsilon N}{2s}\right)(u_n - t_\varepsilon) + t_\varepsilon - \frac{N}{2s}\frac{F(t_\varepsilon)}{f(t_\varepsilon)} \ge \left(1 - \frac{\varepsilon N}{2s}\right)(u_n - t_\varepsilon)\,,$$

having applied (f_3) in the last inequality. This implies

$$u_n(x) \le t_\varepsilon + \frac{v_n(x)}{1-\bar{\varepsilon}} \quad \text{for all } x \in \mathbb{R}^N, \tag{35}$$

where $\bar{\varepsilon} := \frac{\varepsilon N}{2s}$. Hence, by (f_1)-(f_2) we deduce that for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} F(u_n)^\kappa \,\mathrm{d}x &\le \int_{\{|u_n|<t_\varepsilon\}} F(u_n)^\kappa \,\mathrm{d}x + \int_{\{|u_n|\ge t_\varepsilon\}} F(u_n)^\kappa \,\mathrm{d}x \\
&\le C_\varepsilon \|u_n\|^{\frac{N}{s}\kappa} + C_\varepsilon \int_{\{|u_n|\ge t_\varepsilon\}} \Phi_{N,s}\left(\alpha_*\kappa\theta\left(t_\varepsilon + \frac{v_n}{1-\bar{\varepsilon}}\right)^{\frac{N}{N-s}}\right)\mathrm{d}x \\
&\le C_\varepsilon \|u_n\|^{\frac{N}{s}\kappa} + C_\varepsilon \int_{\mathbb{R}^N} \Phi_{N,s}\left(\alpha_*\kappa\theta(1+\bar{\varepsilon})\left(\frac{v_n}{1-\bar{\varepsilon}}\right)^{\frac{N}{N-s}}\right)\mathrm{d}x
\end{aligned} \tag{36}$$

with $\theta > 1$ by Lemma 2.3. By (34) there exists $\sigma > 0$ such that $\|v_n\|_V^{\frac{N}{s}} \leq \gamma_N(s,\tau) + \sigma < 1$ for n large enough, therefore

$$\frac{\kappa\theta(1+\bar{\varepsilon})}{(1-\bar{\varepsilon})^{\frac{N}{N-s}}} \|v_n\|_V^{\frac{N}{N-s}} \leq \frac{\kappa\theta(1+\bar{\varepsilon})}{(1-\bar{\varepsilon})^{\frac{N}{N-s}}} (\gamma_N(s,\tau)+\sigma) < 1 \tag{37}$$

for $\kappa \in \left[1, \frac{1}{\gamma_N(s,\tau)}\right)$, for suitable choices of for $\varepsilon > 0$ and $\theta > 1$ small enough. Thanks to Theorem A with $\lambda = \min\{1, \underline{V_0}^{\frac{s}{N}}\}$, this yields the second inequality in (33). Similarly, but more easily, one can also prove the first inequality.

The remaining part of the proof is devoted to justifying of the claim (34) above. Combining the two relations in (12) as in Lemma 13, one has

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{N}{2s} \frac{|u_n(x)-u_n(y)|^{\frac{N}{s}-2}(u_n(x)-u_n(y))}{|x-y|^{2N}} \left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right) \mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{N}{2s}\int_{\mathbb{R}^N} |u_n|^{\frac{N}{s}-2} u_n \frac{F(u_n)}{f(u_n)}\,\mathrm{d}x - \|u_n\|_V^{\frac{N}{s}} - \frac{N}{s} c_\mu = o_n(1)\,.
\end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned}
\|v_n\|_V^{\frac{N}{s}} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|H_N(u_n(x))-H_N(u_n(y))|^{\frac{N}{s}}}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} V(x)\,|H_N(u_n)|^{\frac{N}{s}}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|H_N(u_n(x))-H_N(u_n(y))|^{\frac{N}{s}}}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\frac{N}{2s}|u_n(x)-u_n(y)|^{\frac{N}{s}-2}(u_n(x)-u_n(y))\left(\frac{F(u_n(x))}{f(u_n(x))} - \frac{F(u_n(y))}{f(u_n(y))}\right)}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N} V(x)\left(|H_N(u_n)|^{\frac{N}{s}} + \frac{N}{2s}\int_{\mathbb{R}^N} |u_n|^{\frac{N}{s}-2} u_n \frac{F(u_n)}{f(u_n)}\right)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{N}{s} c_\mu - [u_n]_{s,\frac{N}{s}}^{\frac{N}{s}} - V(x)|u_n|^{\frac{N}{s}}\,\mathrm{d}x + o_n(1)\,.
\end{aligned} \tag{39}$$

Let us denote by $Z_N(u_n)$ the first term in the right-hand side. Using the mean value theorem we infer

$$\begin{aligned}
Z_N(u_n) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Bigg[&\left|1 - \tfrac{N}{2s}\left(1 - \tfrac{Ff'}{f^2}(\theta(x,y))\right)\right|^{\frac{N}{s}} \\
&+ \tfrac{N}{2s}\left(1 - \tfrac{Ff'}{f^2}(\theta(x,y))\right)\Bigg] \frac{|u_n(x)-u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

for some $\theta(x,y) \in (\min\{u_n(x), u_n(y)\}, \max\{u_n(x), u_n(y)\})$. Splitting $\mathbb{R}^{2N} = A \cup (\mathbb{R}^{2N} \setminus A)$ and $Z_N(u_n) = Z_A(u_n) + Z_{\mathbb{R}^{2N}\setminus A}(u_n)$ accordingly, we estimate the two terms separately. Since

$$\left|1 - \tfrac{N}{2s}\left(1 - \tfrac{Ff'}{f^2}(\theta)\right)\right|^{\frac{N}{s}} \le 1 - \tfrac{N}{2s}\left(1 - \tfrac{Ff'}{f^2}(\theta)\right),$$

one infers

$$Z_{\mathbb{R}^{2N}\setminus A}(u_n) \le \int_{\mathbb{R}^{2N}\setminus A} \frac{|u_n(x) - u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\, \mathrm{d}x\, \mathrm{d}y\,. \tag{40}$$

On the other hand, applying [13, Lemma 5.9] with $q = \frac{N}{s}$ and $W = \frac{N}{2s}\left(\frac{Ff'}{f^2}(\theta) - 1\right) > 0$,

$$Z_A(u_n) - \int_A \frac{|u_n(x) - u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\, \mathrm{d}x\, \mathrm{d}y \le \frac{\frac{N}{s}!}{\frac{N}{s} - \lfloor \frac{N}{s} \rfloor} \int_A \frac{W}{1-W} \frac{|u_n(x) - u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\, \mathrm{d}x\, \mathrm{d}y\,.$$

Imposing

$$\frac{F(t)f'(t)}{f^2(t)} < 1 + \frac{s}{N} \tag{41}$$

one finds $\|W\|_\infty < \frac{1}{2}$, from which

$$\begin{aligned} Z_A(u_n) - \int_A \frac{|u_n(x) - u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\, \mathrm{d}x\, \mathrm{d}y &\le \frac{\frac{N}{s}!}{\frac{N}{s} - \lfloor \frac{N}{s} \rfloor} \int_A \frac{W}{1-W} \frac{|u_n(x) - u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\, \mathrm{d}x\, \mathrm{d}y \\ &\le \frac{2\frac{N}{s}!}{\frac{N}{s} - \lfloor \frac{N}{s} \rfloor} \|W\|_\infty \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{\frac{N}{s}}}{|x-y|^{2N}}\, \mathrm{d}x\, \mathrm{d}y \\ &\le \frac{2s\frac{N}{s}!}{\left(\frac{N}{s} - \lfloor \frac{N}{s} \rfloor\right)\left(\tau - \left(1 - \frac{2}{N}\right)s\right)} \|W\|_\infty\,, \end{aligned}$$

having used (32). This, combined with (40), yields

$$Z(u_n) - [u_n]_{s,\frac{N}{s}}^{\frac{N}{s}} \le \frac{2s\frac{N}{s}!\|W\|_\infty}{\left(\frac{N}{s} - \lfloor \frac{N}{s} \rfloor\right)\left(\tau - \left(1 - \frac{2}{N}\right)s\right)}\,. \tag{42}$$

Moreover, since $s-\tau<\frac{2}{N}s$, from (f_3) one has

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|H_N(u_n)|^{\frac{N}{s}}\,\mathrm{d}x &= \int_{\mathbb{R}^N} V(x)|u_n|^{\frac{N}{s}}\left|1-\frac{N}{2s}\frac{F(u_n)}{f(u_n)u_n}\right|^{\frac{N}{s}}\mathrm{d}x \\ &\le -\frac{N}{2s}\int_{\mathbb{R}^N} V(x)|u_n|^{\frac{N}{s}}\frac{F(u_n)}{f(u_n)u_n}\,\mathrm{d}x\,. \end{aligned} \tag{43}$$

Combining (43) and (42) with (39), one finally gets

$$\|v_n\|_V^{\frac{N}{s}} \le \frac{2s\frac{N}{s}!\|W\|_\infty}{\left(\frac{N}{s}-\lfloor\frac{N}{s}\rfloor\right)\left(\tau-\left(1-\frac{2}{N}\right)s\right)} + \frac{N}{s}c_\mu + o_n(1)\,.$$

Since $c_\mu<\frac{s}{2N}$ by Lemma 3.6, and recalling the definition of W, we obtain (34), provided

$$\left\|\frac{Ff'}{f^2}\right\|_\infty < 1+\frac{\frac{N}{s}-\lfloor\frac{N}{s}\rfloor}{\frac{N}{s}\frac{N}{s}!}\left(\frac{\tau\left(1-\frac{2}{N}\right)s}{2s}\right). \tag{44}$$

Defining now

$$\mu_N(s,\tau) := \min\left\{\frac{s}{N},\ \frac{\frac{N}{s}-\lfloor\frac{N}{s}\rfloor}{\frac{N}{s}\frac{N}{s}!}\left(\frac{\tau\left(1-\frac{2}{N}\right)s}{2s}\right)\right\}, \tag{45}$$

assumption (f_3) guarantees that both conditions (41) and (44) are satisfied and the proof is concluded. □

We are finally in a position to show the existence of a critical point of mountain pass type for the approximating functional J_μ.

PROPOSITION 3.8. *Assume that* (f_1)*–*(f_5) *hold. For all* $\mu\in(0,1)$ *sufficiently small there exists a positive* $u_\mu\in W^{s,\frac{N}{s}}_{rad}(\mathbb{R}^N)$ *such that* $J'_\mu(u_\mu)=0$.

Proof. Having in hand the results obtained so far in this section, the proof follows the lines of [13, Proposition 5.10]. Let us just resume here the main steps.

By Lemma 3.1 and Remark 3.4, for all $\mu\in(0,1)$ there exists a nonnegative PS-sequence $(u_n^\mu)_n\subset W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ of J_μ at level c_μ, which is bounded in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ by Lemma 3.3. Hence there exists a nonnegative $u_\mu\in W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{aligned} &u_n^\mu \rightharpoonup u_\mu \quad \text{in } W^{s,\frac{N}{s}}(\mathbb{R}^N),\\ &u_n^\mu \to u_\mu \quad \text{a.e. in } \mathbb{R}^N \text{ and in } L^p(\mathbb{R}^N) \text{ for all } p\in\left(\tfrac{N}{s},+\infty\right). \end{aligned} \tag{46}$$

First, we prove that

$$\int_{\mathbb{R}^N} F(u_n^\mu)\,\mathrm{d}x \to \int_{\mathbb{R}^N} F(u_\mu)\,\mathrm{d}x\,. \tag{47}$$

Indeed, by the mean value theorem, there exists $\tau_n(x) \in (0,1)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |F(u_n^\mu) - F(u_\mu)|\,\mathrm{d}x = \int_{\mathbb{R}^N} |f\left(u_\mu + \tau_n(x)(u_n^\mu - u_\mu)\right)(u_n^\mu - u_\mu)|\,\mathrm{d}x \\
&\le \varepsilon \int_{\mathbb{R}^N} (|u_n^\mu| + |u_\mu|)^{\frac{N}{s}-1} |u_n^\mu - u_\mu|\,\mathrm{d}x \\
&\quad + C_\varepsilon \int_{\mathbb{R}^N} (|u_n^\mu| + |u_\mu|)^{\frac{N}{s}-1} \Phi_{N,s}\Big(\alpha_* |u_\mu + \tau_n(x)(u_n^\mu - u_\mu)|^{\frac{N}{s}}\Big) |u_n^\mu - u_\mu|\,\mathrm{d}x \\
&\le \varepsilon \left(\|u_n^\mu\|_{\frac{N}{s}} + \|u_\mu\|_{\frac{N}{s}}\right) \|u_n^\mu - u_\mu\|_{\frac{N}{s}} \\
&\quad + C_\varepsilon \int_{\{u_n^\mu > u_\mu\}} |u_n^\mu|^{\frac{N}{s}-1} \Phi_{N,s}(\alpha_* |u_n^\mu|^{\frac{N}{s}}) |u_n^\mu - u_\mu| \\
&\quad + C_\varepsilon \int_{\{u_n^\mu \le u_\mu\}} |u_\mu|^{\frac{N}{s}-1} \Phi_{N,s}(\alpha_* |u_n|^{\frac{N}{s}}) |u_n^\mu - u_\mu| \\
&=: C\varepsilon + S_1 + S_2
\end{aligned}$$

by (32). We have

$$\begin{aligned}
S_1 \lesssim \left(\int_{\mathbb{R}^N} |u_n^\mu|^{(\frac{N}{s}-1)\theta'\eta}\right)^{\frac{1}{\theta'\eta}} \left(\int_{\mathbb{R}^N} \Phi_{N,s}\left(\alpha_* r \left| t_\varepsilon + \frac{v_n}{1-\bar{\varepsilon}}\right|^{\frac{N}{s}}\right)\right)^{\frac{1}{\theta'\eta'}} \\
\cdot \left(\int_{\mathbb{R}^N} |u_n^\mu - u_\mu|^\theta\right)^{\frac{1}{\theta}} = o_n(1),
\end{aligned}$$

with $r > \theta'\eta'$, by choosing $\theta > \frac{N}{s}$ and $\left(\frac{N}{s} - 1\right)\theta'\eta > \frac{N}{s}$. The uniform boundedness of the second term can be proved arguing as in the proof of Lemma 3.7. The term S_2 can be similarly handled.

Next, we show by means of Lemma 3.7 that there exists $C > 0$ independent of n and $\mu \in (0, \mu_0)$ such that for all $x \in \mathbb{R}^N$ one has

$$\int_{\mathbb{R}^N} \frac{F(u_n(y))}{|x-y|^\mu}\,\mathrm{d}y \le C\,. \tag{48}$$

Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{F(u_n(y))}{|x-y|^\mu}\,\mathrm{d}y &= \int_{\{|x-y|\ge 1\}} \frac{F(u_n^\mu(y))}{|x-y|^\mu}\,\mathrm{d}y + \int_{\{|x-y|<1\}} \frac{F(u_n^\mu(y))}{|x-y|^\mu}\,\mathrm{d}y \\
&\le \int_{\mathbb{R}^N} F(u_n^\mu) + \left(\int_{\{|x-y|<1\}} \frac{\mathrm{d}y}{|x-y|^{\mu q}}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} F(u_n^\mu)^{q'}\right)^{\frac{1}{q'}}.
\end{aligned}$$

Choosing q sufficiently large so that $q' \in \left(1, \frac{1}{\gamma_N(s,\tau)}\right)$, the first and last terms are bounded by Lemma 3.7; then, it is sufficient to choose μ_0 sufficiently small so that $|\cdot|^{\mu q} \in L^1(B_1(0))$ for all $\mu \in (0, \mu_0)$. As a consequence, one deduces that

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * F(u_n^\mu)\right) F(u_n^\mu)\,\mathrm{d}x \to \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * F(u_\mu)\right) F(u_\mu)\,\mathrm{d}x\,. \tag{49}$$

Indeed, first by $J'_\mu(u_n)[u_n] = o_n(1)$ and Lemmas 3.3 and 3.7 one infers that

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * F(u_n)\right) f(u_n)u_n \le C$$

uniformly with respect to both n and $\mu \in (0,1)$. This implies that the queues of both integrals in (49) are small, by using (8). On the other hand, the convergence in the interior part of the integrals in (49) can be proved via the dominated convergence theorem relying on (48) and [25, Lemma 2.1]; see [39, Lemma 3.6] for more details.

In a similar way one can also prove that for all $\varphi \in W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ one has

$$\int_{\mathbb{R}^N} f(u_n^\mu)\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^N} f(u_\mu)\varphi\,\mathrm{d}x \tag{50}$$

and

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * F(u_n^\mu)\right) f(u_n^\mu)\varphi\,\mathrm{d}x \to \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^\mu} * F(u_\mu)\right) f(u_\mu)\varphi\,\mathrm{d}x\,. \tag{51}$$

Combining then (47), (50), (51), and the weak convergence $u_n^\mu \rightharpoonup u_\mu$ in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$, we deduce that u_μ is a critical point of J_μ. By exploiting the monotonicity of the operator $(-\Delta)^s_{\frac{N}{s}}$, one can also get $u_n^\mu \to u_\mu$ strongly in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$, which implies, by the continuity of J_μ, that $J_\mu(u_n^\mu) \to J_\mu(u_\mu)$ and so $J_\mu(u_\mu) = c_\mu \ge \underline{c}$ by Remark 3.2. This readily implies that $u_\mu \ne 0$. Its positivity follows by the strong maximum principle for the $p-$fractional Laplacian [22, Theorem 1.4]. ☐

By Proposition 3.8 we obtained a positive critical point for J_μ for all μ small, say $\mu \in (0, \bar{\mu})$. Note that all relevant estimates obtained so far, in particular (32), but also (33), (48), as well as the one for the mountain pass level of Lemma 3.6, are independent of μ. Therefore, thanks to the convergences (46), (47), and $u_n^\mu \to u_\mu$ in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$, one can further get the existence of $u_0 \in W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$\begin{aligned} &u_\mu \rightharpoonup u_0 \quad \text{in } W^{s,\frac{N}{s}}(\mathbb{R}^N),\\ &u_\mu \to u_0 \quad \text{a.e. in } \mathbb{R}^N \text{ and in } L^p(\mathbb{R}^N) \text{ for all } p \in \left(\tfrac{N}{s}, +\infty\right). \end{aligned} \tag{52}$$

In light of Lemma 3.7, since the constants are uniform with respect to μ, it is easy to infer that

$$\int_{\mathbb{R}^N} F(u_\mu)\,\mathrm{d}x \le C \quad \text{and} \quad \int_{\{|x-y|\le 1\}} \frac{F(u_\mu(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\,\mathrm{d}y \le C \tag{53}$$

for $\omega \in \big[1, \frac{1}{\gamma_N(s,\tau)}\big)$. Here, to prove the second inequality from the first, one argues as for (48). Actually, closely following the strategy in [13, Lemma 5.11], which is based on (35)-(36), one may also prove that

$$\int_{\{|x-y|\le 1\}} \frac{F(u_\mu(y))}{|x-y|^{\frac{4(\omega-1)}{3\omega}}}\,\mathrm{d}y \to 0 \qquad \text{as} \ \ |x| \to +\infty\,, \tag{54}$$

uniformly for $\mu \in (0,\bar{\mu})$. Before we proceed, we need some regularity properties for the solution sequence $(u_\mu)_\mu$.

LEMMA 3.9. *Let $\mu \in (0,\bar{\mu})$ and $u_\mu \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$ be a weak solution of* (9), *then there exists $C > 0$ independent of μ such that $\|u_\mu\|_\infty \le C$. Furthermore, there exists $R > 0$ such that for $|x| \ge R$,*

$$|u_\mu(x)| \lesssim \frac{1}{1+|x|^{\frac{s(2N+3)}{2(N-s)}}} \tag{55}$$

uniformly for $\mu \in (0, \frac{4(\omega-1)}{3\omega})$.

Proof. One can prove the boundedness in $L^\infty(\mathbb{R}^N)$ of the sequence $(u_\mu)_\mu$ following the approach of [36] for the $\big(s, \frac{N}{s}\big)$-fractional Schrödinger equation, which exploits a Moser iteration technique. It is easy to adapt to our setting the computations in [13, Lemma 5.12] which are detailed for the planar case $N = 2$.

In order to prove the decay estimate, we start by showing that there exist $C, C_0 > 0$ such that for all $x \in \mathbb{R}^N$ one has

$$\int_{\mathbb{R}^N} G_\mu(x-y)F(u_\mu(y))\,\mathrm{d}y \le \frac{C}{\mu}\left(\left(\frac{|x|}{2}\right)^{-\mu} - 1\right) + C_0\,. \tag{56}$$

Indeed, first

$$\int_{\left\{|y|\le \frac{|x|}{2}\right\}} G_\mu(x-y)F(u_\mu(y))\,\mathrm{d}y \le \frac{1}{\mu}\left(\left(\tfrac{|x|}{2}\right)^{-\mu} - 1\right)\int_{\mathbb{R}^N} F(u_\mu)\,\mathrm{d}y \le \frac{C}{\mu}\left(\left(\tfrac{|x|}{2}\right)^{-\mu} - 1\right) \tag{57}$$

by (53). On the other hand,

$$\int_{\left\{|y|\ge \frac{|x|}{2}\right\}\cap B_1(x)^c} G_\mu(x-y)F(u_\mu(y))\,\mathrm{d}y \le 0\,, \tag{58}$$

since in $B_1(0)^c$ one has $G_\mu(z) \le 0$, while

$$\int_{\left\{|y|\ge\frac{|x|}{2}\right\}\cap B_1(x)} G_\mu(x-y)F(u_\mu(y))\,\mathrm{d}y \lesssim \int_{B_1(0)} G_\mu(z)\,\mathrm{d}z = C_0 < +\infty\,. \quad (59)$$

The estimate (56) follows by combining (57)-(59). Since the right-hand side of (56) pointwisely converges to $-C\log\frac{|x|}{2}+C_0$, there exists $R_1>0$ independent of μ small such that for all $\varphi \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$ such that $\varphi \ge 0$ and $\operatorname{supp}\varphi \subset B_{R_1}(0)^c$ one has

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{|u_\mu(x)-u_\mu(y)|^{\frac{N}{s}-2}(u_\mu(x)-u_\mu(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \\ &\qquad + \int_{\mathbb{R}^N} V(x)|u_\mu|^{\frac{N}{s}-2}u_\mu\varphi\,\mathrm{d}x \\ &= C_N\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} G_\mu(x-y)F(u_\mu(y))\,\mathrm{d}y\right) f(u_\mu(x))\varphi(x)\,\mathrm{d}x < 0 \quad (60)\end{aligned}$$

as $|x|>R_1$. Defining now

$$w(x) = \frac{1}{1+|x|^a} \quad \text{with } a = \frac{s(2N+3)}{2(N-s)}\,, \quad (61)$$

since $a > \frac{Ns}{N-s}$, by [23, Lemma 7.1] there exists $R_2>0$ such that

$$(-\Delta)^s_{\frac{N}{s}} w(x) \lesssim \frac{1}{|x|^{2N}} \quad \text{for all } |x| > R_2\,, \quad (62)$$

therefore

$$(-\Delta)^s_{\frac{N}{s}} w(x) + V(x)|w|^{\frac{N}{s}-2}w \ge 0 \quad \text{for all } |x| > \widetilde{R}_2 \quad (63)$$

for a suitable $\widetilde{R}_2 \ge R_2$. Since $\|u_\mu\|_\infty \le C$ for all $\mu \in (0,\bar\mu)$, there exists $C_1>0$ such that

$$\psi(x) := u_\mu(x) - C_1 w(x) \le 0 \qquad \text{for} \quad |x| = R_3 := \max\{R_1, \widetilde{R}_2\}.$$

Define now $\widetilde{\psi} := \psi^+\chi_{\mathbb{R}^N\setminus B_{R_3}(0)}$, which belongs to $W^{s,\frac{N}{s}}_{\mathrm{rad}}(\mathbb{R}^N)$ and has $\operatorname{supp}\widetilde{\psi} \subset B_{R_1}(0)^c$ by construction, use it as a test function in (60) and, arguing as in the final part of the proof of [13, Lemma 3.4], one finds $\widetilde{\psi} > 0$ in $\mathbb{R}^N \setminus B_{R_3}(0)$, that is, the decay estimate in (55). □

Now we are in a position to prove the existence of a nontrivial critical point for the original functional J. In view of the results obtained so far, the strategy closely follows the one used in [13], see also [39]. Here we report only the main steps.

Proof of Theorem 1.4. Since $J'_\mu(u_\mu) = 0$, (60) holds for any test function $\varphi \in W^{s,\frac{N}{s}}_{\rm rad}(\mathbb{R}^N)$. The left hand-side converges to the respective for u_0 thanks to the weak convergence in (52). The right-hand side will be handled via dominated convergence theorem, by using Lemma 3.9. Let us split the argument according to $|x-y| \gtreqless 1$.

If $|x-y| \leq 1$, we define

$$\begin{aligned}\underline{L_\mu}(x,y) &= G_\mu(x-y)F(u_\mu(y))f(u_\mu(x))\varphi(x)\chi_{\{|x-y|\leq 1\}}(x,y)\\ &\leq C_\omega|x-y|^{-\frac{4(\omega-1)}{3\omega}}F(u_\mu(y))f(u_\mu(x))\varphi(x)\chi_{\{|x-y|\leq 1\}}(x,y) =: h(u_\mu)(x,y)\end{aligned}$$

and, thanks to (53) and the fact that $\int_{\mathbb{R}^N} f(u_\mu)\varphi\,\mathrm{d}x \leq C$ uniform with respect to μ, one finds that $h(u_\mu)$ is uniformly bounded in $L^1(\mathbb{R}^N)$. Since in addition $u_\mu \to u_0$ a.e. in $\mathbb{R}^N$, by [25, Lemma 2.1] one gets $h(u_\mu) \to h(u_0)$, and in turns

$$\underline{L_\mu}(x,y) \to \log\frac{1}{|x-y|}F(u_0(y))f(u_0(x))\varphi(x)\chi_{\{|x-y|\leq 1\}}(x,y) \quad \text{in } L^1(\mathbb{R}^N)\,. \tag{64}$$

On the other hand, if $|x-y| > 1$ one has $0 \geq G_\mu(x-y) = -|x-y|^{\tau\mu}\log|x-y|$ for some $\tau = \tau(x,y)$ by the mean value theorem, and, since $\|u_\mu\|_\infty \leq C$ by Lemma 3.9, we may estimate

$$\begin{aligned}\overline{L_\mu} &:= G_\mu(x-y)F(u_\mu(y))f(u_\mu(x))\varphi(x)\chi_{\{|x-y|>1\}}(x,y)\\ &\lesssim (|x|+|y|)|u_\mu(y)|^{\frac{N}{s}}|u_\mu(x)|^{\frac{N}{s}-1}|\varphi(x)|\chi_{\{|x-y|>1\}}(x,y)\\ &\lesssim \frac{(|x|+|y|)|\varphi(x)|\chi_{\{|x-y|>1\}}(x,y)}{\left(1+|y|^{a\frac{N}{s}}\right)\left(1+|x|^{a\left(\frac{N}{s}-1\right)}\right)}\\ &\lesssim \frac{\varphi(x)}{1+|y|^{a\left(\frac{N}{s}-1\right)}}\,.\end{aligned}$$

Here we used (f_1), the decay estimates in Lemma 3.9 and the compact support of φ. Since $x \mapsto |x|^{-a\left(\frac{N}{s}-1\right)}$ is integrable in $\mathbb{R}^N \setminus B_1(0)$, thanks to the choice of a in (61), by the dominated convergence theorem we conclude that

$$\overline{L_\mu}(x,y) \to \log\frac{1}{|x-y|}F(u_0(y))f(u_0(x))\varphi(x)\chi_{\{|x-y|>1\}}(x,y) \quad \text{in } L^1(\mathbb{R}^N)\,. \tag{65}$$

Combining (64) and (65) with (52), we conclude that u_0 is a critical point of J.

The proof of (5) strictly follows the argument in the proof of [13, Theorem 1.6], being based on Fatou's Lemma, (53), (32), and Lemma 3.6.

It remains to show that u_0 is nontrivial. Suppose the contrary, then, similarly to (47) one can show that $\int_{\mathbb{R}^N} f(u_\mu)u_\mu \to 0$ and hence

$$\begin{aligned}
0 = J'_\mu(u_\mu)[u_\mu] &= \|u_\mu\|_V^{\frac{N}{s}} - C_N \int_{\mathbb{R}^N} (G_\mu(\cdot) * F(u_\mu))\, f(u_\mu)u_\mu \,\mathrm{d}x \\
&\gtrsim \|u_\mu\|_V^{\frac{N}{s}} - C_N \int\int_{\{|x-y|\le 1\}} \frac{F(u_\mu(y))f(u_\mu(x))u_\mu(x)}{|x-y|^{\frac{4(\omega-1)}{3\omega}}} \,\mathrm{d}x\,\mathrm{d}y \\
&\ge \|u_\mu\|_V^{\frac{N}{s}} - \int_{\mathbb{R}^N} f(u_\mu)u_\mu = \|u_\mu\|_V^{\frac{N}{s}} + o_\mu(1)\,.
\end{aligned}$$

This yields $u_\mu \rightharpoonup 0$ in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$. But then

$$\begin{aligned}
\underline{c} &\le c_\mu + o_\mu(1) = J_\mu(u_\mu) \\
&\le -C_N \int\int_{\{|x-y|\ge 1\}} G_\mu(x-y)F(u_\mu(y))F(u_\mu(x))\,\mathrm{d}x\,\mathrm{d}y + o_\mu(1) \\
&\le C_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \log|x-y| F(u_\mu(y))F(u_\mu(x))\,\mathrm{d}x\,\mathrm{d}y + o_\mu(1) \\
&\lesssim \|F(u_\mu)\|_1 \int_{\mathbb{R}^N} |x| F(u_\mu(x))\,\mathrm{d}x + o_\mu(1) \\
&\lesssim R\|F(u_\mu)\|_1^2 + \int_{\{|x|\ge R\}} \frac{|x|}{1+|x|^{a\frac{N}{s}}}\,\mathrm{d}x + o_\mu(1) \le \varepsilon + o_\mu(1),
\end{aligned}$$

by (47) and choosing a sufficiently large R. This is a contradiction, and we conclude the proof. $\square$

4. From Choquard to Schrödinger-Poisson: Proof of Theorem 1.5

Theorem 1.4 yields a positive solution u to the Choquard equation (Ch_s). Next we show that the pair (u, ϕ_u), where we recall that $\phi_u := C_N \log\frac{1}{|\cdot|} * F(u)$, is indeed a solution of the system (SP_s) in the sense of Definition 1.1. This step is often neglected in the literature. In fact this can be done rigorously as follows, based on a characterisation of the distributional solutions of the Poisson equation by Hyder [28]. The idea is to compare ϕ_u with a model solution which we known to solve in the sense of distributions the Poisson equation in (SP_s), proving that the two solutions may differ only by a constant. The key point is to show that

$$\int_{\mathbb{R}^N} \log(1+|x|)F(u(x))\,\mathrm{d}x \le C\,, \tag{66}$$

which mimics the logarithmic-weighted Pohožaev-Trudinger inequality in [9, 14], but just for solutions.

Proof of Theorem 1.5. As in the proof of Lemma 3.9, following a Moser iteration technique, it is possible to prove that any positive weak solution of (Ch_s) belongs to $L^\infty(\mathbb{R}^N)$ and that there exists $R > 0$ such that

$$u(x) \lesssim \frac{1}{1+|x|^{\frac{s(2N+3)}{2(N-s)}}} \quad \text{for all } |x| > R\,. \tag{67}$$

Note that the proof of (67) differs from the one in Lemma 3.9 by the kernel, which now is logarithmic; however, since $G_\mu(\cdot)$ is approximating $\log \frac{1}{|\cdot|}$, the strategy to get (67) is very similar. From (67) and the boundedness in $\mathbb{R}^N$ of u, by (f_1) one may show that

$$F(u(x)) \lesssim \frac{1}{1+|x|^{\frac{N(2N+3)}{2(N-s)}}} \quad \text{for all } |x| > R\,. \tag{68}$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^N} \log(1+|x|)F(u)\,\mathrm{d}x \\
&\quad\le \|F(u)\|_\infty \int_{\{|x|<R\}} \log(1+|x|)\,\mathrm{d}x + C\int_{\{|x|\ge R\}} \frac{\log(1+|x|)}{1+|x|^{\frac{N(2N+3)}{2(N-s)}}}\,\mathrm{d}x \\
&\qquad\qquad \le C + C\int_R^{+\infty} \rho^{N-1-\frac{N(2N+3)}{2(N-s)}} \log\rho\,\mathrm{d}\rho < +\infty\,,
\end{aligned}$$

that is (66). With this in hand, we next show that $\phi_u \in L_\gamma(\mathbb{R}^N)$ for all $\gamma > 0$. Indeed,

$$\begin{aligned}
&C_N^{-1}\int_{\mathbb{R}^N} \frac{|\phi_u(x)|}{1+|x|^{N+2\gamma}}\,\mathrm{d}x \le \int_{\mathbb{R}^N} F(u(y))\left(\int_{\mathbb{R}^N} \left|\log\frac{1}{|x-y|}\right| \frac{1}{1+|x|^{N+2\gamma}}\,\mathrm{d}x\right)\mathrm{d}y \\
&\le \int_{\mathbb{R}^N} F(u(y))\left(\int_{\{|x-y|>1\}} \frac{\log|x-y|}{1+|x|^{N+2\gamma}}\,\mathrm{d}x + \int_{\{|x-y|\le 1\}} \log\frac{1}{|x-y|}\,\mathrm{d}x\right)\mathrm{d}y \\
&\le \int_{\mathbb{R}^N} F(u(y))\bigg(\int_{\mathbb{R}^N} \frac{\log(1+|x|)}{1+|x|^{N+2\gamma}}\,\mathrm{d}x \\
&\quad + \log(1+|y|)\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{1+|x|^{N+2\gamma}} + \|\log(\cdot)\|_{L^1(B_1(0))}\bigg)\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^N} F(u(y))\,(1+\log(1+|y|))\,\mathrm{d}y < +\infty\,,
\end{aligned}$$

thanks to (66) and to the simple inequality $\log|x-y| \le \log(1+|x|)+\log(1+|y|)$.

By [28, Lemma 2.3] the function

$$\widetilde{v}_u(x) := C_N\int_{\mathbb{R}^N} \log\left(\frac{1+|y|}{|x-y|}\right) F(u(y))\,\mathrm{d}y$$

belongs to $W^{N-1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and solves $(-\Delta)^{\frac{N}{2}}\widetilde{v}_u = F(u)$ in $\mathbb{R}^N$ in the sense of Definition 1.1. Moreover, by (66) it is easy to verify that

$$\phi_u(x) = \widetilde{v}_u(x) + C_N \int_{\mathbb{R}^N} \log(1+|y|)F(u(y))\,\mathrm{d}y = \widetilde{v}_u(x) + C\,.$$

This guarantees that $\phi_u \in W^{N-1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and solves (SP_s) in the sense of Definition 1.1 by [28, Lemma 2.4], for which all such solutions of $(-\Delta)^{\frac{N}{2}}\phi = \mathfrak{f}$ in $\mathbb{R}^N$ are of the form $\phi = \widetilde{v}_u + p$ with p polynomial of degree at most $N-1$. The decay behaviour for ϕ_u in (6) can be proved following the approach of [18], as done in the case $N=2$ in the proof of [13, Theorem 1.4]. It remains to show the Hölder continuity of u. To this aim we take advantage of the following local estimate of the Hölder seminorm, obtained in [29, Corollary 5.5], provided $|(-\Delta)^s_{\frac{N}{s}} u| \le K$ weakly in $B_{2R}(x_0)$:

$$\begin{aligned}&\widetilde{C}R^{\nu}[u]_{C^{\nu}(B_R(x_0))}\\ &\le (KR^N)^{\frac{s}{N-s}} + \|u\|_{L^\infty(B_{2R}(x_0))} + R^N\left(\int_{\mathbb{R}^N\setminus B_{2R}(x)} \frac{|u(y)|^{\frac{N}{s}-1}}{|x-y|^{2N}}\,\mathrm{d}y\right)^{\frac{s}{N-s}}, \qquad (69)\end{aligned}$$

where $\widetilde{C}$ is a universal constant. Note that, since $u \in L^\infty(\mathbb{R}^N)$, it is easy to see that the right-hand side of (69) is bounded with a constant depending just on R, N, s, K. The assumption $|(-\Delta)^s_{\frac{N}{s}} u| \le K$ may be verified with similar computations as in [13, Lemma 3.2], for the bound from above, and, basing on (66)-(67), as in [13, Lemma 3.6] for the one from below. □

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