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Duality for a properly efficient solution of bilevel multiobjective fractional programming problems with extremal-value function

Ahmed Rikouane and Mohamed Laghdir

Abstract. *In this paper, we establish some results regarding the optimality conditions and duality properties for properly efficient solutions of a constrained bilevel multiobjective fractional programming problem* (P) *with an extremal-value function. These results are obtained by applying a parametric approach to reduce the problem* (P) *to a parametric problem* (P^μ) *with* $\mu \in \mathbb{R}^p$*, and we obtain optimality conditions for properly efficient solutions for these problems. Furthermore, we define a dual problem of* (P^μ) *and we establish some results on duality.*

Keywords: Bilevel programming, Multiobjective fractional programming, Properly efficient solution, Optimality conditions, Conjugate duality..
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1. Introduction

Bilevel programming is a mathematical optimization framework that involves two levels of optimization problems, where the feasible set and/or the objective function of the first (or upper-level) problem is determined implicitly by the second (or lower-level) problem. In a bilevel programming problem, when the upper-level problem contains the optimal value function of the lower-level problem in its objective and/or constraint functions, it is called a bilevel programming problem with an extremal-value function.

During the past few years, many studies have been presented on the optimality conditions and the duality results for this problem. For example, Dempe [8] developed necessary and sufficient optimality conditions for bilevel programming problems by using differential stability results for parametric optimization problems. Aboussoror and Adly [1] considered a bilevel nonlinear optimization problem with an extremal-value function and obtained necessary and sufficient optimality conditions under constraint qualifications and via the Fenchel-Lagrange duality approach. Wang et al. [24] considered a bilevel multiobjective program with extremal-value function, and obtained optimality conditions and duality results under a generalized Slater-type constraint qualification. For many applications of such a problem, one can see, for example, Ahmad et al. [2], Dempe [9], and Yin [28].

Multiobjective programming problems (also called vector programming) which are optimization problems involving several objectives functions have been the subject of extensive study in the recent literature. Multiobjective fractional programming problems refers to vector optimization problems where the objective functions are quotients. The multiobjective Fractional programming problems play a crucial role in various fields such as transportation, production, information theory, and numerical analysis. The study of multiobjective fractional programming problems has received a great deal of attention in the recent past, one can consult [3, 4, 5, 17, 21] and the references therein. Bector et al. [3] studied duality for multiobjective programming problems having nonlinear constraints through a linearization approach. Bhatnagar [4] established necessary and sufficient optimality conditions for efficient and properly efficient solutions, and proved some duality results for multiobjective Schaible type dual. Bot et al. [5] extended the work of Wanka and Bot [26] A new duality approach for multiobjective convex optimization problems to a Duality for multiobjective fractional programming problems, they have used the transformation proposed by Dinkelbach [10] to reduce the problem considered to a more conventional form, and obtained a necessary and sufficient optimality conditions and some duality results for these problems. Recently, Moustaid et al. [17] established sequential approximate weak optimality conditions for multiobjective fractional programming problems via sequential subdifferential calculus. In multiobjective (fractional) programming, an optimization problem may have no optimal solutions. To address this issue, some researchers introduced the concept of efficient solutions. Pareto [18] was the first one to introduce the idea of Pareto efficiency to study some problems in economics. Based on Pareto's idea, Koopmans [15] introduced the notion of efficient solution of multiobjective optimization problems. Usually, the set of efficient solutions is so big that it may contain anomalous or undesirable points. To eliminate certain efficient points with unwanted properties, Geoffrion [12] introduced the notion of properly efficient solution.

In this present work, we have combined these two important problems, the multiobjective fractional programming problem and the bilevel programming problem to develop optimality conditions and prove some duality results for bilevel multiobjective fractional programming problem with an extremal-value function

$$(P) \quad \mathrm{v}-\min_{x\in C}\left\{\frac{f_1(x,\upsilon(x))}{g_1(x,\upsilon(x))},\ldots,\frac{f_p(x,\upsilon(x))}{g_p(x,\upsilon(x))}\right\},$$

where $C := \{x \in X,\ G(x,\upsilon(x)) \leq_{\mathbb{R}^q_+} 0\}$ and $\upsilon(x)$ is the optimal value of the following problem parametrized by x

$$(P_x) \qquad \min_{y\in A} f(x,y).$$

Hence the symbol "v-min" stands for vector minimization and minimality is taken in terms of efficient solutions and properly efficient solutions as defined in the next section. The study of multiobjective fractional programming problems has received a great deal of attention in the recent past. Some of the authors like Kohli [14], Rikouane [19], Bouibed et al. [7] and others. These types of problems have applications in several fields and the modeling of certain real-life situations (see [25]) leads to bilevel multiobjective fractional bilevel optimization problems, further motivating us to study this class of problems.

In this paper, motivated by the works mentioned above, our main goal is to establish necessary and sufficient optimality conditions and prove some duality results of the above problem in terms of properly efficient solutions. To do this, we often use a parametric approach proposed by Dinkelbach [10] to transform the problem (P) into the nonfractional multiobjective bilevel convex optimization problem (P^μ) with a parameter $\mu \in \mathbb{R}^p$. Subsequently, we employ linearization and scalarization approaches to transform the problem (P^μ) to a more conventional form. Under this case, we derive necessary and sufficient optimality conditions for efficient properly solutions. So we use the previous results to define a dual problem of (P^μ). Under some convexity and monotonicity assumptions, the weak and strong duality theorems are established.

The rest of the presented paper have the following structure: Section 2 is devoted to present some basic definitions, notations and auxiliary results describing important properties of conjugate functions that be used later in the paper. In Section 3, some necessary and sufficient conditions for a feasible point to be properly efficient are established. In Section 4, we apply the previous results to obtain some duality results. In Section 5, we present an example illustrating the main result obtained. Finally, in Section 6, we submite the conclusion and discussions of the paper.

2. Preliminaries and definitions

In this section, we recall some basic definitions, notations and preliminary results from convex analysis which will be used throughout this paper.

Let X be nonempty subset of $\mathbb{R}^n$. We denote by $\mathrm{ri}(X)$ the relative interior of the set X, and by $\mathbb{R}^p_+$ the non-negative orthant of $\mathbb{R}^p$ defined by

$$\mathbb{R}^p_+ := \{u = (u_1, \dots, u_p) \in \mathbb{R}^p,\ u_i \geq 0,\ i = 1, \dots, p\},$$

For $x, y \in \mathbb{R}^p$, we define $x \leq_{\mathbb{R}^p_+} y$ (or $y \geq_{\mathbb{R}^p_+} x$) if $y - x \in \mathbb{R}^p_+$. To $\mathbb{R}^p$, we attach an abstract maximal element with respect to $\leq_{\mathbb{R}^p_+}$, denoted by

$+\infty_{\mathbb{R}^p}$, verifying the following operations and conventions: $y \leq_{\mathbb{R}^p_+} +\infty_{\mathbb{R}^p}$, $y + (+\infty_{\mathbb{R}^p}) = (+\infty_{\mathbb{R}^p}) + y := +\infty_{\mathbb{R}^p}$ and $\alpha \cdot (+\infty_{\mathbb{R}^p}) = +\infty_{\mathbb{R}^p}$, for all $y \in \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ and all $\alpha \geq 0$.

For a nonempty subset X of $\mathbb{R}^n$, we denote by δ_X and σ_X the indicator and the support functions of X, respectively, defined on $\mathbb{R}^n$ by

$$\delta_X(x) := \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{otherwise} \end{cases}, \quad \sigma_X(p^*) = \sup_{x \in X} p^{*T} x, \quad \forall p^* \in \mathbb{R}^n.$$

where $x^T y$ denotes the inner product of the vectors $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$.

For a given function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we consider its effective domain

$$\mathrm{dom}(f) := \{x \in \mathbb{R}^n, \ f(x) < +\infty\}$$

and call the function f proper if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is said to be $\mathbb{R}^n_+$-increasing if for each $x, y \in \mathbb{R}^n$, we have

$$x \leq_{\mathbb{R}^n_+} y \Longrightarrow f(x) \leq f(y).$$

The function defined by

$$f^* : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}, \ \ f^*(x^*) := \sup_{x \in \mathbb{R}^n} \{x^{*T} x - f(x)\}.$$

is called the conjugate function of f. We have the so-called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq x^{*T} x, \quad \forall x, x^* \in \mathbb{R}^n. \tag{1}$$

It is well known that for a non-negative real number λ,

$$(\lambda f)^*(x^*) := \begin{cases} \lambda f^*(\frac{x^*}{\lambda}), & \text{if } \lambda > 0, \\ \delta_{\{0\}}(x^*), & \text{if } \lambda = 0. \end{cases}$$

Let $g : \mathbb{R}^q \longrightarrow \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ be a given vector valued function. The function g is called $\mathbb{R}^p_+$-convex if for all $x, y \in \mathbb{R}^q$ and all $t \in [0, 1]$ we have

$$g(tx + (1-t)y) \leq_{\mathbb{R}^p_+} tg(x) + (1-t)g(y).$$

Furthermore, g is called $(\mathbb{R}^q_+, \mathbb{R}^p_+)$-increasing if for each $x, y \in \mathbb{R}^q$ we have

$$x \leq_{\mathbb{R}^q_+} y \Longrightarrow g(x) \leq_{\mathbb{R}^p_+} g(y).$$

Given a vector valued mapping $h : \mathbb{R}^n \longrightarrow \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$, the composed vector valued mapping $g \circ h : \mathbb{R}^n \longrightarrow \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), & \text{if } x \in \mathrm{dom}(h), \\ +\infty_{\mathbb{R}^p}, & \text{otherwise.} \end{cases}$$

The following lemma provide some conditions that guarantee the convexity of composed vector valued mappings.

LEMMA 2.1 ([16]). *Let* $h : \mathbb{R}^n \longrightarrow \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ *and* $g : \mathbb{R}^q \longrightarrow \mathbb{R}^p \cup \{+\infty_{\mathbb{R}^p}\}$ *be two vector valued mappings. If* g *is* $\mathbb{R}^p_+$*-convex and* $(\mathbb{R}^q_+, \mathbb{R}^p_+)$*-increasing on* $dom(g)$*, and* h *is* $\mathbb{R}^q_+$*-convex with* $h(dom(h)) \subseteq dom(g)$*, then the composed vector mapping* $g \circ h$ *is* $\mathbb{R}^p_+$*-convex.*

Let us recall the following lemmas can be found in [20, 27] which are used in our later results.

LEMMA 2.2 ([20, Theorem 16.4]). *Let* $g_i : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ $(i = 1, \ldots, m)$ *be proper convex functions. If* $\cap_{i=1}^m ri(dom(g_i)) \neq \emptyset$*, then*

(i) $\left(\sum_{i=1}^m g_i\right)^* (x^*) = \inf \left\{\sum_{i=1}^m g_i^*(x_i^*) :\ x^* = \sum_{i=1}^m x_i^*\right\}$*;*

(ii) for all $x^* \in \mathbb{R}^n$*, the infimum in (i) is attained.*

LEMMA 2.3 ([27, Theorem 2]). *Let* $h = (h_1, \ldots, h_n)$ *with* $h_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ $(i = 1, \ldots, n)$ *be convex functions, and* $g : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ *be proper convex and* $\mathbb{R}^n_+$*-increasing function. If* $h(\cap_{i=1}^n dom(h_i)) \cap int(dom(g)) \neq \emptyset$*, then*

$$(g \circ h)^*(x^*) = \inf_{r \in \mathbb{R}^n_+} \left\{ g^*(r) + \left(\sum_{i=1}^n r_i h_i\right)^* (x^*) \right\},$$

where for any $x^* \in \mathbb{R}^m$ *the infimum is attained.*

Let $\overline{x}$ be a feasible point of (P) i.e., $\overline{x} \in C$ and $\upsilon(\overline{x})$ is the optimal value of the lower level problem $(P_{\overline{x}})$. The set of feasible solutions of (P) will be represented by Ω in what follows, that is

$$\Omega := \{x \in X,\ G(x, \upsilon(x)) \leq_{\mathbb{R}^q_+} 0 \text{ and } \upsilon(x) \text{ is the optimal value of } (P_x)\}.$$

DEFINITION 2.4. *An element* $\overline{x} \in \Omega$ *is said to be efficient solution for* (P) *if there is no* $x \in \Omega$ *such that*

$$\frac{f_i(x, \upsilon(x))}{g_i(x, \upsilon(x))} \leq \frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))}, \text{ for each } i \in \{1, \ldots, p\},$$
$$\frac{f_j(x, \upsilon(x))}{g_j(x, \upsilon(x))} < \frac{f_j(\overline{x}, \upsilon(\overline{x}))}{g_j(\overline{x}, \upsilon(\overline{x}))}, \text{ for some one } j \in \{1, \ldots, p\};$$

DEFINITION 2.5 ([12]). *An element* $\overline{x} \in \Omega$ *is said to be properly efficient solution for* (P)*, in the sense of Geoffrion, if and only if (a)* $\overline{x}$ *is efficient solution for* (P)*; (b) there exists a scalar* $M > 0$ *such that for each* i*, we have*

$$\frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))} - \frac{f_i(x, \upsilon(x))}{g_i(x, \upsilon(x))} \leq M \left(\frac{f_j(x, \upsilon(x))}{g_j(x, \upsilon(x))} - \frac{f_j(\overline{x}, \upsilon(\overline{x}))}{g_j(\overline{x}, \upsilon(\overline{x}))} \right),$$

for some j *such that* $\frac{f_j(\overline{x}, \upsilon(\overline{x}))}{g_j(\overline{x}, \upsilon(\overline{x}))} < \frac{f_j(x, \upsilon(x))}{g_j(x, \upsilon(x))}$*, whenever* $x \in \Omega$ *and* $\frac{f_i(x, \upsilon(x))}{g_i(x, \upsilon(x))} < \frac{f_i(\overline{x}, \upsilon(\overline{x}))}{g_i(\overline{x}, \upsilon(\overline{x}))}$.

3. Problem formulation

In this section, we consider the following bilevel multiobjective fractional programming problem with an extremal-value function

$$(P) \qquad \mathrm{v}-\min_{x\in C}\left\{\frac{f_1(x,\upsilon(x))}{g_1(x,\upsilon(x))},\dots,\frac{f_p(x,\upsilon(x))}{g_p(x,\upsilon(x))}\right\},$$

where $C := \{x \in X,\ G(x,\upsilon(x)) \leq_{\mathbb{R}^q_+} 0\} \neq \emptyset$ and $\upsilon(x)$ is the optimal value of the lower level problem

$$(P_x) \qquad \min_{y\in A} f(x,y),$$

Here, X is a nonempty convex subset of $\mathbb{R}^n$, A is a nonempty subset of $\mathbb{R}^m$ compact and convex, $f_i, -g_i : \mathbb{R}^n\times\mathbb{R} \longrightarrow \mathbb{R},\ i=1,\dots,p,\ G_j : \mathbb{R}^n\times\mathbb{R} \longrightarrow \mathbb{R},\ j=1,\dots,q$ are convex functions and $\mathbb{R}^{n+1}_+$-increasing, $f : \mathbb{R}^n\times\mathbb{R}^m \longrightarrow \mathbb{R}$ is a convex function. Moreover, we assume that for any $x\in C,\ f_i(x,\upsilon(x)) \geq 0,\ i=1,\dots,p$ and the following additional hypothesis

$$\exists a>0, b>0, \text{ such that } a \leq g_i(x,\upsilon(x)) \leq b \text{ for all } i=1,\dots,p \text{ and } x\in C. \quad (2)$$

We mention that the functions $f_i,\ g_i,\ i=1,\dots,p$ and $G_j,\ j=1,\dots,q$ are all continuous since $\mathrm{int}(\mathrm{dom}(f_i)) = \mathrm{int}(\mathrm{dom}(g_i)) = \mathbb{R}^{n+1},\ i=1,\dots,p$ and $\mathrm{int}(\mathrm{dom}(G_j)) = \mathbb{R}^{n+1},\ j=1,\dots,q$. Moreover, we can conclude that f is a continuous function since it is convex [20, Corollary 10.1.1], and so the function $\upsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ is finite, convex, continuous and for each $x\in\mathbb{R}^n$ there exists $y\in A$ such that $\upsilon(x) = f(x,y)$.

We now associate with (P) the following parametric nonfractional bilevel multiobjective programming problem (P^μ) for some $\mu\in\mathbb{R}^p_+$, following the parametric approach of Dinkelbach [10].

$$(P^\mu) \qquad \mathrm{v}-\min_{x\in C}\left\{f_1(x,\upsilon(x))-\mu_1 g_1(x,\upsilon(x)),\dots,f_p(x,\upsilon(x))-\mu_p g_p(x,\upsilon(x))\right\}.$$

where

$$\mu_i := \frac{f_i(\overline{x},\upsilon(\overline{x}))}{g_i(\overline{x},\upsilon(\overline{x}))},\ i=1,\dots,p \text{ and } \overline{x}\in\Omega.$$

By introducing the following auxiliary mappings

$$\begin{aligned} F :\ & \mathbb{R}^n\times\mathbb{R} \longrightarrow \mathbb{R}^p \\ & (x,y) \longrightarrow F(x,y) := (f_1(x,y)-\mu_1 g_1(x,y),\dots,f_p(x,y)-\mu_p g_p(x,y)), \end{aligned}$$

$$h :\ \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}, \qquad x \longrightarrow h(x) := (x,\upsilon(x)).$$

It is clear that the function h is $\mathbb{R}^{n+1}$-convex, continuous and $h(\mathrm{dom}h) \subseteq \mathbb{R}^{n+1}$, since the function $\upsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$ is finite, convex and continuous. The form of the problem (P^μ_λ) will be as follows:

$$(P^\mu) \qquad \mathrm{v}-\min_{x\in C} F(h(x)).$$

REMARK 3.1. With the assumptions listed above on $f_i, g_i,\ i = 1, \ldots, p$ and h and by Lemma 2.1, we see easily that the composed vector valued mapping $F \circ h : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^p$ is $\mathbb{R}^{n+1}$-convex.

DEFINITION 3.2. *An element $\overline{x} \in \Omega$ is said to be*
- efficient solution for (P^μ) if there is no $x \in \Omega$ such that

$$F_i(h(x)) \leq F_i(h(\overline{x})),\ \text{for each}\ i \in \{1, \ldots, p\},$$
$$F_j(h(x)) < F_j(h(\overline{x})),\ \text{for some one}\ j \in \{1, \ldots, p\};$$

- properly efficient solution for (P^μ) (in the sense of Geoffrion) if and only if (a) $\overline{x}$ is efficient solution for (P^μ); (b) there exists a scalar $M > 0$ such that for each i, we have

$$F_i(h(\overline{x})) - F_i(h(x)) \leq M(F_j(h(x)) - F_j(h(\overline{x}))),$$

for some j such that $F_j(h(\overline{x})) < F_j(h(x))$, whenever $x \in \Omega$ and $F_i(h(x)) < F_i(h(\overline{x}))$.

It is well known that the set of properly efficient solutions of (P^μ) is related to the optimal solutions of the following scalar problem (see [12]).

$$(P^\mu_\lambda) \quad \begin{cases} \min \lambda^T F(h(x)), \\ x \in C, \end{cases}$$

where

$$\lambda \in \Lambda^+ := \left\{ (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p : \text{ all } \lambda_i > 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1 \right\}.$$

A point $\overline{x} \in \Omega$ is called an optimal solution of the scalar problem (P^μ_λ) if

$$\lambda^T F(h(\overline{x})) \leq \lambda^T F(h(x)),\ \forall x \in \Omega.$$

We will need the following lemma.

LEMMA 3.3. *([21, Theorem 2]) The point $\overline{x} \in \Omega$ is an efficient solution of (P) if and only if $\overline{x}$ is an efficient solution of (P^μ) where $\mu = (\mu_1, \ldots, \mu_p)$ and $\mu_i = \frac{f_i(\overline{x}, v(\overline{x}))}{g_i(\overline{x}, v(\overline{x}))},\ i = 1, \ldots, p$.*

The following lemma gives the relationship linking (P), (P^μ) and (P^μ_λ) which will be useful for our purposes.

LEMMA 3.4. *Suppose that $\overline{x} \in \Omega$. Then the following statements are equivalent*

(i) $\overline{x}$ is a properly efficient solution for problem (P) ;

(ii) $\overline{x}$ *is a properly efficient solution for problem* (P^μ) *where* $\mu_i = \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))}$, $i = 1, \ldots, p$;

(iii) $\overline{x}$ *is an optimal solution for problem* (P^μ_λ) *for some* $\lambda \in \Lambda^+$ *and* $\mu_i = \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))}$, $i = 1, \ldots, p$.

Proof. $(i) \Longrightarrow (ii)$ Assume that $\overline{x} \in \Omega$ is a properly efficient solution of (P), then by Definition 2.5, it follows that (a) $\overline{x}$ is efficient solution for (P); (b) there exists a scalar $M > 0$ such that for each i, we have

$$\frac{f_i(\overline{x}, v(\overline{x}))}{g_i(\overline{x}, v(\overline{x}))} - \frac{f_i(x, v(x))}{g_i(x, v(x))} \leq M \left(\frac{f_j(x, v(x))}{g_j(x, v(x))} - \frac{f_j(\overline{x}, v(\overline{x}))}{g_j(\overline{x}, v(\overline{x}))} \right),$$

for some j such that $\frac{f_j(\overline{x},v(\overline{x}))}{g_j(\overline{x},v(\overline{x}))} < \frac{f_j(x,v(x))}{g_j(x,v(x))}$, whenever $x \in \Omega$ and $\frac{f_i(x,v(x))}{g_i(x,v(x))} < \frac{f_i(\overline{x},v(\overline{x}))}{g_i(\overline{x},v(\overline{x}))}$. Then by Lemma 3.3, and the assumption (a), we obtain that $\overline{x}$ is an efficient solution of (P^μ). By utilizing the fact that $f_i(\overline{x}, v(\overline{x})) - \mu_i g_i(\overline{x}, v(\overline{x})) = 0$ and $g_i(x, v(x)) > 0$, for any $x \in \Omega$ and $i = 1, \ldots, p$ the condition (b) can be rewritten equivalently as: there exists a scalar $M > 0$ such that for each $i \in \{1, \ldots, p\}$, we have

$$\begin{aligned}(f_i - \mu_i g_i)(\overline{x}, v(\overline{x})) &- (f_i - \mu_i g_i)(x, v(x)) \\ &\leq M \frac{g_i(x, v(x))}{g_j(x, v(x))}((f_j - \mu_i g_j)(x, v(x)) - (f_j - \mu_j g_j)(\overline{x}, v(\overline{x}))).\end{aligned}$$

Since the functions g_i for any $i \in \{1, \ldots, p\}$ satisfy the condition (2) i.e. there exist $a > 0$, $b > 0$ such that $a \leq g_i(x, v(x)) \leq b$ for all $i \in \{1, \ldots, p\}$, $x \in \Omega$ and by setting $M_1 := \frac{Mb}{a}$, we have

$$\begin{aligned}(f_i - \mu_i g_i)(\overline{x}, v(\overline{x})) &- (f_i - \mu_i g_i)(x, v(x)) \\ &\leq M_1((f_j - \mu_j g_j)(x, v(x)) - (f_j - \mu_j g_j)(\overline{x}, v(\overline{x}))),\end{aligned}$$

i.e.

$$F_i(h(\overline{x})) - F_i(h(x)) \leq M_1(F_j(h(x)) - F_j(h(\overline{x}))).$$

Thus, by Definition 3.2, it follows that $\overline{x}$ is a properly efficient solution for problem (P^μ). $(ii) \Longrightarrow (i)$ The proof can be proven in the same way as $(i) \Longrightarrow (ii)$, so that (i) is equivalent to (ii). $(ii) \Longleftrightarrow (iii)$ The proof deduces directly from [13, Theorem 3.4]. Hence, the proof is complete. □

Given that the problem (P^μ_λ) is evidently a composite convex optimization problem, we can proceed to formulate its Lagrangian dual problem in the following manner (for more details see [1, 6, 26, 20]).

$$(D^\mu_\lambda) \ \sup_{r \in \mathbb{R}^q_+} \inf_{x \in \mathbb{R}^n} \left\{ (\lambda^T F + r^T G)(h(x)) + \delta_X(x) \right\}, \text{ where } \lambda \in \Lambda^+.$$

It is easy to check that h_i is a finite convex function for every $i \in \{1, \ldots, n+1\}$, and since $\lambda \in \Lambda^+$ and $r \in \mathbb{R}^q_+$, thus, the function $\lambda^T F + r^T G$ is finite valued convex and $\mathbb{R}^{n+1}_+$-increasing. The dual problem (D^μ_λ) can be deduced from Lemmas 2.2 and 2.3, which implies that its form will be as follows:

$$\sup_{(r,s,w,t)\in Y} \left\{ -\sum_{i=1}^{p} \lambda_i F_i^*(w_i) - (r^T G)^* \Big(s - \sum_{i=1}^{p} \lambda_i w_i\Big) - (s^T h)^*(t) - \sigma_X(-t) \right\}$$

where

$$Y = \Big\{(r, s, w, t), r \in \mathbb{R}^q_+, s \in \mathbb{R}^{n+1}_+, t \in \mathbb{R}^n, w = (w_1, \ldots, w_p), \\ w_i \in \mathbb{R}^{n+1}, i = 1, \ldots, p\Big\}.$$

Since the functions f_i, $(-\mu_i g_i)$, $i = 1, \ldots, p$ satisfy all the assumptions of Lemma 2.2, then for each $i \in \{1, \ldots, p\}$, we have

$$\begin{aligned} F_i^*(w_i) &= (f_i + (-\mu_i g_i))^*(w_i) \\ &= \inf\{f_i^*(u_i) + (-\mu_i g_i)^*(v_i),\ u_i + v_i = w_i\} \\ &= -\sup\{-f_i^*(u_i) - (-\mu_i g_i)^*(v_i),\ u_i + v_i = w_i\}. \end{aligned}$$

Hence, the problem (D^μ_λ) can be rewritten as

$$\sup_{(r,s,u,v,t)\in Y^\lambda} \left\{ -\sum_{i=1}^{p} \lambda_i [f_i^*(u_i) + (-\mu_i g_i)^*(v_i)] \\ - (r^T G)^* \Big(s - \sum_{i=1}^{p} \lambda_i (u_i + v_i)\Big) - (s^T h)^*(t) - \sigma_X(-t) \right\}$$

where

$$Y^\lambda := \Big\{(r, s, u, v, t),\ r \in \mathbb{R}^q_+,\ s \in \mathbb{R}^{n+1}_+,\ t \in \mathbb{R}^n, u = (u_1, \ldots, u_p), \\ v = (v_1, \ldots, v_p),\ u_i, v_i \in \mathbb{R}^{n+1}, i = 1, \ldots, p\Big\}.$$

We denote by val(P^μ_λ) and val (D^μ_λ) the optimal values of the problem (P^μ_λ) and (D^μ_λ) , respectively. The weak duality always holds, i.e.

$$\text{val}(P^\mu_\lambda) \geq \text{val}(D^\mu_\lambda). \tag{3}$$

To prove strong duality and establish optimality conditions in the following, a constraint qualification is necessary:

$$(CQ)\ \exists \overline{x} \in \text{ri}(X) \text{ such that } \begin{cases} G_i(h(\overline{x})) \leq 0, & \text{if } i \in L \\ G_i(h(\overline{x})) < 0, & \text{if } i \in N, \end{cases}$$

where $L = \{i \in \{1, \ldots, q\} : \ G_i \circ h \ \text{ is an affine function}\}$ and $N = \{1, \ldots, q\} \backslash L$.

In what follows, we will need the following strong duality theorem (see [6, Theorem 3.2]).

THEOREM 3.5 (Strong duality for (P_λ^μ)). *If (CQ) is fulfilled and $val(P_\lambda^\mu)$ is finite, then the problem (D_λ^μ) has an optimal solution and it holds*

$$val(P_\lambda^\mu) = val(D_\lambda^\mu).$$

Now, we derive a necessary and sufficient optimality conditions for the problem (P_λ^μ) and its dual (D_λ^μ) under a constraint qualification.

THEOREM 3.6. *(1) Let $\overline{x} \in C$, $\lambda \in \Lambda^+$ and $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{R}_+^p$ with $\mu_i = \frac{f_i(\overline{x}, v(\overline{x}))}{g_i(\overline{x}, v(\overline{x}))}$, $i = 1, \ldots p$. Suppose that the constraint qualification (CQ) is satisfied at $\overline{x}$. If $\overline{x}$ is an optimal solution of (P_λ^μ), then there exists an optimal solution $(\overline{r}, \overline{u}, \overline{v}, \overline{s}, \overline{t}) \in Y^\lambda$ to the dual problem (D_λ^μ), such that the following optimality conditions hold*

(i) $f_i(h(\overline{x})) + f_i^*(\overline{u}_i) = \overline{u}_i^T h(\overline{x}), \ \forall i \in \{1, \ldots, p\}$,

(ii) $(-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\overline{v}_i) = \overline{v}_i^T h(\overline{x}), \ \forall i \in \{1, \ldots, p\}$,

(iii) $\overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^* \Big(\overline{s} - \sum_{i=1}^{p} \lambda_i (\overline{u}_i + \overline{v}_i) \Big) = \Big(\overline{s} - \sum_{i=1}^{p} \lambda_i (\overline{u}_i + \overline{v}_i) \Big)^T h(\overline{x})$,

(iv) $\overline{s}^T h(\overline{x}) + (\overline{s}^T h)^*(\overline{t}) = \overline{t}^T \overline{x}$,

(v) $\sigma_X(-\overline{t}) = -\overline{t}^T \overline{x}$,

(vi) $\overline{r}^T G(h(\overline{x})) = 0$.

(2) Let $\overline{x} \in C$, $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{R}_+^p$ with $\mu_i = \frac{f_i(\overline{x}, v(\overline{x}))}{g_i(\overline{x}, v(\overline{x}))}$, $i = 1, \ldots p$ and for a given $\lambda \in \Lambda^+$, assume that $\overline{x} \in \Omega$ and $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}) \in Y^\lambda$ satisfy the condition $(i) - (vi)$. Then $\overline{x}$ is an optimal solution to (P_λ^μ), $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t})$ is an optimal solution to (D_λ^μ) and $val(P_\lambda^\mu) = val(D_\lambda^\mu)$.

Proof. (1) Let $\overline{x}$ be an optimal solution of (P_λ^μ). According to Theorem 3.5, there exists $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}) \in Y^\lambda$ optimal solution to (D_λ^μ), such that

$$\begin{aligned} \lambda^T F(h(\overline{x})) = &- \sum_{i=1}^{p} \lambda_i [f_i^*(\overline{u}_i) + (-\mu_i g_i)^*(\overline{v}_i)] \\ &- (\overline{r}^T G)^* \Big(\overline{s} - \sum_{i=1}^{p} \lambda_i (\overline{u}_i + \overline{v}_i) \Big) - (\overline{s}^T h)^*(\overline{t}) - \sigma_X(-\overline{t}) \end{aligned} \tag{4}$$

The last equality is equivalent to

$$
\begin{aligned}
0 = &\left\{\sum_{i=1}^{p}\lambda_i[f_i(h(\overline{x})) + f_i^*(\overline{u}_i) - \overline{u}_i^T h(\overline{x})]\right\} \\
&+ \left\{\sum_{i=1}^{p}\lambda_i[(-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\overline{v}_i) - \overline{v}_i^T h(\overline{x})]\right\} \\
&+ \left\{\overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i=1}^{p}\lambda_i(\overline{u}_i + \overline{v}_i)\Big) - \Big(\overline{s} - \sum_{i=1}^{p}\lambda_i(\overline{u}_i + \overline{v}_i)\Big)^T h(\overline{x})\right\} \\
&\quad + \left\{\overline{s}^T h(\overline{x}) + (\overline{s}^T h)^*(\overline{t}) - \overline{t}^T \overline{x}\right\} + \left\{\sigma_X(-\overline{t}) + \overline{t}^T \overline{x}\right\} - \overline{r}^T G(h(\overline{x}). \qquad (5)
\end{aligned}
$$

It follows from the Young-Fenchel inequality (1), the following inequalities hold:

$$
\begin{cases}
f_i(h(\overline{x})) + f_i^*(\overline{u}_i) - \overline{u}_i^T h(\overline{x}) \geq 0, \qquad \forall i \in \{1, \ldots, p\}; \\
(-\mu_i g_i)(h(\overline{x})) + (-\mu_i g_i)^*(\overline{v}_i) - \overline{v}_i^T h(\overline{x}) \geq 0, \qquad \forall i \in \{1, \ldots, p\}; \\
\overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i=1}^{p}\lambda_i(\overline{u}_i + \overline{v}_i)\Big) - \Big(\overline{s} - \sum_{i=1}^{p}\lambda_i(\overline{u}_i + \overline{v}_i)\Big)^T h(\overline{x}) \geq 0; \\
\overline{s}^T h(\overline{x}) + (\overline{s}^T h)^*(\overline{t}) - \overline{t}^T \overline{x} \geq 0; \\
\sigma_X(-\overline{t}) + \overline{t}^T \overline{x} \geq 0.
\end{cases} \qquad (6)
$$

Since $\overline{r} \in \mathbb{R}_+^q$ and $\overline{x} \in C$, there is $-\overline{r}^T G(h(\overline{x})) \geq 0$. By the inequalities (6), it follows that all the terms of the sum in (5) must be equal to 0. Then, we can conclude the relations $(i) - (vi)$.

(2) Assume that $\overline{x} \in \Omega$ and $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}) \in Y^\lambda$ satisfy the condition $(i) - (vi)$. Then, we get

$$
\begin{aligned}
-\sum_{i=1}^{p}\lambda_i[f_i^*(\overline{u}_i) + (-\mu_i g_i)^*(\overline{v}_i)] - (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i=1}^{p}\lambda_i(\overline{u}_i + \overline{v}_i)\Big) \\
- (\overline{s}^T h)^*(\overline{t}) - \sigma_X(-\overline{t}) = \lambda^T F(h(\overline{x})),
\end{aligned}
$$

since $\mathrm{val}(P_\lambda^\mu) = \inf(P_\lambda^\mu)$ and $\mathrm{val}(D_\lambda^\mu) = \max(D_\lambda^\mu)$, we obtain

$$
\begin{aligned}
\mathrm{val}(D_\lambda^\mu) &\geq -\sum_{i=1}^{p}\lambda_i[f_i^*(\overline{u}_i) + (-\mu_i g_i)^*(\overline{v}_i)] - (\overline{r}^T G)^*\Big(\overline{s} - \sum_{i=1}^{p}\lambda_i(\overline{u}_i + \overline{v}_i)\Big) \\
&\qquad - (\overline{s}^T h)^*(\overline{t}) - \sigma_X(-\overline{t}) \\
&= \lambda^T F(h(\overline{x})) \geq \mathrm{val}(P_\lambda^\mu).
\end{aligned}
$$

It follows from (3) that

$$\mathrm{val}(P_\lambda^\mu) = \mathrm{val}(D_\lambda^\mu).$$

Then, this proves that the equality (4) results and shows that $\overline{x}$ is a solution to (P^{μ}_{λ}) and $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t})$ is a solution to (D^{μ}_{λ}). □

4. The multiobjective dual problem

In this section, we aim to establish the weak and strong duality theorems for the problem (P^{μ}) and its corresponding dual multiobjective optimization problem (D^{μ}) defined by

$$(D^{\mu}) \qquad \begin{cases} \text{v} - \max H(r, s, u, v, t, \lambda, \alpha), \\ \text{s.t} \quad (r, s, u, v, t, \lambda, \alpha) \in B \end{cases}$$

where

$$H(r, s, u, v, t, \lambda, \alpha) = \begin{pmatrix} H_1(r, s, u, v, t, \lambda, \alpha) \\ \vdots \\ H_p(r, s, u, v, t, \lambda, \alpha) \end{pmatrix}$$

with for all $i \in \{1, \ldots, p\}$

$$\begin{aligned} H_i(r, s, u, v, t, \lambda, \alpha) &= -f_i^*(u_i) - (-\mu_i g_i)^*(v_i) \\ &- \frac{1}{p\lambda_i}\Big[(r^T G)^*\Big(s - \sum_{i=1}^{p} \lambda_i(u_i + v_i)\Big) + (s^T h)^*(t) + \sigma_X(-t)\Big] + \alpha_i, \end{aligned}$$

and the set of constraints

$$\begin{aligned} B := \Big\{ &(r, s, u, v, t, \lambda, \alpha) : \ r \in \mathbb{R}^q_+, s \in \mathbb{R}^{n+1}_+, t \in \mathbb{R}^n, u = (u_1, \ldots, u_p), \\ &v = (v_1, \ldots, v_p), u_i, v_i \in \mathbb{R}^{n+1}, i = 1, \ldots, p, \lambda \in \Lambda^+, \\ &\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p, \sum_{i=1}^{p} \lambda_i \alpha_i = 0 \Big\}. \end{aligned}$$

Definition 4.1. *An element* $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \in B$ *is said to be an efficient solution of the problem* (D^{μ})*, if there is no* $(r, s, u, v, t, \lambda, \alpha) \in B$ *such that* $H(r, s, u, v, t, \lambda, \alpha) \geq_{\mathbb{R}^p_+} H(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha})$ *with*

$$H_i(r, s, u, v, t, \lambda, \alpha) > H_i(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}), \text{ for some } i \in \{1, \ldots, p\}.$$

The following theorem states the weak duality assertion between the bilevel multiobjective problem (P^{μ}) and its dual (D^{μ}).

Theorem 4.2 (Weak duality for (P^{μ})). *There is no* $(r, s, u, v, t, \lambda, \alpha) \in B$ *and no* $x \in \Omega$ *such that* $F(h(x)) \leq_{\mathbb{R}^p_+} H(r, s, u, v, t, \lambda, \alpha)$ *and* $F_i(h(x)) < H_i(r, s, u, v, t, \lambda, \alpha)$ *for some* $i \in \{1, \ldots, p\}$.

Proof. Assume that there exist $x \in \Omega$ and $(r, s, u, v, t, \lambda, \alpha) \in B$ satisfying that $F(h(x)) \leq_{\mathbb{R}^p_+} H(r, s, u, v, t, \lambda, \alpha)$ and $F_i(h(x)) < H_i(r, s, u, v, t, \lambda, \alpha)$ for some $i \in \{1, \ldots, p\}$. This means that

$$\lambda^T F(h(x)) < \lambda^T H(r, s, u, v, t, \lambda, \alpha), \ \forall \lambda \in \Lambda^+. \tag{7}$$

On the other hand,

$$\begin{aligned}
\lambda^T H(r, s, u, v, t, \lambda, \alpha) &= \sum_{i=1}^{p} \lambda_i^T H_i(r, s, u, v, t, \lambda, \alpha) \\
&= -\sum_{i=1}^{p} \lambda_i [f_i^*(u_i) + (-\mu_i g_i)^*(v_i)] \\
&\quad - (r^T G)^*(s - \sum_{i=1}^{p} \lambda_i(u_i + v_i)) - (s^T h)^*(t) - \sigma_X(-t) \\
&\leq \lambda^T F(h(x)).
\end{aligned}$$

Then, the inequality $\lambda^T F(h(x)) \geq \lambda^T H(r, s, u, v, t, \lambda, \alpha)$ contradicts the relation (7). Thus, the weak duality between (P^μ) and (D^μ) holds. □

The following theorem provides the strong duality between the problem (P^μ) and its dual (D^μ).

THEOREM 4.3 (Strong duality for (P^μ)). *Let (CQ) be fulfilled and $\overline{x} \in \Omega$ be a properly efficient solution to (P^μ). Then an efficient solution $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \in B$ to the dual problem (D^μ) exists and the strong duality is true, i.e.*

$$F(h(\overline{x})) = H(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}).$$

Proof. According to Lemma 3.4, a point $\overline{x} \in \Omega$ is a properly efficient solution for (P^μ) if and only if there exists $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_p) \in \Lambda^+$ such that $\overline{x}$ solves the scalar problem $(P^\mu_{\overline{\lambda}})$. Since the constraint qualification (CQ) is fulfilled and, so, the Theorem 3.6 assures the existence of an optimal solution $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}) \in Y^{\overline{\lambda}}$ to the dual problem $(D^\mu_{\overline{\lambda}})$ such that the optimality conditions $(i) - (vi)$ are fulfilled. Let us define for $i = 1, \ldots, p$

$$\overline{\alpha}_i = \frac{1}{p\overline{\lambda}_i} \left((\overline{r}^T G)^* \Big(\overline{s} - \sum_{i=1}^{p} \overline{\lambda}_i (\overline{u}_i + \overline{v}_i) \Big) + (\overline{s}^T h)^*(\overline{t}) + \sigma_X(-\overline{t}) \right) + (\overline{u}_i + \overline{v}_i)^T h(\overline{x}).$$

By using the optimality conditions established in Theorem 3.6, we obtain

$$\begin{aligned}
\sum_{i=1}^{p}\overline{\lambda}_i\overline{\alpha}_i &= (\overline{r}^T G)^*(\overline{s}-\sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i)) + (\overline{s}^T h)^*(\overline{t}) + \sigma_X(-\overline{t}) \\
&\qquad + \sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i)^T h(\overline{x}) \\
&= (\overline{r}^T G)^*(\overline{s}-\sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i)) + \overline{x}^T\overline{t} - \overline{s}^T h(\overline{x}) - \overline{x}^T\overline{t} \\
&\qquad + \sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i)^T h(\overline{x}) \\
&= \overline{r}^T G(h(\overline{x})) + (\overline{r}^T G)^*(\overline{s}-\sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i)) - (\overline{s}-\sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i))^T h(\overline{x}) \\
&= 0.
\end{aligned}$$

Then, we proved that the element $(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha})$ is feasible to (D^μ). Now, we show that $F(h(\overline{x})) = H(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha})$. Applying Theorem 3.6, we have for $i = 1,\ldots,p$,

$$\begin{aligned}
H_i(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha}) &= -f_i^*(\overline{u}_i) - (-\mu_i g_i)^*(\overline{v}_i) \\
&\qquad - \frac{1}{p\overline{\lambda}_i}\Big[(\overline{r}^T G)^*\Big(\overline{s}-\sum_{i=1}^{p}\overline{\lambda}_i(\overline{u}_i+\overline{v}_i)\Big) + (\overline{s}^T h)^*(\overline{t}) + \sigma_X(-\overline{t})\Big] + \overline{\alpha}_i \\
&= -f_i^*(\overline{u}_i) - (-\mu_i g_i)^*(\overline{v}_i) + (\overline{u}_i+\overline{v}_i)^T h(\overline{x}) \\
&= F_i(h(\overline{x})).
\end{aligned}$$

According to Theorem 4.2 (weak duality), it follows that $(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha})$ is an efficient solution of (D^μ) and

$$H(\overline{r},\overline{s},\overline{u},\overline{v},\overline{t},\overline{\lambda},\overline{\alpha}) = F(h(\overline{x})). \qquad \square$$

Now, we give an example illustrating Theorem 4.3.

5. An example

Let us consider the following bilevel multiobjective fractional problem

$$(P) \quad \mathrm{v}-\min_{x\in C}\left\{\frac{f_1(x,\upsilon(x))}{g_1(x,\upsilon(x))}, \frac{f_2(x,\upsilon(x))}{g_2(x,\upsilon(x))}\right\},\ C := \{x\in X,\ G(x,\upsilon(x)) \le 0\},$$

where $f_i, g_i : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R},\ (i = 1, 2)$, $G : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned} f_1(x,t) &= x - t + 3, & g_1(x,t) &= -x - t + 1\\ f_2(x,t) &= 2x + t + 3, & g_2(x,t) &= -2x - t + 1\\ f(x,y) &= -x + y^2 - 1, & G(x,t) &= 2x + t. \end{aligned}$$

Let $X = \mathbb{R}_+$, $A = [0, 1]$. For any $x \in \mathbb{R}$, we have

$$v(x) := \inf_{y\in A} f(x,y) = -x - 1 \text{ and } h(x) := (x, v(x)) = (x, -x - 1).$$

Herein, the set Ω of the feasible solutions of (P) is given by

$$\Omega := \{x \in X,\ x - 1 \leq 0\} = [0, 1].$$

Subsequently, we can to formulate our problem (P) as follows

$$(P) \quad \mathrm{v}-\min_{x\in\Omega} \left(x + 2, \frac{x + 2}{-x + 2} \right),$$

The corresponding parametric problem (P^μ) will be

$$(P^\mu) \quad \mathrm{v}-\min_{x\in\Omega} \left(2x + 4 - 2\mu_1, (1 + \mu_2)x + 2 - 2\mu_2\right),$$

Clearly, $f_i, -g_i,\ i = 1, 2$ and G are convex functions and $\mathbb{R}^2_+$-increasing. Moreover, one can see that $f_i(x, v(x)) \geq 0$ and $0 < a = 1 \leq g_i(x, v(x)) \leq 2 = b$ for all $x \in C$, $i = 1, 2$.
To formulate the dual problem (D^μ), we need to determine the conjugate functions as presented below.

$$f_1^*(x^*, t^*) = \begin{cases} -3, & x^* = 1,\ t^* = -1,\\ +\infty, & \text{otherwise.} \end{cases}$$

$$(-\mu_1 g_1)^*(x^*, t^*) = \begin{cases} \mu_1, & x^* = \mu_1,\ t^* = \mu_1,\\ +\infty, & \text{otherwise.} \end{cases}$$

$$f_2^*(x^*, t^*) = \begin{cases} -3, & x^* = 2,\ t^* = 1,\\ +\infty, & \text{otherwise.} \end{cases}$$

$$(-\mu_2 g_2)^*(x^*, t^*) = \begin{cases} \mu_2, & x^* = 2\mu_2,\ t^* = \mu_2,\\ +\infty, & \text{otherwise.} \end{cases}$$

$$(s^T h)^*(x^*) = \begin{cases} s_2, & x^* = s_1 - s_2,\\ +\infty, & \text{otherwise.} \end{cases}$$

$$(rG)^*(x^*,t^*) = \begin{cases} 0, & x^* = 2r,\ t^* = r, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$\sigma_X(x^*) = \begin{cases} 0, & x^* \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Consequently, the dual problem (D^μ) takes on the following form

$$(D^\mu) \quad \text{v} - \max_{(r,s,u,v,t,\lambda,\alpha)\in B} H(r,s,u,v,t,\lambda,\alpha) = \begin{pmatrix} H_1(r,s,u,v,t,\lambda,\alpha) \\ H_2(r,s,u,v,t,\lambda,\alpha) \end{pmatrix},$$

where

$$H_i(r,s,u,v,t,\lambda,\alpha) = -f_i^*(u_i) - (-\mu_i g_i)^*(v_i)$$
$$- \frac{1}{2\lambda_i}\Big[(rG)^*\Big(s - \sum_{i=1}^{2} \lambda_i(u_i+v_i)\Big) + (s^T h)^*(t) + \sigma_X(-t)\Big] + \alpha_i,\ (i=1,2).$$

It is evident that the two objective functions of the dual problem are greater than $-\infty$ if and only if $u_1 = (1,-1),\ u_2 = (2,1),\ v_1 = (\mu_1,\mu_1),\ v_2 = (2\mu_2,\mu_2),\ s - \sum_{j=1}^{2} \lambda_j(u_j+v_j) = (2r,r),\ t = s_1 - s_2 \geq 0$. Then, the dual problem (D^μ) becomes

$$(D) \quad \text{v} - \max_{(r,s,u,v,t,\lambda,\alpha)\in B} \begin{pmatrix} H_1(r,s,u,v,t,\lambda,\alpha) \\ H_2(r,s,u,v,t,\lambda,\alpha) \end{pmatrix},$$

where

$$H_1(r,s,u,v,t,\lambda,\alpha) = 3 - \mu_1 - \frac{s_2}{2\lambda_1} + \alpha_1,$$

$$H_2(r,s,u,v,t,\lambda,\alpha) = 3 - \mu_2 - \frac{s_2}{2\lambda_2} + \alpha_2,$$

and

$$B := \Big\{(r,s,u,v,t,\lambda,\alpha):\ r \geq 0,\ s = (s_1,s_2) \in \mathbb{R}^2_+,\ t = s_1 - s_2 \geq 0,$$
$$u = (u_1,u_2),\ v = (v_1,v_2),\ u_1 = (1,-1),\ u_2 = (2,1),\ v_1 = (\mu_1,\mu_1),$$
$$v_2 = (2\mu_2,\mu_2),\ s - \sum_{j=1}^{2} \lambda_j(u_j+v_j) = (2r,r), \alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2,$$
$$\lambda = (\lambda_1,\lambda_2) \in \Lambda^+,\ \sum_{j=1}^{2} \lambda_j\alpha_j = 0\Big\}.$$

It is easy to check that the feasible point $\overline{x} = 0$ is a properly efficient solution to the problem (P^{μ}) and that (CQ) is fulfilled (since $G(h(\overline{x})) = G(0, -1) = -1 \leq 0$). By the Theorem 4.3, there exists an efficient solution $(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}) \in B$ to the dual problem (D^{μ}) such that

$$F(h(\overline{x})) = H(\overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \overline{\lambda}, \overline{\alpha}),$$

where

$$\mu = (\mu_1, \mu_2) = \left(\frac{f_1(\overline{x}, v(\overline{x}))}{g_1(\overline{x}, v(\overline{x}))}, \frac{f_2(\overline{x}, v(\overline{x}))}{g_2(\overline{x}, v(\overline{x}))} \right) = (2, 1).$$

An efficient solution for (D^{μ}) can be found through simple calculations as follows.

$$\overline{r} = 0,\ \overline{s} = (\overline{s}_1, \overline{s}_2) = (\tfrac{7}{2}, \tfrac{3}{2}),\ \overline{t} = \overline{s}_1 - \overline{s}_2 = 2,\ \overline{u} = (\overline{u}_1, \overline{u}_2) = ((1, -1), (2, 1)),$$
$$\overline{v} = (\overline{v}_1, \overline{v}_2) = ((2, 2), (2, 1)), \overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2) = (\tfrac{1}{2}, \tfrac{1}{2}) \text{ and } \overline{\alpha} = (0, 0)\,.$$

6. Conclusion and Discussions

In this paper, we established necessary and sufficient optimality conditions and duality results for constrained bilevel multiobjective fractional programming problems with extremal-value function characterizing a properly efficient solution in terms of the conjugate duality approach of the data functions. These results are obtained by introducing a second multiobjective (convex/nonfractional) programming problem (P^{μ}) that is, in some sense, equivalent to the previous problem under consideration. Afterwards, we introduced a scalar optimization problem (P^{μ}_{λ}), with $\lambda \in \Lambda^{+}$, by use of a scalarization technique. Following this, the necessary and sufficient optimality conditions are established under a constraint qualification. Thanks to the previous results, the dual problem of (P^{μ}) is constructed, and weak and strong duality results between (P^{μ}) and its dual are proved.
For future research, we will try to study the same problem (P) where all the data functions $g_1, \ldots, g_m$ are supposed convex. In this case, by using a parametric approach we transform the problem (P) equivalently as a DC programming problem. Also, we will attempt to examine the robustness of the problem (P) from optimality conditions and duality point of view.

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Authors' addresses:

Ahmed Rikouane
Department of Mathematics
Faculty of sciences
University Ibnou Zohr
Agadir, Morocco.
E-mail: `rikouaneah@gmail.com`

Mohamed Laghdir
Faculty of sciences
University Chouaib Doukkali
El Jadida, Morocco.
E-mail: `laghdirm@gmail.com`

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