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Statistical Convergence Restricted by Weight Functions and its Application in the Variation of γ-covers

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Abstract. *In this paper, we utilize a weight function g to regulate the pace of the statistical convergence in a topological space. We extend the notion of statistical convergence to weighted statistical convergence by utilizing the weighted density. Using this intriguing idea of convergence, a new variation of γ-covers (referred to as s_g-γ cover) is introduced. Subsequently some topological analysis are conducted on the class of s_g-γ coverings. It is demonstrated that the new class of s_g-γ coverings lies some where between the class of γ-covers and the class of s-γ cover classes.*

Keywords: Keywords: Weighted density, weighted statistical convergence, γ-covers, s_g-γ-covers.
MS Classification 2020: MSClassification: 54D20, 54B20, 54C35.

1. Introduction

Fast [16] as well as Schoenberg [22] provided some fundamental characteristics of statistical convergence and also examined the notion as a summability technique. Let $S \subseteq \mathbb{N}$ and let $\mathbb{N}$ be the set of all natural numbers. The symbol $\delta(S)$ is employed to denote the natural density (also called asymptotic density) of the set S, where

$$\delta(S) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in S\}|}{n}.$$

In 1951, Fast [16] expanded the notion of convergence to include statistical convergence for sequences of real numbers. In 2008, Di Maio and Kočinac introduced the concept of statistical convergence in a topological space. A sequence $\{a_n : n \in \mathbb{N}\}$ in a topological space X is considered statistically convergent to $a \in X$ if, for every neighborhood U of a, the natural density of the set $\{n \in \mathbb{N} : a_n \notin U\}$ is zero. In their study, Di Maio and Kočinac [14] extended the idea of γ cover to include statistical γ cover. Recent investigations on covering properties can be found in [2, 3, 4, 5, 6, 8, 9, 12, 13, 20]. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is referred to as a statistical γ cover if, for every $x \in X$, the natural density of the set $\{n \in \mathbb{N} : x \notin U_n\}$ is equal to zero.

The authors of [10] suggested a modified form of asymptotic density in which $n^\alpha, 0 < \alpha < 1$ was used in place of n which has been stated as the statistical convergence of order α. The symbol $\delta^\alpha(S)$ is utilized to represent the set S's natural density of order α, given by

$$\delta^\alpha(S) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in S\}|}{n^\alpha}.$$

By using this variation Bhunia et. al. [10] has extened the concept of statistical convergence to statistical convergence of order α where the parameter α controls the rate of statistical convergence.

A mathematical notion known as weighted statistical convergence expands the conventional ideas of statistical convergence by adding a weight function that gives distinct data points having varying degrees of significance. A mapping g defined in the form $g : \mathbb{N} \longrightarrow [0, \infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{g(n)} \neq 0$ are called weight functions.

The authors of [1] defined a more general type of natural density by substituting n^α with a weigt function g. The notation $\delta_g(S)$ of the same set $S \subset \mathbb{N}$ defines the weighted density by the formula,

$$\delta_g(S) = \lim_{n\to\infty} \frac{|\{k \leq n : k \in S\}|}{g(n)}.$$

Using weighted density authors of [1] has studied weighted statistical convergence of real sequences. In this paper, we introduce and investigave weighted statistical convergence in topology. Moreover following the path of Di Maio and Kočinac we investigate some topological features weighted statistical veriation of γ covers.

2. Preliminaries

This section outlines many prerequisite topics for the convenient reference of the readers. Throughout the paper no separation axiom has been asumed otherwise stated. For usual symbols and notions we follow [15].

Definition 2.1 ([18]). *A countable open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *is called* γ *cover if for every* $x \in X, \{n \in \mathbb{N} : x \notin U_n\}$ *is finite .*

Definition 2.2 ([14]). *A countable open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *is called an statistical* γ *cover if for each* $x \in X$*, the set* $\{n \in \mathbb{N} : x \notin U_n\}$ *has natural density 0.*

Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of families of subsets of an infinite set X. Then, Definition 2.3 ([21]). *$S_1(\mathcal{A},\mathcal{B})$ denotes the selection principle that for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there exists a sequence $\{B_n : n \in \mathbb{N}\}$ such that $B_n \in A_n$, for each n and $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.*

Definition 2.4 ([21]). *$S_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there exists a sequence $\{B_n : n \in \mathbb{N}\}$ of finite sets such that $B_n \subseteq A_n$, for each n and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.*

It is easy to verify that $g, f : \mathbb{N} \longrightarrow [0,\infty)$ defined as $g(n) = \ln(1+n)$ and $f(n) = n^{\alpha}, 0 < \alpha < 1$ are weight functions. In this paper, these weight functions will be utilized during the development of several examples.

3. Main Results

Definition 3.1. *Let g be a weight function. A sequence $\{x_n : n \in \mathbb{N}\}$ is said to be g-statistical convergence to x_0 in a topological space (X,τ), if for every neighbourhood U of x_0, $\delta_g(\{n \in \mathbb{N} : x_n \notin U\}) = 0$.*

In this case we can write,$s_g\text{-}\lim_{n\to\infty} x_n = x_0$.

Theorem 3.2. *Every s_g-convergence sequence is s-convergence if the limit $\lim_{n\to\infty} \frac{n}{g(n)}$ exists.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a s_g-convergent sequence in a topological space (X,τ) such that

$$s_g\text{-}\lim_{n\to\infty} x_n = x_0.$$

Now for every neighbourhood U of x_0,

$$\begin{aligned}
&\implies \delta_g(\{n \in \mathbb{N} : x_n \notin U\}) = 0\\
&\implies \delta_g(A) = 0, \text{ where } \{n \in \mathbb{N} : x_n \notin U\} = A\\
&\implies \lim_{n\to\infty} \frac{|\{k \in A : k \le n\}|}{g(n)} = 0\\
&\implies \lim_{n\to\infty} \frac{|\{k \in A : k \le n\}|}{n} \times \lim_{n\to\infty} \frac{n}{g(n)} = 0
\end{aligned}$$

So,

$$\lim_{n\to\infty} \frac{|\{k \in A : k \le n\}|}{n} = 0. \left[\text{since, } \lim_{n\to\infty} \frac{n}{g(n)} \neq 0\right]$$
$$\implies \delta(A) = 0 \qquad \implies \delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$$

Thus,$s\text{-}\lim_{n\to\infty} x_n = x_0$. Hence the theorem. □

If $g(n) = n$, then the concept of natural density and g-weighted density coincides. Hence s-convergence and s_g-convergence also coincides.
There exists a sequence which is statistical convergence but not s_g-convergence.

EXAMPLE 3.3. Let $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$ then (X, τ) is a topological space. Let us consider the weight function $g(n) = \ln(1 + n)$ and

$$x_n = \begin{cases} a, & \text{if } n = m^2,\ m \in \mathbb{N} \\ b, & \text{otherwise.} \end{cases}$$

Open neighbourhood of a are $U_1 = X$ and $U_2 = \{a\}$ for the neighbourhood U_2 of a, $\{n \in \mathbb{N} : x_n \notin U_2\} = \{1, 4, 9, 16, 25, \cdots\}$

$$\implies \delta_g(\{n \in \mathbb{N} : x_n \notin U_2\}) = \infty \neq 0.$$
$$\therefore x_n \overset{s_g-lim}{\not\to} a.$$

Open neighbourhood of b are $V_1 = X$ and $V_2 = \{b\}$ for the neighbourhood V_2 of b, $\{n \in \mathbb{N} : x_n \notin V_2\} = \{2, 3, 5, 6, 7, 8, 10, \cdots\}$

$$\implies \delta_g(\{n \in \mathbb{N} : x_n \notin V_2\}) = \infty \neq 0.$$
$$\therefore x_n \overset{s_g-lim}{\not\to} b.$$

So, $\{x_n\}$ is not s_g-convergence.

But, for the neighbourhood U_1 of a, $\{n \in \mathbb{N} : x_n \notin U_1\} = \emptyset$.

$$\implies \delta(\{n \in \mathbb{N} : x_n \notin U_1\}) = 0.$$

Also for the neighbourhood U_2 of a, $\{n \in \mathbb{N} : x_n \notin U_2\} = \{1, 4, 9, 16, 25, \cdots\}$

$$\implies \delta(\{n \in \mathbb{N} : x_n \notin U_2\}) = 0.$$

So, $\{x_n\}$ is statistical convergent and its statistical limit is a.

THEOREM 3.4. *In a first countable space, if a sequence $\{x_n : n \in \mathbb{N}\}$ is s_g-convergent to x_0, then there exists a $\Delta = \{n_1 < n_2 < n_3 < \cdots\} \subseteq \mathbb{N}$ such that*

$$\delta_g(\Delta) = \lim_{n\to\infty} \frac{n}{g(n)} \text{ and } \lim_{n\to\infty, m\in\mathbb{N}} x_n = x_0.$$

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in a first countable space X such that

$$s_g\text{-}\lim_{n\to\infty} x_n = x_0.$$

We fix a countable decreasing local base $B_{1,x_0} \supseteq B_{2,x_0} \supseteq B_{3,x_0} \supseteq \cdots$. For every $m \in \mathbb{N}$ we set $\Delta_m = \{n \in \mathbb{N} : x_n \in B_{m,x_0}\}$. Clearly, $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3 \supseteq \cdots$.

But

$$s_g\text{-}\lim_{n\to\infty} x_n = x_0.$$
$$\therefore \delta_g(\{n \in \mathbb{N} : x_n \notin B_{m,x_0}\}) = 0 \ \ \forall m \in \mathbb{N}$$
$$\implies \delta_g(\Delta_m^{\complement}) = 0 \ \ \forall m \in \mathbb{N}$$
$$\implies \delta_g(\Delta_m) = \lim_{n\to\infty} \frac{n}{g(n)} \ \ \forall m \in \mathbb{N}\,.$$

Now, $\left\{\frac{i}{g(i)} : i \in \mathbb{N}\right\}$ is a sequence of positive numbers. Take $K_1 \in \Delta_1$ arbitrarily. Since

$$\delta_g(\Delta_2) = \lim_{n\to\infty} \frac{n}{g(n)}\,,$$

there exists a $K_2 \in \Delta_2$ such that $K_2 > K_1$ and for all $n > K_2$

$$\frac{|\{i \in \Delta_2 : i \leq n\}|}{g(n)} > \frac{2}{g(2)}\,.$$

Since

$$\delta_g(\Delta_3) = \lim_{n\to\infty} \frac{n}{g(n)}\,,$$

there exists a $K_3 \in \Delta_3$ such that $K_3 > K_2$ and for all $n > K_3$

$$\frac{|\{i \in \Delta_3 : i \leq n\}|}{g(n)} > \frac{3}{g(3)}\,.$$

Proceeding in this way, we will have a sequence $K_1 < K_2 < K_3 < \cdots < K_j < \cdots$ of positive numbers such that $K_j \in \Delta_j$ $(j = 1, 2, 3, \cdots)$ and for all $n > K_j$

$$\frac{|\{i \in \Delta_j : i \leq n\}|}{g(n)} > \frac{j}{g(j)}\,.$$

Let Δ be the set which contains all the natural number in the interval $[1, K_1]$ and also contains all the natural number of the interval $[K_p, K_p + 1]$ which belongs to $\Delta_p (p = 1, 2, 3, \cdots)$. Let $\Delta = \{n_1 < n_2 < n_3 < \cdots\}$

$$\frac{|\{i \in \Delta : i \leq n\}|}{g(n)} \geq \frac{|\{i \in \Delta_m : i \leq n\}|}{g(n)} > \frac{j}{g(j)}$$
$$\therefore \lim_{n\to\infty} \frac{|\{i \in \Delta : i \leq n\}|}{g(n)} = \lim_{n\to\infty} \frac{n}{g(n)} \ \ \forall n \in \mathbb{N} \ \ \textit{where} \ \ K_i \leq n \leq K_{i+1}$$

Thus,

$$\delta_g(\Delta) = \lim_{n\to\infty} \frac{n}{g(n)}\,.$$

Let V be a neighbourhood of x_0 and $B_{i,x_0} \subset V$. If $n \in \Delta$ and $n > K_j$ then there is a $t \geq i$ with $K_t \leq n \leq K_{t+1}$ therefore, $n \in \Delta_t$. For every $n \in \Delta$, $n \geq K_i$ $x_n \in B_{t,x_0} \subset B_{i,x_0} \subset V$

$$\lim_{n\to\infty, m\in\mathbb{N}} x_n = x_0.$$

Hence the theorem. □

EXAMPLE 3.5. s_g-limit of an s_g-convergent sequence may not be unique.
Let $X = \{p, q, r\}$ and $\tau = \{\emptyset, X, \{p, q\}, \{r\}\}$ then (X, τ) is a topological space. Consider $g(n) = \ln(1+n)$ be the weight function and the sequence $\{x_n : n \in \mathbb{N}\}$ where

$$x_n = \begin{cases} p, & \text{if } n = m^m,\ m \in \mathbb{N} \\ q, & \text{otherwise.} \end{cases}$$

Open neighbourhood of p are $W_1 = X$ and $W_2 = \{p, q\}$, $\{n \in \mathbb{N} : x_n \notin W_1\} = \emptyset$ which implies $\delta_g(\{n \in \mathbb{N} : x_n \notin W_1\}) = 0$.

Also $\{n \in \mathbb{N} : x_n \notin W_2\} = \{1, 4, 27, 256, \cdots\}$

$$\begin{aligned} \delta_g(\{n \in \mathbb{N} : x_n \notin W_2\}) &= \delta_g(\{1, 4, 27, \cdots\}) \\ &= \lim_{n\to\infty} \frac{|\{K \in A : K \leq n\}|}{g(n)}, \text{ where } A = \{1, 4, 27, \cdots\} \\ &= \lim_{n\to\infty} \frac{n}{\ln(1+n)} = 0 \end{aligned}$$

Thus, for every neighbourhood W of p $\delta_g(\{n \in \mathbb{N} : x_n \notin W\}) = 0$. So,

$$x_n \overset{s_g-lim}{\longrightarrow} p.$$

But the neighbourhoods of p are the only neighbourhoods of q. Therefore for every open neighbourhood of q, $\delta_g(\{n \in \mathbb{N} : x_n \notin W\}) = 0$. So,

$$x_n \overset{s_g-lim}{\longrightarrow} q.$$

Thus the limit of s_g-limit of an s_g-convergent sequence may not be unique.

THEOREM 3.6. *In a Hausdorff space, limit of an s_g-convergent sequence is unique.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in a Hausdorff space X such that

$$x_n \overset{s_g-lim}{\longrightarrow} p \text{ and } x_n \overset{s_g-lim}{\longrightarrow} q \text{ with } p \neq q\,.$$

Since X is a Hausdorff space, there exist $A, B \in \tau$ such that $p \in A$ and $q \in B$ and $A \cap B = \emptyset$.

But $\delta_g(\{n \in \mathbb{N} : x_n \notin A\}) = 0$ which implies $\delta_g(\{n \in \mathbb{N} : x_n \in X \setminus A\}) = 0$. But $B \subseteq X \setminus A$ and since $A \cap B = \emptyset$ and hence

$$\begin{aligned} \delta_g(\{n \in \mathbb{N} : x_n \in B\}) &\leq \delta_g(\{n \in \mathbb{N} : x_n \in X \setminus A\}) = 0 \\ &\implies \delta_g(\{n \in \mathbb{N} : x_n \in B\}) = 0 \\ &\implies \delta_g(\{n \in \mathbb{N} : x_n \notin B\}) = \lim_{n\to\infty} \frac{n}{g(n)} \neq 0\,, \end{aligned}$$

which contradicts $x_n \overset{s_g-lim}{\longrightarrow} q$. $\therefore p = q$ and hence in a Hausdorff space limit of an s_g-limit of an s_g-convergent sequence is unique. □

Example 3.7. Subsequence of s_g-convergent sequence may not be s_g-convergent. Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then (X, τ) is a topological space. Let us consider the weight function $g(n) = \ln(1 + n)$ and the sequence $\{\alpha_n : n \in \mathbb{N}\}$ where

$$\alpha_n = \begin{cases} b, & \text{if } n = m^m,\ m \in \mathbb{N} \\ a, & \text{otherwise.} \end{cases}$$

for every open neighbourhood U of a $\delta_g(\{n \in \mathbb{N} : \alpha_n \notin U\}) = 0$. $\therefore$ $\{\alpha_n\}$ is s_g-convergent sequence.

On the other hand, considering the subsequence $\{\alpha_{n_i} : i \in \mathbb{N}\}$ of $\{\alpha_n : n \in \mathbb{N}\}$

$$\alpha_{n_i} = \begin{cases} \alpha_{i^i}, & \text{if i is odd} \\ \alpha_{i^i+1}, & \text{if i is even.} \end{cases}$$

Here, for the open neighbourhood $V = \{a\}$, $\delta_g(\{n \in \mathbb{N} : \alpha_{n_i} \notin V\}) = \delta_g(S)$ where $S = \{n \in \mathbb{N} : \alpha_{n_i} \notin V\}$.

$$\begin{aligned} \delta_g(\{n \in \mathbb{N} : \alpha_{n_i} \notin V\}) &= \lim_{n\to\infty} \frac{|\{K \in S : K \leq n\}|}{g(n)} \\ &\implies \lim_{n\to\infty} \frac{n}{\ln(1+2n)} = \infty \end{aligned}$$

$\therefore$ the sequence $\{\alpha_{n_i} : i \in \mathbb{N}\}$ is not s_g-convergent sequence.

Example 3.8. Consider $X = \{p, q\}$, $\tau = p(X)$ and the sequence $\{a_n : n \in \mathbb{N}\}$ is given by

$$a_n = \begin{cases} p, & \text{if } n = f(m),\ m \in \mathbb{N} \\ q, & \text{otherwise.} \end{cases}$$

where $f : \mathbb{N} \longrightarrow \mathbb{R}$ is an invertible weight function. The sequence $\{a_n : n \in \mathbb{N}\}$ is s_g-convergent if $g^{-1}(n) < f^{-1}(n)$ $\forall n \in \mathbb{N}$ and does not converge otherwise;

where g is an invertible weight function. $U = \{q\}$ is the smallest neighbourhood of q and

$$\{n \in \mathbb{N} : a_n \notin U\} = \{f(m) \in \mathbb{N} : m \in \mathbb{N}\} = A(say)$$
$$\delta_g(A) = \lim_{n\to\infty, n\in A} \frac{|\ \{k \in A : k \leq n\}\ |}{g(n)} = \lim_{n\to\infty} \frac{n}{g(f^{-1}(n))}.$$

So,

$$\delta_g(A) = \begin{cases} 0, & \text{if } g^{-1}(n) < f^{-1}(n) \ \ \forall n \in \mathbb{N} \\ 1, & \text{if } f^{-1}(n) = g^{-1}(n) \ \ \forall n \in \mathbb{N} \\ \infty, & \text{if } g^{-1}(n) > f^{-1}(n) \ \ \forall n \in \mathbb{N} \end{cases}$$

$\therefore$ the sequence $\{a_n : n \in \mathbb{N}\}$ is s_g-convergent for $g^{-1}(n) < f^{-1}(n)$ $\forall n \in \mathbb{N}$.

We adopt the following notations:
M_0 :the collection of all statistically convergent sequences in a space (X,τ).
${}_gM_0$: the collection of all s_g-convergent sequences in a space (X,τ).

NOTE. If $g : \mathbb{N} \longrightarrow \mathbb{N}$ is a map such that $g(n) = n^\alpha, 0 < \alpha < 1$ then statistically convergence of order α is equal to the weight function.

THEOREM 3.9. *Let $f(x)$ and $g(x)$ be two weight functions. If $f(n) < g(n)$ $\forall n \in \mathbb{N}$, then ${}_gM_0 \subseteq {}_fM_0$.*

Proof. Let $f(n) < g(n)$ $\forall n \in \mathbb{N}$ and $\{a_n : n \in \mathbb{N}\}$ be a sequence in X. For every neighbourhood U of $a \in X$,

$$\frac{|\{K \leq n : a_K \notin U\}|}{g(n)} \leq \frac{|\{K \leq n : a_K \notin U\}|}{f(n)}$$
$$\therefore {}_gM_0 \subseteq {}_fM_0.$$

Hence the theorem. □

DEFINITION 3.10. *A subset $P \subseteq \mathbb{N}$ is said to be statistical g-dense (or s_g-dense) if*

$$\delta_g(P) = \lim_{n\to\infty} \frac{n}{g(n)}.$$

It is to be mentioned that union of two s_g-dense set is s_g-dense. The set $2\mathbb{N}$ and ($\mathbb{N} \setminus 2\mathbb{N}$) both are s_g-dense. But ($\mathbb{N} \setminus 2\mathbb{N}) \cap 2\mathbb{N} = \emptyset$ is not s_g-dense.

DEFINITION 3.11. *A subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of a sequence $\{x_n : n \in \mathbb{N}\}$ is said to be s_g-dense if $\{n \in \mathbb{N} : x_n \in \{x_{n_k} : k \in \mathbb{N}\}\}$ is s_g-dense.*

THEOREM 3.12. *In a topological space (X, τ), a sequence is s_g-convergent if and only if its s_g-dense subsequence is s_g-convergent.*

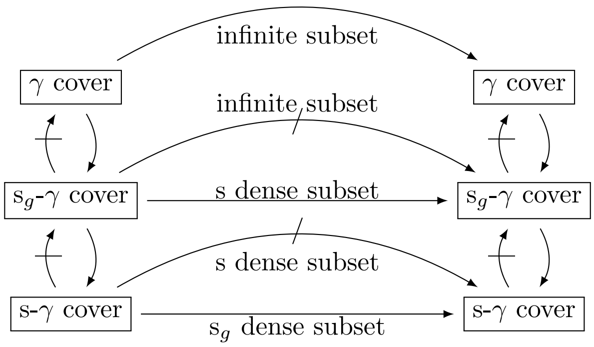

Figure 1: The relationship between different forms of γ covers is examined as dense subsets vary.

Proof. Let g be a weight function and $\{a_n : n \in \mathbb{N}\}$ be a s_g-convergent sequence in a topological space (X, τ). Now,

$$\lim_{n\to\infty} \frac{|\{k \le n : a_k \in \{a_n : n \in \mathbb{N}\}\}|}{g(n)} = \lim_{n\to\infty} \frac{n}{g(n)}$$

$\therefore$ Every sequence in s_g-dense in itself. So, $\{a_n : n \in \mathbb{N}\}$ is a s_g-dense subsequence of itself which is s_g-convergent.

Conversely, let $a_n \overset{s_g\text{-}lim}{\longrightarrow} a$ and $\{a_{n_k} : k \in \mathbb{N}\}$ be a s_g-dense subsequence of $\{a_n : n \in \mathbb{N}\}$ and $\{a_{n_k} : k \in \mathbb{N}\}$ does not s_g-converge. $\therefore$ for every $p \in X$ there exists a neighbourhood U of p such that $\delta_g(\{k \in \mathbb{N} : a_{n_k} \notin U\}) \neq 0$.

$$\delta_g(\{n \in \mathbb{N} : a_n \notin U\}) \ge \delta_g(\{k \in \mathbb{N} : a_{n_k} \notin U\}) \neq 0$$

$\implies \{a_n : n \in \mathbb{N}\}$ does not converge, which is a contradiction. $\therefore \{a_{n_k} : k \in \mathbb{N}\}$ must converge.

Hence the theorem. □

We know that infinite subset of an γ cover is a γ-cover and s-dense subset of an s-γ cover is an s-γ cover. So from our study we conclude the diagram given in Figure 1.

DEFINITION 3.13. *A countable open cover $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$ of a topological space X is said to be s_g-γ cover if for each $x \in X$, the set $\{n \in \mathbb{N} : x \notin W_n\}$ has g-weighted density 0 i.e. $\delta_g(\{n \in \mathbb{N} : x \notin W_n\}) = 0$.*

NOTATIONS. In this section, the following symbols are assumed:
Γ: the collection of all γ covers of a space X.

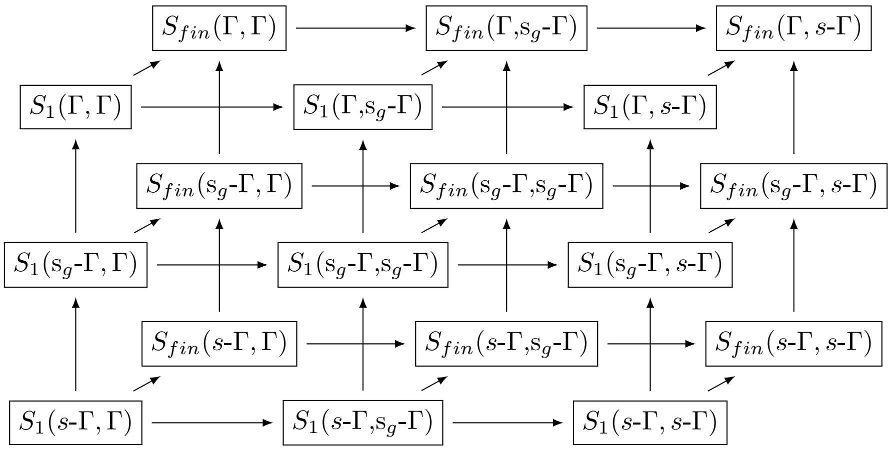

Figure 2: Classical selection principles under the variation of γ covers.

s-Γ: the collection of all statistical γ-covers (or s-γ cover) of a space X.
s_g-Γ: the collection of all weighted statistical γ-covers (or s_g-γ cover) of a space X.

Theorem 3.14. $\Gamma \subseteq s_g\text{-}\Gamma \subseteq s\text{-}\Gamma$.

Proof. Let $\mathcal{W} = \{w_n : n \in \mathbb{N}\} \in \Gamma$.

Then, for every $x \in X$, the set $\{n \in \mathbb{N} : x \notin W_n\}$ is finite. But g weighted density of finite set is always 0. i.e. $\delta_g(\{n \in \mathbb{N} : x \notin W_n\}) = 0 \ \forall x \in X$. $\implies \mathcal{W} = \{w_n : n \in \mathbb{N}\} \in s_g\text{-}\Gamma$. $\therefore \Gamma \subseteq s_g\text{-}\Gamma$.

By theorem 3.2, $s_g\text{-}\Gamma \subseteq s\text{-}\Gamma$. Hence $\Gamma \subseteq s_g\text{-}\Gamma \subseteq s\text{-}\Gamma$. □

Since every singleton set is a finite set, selection principle $S_1(\mathcal{A},\mathcal{B})$ implies the selection principle $S_{fin}(\mathcal{A},\mathcal{B})$ and we know that $S_1(\mathcal{A},\mathcal{B})$ and $S_{fin}(\mathcal{A},\mathcal{B})$ are monotonic in the first collection and anti-monotonic in the second collection, we have the implication diagram Figure 2.

Example 3.15. There exists a s-γ cover which is not an s_g-γ cover. Also there exists a s_g-γ cover which is not a γ-cover.

Let $X = \{a,b\}$, $\tau = \{\emptyset, \{b\}, X\}$ then $\{X,\tau\}$ is a topological space. Considering the weight function $g(n) = \ln(1+2n)$ and the cover $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$ where

$$w_n = \begin{cases} \{b\}, & \text{if } n \in \{n^3 : n \in \mathbb{N}\} \\ X, & \text{otherwise} \end{cases}$$

For every $x \in X$, $\delta(\{n \in \mathbb{N} : x \notin w_n\}) = 0 \implies \mathcal{W} = \{w_n : n \in \mathbb{N}\} \in s\text{-}\Gamma$.

But,

$$\delta_g(\{n \in \mathbb{N} : a \notin w_n\}) = \delta_g\{1, 8, 27, \cdot, \cdot, \cdot\} = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{\ln(1+2n)}\,.$$

For every $x \in X$, $\delta_g(\{n \in \mathbb{N} : x \notin w_n\}) \neq 0 \implies \mathcal{W} = \{w_n : n \in \mathbb{N}\} \notin s_g\text{-}\Gamma$.

Considering another cover $\mathcal{U} = \{u_n : n \in \mathbb{N}\}$ where

$$u_n = \begin{cases} \{a\}, & \text{if } n = n^m : m \in \mathbb{N} \\ X, & \text{otherwise} \end{cases}$$

for every $x \in X$, $\delta_g(\{n \in \mathbb{N} : x \notin u_n\}) = 0. \implies \mathcal{U} = \{u_n : n \in \mathbb{N}\} \in s_g\text{-}\Gamma$.

But, $\{n \in \mathbb{N} : b \notin u_n\} = \{1, 4, 27, 64, \cdot, \cdot, \cdot\}$ is not finite. $\implies \mathcal{U} = \{u_n : n \in \mathbb{N}\} \notin \Gamma$.

Hence there exists a s-γ cover which is not an s_g-γ cover. Also there exists a s_g-γ cover which is not a γ-cover.

Example 3.16. Subcover of s_g-γ cover may not be an s_g-γ cover.

Let $X = \{a, b\}$ and $\tau = \{\emptyset, X, \{b\}\}$. Then (X, τ) is a topological space.

Considering weight function $g(n) = \ln(1 + 2n)$ and the cover $\mathcal{U} = \{u_n : n \in \mathbb{N}\}$ where

$$u_n = \begin{cases} \{b\}, & \text{if } n = n^m : m \in \mathbb{N} \\ X, & \text{otherwise} \end{cases}$$

for every $x \in X$, $\{n \in \mathbb{N} : x \notin u_n\} = \emptyset. \implies \delta_g(\{n \in \mathbb{N} : x \notin u_n\}) = \delta_g(\{1, 4, 27, 64, \cdot, \cdot, \cdot\}) = 0. \implies \mathcal{U} = \{u_n : n \in \mathbb{N}\} \in s_g\text{-}\Gamma$.

On the other hand, for the subcover $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V} = \{v_n : n \in \mathbb{N}\}$ where

$$v_n = \begin{cases} u_2, & \text{if n=2} \\ u_{n^n}, & \text{otherwise} \end{cases}$$

$\implies \delta_g(\{n \in \mathbb{N} : a \notin v_n\}) = \delta_g(\{1, 3, 4, 5, \cdot, \cdot, \cdot\}) = \infty. \implies \mathcal{V} = \{v_n : n \in \mathbb{N}\} \notin s_g\text{-}\Gamma$.

Hence Sub cover of s_g-γ cover may not be an s_g-γ cover.

Theorem 3.17. *If $\mathcal{U}$ and $\mathcal{V}$ are two s_g-γ covers of a topological space (X, τ). Then $\mathcal{U} \bigsqcup \mathcal{V} = \{u_i \cup v_i : u_i \in \mathcal{U}$ and $v_i \in \mathcal{V}$ and $i \in \mathbb{N}\}$ is also an s_g-γ cover.*

Proof. Let $\mathcal{U} = \{u_i : i \in \mathbb{N}\}$ and $\mathcal{V} = \{v_i : i \in \mathbb{N}\}$ be two s_g-γ covers of a topological space (X, τ).

Then for each $x \in X$, $\delta_g(\{i \in \mathbb{N} : x \notin u_i\}) = 0$ and $\delta_g(\{i \in \mathbb{N} : x \notin v_i\}) = 0$. But,

$$\implies \{i \in \mathbb{N} : x \notin u_i \cup v_i\} \subseteq \{i \in \mathbb{N} : x \notin u_i\}.$$
$$\implies \delta_g(\{i \in \mathbb{N} : x \notin u_i \cup v_i\}) \leq \delta_g(\{i \in \mathbb{N} : x \notin u_i\}) = 0.$$
$$\implies \delta_g(\{i \in \mathbb{N} : x \notin u_i \cup v_i\}) = 0\,.$$

$\therefore \mathcal{U} \bigsqcup \mathcal{V} = \{u_i \cup v_i : i \in \mathbb{N}\} \in s_g\text{-}\gamma$. Hence $\mathcal{U} \bigsqcup \mathcal{V}$ is an s_g-γ cover of (X, τ). □

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