

# On the dynamics of non-autonomous systems in a neighborhood of a homoclinic trajectory

ALESSANDRO CALAMAI, MATTEO FRANCA,  
AND MICHAL POSPÍŠIL

**ABSTRACT.** *This article is devoted to the study of a 2-dimensional piecewise smooth (but possibly) discontinuous dynamical system, subject to a non-autonomous perturbation; we assume that the unperturbed system admits a homoclinic trajectory  $\vec{\gamma}(t)$ . Our aim is to analyze the dynamics in a neighborhood of  $\vec{\gamma}(t)$  as the perturbation is turned on, by defining a Poincaré map and evaluating fly time and space displacement of trajectories performing a loop close to  $\vec{\gamma}(t)$ .*

*Besides their intrinsic mathematical interest, these results can be thought of as a first step in the analysis of several interesting problems, such as the stability of a homoclinic trajectory of a non-autonomous ODE and a possible extension of Melnikov chaos to a discontinuous setting.*

**Keywords:** piecewise smooth systems, homoclinic trajectories, non-autonomous dynamical systems, exponential dichotomy, non-autonomous perturbation.

**MS Classification 2020:** MSCClassification, Primary 34C37, 34A36, 34D10, 37G20; Secondary 37C60.

## 1. Introduction

This article is devoted to the study of a 2-dimensional piecewise smooth (but possibly) discontinuous dynamical system, subject to a non-autonomous perturbation, namely

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad (1)$$

where  $\vec{f}$  and  $\vec{g}$  are bounded together with their derivatives up to the  $r$ -th order,  $r > 1$ , and  $\varepsilon \geq 0$  is a small parameter.

In particular we assume that the origin  $\vec{0} = (0, 0)$  is a critical point for (1) for any  $\varepsilon$  and that there is a trajectory  $\vec{\gamma}(t)$  homoclinic to  $\vec{0}$  when  $\varepsilon = 0$ .

In this context, classical Melnikov theory allows us to define explicitly a function  $\mathcal{M}(\tau)$ , the *Melnikov function*, see (9), such that the existence of non-degenerate zeros of  $\mathcal{M}(\tau)$  is a sufficient condition for the persistence of the

homoclinic trajectory, say  $\vec{x}_b(t, \varepsilon)$ , and the insurgence of chaotic phenomena if we also assume that  $\vec{g}$  is periodic in  $t$ . In fact, after the pioneering work of Melnikov [28], in the smooth case, the problem of detecting the occurrence of chaotic solutions for non-autonomous dynamical systems is now well studied, see e.g. [21, 29, 31, 32]. In this paper we want to deepen our knowledge of the dynamics close to  $\vec{\gamma}(t)$  when the perturbation is turned on. Namely, let  $L^0$  be a curve transversal to  $\{\vec{\gamma}(t) \mid t \in \mathbb{R}\}$ , our purpose is to define a Poincaré map from a compact connected subset  $A^{\text{fwd},+}(\tau)$  of  $L^0$  back to  $L^0$  and to evaluate space displacement and fly time when  $\varepsilon \neq 0$ .

Our work takes motivation and inspiration from the study of the asymptotic stability of periodic and homoclinic trajectories of an ordinary differential equation (ODE). Borg, Hartman and Leonov (among others), see e.g. [6, 23, 26], have given necessary and sufficient conditions for the existence, the uniqueness and the asymptotic stability of a periodic orbit of an autonomous  $n \geq 2$  dimensional ODE, and criteria to determine their basin of attraction. More recently this kind of study has been addressed in a non-autonomous context, extending Borg's criterion to almost periodic trajectories of an almost periodic differential equation, see [18–20]: in all these problems the study of the Poincaré map in a neighborhood of the periodic solutions plays a key role.

The case of homoclinic trajectories is more tricky. Consider (1) when  $\varepsilon = 0$ : the problem of the stability of the homoclinic trajectory  $\vec{\gamma}(t)$  has been completely solved by the so called Dulac sequence, see e.g. [31, §13] and references therein. That is, if  $\text{div}[\mathbf{f}_{\mathbf{x}}(\vec{0})] < 0$ , or  $\text{div}[\mathbf{f}_{\mathbf{x}}(\vec{0})] = 0$  and  $\int_{\vec{\gamma}} \text{div}[\mathbf{f}_{\mathbf{x}}] ds < 0$ , then  $\vec{\gamma}(t)$  is stable from inside, while if  $\text{div}[\mathbf{f}_{\mathbf{x}}(\vec{0})] > 0$ , or  $\text{div}[\mathbf{f}_{\mathbf{x}}(\vec{0})] = 0$  and  $\int_{\vec{\gamma}} \text{div}[\mathbf{f}_{\mathbf{x}}] ds > 0$ , then  $\vec{\gamma}(t)$  is unstable from inside, see [31, §13] for more details. Once again the study of the Poincaré map “inside”  $\Gamma = \{\vec{\gamma}(t) \mid t \in \mathbb{R}\} \cup \{\vec{0}\}$  is crucial for this analysis; as far as the authors are aware there have been no attempts to extend the theory in a non-autonomous context: with this paper we try to fill this gap.

The first difficulty one faces in doing so is that we do not have anymore a proper Poincaré map, but we are forced to consider a map starting from a time dependent transversal,  $A^{\text{fwd},+}(\tau)$ , which has as endpoint the intersection  $\vec{P}_s(\tau)$  between the stable leaf  $\tilde{W}^s(\tau)$  and the transversal  $L^0$ , to end up close to  $\vec{P}_u(\tau)$ , the intersection of the unstable leaf  $\tilde{W}^u(\tau)$  and  $L^0$ . Further we have to split our argument in the analysis of four different subpaths: firstly from  $L^0$  to a transversal to  $\tilde{W}^s(\tau)$  close to the origin, denoted by  $\tilde{S}^+$ , secondly from  $\tilde{S}^+$  to a curve  $\Omega^0$  passing through the origin, thirdly from  $\Omega^0$  to a transversal to  $\tilde{W}^u(\tau)$  close to the origin,  $\tilde{S}^-$ , finally from  $\tilde{S}^-$  back to  $L^0$ . In the first path we use a (nontrivial) fixed point argument which reminds the one used in [23] for periodic trajectories, in the second one we decompose the trajectory as a stable part, a linear one and a remainder and we use a (tricky) fixed point argument which combines exponential dichotomy with some ideas borrowed from [31, §13], and

we estimate fly time space displacement adapting in a nontrivial way [31, §13]. Then we conclude with an inversion of time argument.

This contribution will be essential in proving some results concerning the analogues of the *stability from inside* of  $\Gamma$  in a non-autonomous context which will be addressed in a forthcoming paper.

Further we plan to use these ideas to study the possibility to establish a sub-harmonic theory for Melnikov, i.e., the possibility that we may get a large number of homoclinic trajectories for  $\varepsilon \neq 0$  even if we have just one nondegenerate zero of  $\mathcal{M}(\tau)$ .

Moreover we believe this result could be used in a system characterized by the presence of a chaotic pattern to find *safe regions*, i.e., subsets of initial conditions in a neighborhood of  $\Gamma$ , which cannot exhibit chaotic behavior. See §6 for more details concerning these three possible developments.

In fact all the results of this paper are obtained in a discontinuous piecewise smooth setting, i.e., we consider

$$\dot{\vec{x}} = \vec{f}^\pm(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega^\pm, \tag{PS}$$

where  $\Omega^\pm = \{\vec{x} \in \Omega \mid \pm G(\vec{x}) > 0\}$ ,  $\Omega^0 = \{\vec{x} \in \Omega \mid G(\vec{x}) = 0\}$ ,  $\Omega \subset \mathbb{R}^2$  is an open set,  $G$  is a  $C^r$ -function on  $\Omega$  with  $r > 1$  such that 0 is a regular value of  $G$ . Next,  $\varepsilon \in \mathbb{R}$  is a small parameter, and  $\vec{f}^\pm \in C_b^r(\Omega^\pm \cup \Omega^0, \mathbb{R}^2)$ ,  $\vec{g} \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R}, \mathbb{R}^2)$  and  $G \in C_b^r(\Omega, \mathbb{R})$ , i.e., the derivatives of  $\vec{f}^\pm$ ,  $\vec{g}$  and  $G$  are uniformly continuous and bounded up to the  $r$ -th order, respectively, if  $r \in \mathbb{N}$ , and up to  $r_0$  if  $r = r_0 + r_1$  with  $r_0 \in \mathbb{N}$  and  $0 < r_1 < 1$  and the  $r_0$ -th derivatives are  $r_1$  Hölder continuous. Further, we assume that both the systems  $\dot{\vec{x}} = \vec{f}^\pm(\vec{x})$  admit the origin  $\vec{0} \in \mathbb{R}^2$  as a fixed point, and that  $\vec{0}$  lies on the discontinuity level  $\Omega^0$ .

Even in this discontinuous framework, the existence of a non-degenerate zero of the Melnikov function  $\mathcal{M}(\tau)$  guarantees the persistence to perturbation of the homoclinic trajectory, see [8]. Nevertheless, a geometrical obstruction forbids chaotic phenomena whenever we have sliding close to the origin, see [15]: this is indeed quite unexpected since this condition, together with periodicity in  $t$  of  $\vec{g}$ , are enough to guarantee the existence of a chaotic pattern in a smooth context, see e.g. [29], and in a piecewise smooth context if  $\vec{0} \notin \Omega^0$ , see e.g. [2–4]. We plan to use Theorems 4.2 and 4.3 in a forthcoming paper to show that the usual Melnikov conditions guarantees chaos as in the smooth setting if the geometrical obstruction is removed.

Discontinuous problems are motivated by several physical applications, for instance mechanical systems with impacts, see e.g. [7], power electronics when we have state dependent switches [1], walking machines [17], relay feedback systems [5], biological systems [30]; see also [14, 27] and the references therein. Further they are also a good source of examples since it is somehow easy to produce piecewise linear systems exhibiting an explicitly known homoclinic

trajectory, so giving rise to chaos if subject to perturbations, whereas this is not an easy task in general for the smooth case (especially if the system is not Hamiltonian).

But why shall one insist on studying the case where the critical point lies on the discontinuity surface  $\Omega^0$ ? In many real applications this is what really happens, e.g. in systems with dry friction.

The paper is divided as follows: in §2 we collect the main assumptions used in the paper and we define the stable and unstable leaves; in §3 we construct the Poincaré map made up by trajectories of the perturbed problem performing a loop close to  $\vec{\gamma}(t)$ ; in §4 we state the main results of the paper, i.e., Theorems 4.2 and 4.3, which are proved in §5. In §6 we give some hints concerning future development of this work which are in preparation. In the Appendix we sketch the proof of two auxiliary lemmas which are used to construct the Poincaré map.

## 2. Preliminary constructions and notation

First we give a notion of solution and we collect the basic assumptions which we assume through the whole paper. By a *solution* of (PS) we mean a continuous, piecewise  $C^r$  function  $\vec{x}(t)$  that satisfies

$$\dot{\vec{x}}(t) = \vec{f}^+(\vec{x}(t)) + \varepsilon \vec{g}(t, \vec{x}(t), \varepsilon), \quad \text{whenever } \vec{x}(t) \in \Omega^+, \quad (\text{PS}+)$$

$$\dot{\vec{x}}(t) = \vec{f}^-(\vec{x}(t)) + \varepsilon \vec{g}(t, \vec{x}(t), \varepsilon), \quad \text{whenever } \vec{x}(t) \in \Omega^-. \quad (\text{PS}-)$$

Moreover, if  $\vec{x}(t_0)$  belongs to  $\Omega^0$  for some  $t_0$ , then we assume either  $\vec{x}(t) \in \Omega^-$  or  $\vec{x}(t) \in \Omega^+$  for  $t$  in some left neighborhood of  $t_0$ , say  $]t_0 - \tau, t_0[$  with  $\tau > 0$ . In the first case, the left derivative of  $\vec{x}(t)$  at  $t = t_0$  has to satisfy  $\dot{\vec{x}}(t_0^-) = \vec{f}^-(\vec{x}(t_0)) + \varepsilon \vec{g}(t_0, \vec{x}(t_0), \varepsilon)$ ; while in the second case,  $\dot{\vec{x}}(t_0^-) = \vec{f}^+(\vec{x}(t_0)) + \varepsilon \vec{g}(t_0, \vec{x}(t_0), \varepsilon)$ . A similar condition is required for the right derivative  $\dot{\vec{x}}(t_0^+)$ . We stress that, in this paper, we do not consider solutions of equation (PS) that belong to  $\Omega^0$  for  $t$  in some nontrivial interval, i.e., sliding solutions.

### Notation

Throughout the paper we will use the following notation. We denote scalars by small letters, e.g.  $a$ , vectors in  $\mathbb{R}^2$  with an arrow, e.g.  $\vec{a}$ , and  $n \times n$  matrices by bold letters, e.g.  $\mathbf{A}$ . By  $\vec{a}^*$  and  $\mathbf{A}^*$  we mean the transpose of the vector  $\vec{a}$  and of the matrix  $\mathbf{A}$ , resp., so that  $\vec{a}^* \vec{b}$  denotes the scalar product of the vectors  $\vec{a}$ ,  $\vec{b}$ . We denote by  $\|\cdot\|$  the Euclidean norm in  $\mathbb{R}^2$ , while for matrices we use the functional norm  $\|\mathbf{A}\| = \sup_{\|\vec{w}\| \leq 1} \|\mathbf{A}\vec{w}\|$ . We will use the shorthand notation  $\mathbf{f}_x = \frac{\partial \mathbf{f}}{\partial x}$  unless this may cause confusion.

In the whole paper we denote by  $\bar{x}(t, \tau; \bar{P})$  the trajectory of (PS) passing through  $\bar{P}$  at  $t = \tau$  (evaluated at  $t$ ). If  $D$  is a set and  $\delta > 0$ , we define

$$B(D, \delta) := \{\bar{Q} \in \mathbb{R}^2 \mid \exists \bar{P} \in D : \|\bar{Q} - \bar{P}\| < \delta\} = \cup\{B(\bar{P}, \delta) \mid \bar{P} \in D\}.$$

We list here some hypotheses which we assume through the whole paper.

**F0** We have  $\bar{0} \in \Omega^0$ ,  $\bar{f}^\pm(\bar{0}) = \bar{0}$ , and the eigenvalues  $\lambda_s^\pm, \lambda_u^\pm$  of  $\mathbf{f}_x^\pm(\bar{0})$  are such that  $\lambda_s^\pm < 0 < \lambda_u^\pm$ .

Denote by  $\bar{v}_s^\pm, \bar{v}_u^\pm$  the normalized eigenvectors of  $\mathbf{f}_x^\pm(\bar{0})$  corresponding to  $\lambda_s^\pm, \lambda_u^\pm$ . Let us set

$$c_u^{\pm, \pm} = [\bar{\nabla}G(\bar{0})]^* \bar{v}_u^\pm, \quad c_s^{\pm, \pm} = [\bar{\nabla}G(\bar{0})]^* \bar{v}_s^\pm. \quad (2)$$

We assume that the eigenvectors  $\bar{v}_s^\pm, \bar{v}_u^\pm$  are not orthogonal to  $\bar{\nabla}G(\bar{0})$ , i.e., the constants  $c_u^{\pm, \pm}$  and  $c_s^{\pm, \pm}$  are nonzero. To fix the ideas we require

**F1**  $c_u^{\pm, -} < 0 < c_u^{\pm, +}$ ,  $c_s^{\pm, -} < 0 < c_s^{\pm, +}$ .

Moreover we require a further condition on the mutual positions of the directions spanned by  $\bar{v}_s^\pm, \bar{v}_u^\pm$ .

More precisely, set  $\mathcal{T}_u^\pm := \{c\bar{v}_u^\pm \mid c \geq 0\}$ , and denote by  $\Pi_u^1$  and  $\Pi_u^2$  the disjoint open sets in which  $\mathbb{R}^2$  is divided by the polyline  $\mathcal{T}^u := \mathcal{T}_u^+ \cup \mathcal{T}_u^-$ . We require that  $\bar{v}_s^+$  and  $\bar{v}_s^-$  lie on “opposite sides” with respect to  $\mathcal{T}^u$ . Hence, to fix the ideas, we assume:

**F2**  $\bar{v}_s^+ \in \Pi_u^1$  and  $\bar{v}_s^- \in \Pi_u^2$ .

We emphasize that if **F2** holds, there is no sliding on  $\Omega^0$  close to  $\bar{0}$ . On the other hand, sliding might occur when both  $\bar{v}_s^\pm$  lie in  $\Pi_u^1$ , or they both lie in  $\Pi_u^2$ , see [15, §3].

**REMARK 2.1.** We point out that it is the *mutual position of the eigenvectors* that plays a role in the argument. In the paper we fix a particular situation for definiteness; however, by reversing all the directions, one may obtain equivalent results. Moreover, in the continuous case, i.e., for equation (1), then  $\mathcal{T}^u$  is a line and  $\Pi_u^1, \Pi_u^2$  are halfplanes. In fact, all smooth systems satisfy **F2**. On the other hand, if assumption **F2** is replaced with the opposite condition, that is  $\bar{v}_s^+$  and  $\bar{v}_s^-$  lie “on the same side” with respect to  $\mathcal{T}^u$ , then it was shown in [15] that generically chaos cannot occur, while new bifurcation phenomena, involving continua of sliding homoclinic trajectories, may arise.

**K** For  $\varepsilon = 0$  there is a unique solution  $\bar{\gamma}(t)$  of (PS) homoclinic to the origin such that

$$\bar{\gamma}(t) \in \begin{cases} \Omega^-, & t < 0, \\ \Omega^0, & t = 0, \\ \Omega^+, & t > 0. \end{cases}$$

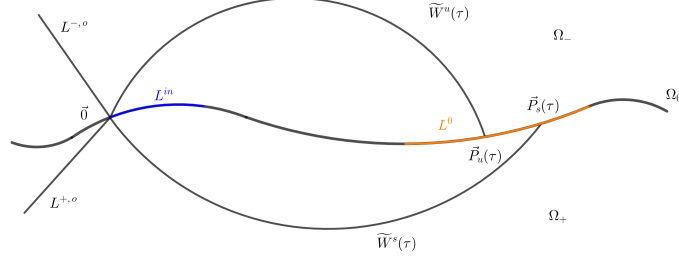


Figure 1: Stable and unstable leaves (the superscript “out” is denoted as “o” for short).

Furthermore,  $(\vec{\nabla}G(\vec{\gamma}(0)))^* \vec{f}^\pm(\vec{\gamma}(0)) > 0$ .

Recalling the orientation of  $\vec{v}_s^\pm, \vec{v}_u^\pm$  chosen in **F1**, we assume w.l.o.g. that

$$\lim_{t \rightarrow -\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = \vec{v}_u^- \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = -\vec{v}_s^+. \quad (3)$$

Concerning the perturbation term  $\vec{g}$ , we assume the following:

**G**  $\vec{g}(t, \vec{0}, \varepsilon) = \vec{0}$  for any  $t, \varepsilon \in \mathbb{R}$ .

Hence, the origin is a critical point for the perturbed problem too.

We recall that  $\Gamma := \{\vec{\gamma}(t) \mid t \in \mathbb{R}\} \cup \{\vec{0}\}$ ; let us denote by  $E^{\text{in}}$  the open set enclosed by  $\Gamma$ , and by  $E^{\text{out}}$  the open set complementary to  $E^{\text{in}} \cup \Gamma$ .

Further, for any fixed  $\delta > 0$ , we set (see Figure 1):

$$\begin{aligned} L^0 &= L^0(\delta) := \{\vec{Q} \in \Omega^0 \mid \|\vec{Q} - \vec{\gamma}(0)\| < \delta\}, \\ L^{\text{in}} &= L^{\text{in}}(\delta) := \{\vec{Q} \in (\Omega^0 \cap E^{\text{in}}) \mid \|\vec{Q}\| < \delta\}, \\ L^{-,\text{out}} &= L^{-,\text{out}}(\delta) := \{\vec{Q} = d(\vec{v}_u^- + \vec{v}_s^-) \mid 0 \leq d \leq \delta\}, \\ L^{+,\text{out}} &= L^{+,\text{out}}(\delta) := \{\vec{Q} = d(\vec{v}_u^+ + \vec{v}_s^+) \mid 0 \leq d \leq \delta\}. \end{aligned} \quad (4)$$

Now, we define the stable and the unstable leaves  $W^s(\tau)$  and  $W^u(\tau)$  of (PS).

Assume first for simplicity that the system is *smooth*; that is consider (1) and suppose that **F0** holds true. Then, following [16, Appendix], which is based on [24, Theorem 2.16], we can define local and global stable and unstable leaves

as follows:

$$\begin{aligned}
W_{loc}^u(\tau) &:= \{\vec{P} \in B(\vec{0}, \delta) \mid \vec{x}(t, \tau; \vec{P}) \in B(\vec{0}, \delta) \text{ for } t \leq \tau, \\
&\quad \lim_{t \rightarrow -\infty} \vec{x}(t, \tau; \vec{P}) = \vec{0}\}, \\
W_{loc}^s(\tau) &:= \{\vec{P} \in B(\vec{0}, \delta) \mid \vec{x}(t, \tau; \vec{P}) \in B(\vec{0}, \delta) \text{ for } t \geq \tau, \\
&\quad \lim_{t \rightarrow +\infty} \vec{x}(t, \tau; \vec{P}) = \vec{0}\}, \\
W^u(\tau) &:= \{\vec{P} \in \mathbb{R}^2 \mid \lim_{t \rightarrow -\infty} \vec{x}(t, \tau; \vec{P}) = \vec{0}\}, \\
W^s(\tau) &:= \{\vec{P} \in \mathbb{R}^2 \mid \lim_{t \rightarrow +\infty} \vec{x}(t, \tau; \vec{P}) = \vec{0}\}.
\end{aligned} \tag{5}$$

From [24, Theorem 2.16] it follows that  $W_{loc}^u(\tau)$  and  $W_{loc}^s(\tau)$  are  $C^r$  embedded 1-dimensional manifolds if  $\delta > 0$  is small enough, while  $W^u(\tau)$  and  $W^s(\tau)$  are  $C^r$  immersed 1-dimensional manifolds, i.e., they are the image of  $C^r$  curves. Note that they also depend on  $\varepsilon$  but we leave this dependence unsaid. The manifolds  $W^u(\tau)$  and  $W^s(\tau)$  are the sets of all the initial conditions of the trajectories converging to the origin in the past and in the future, respectively, and they are not invariant for the flow of (1). However, if  $\vec{P} \in W^u(\tau)$  then  $\vec{x}(t, \tau; \vec{P}) \in W^u(t)$  for any  $t, \tau \in \mathbb{R}$ . Analogously for  $W^s(\tau)$ .

We emphasize that, choosing  $\delta > 0$  small enough, we can assume that  $W_{loc}^u(\tau)$  and  $W_{loc}^s(\tau)$  are graphs on the respective tangent spaces; further  $W^u(\tau)$  and  $W^s(\tau)$  are constructed from  $W_{loc}^u(\tau)$  and  $W_{loc}^s(\tau)$  using the flow of (1) as follows

$$\begin{aligned}
W^u(\tau) &= \cup_{T \leq \tau} \{x(\tau, T; \vec{Q}) \mid \vec{Q} \in W_{loc}^u(T)\}, \\
W^s(\tau) &= \cup_{T \geq \tau} \{x(\tau, T; \vec{Q}) \mid \vec{Q} \in W_{loc}^s(T)\}.
\end{aligned} \tag{6}$$

Denote by  $W_{loc}^{u,\pm}(\tau) = W_{loc}^u(\tau) \cap (\Omega^\pm \cup \Omega^0)$  and by  $W_{loc}^{s,\pm}(\tau) = W_{loc}^s(\tau) \cap (\Omega^\pm \cup \Omega^0)$ . Assume further **F1**, **K** and follow  $W^u(\tau)$  (respectively  $W^s(\tau)$ ) from the origin towards  $L^0(\sqrt{\varepsilon})$ : then it intersects  $L^0(\sqrt{\varepsilon})$  transversely in a point denoted by  $\vec{P}_u(\tau)$  (respectively by  $\vec{P}_s(\tau)$ ). In fact,  $\vec{P}_u(\tau)$  and  $\vec{P}_s(\tau)$  are  $C^r$  functions of  $\varepsilon$  and  $\tau$ ; hence  $\vec{P}_u(\tau) = \vec{P}_s(\tau) = \vec{\gamma}(0)$  if  $\varepsilon = 0$  for any  $\tau \in \mathbb{R}$ .

We denote by  $\tilde{W}^u(\tau)$  the branch of  $W^u(\tau)$  between the origin and  $\vec{P}_u(\tau)$  (a path), and by  $\tilde{W}^s(\tau)$  the branch of  $W^s(\tau)$  between the origin and  $\vec{P}_s(\tau)$ , in both the cases including the endpoints. Since  $\tilde{W}^u(\tau)$  and  $\tilde{W}^s(\tau)$  coincide with  $\Gamma \cap (\Omega^- \cup \Omega^0)$  and  $\Gamma \cap (\Omega^+ \cup \Omega^0)$  if  $\varepsilon = 0$ , respectively, and vary in a  $C^r$  way, see [24, Theorem 2.16] or [16, Appendix], we find  $\tilde{W}^u(\tau) \subset (\Omega^- \cup \Omega^0)$  and  $\tilde{W}^s(\tau) \subset (\Omega^+ \cup \Omega^0)$ , for any  $\tau \in \mathbb{R}$  and any  $0 \leq \varepsilon \leq \varepsilon_0$ . Further

$$\begin{aligned}
\vec{Q}_u \in \tilde{W}^u(\tau) &\Rightarrow \vec{x}(t, \tau; \vec{Q}_u) \in \tilde{W}^u(t) \subset (\Omega^- \cup \Omega^0) \quad \text{for any } t \leq \tau, \\
\vec{Q}_s \in \tilde{W}^s(\tau) &\Rightarrow \vec{x}(t, \tau; \vec{Q}_s) \in \tilde{W}^s(t) \subset (\Omega^+ \cup \Omega^0) \quad \text{for any } t \geq \tau.
\end{aligned} \tag{7}$$

Now, we go back to the general case where (PS) is piecewise smooth but discontinuous. Using the previous construction, we can define also in this case  $W_{loc}^{u,\pm}(\tau)$ ,  $W_{loc}^{s,\pm}(\tau)$ ,  $\tilde{W}^u(\tau)$  and  $\tilde{W}^s(\tau)$ , and they are all  $C^r$ .

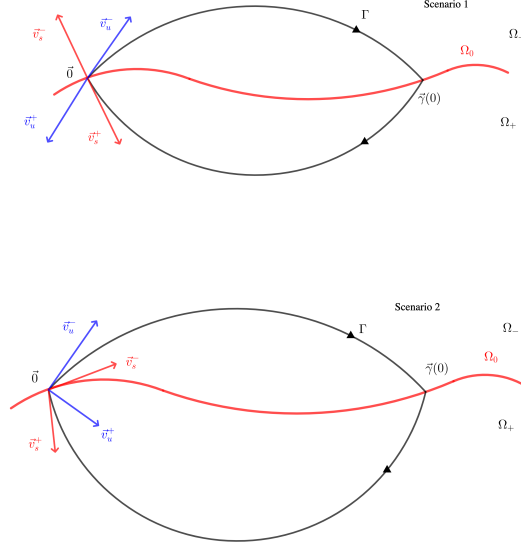


Figure 2: Scenarios 1 and 2. In these two settings there is no sliding close to the origin. Further Melnikov theory guarantees persistence of the homoclinic trajectories [8], and we conjecture that we may have chaotic phenomena.

REMARK 2.2. Consider (PS) and assume **F0**, **F1**, **F2**. Then the manifolds  $W_{loc}^u(\tau)$  and  $W_{loc}^s(\tau)$  defined as in (5) are such that  $W_{loc}^u(\tau) = W_{loc}^{u,+}(\tau) \cup W_{loc}^{u,-}(\tau)$ ,  $W_{loc}^s(\tau) = W_{loc}^{s,+}(\tau) \cup W_{loc}^{s,-}(\tau)$ . Hence,  $W_{loc}^u(\tau)$  and  $W_{loc}^s(\tau)$  are piecewise  $C^r$  manifolds (they might lose smoothness in the origin). Further  $\tilde{W}^u(\tau)$  and  $\tilde{W}^s(\tau)$  are again smooth and have the property (7).

Moreover, if **K** holds then  $\vec{P}_u(\tau)$  and  $\vec{P}_s(\tau)$  are again  $C^r$  in  $\varepsilon$  and  $\tau$ , and  $\vec{P}_u(\tau) = \vec{P}_s(\tau) = \bar{\gamma}(0)$  if  $\varepsilon = 0$  for any  $\tau \in \mathbb{R}$ .

We set

$$\tilde{W}(\tau) := \tilde{W}^u(\tau) \cup \tilde{W}^s(\tau). \quad (8)$$

We will see that  $\tilde{W}(\tau) \subset B(\Gamma, \bar{c}^* \varepsilon)$  for a suitable  $\bar{c}^* > 0$ , see Remark 3.2 below.

Let us define the Melnikov function  $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$  which, for planar piecewise smooth systems as (PS), takes the following form, see e.g. [8]:



$$\begin{aligned} \mathcal{M}(\alpha) = & \int_{-\infty}^0 e^{-\int_0^t \text{tr} \bar{f}_x^-(\bar{\gamma}(s)) ds} \bar{f}^-(\bar{\gamma}(t)) \wedge \bar{g}(t + \alpha, \bar{\gamma}(t), 0) dt \\ & + \int_0^{+\infty} e^{-\int_0^t \text{tr} \bar{f}_x^+(\bar{\gamma}(s)) ds} \bar{f}^+(\bar{\gamma}(t)) \wedge \bar{g}(t + \alpha, \bar{\gamma}(t), 0) dt, \end{aligned} \quad (9)$$

where “ $\wedge$ ” is the wedge product in  $\mathbb{R}^2$  defined by  $\vec{a} \wedge \vec{b} = a_1 b_2 - a_2 b_1$  for any vectors  $\vec{a} = (a_1, a_2)$ ,  $\vec{b} = (b_1, b_2)$ . In fact, also in the piecewise smooth case the function  $\mathcal{M}$  is  $C^r$ .

At this point, we need to distinguish between four possible scenarios, see Figures 2 and 3.

**Scenario 1** Assume **K** and that there is  $\rho > 0$  such that  $d\bar{v}_u^+ \in E^{\text{out}}$ ,  $d\bar{v}_s^- \in E^{\text{out}}$  for any  $0 < d < \rho$ .

**Scenario 2** Assume **K** and that there is  $\rho > 0$  such that  $d\bar{v}_u^+ \in E^{\text{in}}$ ,  $d\bar{v}_s^- \in E^{\text{in}}$  for any  $0 < d < \rho$ .

**Scenario 3** Assume **K** and that there is  $\rho > 0$  such that  $d\bar{v}_u^+ \in E^{\text{in}}$ ,  $d\bar{v}_s^- \in E^{\text{out}}$  for any  $0 < d < \rho$ , so **F2** does not hold.

**Scenario 4** Assume **K** and that there is  $\rho > 0$  such that  $d\bar{v}_u^+ \in E^{\text{out}}$ ,  $d\bar{v}_s^- \in E^{\text{in}}$  for any  $0 < d < \rho$ , so **F2** does not hold.

Notice that **F2** holds in both Scenarios 1 and 2, and our results apply to both the cases. Further, in Scenario 1  $W_{loc}^{u,+}(\tau)$  and  $W_{loc}^{s,-}(\tau)$  both lie in  $E^{\text{out}}$  for any  $\tau \in \mathbb{R}$ , while in Scenario 2  $W_{loc}^{u,+}(\tau)$  and  $W_{loc}^{s,-}(\tau)$  both lie in  $E^{\text{in}}$  for any  $\tau \in \mathbb{R}$ .

In Scenarios 3 and 4 sliding generically occurs in  $\Omega^0$  close to the origin and **F2** does not hold. Notice that Scenarios 1 and 2 have a smooth counterpart while Scenarios 3 and 4 may take place just if the system is discontinuous. We recall once again that in all the four scenarios the existence of a non-degenerate zero of the Melnikov function guarantees the persistence of the homoclinic trajectory, cf. [8], but chaos is generically not possible in Scenarios 3 and 4. We conjecture that chaos is still possible in Scenarios 1 and 2: this will be the object of a future investigation in which the result of this article will be crucial.

**In this paper we will just consider Scenario 1 to fix the ideas, even though Scenario 2 can be handled in a similar way.**

We collect here for convenience of the reader and future reference, the main constants which will play a role in our argument:

$$\begin{aligned} \sigma_+^{\text{fwd}} &= \frac{|\lambda_s^+|}{\lambda_u^+ + |\lambda_s^+|}, & \sigma_-^{\text{fwd}} &= \frac{\lambda_u^- + |\lambda_s^-|}{\lambda_u^-}, & \sigma^{\text{fwd}} &= \sigma_+^{\text{fwd}} \sigma_-^{\text{fwd}}, \\ \sigma_+^{\text{bwd}} &= \frac{1}{\sigma_+^{\text{fwd}}}, & \sigma_-^{\text{bwd}} &= \frac{1}{\sigma_-^{\text{fwd}}}, & \sigma^{\text{bwd}} &= \sigma_+^{\text{bwd}} \sigma_-^{\text{bwd}}, \\ \underline{\sigma} &= \min\{\sigma_+^{\text{fwd}}, \sigma_-^{\text{bwd}}\}, & \bar{\sigma} &= \max\{\sigma_+^{\text{fwd}}, \sigma_-^{\text{bwd}}\}, \end{aligned} \quad (10)$$

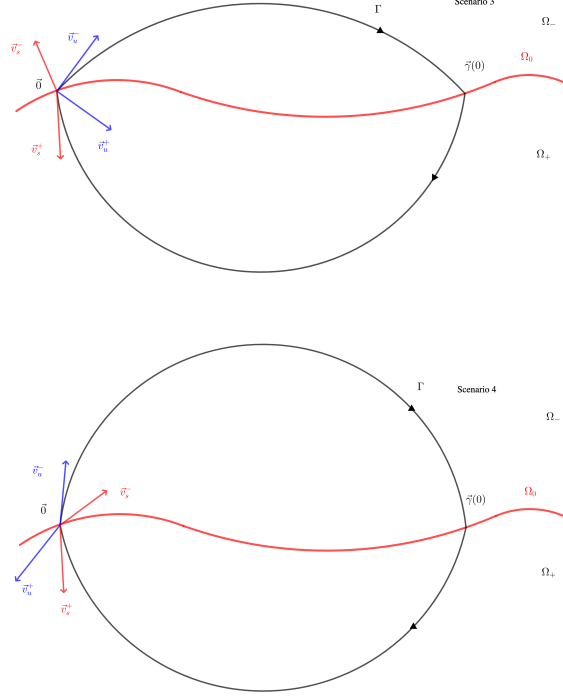


Figure 3: Scenarios 3 and 4: in these settings we have persistence of the homoclinic trajectories but sliding might occur close to the origin. Here our analysis does not apply directly. Further Melnikov theory guarantees persistence of the homoclinic trajectories [8], but chaos is forbidden [15] in general.

$$\begin{aligned}
 \Sigma_+^{\text{fwd}} &= \frac{1}{\lambda_u^+ + |\lambda_s^+|}, & \Sigma_-^{\text{bwd}} &= \frac{1}{\lambda_u^- + |\lambda_s^-|}, \\
 \Sigma^{\text{fwd}} &= \frac{\lambda_u^- + |\lambda_s^+|}{\lambda_u^- (\lambda_u^+ + |\lambda_s^+|)}, & \Sigma^{\text{bwd}} &= \frac{\lambda_u^+ + |\lambda_s^-|}{|\lambda_s^-| (\lambda_u^- + |\lambda_s^-|)}, \\
 \underline{\Sigma} &= \min\{\Sigma^{\text{fwd}}, \Sigma^{\text{bwd}}\}, & \bar{\Sigma} &= \max\{\Sigma^{\text{fwd}}, \Sigma^{\text{bwd}}\}, \\
 \underline{\lambda} &= \min\{\lambda_u^-; \lambda_u^+; |\lambda_s^-|; |\lambda_s^+|\}, & \bar{\lambda} &= \max\{\lambda_u^-; \lambda_u^+; |\lambda_s^-|; |\lambda_s^+|\}.
 \end{aligned}$$

REMARK 2.3. Notice that in the smooth setting we have  $\lambda_u^+ = \lambda_u^-$ , and  $\lambda_s^+ = \lambda_s^-$  so we have the following simplifications

$$\sigma^{\text{fwd}} = \frac{|\lambda_s|}{\lambda_u}, \quad \sigma^{\text{bwd}} = \frac{\lambda_u}{|\lambda_s|}; \quad \Sigma^{\text{fwd}} = \frac{1}{\lambda_u}, \quad \Sigma^{\text{bwd}} = \frac{1}{|\lambda_s|}. \quad (11)$$

### 3. Construction of the Poincaré map

Fix  $\tau \in \mathbb{R}$  and consider  $L^0(\delta)$  given by (4), where  $\delta > 0$  will be chosen below.

The point  $\vec{P}_s(\tau)$  splits  $L^0$  in two parts, say  $A^{\text{fwd},+}(\tau)$  and  $B^{\text{fwd},+}(\tau)$ , respectively “inside” and “outside”, see Figure 4. The purpose of this section is to construct a Poincaré map using the flow of (PS) from  $L^0$  back to itself remaining close to  $\Gamma$ ; i.e.  $\mathcal{P}^{\text{fwd}}(\cdot, \tau) : A^{\text{fwd},+}(\tau) \rightarrow L^0$  and a time map  $\mathcal{T}^{\text{fwd}}(\cdot, \tau) : A^{\text{fwd},+}(\tau) \rightarrow \mathbb{R}$  such that for any  $\vec{P} \in A^{\text{fwd},+}(\tau)$  the trajectory  $\vec{x}(t, \tau; \vec{P})$  will stay in a neighborhood of  $\Gamma$  (in fact in a neighborhood of  $\tilde{W}(t)$ ) for any  $t \in [\tau, \mathcal{T}^{\text{fwd}}(\vec{P}, \tau)]$  and it will cross transversely  $L^0$  for the first time at  $t = \mathcal{T}^{\text{fwd}}(\vec{P}, \tau) > \tau$  in the point  $\mathcal{P}^{\text{fwd}}(\vec{P}, \tau)$ . Further, we want to show that both the maps are  $C^r$ .

Moreover, if  $\vec{P} \in B^{\text{fwd},+}(\tau)$  then there is some  $\mathcal{T}^{\text{out}} = \mathcal{T}^{\text{out}}(\vec{P}, \tau) > \tau$  such that the trajectory will leave a neighborhood of  $\Gamma$  at  $t \geq \mathcal{T}^{\text{out}}$ .

Similarly, the point  $\vec{P}_u(\tau)$  splits the curve  $L^0$  in two parts, say  $A^{\text{bwd},-}(\tau)$  and  $B^{\text{bwd},-}(\tau)$ , respectively “inside” and “outside”, see Figure 5. Using the flow of (PS) (but now going backward in time) we can construct a  $C^r$  Poincaré map  $\mathcal{P}^{\text{bwd}}(\cdot, \tau) : A^{\text{bwd},-}(\tau) \rightarrow L^0$  and a  $C^r$  time map  $\mathcal{T}^{\text{bwd}}(\cdot, \tau) : A^{\text{bwd},-}(\tau) \rightarrow \mathbb{R}$  such that for any  $\vec{P} \in A^{\text{bwd},-}(\tau)$  the trajectory  $\vec{x}(t, \tau; \vec{P})$  will stay in a neighborhood of  $\Gamma$  (in fact in a neighborhood of  $\tilde{W}(t)$ ) for any  $t \in [\mathcal{T}^{\text{bwd}}(\vec{P}, \tau), \tau]$  and it will cross transversely  $L^0$  for the first time at  $t = \mathcal{T}^{\text{bwd}}(\vec{P}, \tau) < \tau$  in the point  $\mathcal{P}^{\text{bwd}}(\vec{P}, \tau)$ .

Moreover if  $\vec{P} \in B^{\text{bwd},-}(\tau)$  then there is some  $\mathcal{T}^{\text{out}}(\vec{P}, \tau) < \tau$  such that the trajectory will leave a neighborhood of  $\Gamma$  at  $t \leq \mathcal{T}^{\text{out}}$ .

Hereafter it is convenient to denote  $\vec{\gamma}^-(t) := \vec{\gamma}(t)$  when  $t \leq 0$  and  $\vec{\gamma}^+(t) := \vec{\gamma}(t)$  when  $t \geq 0$ .

**REMARK 3.1.** Assume **F0**, **K**, then there is a constant  $c_0^* > 0$  such that  $\|\vec{\gamma}^-(t)\| \leq \frac{c_0^*}{4} e^{\lambda_u^- t}$  for any  $t \leq 0$  and  $\|\vec{\gamma}^+(t)\| \leq \frac{c_0^*}{4} e^{\lambda_s^+ t}$  for any  $t \geq 0$ .

We state now a classical result concerning the possibility to estimate the position of the trajectories of the unstable manifold  $\tilde{W}^u(\tau)$  and of the stable manifold  $\tilde{W}^s(\tau)$  using the homoclinic trajectory  $\vec{\gamma}(t)$ . The proof is omitted, see, e.g., the nice introduction of [25], or [21, §4.5].

**REMARK 3.2.** Assume **K** and **F1**, then there is  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon \leq \varepsilon_0$  we have the following. There is  $\bar{c}^* > 0$  such that

$$\begin{aligned} \|\vec{x}(t, \tau; \vec{P}_u(\tau)) - \vec{\gamma}^-(t - \tau)\| &\leq \bar{c}^* \varepsilon && \text{for any } t \leq \tau, \\ \|\vec{x}(t, \tau; \vec{P}_s(\tau)) - \vec{\gamma}^+(t - \tau)\| &\leq \bar{c}^* \varepsilon && \text{for any } t \geq \tau. \end{aligned} \tag{12}$$

By a slight modification of [15, Lemmas 6.4, 6.7] and the argument of [15, §6.2.2] we can construct two auxiliary curves  $Z^{\text{fwd},\text{in}}$ ,  $Z^{\text{fwd},\text{out}}$  with the following

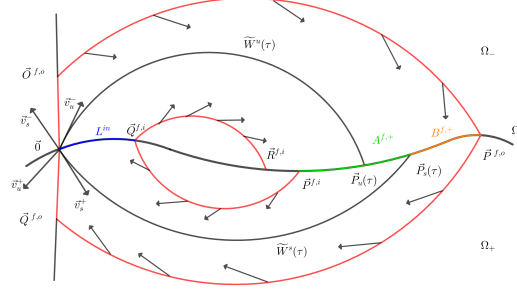


Figure 4: Construction of the set  $K^{\text{fwd}}$  (the superscripts “fwd”, “in”, “out” are denoted as “f”, “i”, “o”, respectively, for short). This picture enables us to control the trajectories of (PS) in a neighborhood of  $\Gamma$  in forward time.

properties, see Figure 4, and also Figures 8, 9. These curves are useful to build up our Poincaré map in forward time.

The curve  $Z^{\text{fwd},\text{in}}$  is a continuous, piecewise- $C^r$  path starting from a point  $\vec{P}^{\text{fwd},\text{in}} \in (L^0 \cap E^{\text{in}})$ , passing through a point  $\vec{Q}^{\text{fwd},\text{in}} \in L^{\text{in}}$  and ending in a point  $\vec{R}^{\text{fwd},\text{in}} \in (L^0 \cap E^{\text{in}})$ , see (4) and the beginning of Section 2.

The branch from  $\vec{P}^{\text{fwd},\text{in}}$  to  $\vec{Q}^{\text{fwd},\text{in}}$  is in  $\Omega^+$ , while the branch from  $\vec{Q}^{\text{fwd},\text{in}}$  to  $\vec{R}^{\text{fwd},\text{in}}$  is in  $\Omega^-$  and they are both  $C^r$ .

Analogously, the curve  $Z^{\text{fwd},\text{out}}$  is a continuous, piecewise- $C^r$  path starting from a point  $\vec{O}^{\text{fwd},\text{out}} \in L^{-,\text{out}}$ , passing through a point  $\vec{P}^{\text{fwd},\text{out}} \in (L^0 \cap E^{\text{out}})$  and ending in a point  $\vec{Q}^{\text{fwd},\text{out}} \in L^{+,\text{out}}$ . The branch from  $\vec{O}^{\text{fwd},\text{out}}$  to  $\vec{P}^{\text{fwd},\text{out}}$  is in  $\Omega^-$ , while the branch from  $\vec{P}^{\text{fwd},\text{out}}$  to  $\vec{Q}^{\text{fwd},\text{out}}$  is in  $\Omega^+$  and they are both  $C^r$ . Further  $Z^{\text{fwd},\text{in}} \subset E^{\text{in}}$  while  $Z^{\text{fwd},\text{out}} \subset E^{\text{out}}$ .

We denote by  $K^{\text{fwd}}$  the compact set enclosed by  $Z^{\text{fwd},\text{out}}$ ,  $L^{+,\text{out}}$ ,  $L^{-,\text{out}}$ ,  $Z^{\text{fwd},\text{in}}$  and the path of  $\Omega^0$  between  $\vec{P}^{\text{fwd},\text{in}}$  and  $\vec{R}^{\text{fwd},\text{in}}$ . We denote by  $K^{\text{fwd},\pm} = K^{\text{fwd}} \cap \Omega^\pm$ . Notice that by construction  $Z^{\text{fwd},\text{in}}$  and  $Z^{\text{fwd},\text{out}}$  do not intersect each other or themselves and that  $\Gamma \subset K^{\text{fwd}}$ .

In order to construct the auxiliary curves  $Z^{\text{fwd},\text{in}}$  and  $Z^{\text{fwd},\text{out}}$ , we need to introduce a small parameter  $\beta$ , which however has to be larger than  $\varepsilon$ , namely:

$$\beta \geq \varepsilon^{\frac{\sigma^{\text{fb}}}{2}}, \quad \sigma^{\text{fb}} = \min\{\sigma^{\text{fwd}}, \sigma^{\text{bwd}}\} < 1. \quad (13)$$

We introduce a further parameter  $\mu$ , whose size has to be small, independently of  $\varepsilon$ . Roughly speaking,  $\mu$  will play the role of controlling the errors in evaluating the positions of the “barriers”  $Z^{\text{fwd},\text{in}}$  and similar, just below, and the space displacement  $\mathcal{P}$  with respect to  $\tilde{W}$ , defined in the main results, Theorems 4.2

and 4.3 below.

Let us define

$$\mu_0 = \frac{1}{4} \min \{ \Sigma_+^{\text{fwd}}, \Sigma_-^{\text{bwd}}, \underline{\sigma}^2 \} \quad (14)$$

Then we have the following.

**LEMMA 3.3.** *Assume **F0, F1, F2, K, G**. We can choose  $\varepsilon_0$  and  $\beta_0$  such that for any  $0 < \varepsilon \leq \varepsilon_0$ , any  $0 < \beta \leq \beta_0$  with  $\beta \geq \varepsilon^{\frac{\sigma^{\text{fb}}}{2}}$  we can define  $Z^{\text{fwd}, \text{in}}$ ,  $Z^{\text{fwd}, \text{out}}$  so that the flow of (PS) on  $Z^{\text{fwd}, \text{in}} \setminus \{ \vec{P}^{\text{fwd}, \text{in}}, \vec{R}^{\text{fwd}, \text{in}} \}$  and on  $Z^{\text{fwd}, \text{out}} \setminus \{ \vec{O}^{\text{fwd}, \text{out}}, \vec{Q}^{\text{fwd}, \text{out}} \}$  aims towards the interior of  $K^{\text{fwd}}$ . Then, for any  $0 < \mu \leq \mu_0$  we find*

$$\begin{aligned} \|\vec{P}^{\text{fwd}, \text{in}} - \vec{\gamma}(0)\| &= \beta, & \|\vec{P}^{\text{fwd}, \text{out}} - \vec{\gamma}(0)\| &= \beta, \\ \beta^{\sigma_+^{\text{fwd}} + \mu} &\leq \|\vec{Q}^{\text{fwd}, \text{in}}\| \leq \beta^{\sigma_+^{\text{fwd}} - \mu}, & \beta^{\sigma_+^{\text{fwd}} + \mu} &\leq \|\vec{Q}^{\text{fwd}, \text{out}}\| \leq \beta^{\sigma_+^{\text{fwd}} - \mu}, \\ \beta^{\sigma^{\text{fwd}} + \mu} &\leq \|\vec{R}^{\text{fwd}, \text{in}} - \vec{\gamma}(0)\| \leq \beta^{\sigma^{\text{fwd}} - \mu}, & & \\ \beta^{\sigma_-^{\text{bwd}} + \mu} &\leq \|\vec{O}^{\text{fwd}, \text{out}}\| \leq \beta^{\sigma_-^{\text{bwd}} - \mu}. & & \end{aligned} \quad (15)$$

Finally we can assume that  $K^{\text{fwd}} \subset B(\Gamma, \beta^{\underline{\sigma} - \mu})$ , and that there is  $c > 0$  such that  $B(\Gamma, c\varepsilon) \cap (Z^{\text{fwd}, \text{in}} \cup Z^{\text{fwd}, \text{out}}) = \emptyset$ .

A sketch of the proof of both Lemma 3.3 and Lemma 3.5 below is postponed to Appendix A.

**REMARK 3.4.** We invite the reader not to focus on the precise values of the exponents appearing in (15). Notice that if  $\lambda_u^+ = \lambda_u^- = |\lambda_s^+| = |\lambda_s^-|$  as, e.g., in the smooth Hamiltonian case inequalities (15) simplify as follows

$$\begin{aligned} \|\vec{P}^{\text{fwd}, \text{in}} - \vec{\gamma}(0)\| &= \beta, & \|\vec{P}^{\text{fwd}, \text{out}} - \vec{\gamma}(0)\| &= \beta, \\ \beta^{\frac{1}{2} + \mu} &\leq \|\vec{Q}^{\text{fwd}, \text{in}}\| \leq \beta^{\frac{1}{2} - \mu}, & \beta^{\frac{1}{2} + \mu} &\leq \|\vec{Q}^{\text{fwd}, \text{out}}\| \leq \beta^{\frac{1}{2} - \mu}, \\ \beta^{1 + \mu} &\leq \|\vec{R}^{\text{fwd}, \text{in}} - \vec{\gamma}(0)\| \leq \beta^{1 - \mu}, & & \\ \beta^{\frac{1}{2} + \mu} &\leq \|\vec{O}^{\text{fwd}, \text{out}}\| \leq \beta^{\frac{1}{2} - \mu}. & & \end{aligned} \quad (16)$$

Further, taking into account that  $0 \leq \mu \leq 1/16$ , we get  $K^{\text{fwd}} \subset B(\Gamma, \beta^{1/4})$ .

Similarly, we construct some other auxiliary curves useful to build up the Poincaré map in backward time. By a modification of [15, Lemmas 6.4, 6.7] and from [15, §6.2.2] we construct the continuous, piecewise- $C^r$  curves  $Z^{\text{bwd}, \text{in}}$ ,  $Z^{\text{bwd}, \text{out}}$  with the following properties, see Figure 5, see also Figures 10 and 11.

The curve  $Z^{\text{bwd}, \text{in}}$  starts from a point  $\vec{P}^{\text{bwd}, \text{in}} \in (L^0 \cap E^{\text{in}})$ , passes through a point  $\vec{Q}^{\text{bwd}, \text{in}} \in L^{\text{in}}$  and ends in a point  $\vec{R}^{\text{bwd}, \text{in}} \in (L^0 \cap E^{\text{in}})$ , so that the branch from  $\vec{P}^{\text{bwd}, \text{in}}$  to  $\vec{Q}^{\text{bwd}, \text{in}}$  is in  $\Omega^-$ , while the branch from  $\vec{Q}^{\text{bwd}, \text{in}}$  to  $\vec{R}^{\text{bwd}, \text{in}}$  is in  $\Omega^+$ , and they are both  $C^r$ .

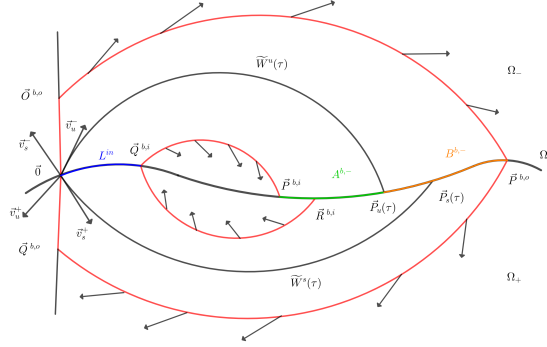


Figure 5: Construction of the set  $K^{\text{bwd}}$  (the superscripts “bwd”, “in”, “out” are denoted as “b”, “i”, “o”, respectively, for short). This picture enables us to control the trajectories of (PS) in a neighborhood of  $\Gamma$  in backward time.

Analogously, the curve  $Z^{\text{bwd},\text{out}}$  starts from a point  $\vec{O}^{\text{bwd},\text{out}} \in L^{-,\text{out}}$ , passes through a point  $\vec{P}^{\text{bwd},\text{out}} \in (L^0 \cap E^{\text{out}})$  and ends in a point  $\vec{Q}^{\text{bwd},\text{out}} \in L^{+,\text{out}}$ , so that the branch from  $\vec{O}^{\text{bwd},\text{out}}$  to  $\vec{P}^{\text{bwd},\text{out}}$  is in  $\Omega^-$ , while the branch from  $\vec{P}^{\text{bwd},\text{out}}$  to  $\vec{Q}^{\text{bwd},\text{out}}$  is in  $\Omega^+$ , and they are both  $C^r$ . Further  $Z^{\text{bwd},\text{in}} \subset E^{\text{in}}$  while  $Z^{\text{bwd},\text{out}} \subset E^{\text{out}}$ .

We denote by  $K^{\text{bwd}}$  the compact set enclosed by  $Z^{\text{bwd},\text{out}}$ ,  $L^{+,\text{out}}$ ,  $L^{-,\text{out}}$ ,  $Z^{\text{bwd},\text{in}}$  and the segment of  $\Omega^0$  between  $\vec{P}^{\text{bwd},\text{in}}$  and  $\vec{R}^{\text{bwd},\text{in}}$ . We denote by  $K^{\text{bwd},\pm} = K^{\text{bwd}} \cap \Omega^\pm$ . Again, by construction,  $Z^{\text{bwd},\text{in}}$  and  $Z^{\text{bwd},\text{out}}$  do not intersect each other or themselves and  $\Gamma \subset K^{\text{bwd}}$ .

**LEMMA 3.5.** *Assume **F0**, **F1**, **F2**, **K**, **G**. We can choose  $\varepsilon_0$  and  $\beta_0$  such that for any  $0 < \varepsilon \leq \varepsilon_0$  and any  $0 < \beta \leq \beta_0$  with  $\beta \geq \varepsilon^{\frac{\sigma_0}{2}}$  we can define  $Z^{\text{bwd},\text{in}}$ ,  $Z^{\text{bwd},\text{out}}$  so that the flow of (PS) on  $Z^{\text{bwd},\text{in}} \setminus \{\vec{P}^{\text{bwd},\text{in}}, \vec{R}^{\text{bwd},\text{in}}\}$  and on  $Z^{\text{bwd},\text{out}} \setminus \{\vec{O}^{\text{bwd},\text{out}}, \vec{Q}^{\text{bwd},\text{out}}\}$  aims towards the exterior of  $K^{\text{bwd}}$ . Then, for any  $0 < \mu \leq \mu_0$  we find*

$$\begin{aligned}
\|\vec{P}^{\text{bwd},\text{in}} - \vec{\gamma}(0)\| &= \beta, & \|\vec{P}^{\text{bwd},\text{out}} - \vec{\gamma}(0)\| &= \beta, \\
\beta^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{Q}^{\text{bwd},\text{in}}\| \leq \beta^{\sigma^{\text{bwd}}-\mu}, & \beta^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{Q}^{\text{bwd},\text{out}}\| \leq \beta^{\sigma^{\text{bwd}}-\mu}, \\
\beta^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{R}^{\text{bwd},\text{in}} - \vec{\gamma}(0)\| \leq \beta^{\sigma^{\text{bwd}}-\mu}, & & \\
\beta^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{O}^{\text{bwd},\text{out}}\| \leq \beta^{\sigma^{\text{bwd}}-\mu}. & & 
\end{aligned} \tag{17}$$

Finally, we can assume that  $K^{\text{bwd}} \subset B(\Gamma, \beta^{\sigma-\mu})$ , and that there is  $c > 0$  such that  $B(\Gamma, c\varepsilon) \cap (Z^{\text{bwd},\text{in}} \cup Z^{\text{bwd},\text{out}}) = \emptyset$ .

Again, the precise values of the exponents appearing in (17) are not so relevant in this paper, and if  $\lambda_u^+ = \lambda_u^- = |\lambda_s^+| = |\lambda_s^-|$ , (17) simplifies analogously

to (16).

We continue with further constructions illustrated again by Figure 4.

From Lemmas 3.3 and 3.5, we know that  $\tilde{W}(\tau) \subset K^{\text{fwd}} \cap K^{\text{bwd}}$ . Further  $\tilde{W}(\tau)$  splits the compact set  $K^{\text{fwd}}(\tau)$  ( $K^{\text{bwd}}(\tau)$ ) in two components, denoted by  $K_A^{\text{fwd}}(\tau)$  and  $K_B^{\text{fwd}}(\tau)$  ( $K_A^{\text{bwd}}(\tau)$  and  $K_B^{\text{bwd}}(\tau)$ ), respectively in the interior and in the exterior of the bounded set enclosed by  $\tilde{W}(\tau)$  and the branch of  $\Omega^0$  between  $\vec{P}_u(\tau)$  and  $\vec{P}_s(\tau)$ .

We denote by  $A^{\text{fwd},+}(\tau)$  the open branch (i.e. without endpoints) of  $\Omega^0$  between  $\vec{P}^{\text{fwd},\text{in}}$  and  $\vec{P}_s(\tau)$ , and by  $A^{\text{fwd},-}(\tau)$  the open branch of  $\Omega^0$  between  $\vec{R}^{\text{fwd},\text{in}}$  and  $\vec{P}_u(\tau)$ . Let  $\vec{Q} \in A^{\text{fwd},+}(\tau)$  and follow  $\vec{x}(t, \tau; \vec{Q})$  forward in  $t$  for  $t \geq \tau$ ; from Lemma 3.3 we get that  $\vec{x}(t, \tau; \vec{Q})$  will stay close to  $\tilde{W}(t)$  until it completes a loop and reaches  $A^{\text{fwd},-}(\mathcal{I}^{\text{fwd}}(\vec{Q}, \tau))$  at  $t = \mathcal{I}^{\text{fwd}}(\vec{Q}, \tau) > \tau$ .

Similarly, we denote by  $A^{\text{bwd},-}(\tau)$  the open branch of  $\Omega^0$  between  $\vec{P}^{\text{bwd},\text{in}}$  and  $\vec{P}_u(\tau)$  and by  $A^{\text{bwd},+}(\tau)$  the open branch of  $\Omega^0$  between  $\vec{R}^{\text{bwd},\text{in}}$  and  $\vec{P}_s(\tau)$ . Let  $\vec{Q} \in A^{\text{bwd},-}(\tau)$  and follow  $\vec{x}(t, \tau; \vec{Q})$  backward in  $t$  for  $t \leq \tau$ ; from Lemma 3.5 we know that  $\vec{x}(t, \tau; \vec{Q})$  will stay close to  $\tilde{W}(t)$  until it completes a loop and reaches  $A^{\text{bwd},+}(\mathcal{I}^{\text{bwd}}(\vec{Q}, \tau))$  at a suitable  $t = \mathcal{I}^{\text{bwd}}(\vec{Q}, \tau) < \tau$ .

Let  $K_A^{\text{fwd},\pm}(\tau) = K_A^{\text{fwd}}(\tau) \cap \Omega^\pm$ ,  $K_A^{\text{bwd},\pm}(\tau) = K_A^{\text{bwd}}(\tau) \cap \Omega^\pm$ , then we have the following.

**LEMMA 3.6.** *Assume **F0, F1, F2, K, G**. Let  $\vec{Q} \in A^{\text{fwd},+}(\tau)$ . Then there are  $\mathcal{I}^{\text{fwd}}(\vec{Q}, \tau) > \tau_1(\vec{Q}, \tau) > \tau$  such that the trajectory  $\vec{x}(t, \tau; \vec{Q}) \in K_A^{\text{fwd},+}(t)$  for any  $\tau < t < \tau_1(\vec{Q}, \tau)$ ,  $\vec{x}(t, \tau; \vec{Q}) \in K_A^{\text{fwd},-}(t)$  for any  $\tau_1(\vec{Q}, \tau) < t < \mathcal{I}^{\text{fwd}}(\vec{Q}, \tau)$ , and it crosses transversely  $\Omega^0$  at  $t \in \{\tau, \tau_1(\vec{Q}, \tau), \mathcal{I}^{\text{fwd}}(\vec{Q}, \tau)\}$ . Hence,*

$$\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau) := \vec{x}(\tau_1(\vec{Q}, \tau), \tau; \vec{Q}) \in L^{\text{in}},$$

$$\mathcal{P}^{\text{fwd}}(\vec{Q}, \tau) := \vec{x}(\mathcal{I}^{\text{fwd}}(\vec{Q}, \tau), \tau; \vec{Q}) \in A^{\text{fwd},-}(\mathcal{I}^{\text{fwd}}(\vec{Q}, \tau)).$$

Analogously, let  $\vec{Q} \in A^{\text{bwd},-}(\tau)$ . Then there are  $\mathcal{I}^{\text{bwd}}(\vec{Q}, \tau) < \tau_{-1}(\vec{Q}, \tau) < \tau$  such that  $\vec{x}(t, \tau; \vec{Q}) \in K_A^{\text{bwd},-}(t)$  for any  $\tau_{-1}(\vec{Q}, \tau) < t < \tau$ ,  $\vec{x}(t, \tau; \vec{Q}) \in K_A^{\text{bwd},+}(t)$  for any  $\mathcal{I}^{\text{bwd}}(\vec{Q}, \tau) < t < \tau_{-1}(\vec{Q}, \tau)$ , and it crosses transversely  $\Omega^0$  at  $t \in \{\tau, \tau_{-1}(\vec{Q}, \tau), \mathcal{I}^{\text{bwd}}(\vec{Q}, \tau)\}$ . Hence

$$\mathcal{P}_-^{\text{bwd}}(\vec{Q}, \tau) := \vec{x}(\tau_{-1}(\vec{Q}, \tau), \tau; \vec{Q}) \in L^{\text{in}},$$

$$\mathcal{P}^{\text{bwd}}(\vec{Q}, \tau) := \vec{x}(\mathcal{I}^{\text{bwd}}(\vec{Q}, \tau), \tau; \vec{Q}) \in A^{\text{bwd},+}(\mathcal{I}^{\text{bwd}}(\vec{Q}, \tau)).$$

From the smoothness of the flow of (PS) it follows that all the functions defined in Lemma 3.6 are  $C^r$ : we add a sketch of the proof for completeness.

REMARK 3.7. The functions  $\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau)$ ,  $\tau_1(\vec{Q}, \tau)$ ,  $\mathcal{P}^{\text{fwd}}(\vec{Q}, \tau)$ ,  $\mathcal{T}^{\text{fwd}}(\vec{Q}, \tau)$ , and  $\mathcal{P}_-^{\text{bwd}}(\vec{R}, \tau)$ ,  $\tau_{-1}(\vec{R}, \tau)$  and  $\mathcal{P}^{\text{bwd}}(\vec{R}, \tau)$ ,  $\mathcal{T}^{\text{bwd}}(\vec{R}, \tau)$  constructed via Lemma 3.6 are  $C^r$  in both the variables, respectively for  $\vec{Q} \in A^{\text{fwd},+}(\tau)$ ,  $\vec{R} \in A^{\text{bwd},-}(\tau)$  and  $\tau \in \mathbb{R}$ .

*Proof of Remark 3.7.* The smoothness of  $\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau)$ ,  $\tau_1(\vec{Q}, \tau)$  follows from the smoothness of the flow of (PS) on  $\Omega^+$ . Let  $\vec{Q}$  be a point in the branch of  $L^{\text{in}}$  between the origin and  $\vec{Q}^{\text{fwd},\text{in}}$ , and  $\tau \in \mathbb{R}$ ; let us denote by  $\tau_2(\vec{Q}, \tau)$  the time such that  $\vec{x}(t, \tau; \vec{Q}) \in \Omega^-$  for any  $\tau < t < \tau_2(\vec{Q}, \tau)$  and it crosses transversely  $\Omega^0$  at  $t = \tau_2(\vec{Q}, \tau)$ . Then  $\tau_2(\vec{Q}, \tau)$  and  $\mathcal{P}_-^{\text{fwd}}(\vec{Q}, \tau) := \vec{x}(\tau_2(\vec{Q}, \tau), \tau; \vec{Q})$  are  $C^r$  in both the variables due to the smoothness of the flow of (PS) on  $\Omega^-$ .

Hence, the maps

$$\begin{aligned}\mathcal{T}^{\text{fwd}}(\vec{Q}, \tau) &= \tau_1(\vec{Q}, \tau) + \tau_2(\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau), \tau_1(\vec{Q}, \tau)), \\ \mathcal{P}^{\text{fwd}}(\vec{Q}, \tau) &= \mathcal{P}_-^{\text{fwd}}(\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau), \tau_1(\vec{Q}, \tau))\end{aligned}$$

are  $C^r$  since they are obtained as a sum and compositions of smooth maps.

The smoothness of  $\mathcal{P}_-^{\text{bwd}}(\vec{R}, \tau)$ ,  $\tau_{-1}(\vec{R}, \tau)$  and  $\mathcal{P}^{\text{bwd}}(\vec{R}, \tau)$ ,  $\mathcal{T}^{\text{bwd}}(\vec{R}, \tau)$  can be shown similarly.  $\square$

In fact, using the argument of Remark 3.7 with some minor changes we can show the following.

REMARK 3.8. Let  $A$  be an open, connected and bounded subset of  $\Omega$ , let  $\tau_2 > \tau_1$  and denote by

$$B(t) = \{\vec{x}(t, \tau_1; \vec{Q}) \mid \vec{Q} \in A\}.$$

Assume that in  $\Omega^0 \cap B(t)$  there are no sliding phenomena for any  $t \in [\tau_1, \tau_2]$ . Then the functions

$$\begin{aligned}\Phi_{\tau_2, \tau_1} : A &\rightarrow B(\tau_2), & \Phi_{\tau_1, \tau_2} : B(\tau_2) &\rightarrow A, \\ \Phi_{\tau_2, \tau_1}(\vec{Q}) &= \vec{x}(\tau_2, \tau_1; \vec{Q}), & \Phi_{\tau_1, \tau_2} &= \Phi_{\tau_2, \tau_1}^{-1}\end{aligned}$$

are homeomorphisms.

Assume further that  $A \cap \Omega^0 = \emptyset$ ,  $B(\tau_2) \cap \Omega^0 = \emptyset$ , and that for any  $\vec{Q} \in A$ , if  $\vec{x}(\bar{t}, \tau_1; \vec{Q}) \in \Omega^0$  for some  $\bar{t} \in (\tau_1, \tau_2)$ , then it crosses  $\Omega^0$  transversely. Then  $\Phi_{\tau_2, \tau_1}$  and  $\Phi_{\tau_1, \tau_2}$  are  $C^r$  diffeomorphisms.

REMARK 3.9. Let us denote by  $\tilde{V}(\tau)$  the compact connected set enclosed by  $\tilde{W}(\tau)$  and by the branch of  $\Omega^0$  between  $\tilde{P}_u(\tau)$  and  $\tilde{P}_s(\tau)$ . If  $\vec{Q} \in A^{\text{fwd},+}(\tau)$ , then  $\vec{x}(t, \tau; \vec{Q}) \in \tilde{V}(t)$  for any  $t \in [\tau, \mathcal{T}^{\text{fwd}}(\vec{Q}, \tau)]$ .

Analogously if  $\vec{Q} \in A^{\text{bwd},-}(\tau)$ , then  $\vec{x}(t, \tau; \vec{Q}) \in \tilde{V}(t)$  for any  $t \in [\mathcal{T}^{\text{bwd}}(\vec{Q}, \tau), \tau]$ .



REMARK 3.10. Denote by  $B^{\text{fwd},+}(\tau)$  the branch of  $\Omega^0$  between  $\vec{P}_s(\tau)$  and  $\vec{P}^{\text{fwd},\text{out}}$  and by  $B^{\text{bwd},-}(\tau)$  the branch of  $\Omega^0$  between  $\vec{P}_u(\tau)$  and  $\vec{P}^{\text{bwd},\text{out}}$ .

Observe that, if  $\vec{Q} \in B^{\text{fwd},+}(\tau)$ , there is  $\tau_1(\vec{Q}) > \tau$  such that  $\vec{x}(t, \tau; \vec{Q}) \in K_B^{\text{fwd},+}(t)$  for any  $\tau < t < \tau_1(\vec{Q})$ , it crosses transversely  $L^{+,\text{out}}$  at  $t = \tau_1(\vec{Q})$  and leaves a neighborhood of  $\Gamma$  at some  $t > \tau_1(\vec{Q})$ .

Analogously, if  $\vec{Q} \in B^{\text{bwd},-}(\tau)$ , there is  $\tau_{-1}(\vec{Q}) < \tau$  such that  $\vec{x}(t, \tau; \vec{Q}) \in K_B^{\text{bwd},-}(t)$  for any  $\tau_{-1}(\vec{Q}) < t < \tau$ , it crosses transversely  $L^{-,\text{out}}$  at  $t = \tau_{-1}(\vec{Q})$  and leaves a neighborhood of  $\Gamma$  at some  $t < \tau_{-1}(\vec{Q})$ .

#### 4. Statement of the main results

In this section we need to measure the distance between the points on  $\Omega^0$ , and to be able to determine their mutual positions. Since  $\Omega^0$  is a regular curve we can define a directed distance for points in  $\Omega^0$  by arc length, once an orientation is fixed. We choose as the positive orientation on  $\Omega^0$  the one that goes from the origin to  $\vec{\gamma}(0)$ . So, for any  $\vec{Q} \in L^0$  we define  $\ell(\vec{Q}) = \int_{\Omega^0(\vec{0}, \vec{Q})} ds > 0$  where  $\Omega^0(\vec{0}, \vec{Q})$  is the (oriented) path of  $\Omega^0$  connecting  $\vec{0}$  with  $\vec{Q}$ , and we define the directed distance

$$\mathcal{D}(\vec{Q}, \vec{P}) := \ell(\vec{P}) - \ell(\vec{Q}) \quad (18)$$

for  $\vec{Q}, \vec{P} \in L^0$ . Notice that  $\mathcal{D}(\vec{Q}, \vec{P}) > 0$  means that  $\vec{Q}$  lies on  $\Omega^0$  between  $\vec{0}$  and  $\vec{P}$ . Now, we introduce some further crucial notation.

**Notation.** We denote by  $\vec{Q}_s(d, \tau)$  the point in  $A^{\text{fwd},+}(\tau)$  such that

$$\mathcal{D}(\vec{Q}_s(d, \tau), \vec{P}_s(\tau)) = d > 0,$$

and by  $\vec{Q}_u(d, \tau)$  the point in  $A^{\text{bwd},-}(\tau)$  such that

$$\mathcal{D}(\vec{Q}_u(d, \tau), \vec{P}_u(\tau)) = d > 0.$$

We introduce a further small parameter  $\delta > 0$ . This parameter can be chosen independently of  $\varepsilon > 0$ , but we need to set  $\delta < \frac{\beta}{2}$ . Then we will always assume  $0 < d \leq \delta$ .

REMARK 4.1. We explicitly notice that, if  $0 < d \leq \delta$  then  $\vec{Q}_s(d, \tau) \in A^{\text{fwd},+}(\tau)$ , i.e., the open branch of  $\Omega^0$  between  $\vec{P}^{\text{fwd},\text{in}}$  and  $\vec{P}_s(\tau)$ . Indeed

$$\|\vec{Q}_s(d, \tau) - \vec{P}_s(\tau)\| \leq |\mathcal{D}(\vec{Q}_s(d, \tau), \vec{P}_s(\tau))| = d \leq \delta < \beta/2;$$

further from  $\|\vec{P}_s(\tau) - \vec{\gamma}(0)\| = O(\varepsilon)$  one can see that

$$\|\vec{Q}_s(d, \tau) - \vec{\gamma}(0)\| \leq \|\vec{P}_s(\tau) - \vec{\gamma}(0)\| + \|\vec{Q}_s(d, \tau) - \vec{P}_s(\tau)\| < \beta,$$

since  $\varepsilon \ll \beta$ ; while by construction  $\|\vec{P}^{\text{fwd},\text{in}} - \vec{\gamma}(0)\| = \beta$ . Hence for  $\varepsilon$  small enough  $\vec{Q}_s(d, \tau) \in A^{\text{fwd},+}(\tau)$ .

Similarly we see that if  $0 < d \leq \delta$  then  $\vec{Q}_u(d, \tau) \in A^{\text{bwd},-}(\tau)$ , i.e.,  $\vec{Q}_u(d, \tau)$  lies on the open branch of  $\Omega^0$  between  $\vec{P}^{\text{bwd},\text{in}}$  and  $\vec{P}_u(\tau)$ .

From Lemma 3.6, we see that for any  $\tau \in \mathbb{R}$  and any  $0 < d \leq \delta$  we can define the maps

$$\begin{aligned} \mathcal{T}_1(d, \tau) &:= \mathcal{T}^{\text{fwd}}(\vec{Q}_s(d, \tau), \tau), & \mathcal{P}_1(d, \tau) &:= \mathcal{P}^{\text{fwd}}(\vec{Q}_s(d, \tau), \tau), \\ \mathcal{T}_{-1}(d, \tau) &:= \mathcal{T}^{\text{bwd}}(\vec{Q}_u(d, \tau), \tau), & \mathcal{P}_{-1}(d, \tau) &:= \mathcal{P}^{\text{bwd}}(\vec{Q}_u(d, \tau), \tau). \end{aligned} \quad (19)$$

Sometimes we will also make use of the maps

$$\begin{aligned} \mathcal{T}_{\frac{1}{2}}(d, \tau) &:= \tau_1(\vec{Q}_s(d, \tau), \tau), & \mathcal{P}_{\frac{1}{2}}(d, \tau) &:= \mathcal{P}_+^{\text{fwd}}(\vec{Q}_s(d, \tau), \tau), \\ \mathcal{T}_{-\frac{1}{2}}(d, \tau) &:= \tau_{-1}(\vec{Q}_u(d, \tau), \tau), & \mathcal{P}_{-\frac{1}{2}}(d, \tau) &:= \mathcal{P}_-^{\text{bwd}}(\vec{Q}_u(d, \tau), \tau). \end{aligned} \quad (20)$$

Notice that by construction the trajectory  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$  is in  $K_A^{\text{fwd},+}(t)$  for any  $\tau < t < \mathcal{T}_{\frac{1}{2}}(d, \tau)$  and it is in  $K_A^{\text{fwd},-}(t)$  for any  $\mathcal{T}_{\frac{1}{2}}(d, \tau) < t < \mathcal{T}_1(d, \tau)$ ; further it intersects transversely  $L^{\text{in}}$  at  $t = \mathcal{T}_{\frac{1}{2}}(d, \tau)$  and  $A^{\text{fwd},-}(\mathcal{T}_1(d, \tau)) \subset L^0$  at  $t = \mathcal{T}_1(d, \tau)$ . Similarly, the trajectory  $\vec{x}(t, \tau; \vec{Q}_u(d, \tau))$  is in  $K_A^{\text{bwd},-}(t)$  for any  $\mathcal{T}_{-\frac{1}{2}}(d, \tau) < t < \tau$  and it is in  $K_A^{\text{bwd},+}(t)$  for any  $\mathcal{T}_{-1}(d, \tau) < t < \mathcal{T}_{-\frac{1}{2}}(d, \tau)$ ; further it intersects transversely  $L^{\text{in}}$  at  $t = \mathcal{T}_{-\frac{1}{2}}(d, \tau)$  and  $A^{\text{bwd},+}(\mathcal{T}_{-1}(d, \tau)) \subset L^0$  at  $t = \mathcal{T}_{-1}(d, \tau)$ .

**THEOREM 4.2.** *Assume **F0**, **F1**, **F2**, **K**, **G** and let  $\vec{f}^\pm$  and  $\vec{g}$  be  $C^r$  with  $r > 1$ . We can find  $\varepsilon_0 > 0$ ,  $\delta > 0$ , such that for any  $0 < \varepsilon \leq \varepsilon_0$ , the functions  $\mathcal{T}_{\pm 1}(d, \tau)$ ,  $\mathcal{P}_{\pm 1}(d, \tau)$  are  $C^r$  when  $0 < d \leq \delta$  and  $\tau \in \mathbb{R}$ . Further, for any  $0 < \mu < \mu_0$  we get*

$$\begin{aligned} d^{\sigma^{\text{fwd}} + \mu} &\leq \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) \leq d^{\sigma^{\text{fwd}} - \mu}, \\ d^{\sigma^{\text{bwd}} + \mu} &\leq \mathcal{D}(\mathcal{P}_{-1}(d, \tau), \vec{P}_s(\mathcal{T}_{-1}(d, \tau))) \leq d^{\sigma^{\text{bwd}} - \mu}, \\ \|\mathcal{P}_{\frac{1}{2}}(d, \tau)\| &\leq d^{\sigma_+^{\text{fwd}} - \mu}, & \|\mathcal{P}_{-\frac{1}{2}}(d, \tau)\| &\leq d^{\sigma_-^{\text{bwd}} - \mu}, \end{aligned} \quad (21)$$

$$\begin{aligned} [\Sigma^{\text{fwd}} - \mu] |\ln(d)| &\leq (\mathcal{T}_1(d, \tau) - \tau) \leq [\Sigma^{\text{fwd}} + \mu] |\ln(d)|, \\ [\Sigma^{\text{bwd}} - \mu] |\ln(d)| &\leq \tau - \mathcal{T}_{-1}(d, \tau) \leq [\Sigma^{\text{bwd}} + \mu] |\ln(d)|, \\ [\Sigma_+^{\text{fwd}} - \mu] |\ln(d)| &\leq (\mathcal{T}_{\frac{1}{2}}(d, \tau) - \tau) \leq [\Sigma_+^{\text{fwd}} + \mu] |\ln(d)|, \\ [\Sigma_-^{\text{bwd}} - \mu] |\ln(d)| &\leq \tau - \mathcal{T}_{-\frac{1}{2}}(d, \tau) \leq [\Sigma_-^{\text{bwd}} + \mu] |\ln(d)|, \end{aligned} \quad (22)$$

and all the expressions in (21) are uniform with respect to any  $\tau \in \mathbb{R}$  and  $0 < \varepsilon \leq \varepsilon_0$ .

In fact, with the same argument developed in §5, we are able to estimate the positions of the trajectories  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$  and  $\vec{x}(t, \tau; \vec{Q}_u(d, \tau))$  for any  $t \in [\tau, \mathcal{T}_1(d, \tau)]$  and  $t \in [\mathcal{T}_{-1}(d, \tau), \tau]$ , respectively.

**THEOREM 4.3.** *Assume **F0**, **F1**, **F2**, **K**, **G** and let  $f^\pm$  and  $\vec{g}$  be  $C^r$  with  $r > 1$ . We can find  $\varepsilon_0 > 0$ ,  $\delta > 0$ , such that for any  $0 < \varepsilon \leq \varepsilon_0$ ,  $0 < d \leq \delta$ ,  $0 < \mu < \mu_0$  and any  $\tau \in \mathbb{R}$  we find*

$$\|\vec{x}(t, \tau; \vec{Q}_s(d, \tau)) - \vec{x}(t, \tau; \vec{P}_s(\tau))\| \leq d^{\sigma_+^{fwd} - \mu} \quad (23)$$

for any  $\tau \leq t \leq \mathcal{T}_{\frac{1}{2}}(d, \tau)$ , and

$$\|\vec{x}(t, \tau; \vec{Q}_s(d, \tau)) - \vec{x}(t, \mathcal{T}_1(d, \tau); \vec{P}_u(\mathcal{T}_1(d, \tau)))\| \leq d^{\sigma_+^{fwd} - \mu} \quad (24)$$

for any  $\mathcal{T}_{\frac{1}{2}}(d, \tau) \leq t \leq \mathcal{T}_1(d, \tau)$ .

Similarly, for any  $0 < \varepsilon \leq \varepsilon_0$ ,  $0 < d \leq \delta$ ,  $0 < \mu < \mu_0$  and any  $\tau \in \mathbb{R}$  we find

$$\|\vec{x}(t, \tau; \vec{Q}_u(d, \tau)) - \vec{x}(t, \tau; \vec{P}_u(\tau))\| \leq d^{\sigma_-^{bwd} - \mu} \quad (25)$$

for any  $\mathcal{T}_{-\frac{1}{2}}(d, \tau) \leq t \leq \tau$ , and

$$\|\vec{x}(t, \tau; \vec{Q}_u(d, \tau)) - \vec{x}(t, \mathcal{T}_{-1}(d, \tau); \vec{P}_s(\mathcal{T}_{-1}(d, \tau)))\| \leq d^{\sigma_-^{bwd} - \mu} \quad (26)$$

for any  $\mathcal{T}_{-1}(d, \tau) \leq t \leq \mathcal{T}_{-\frac{1}{2}}(d, \tau)$ .

## 5. Proofs of Theorems 4.2 and 4.3

In this section we assume all the hypotheses of Theorems 4.2 and 4.3 without further mentioning, and we adopt the notation introduced in §3; we recall that  $f^\pm$  and  $\vec{g}$  are  $C^r$  with  $r \geq 1 + \alpha$  for some  $0 < \alpha \leq 1$ , i.e., their derivatives are Hölder continuous.

For simplicity we denote by

$$\vec{F}^\pm(\vec{x}, t, \varepsilon) = f^\pm(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega^\pm \cup \Omega^0$$

and we assume the following:

**H** There are  $0 < \alpha \leq 1$  and  $N_\alpha > 0$  such that, for  $\|\vec{h}\| < 1$ ,

$$\sup\{\|\mathbf{F}_x^\pm(\vec{P} + \vec{h}, t, \varepsilon) - \mathbf{F}_x^\pm(\vec{P}, t, \varepsilon)\| \mid \vec{P} \in B(\mathbf{\Gamma}, 1), t \in \mathbb{R}, |\varepsilon| \leq 1\} \leq N_\alpha \|\vec{h}\|^\alpha.$$

The proofs of Theorems 4.2 and 4.3 is technical and lengthy and it is divided in several steps.

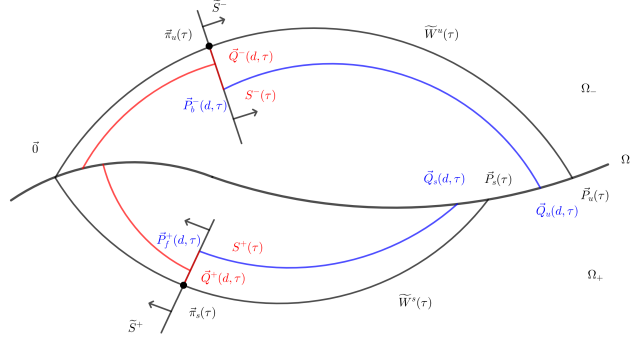


Figure 6: A scheme of the proofs of Lemmas 4.2 and 4.3.

Adapting [15] we introduce a small auxiliary parameter  $\varpi$ ,  $\varpi > \delta$ , see Remark 5.1, whose value will be fixed below, and the following set depending on it:

$$\tilde{S}^+ := \left\{ d\vec{v}_u^+ + \frac{\vec{v}_s^+}{|\ln(\varpi)|} \mid |d| \leq \frac{1}{|\ln(\varpi)|} \right\}, \quad (27)$$

see Figure 6. Let us denote by  $\tilde{\pi}_s(\tau)$  the unique intersection between  $\tilde{W}^s(\tau)$  and  $\tilde{S}^+$ , whose existence follows from the fact that  $\tilde{W}^s(\tau)$  is locally a graph on its tangent, see also Remark 5.14. Moreover we let  $\vec{y}_s(\theta) = \vec{x}(\theta + \tau, \tau; \tilde{\pi}_s(\tau))$ ; notice that  $\vec{y}_s(\theta) \in K^{\text{fwd},+}$  for any  $\theta \geq 0$  and  $\lim_{\theta \rightarrow +\infty} \vec{y}_s(\theta) = \vec{0}$ , see Figure 4. Then we denote by

$$S^+(\tau) := \left\{ -d\vec{v}_u^+ + \tilde{\pi}_s(\tau) \mid 0 < d \leq \delta \right\}. \quad (28)$$

Since  $\tilde{\pi}_s(\tau) = \frac{\vec{v}_s^+}{|\ln(\varpi)|} + O\left(\frac{1}{|\ln(\varpi)|^{1+\alpha}}\right)$ , then  $S^+(\tau) \subset \tilde{S}^+$ .

Analogously we denote by

$$\tilde{S}^- := \left\{ d\vec{v}_s^- + \frac{\vec{v}_u^-}{|\ln(\varpi)|} \mid |d| \leq \frac{1}{|\ln(\varpi)|} \right\}. \quad (29)$$

Let  $\tilde{\pi}_u(\tau)$  be the unique intersection between  $\tilde{W}^u(\tau)$  and  $\tilde{S}^-$ , whose existence follows from the fact that  $\tilde{W}^u(\tau)$  is locally a graph on its tangent. We let  $\vec{y}_u(\theta) = \vec{x}(\theta + \tau, \tau; \tilde{\pi}_u(\tau))$ , and we observe that  $\vec{y}_u(\theta) \in K^{\text{bwd},-}$  for any  $\theta \leq 0$  and  $\lim_{t \rightarrow -\infty} \vec{y}_u(\theta) = \vec{0}$ . Then we set

$$S^-(\tau) := \left\{ -d\vec{v}_s^- + \tilde{\pi}_u(\tau) \mid 0 < d \leq \delta \right\}. \quad (30)$$

Again by construction  $S^-(\tau) \subset \tilde{S}^-$ .

Roughly speaking, to prove the results of this section we consider a trajectory performing a loop and we estimate flight time and space displacement: for this purpose we distinguish four different parts of the loop. Firstly in §5.1 we follow the trajectories from  $L^0$  to  $\tilde{S}^+$  forward in time (and we deduce what happens backward in time from  $\tilde{S}^+$  to  $L^0$ ): we get estimates of the fly time and the displacement with respect to  $\tilde{W}^s$ . Then with the same argument we follow the trajectories from  $L^0$  to  $\tilde{S}^-$  backward in time (and forward from  $\tilde{S}^-$  to  $L^0$ ) and we estimate fly time and displacement with respect to  $\tilde{W}^u$ . In the proofs we will use the fact that the trajectories are close to  $\tilde{W}$ , which is in turn close to  $\Gamma$ .

Then, in §5.2 we follow the trajectories going from  $\tilde{S}^+$  to  $L^{\text{in}}$ , forward and backward in time, and we evaluate fly time and displacement from  $\tilde{W}^s$ . Then in §5.3, with an argument analogous to the one of §5.2, we follow trajectories going from  $\tilde{S}^-$  to  $L^{\text{in}}$ , forward and backward in time and we evaluate fly time and displacement from  $\tilde{W}^u$ . Eventually, putting all these results together, we get an estimate of the distance of the trajectories from  $\tilde{W}$  and we prove Theorems 4.2 and 4.3.

REMARK 5.1. In this section we introduce several small parameters, which need to be chosen in the right order.

Equation (PS) assigns  $\varepsilon > 0$ , which measures the size of the perturbation, and the constant  $\mu_0$  which depends only on the eigenvalues  $\lambda_s^\pm$ ,  $\lambda_u^\pm$ , see (14).

Then we introduce a parameter  $\beta > \varepsilon^{\sigma^{\text{fb}}/2}$  and, via Lemma 3.6, we deduce the existence of  $\varepsilon_0$  and  $\beta_0$  such that for any  $0 < \varepsilon \leq \varepsilon_0$  and any  $0 < \varepsilon^{\sigma^{\text{fb}}/2} < \beta \leq \beta_0$  we can construct the maps  $\mathcal{P}_+^{\text{fwd}}$ ,  $\mathcal{P}^{\text{fwd}}$ ,  $\mathcal{P}_-^{\text{bwd}}$ ,  $\mathcal{P}^{\text{bwd}}$ .

Nextly we set  $\mu \in ]0, \mu_0]$  which measures the size of the errors that we make in the estimates of flight time and displacement appearing in Theorems 4.2 and 4.3.

In this section we introduce a further small parameter  $\varpi \in ]0, 1[$  which measures the distance between the origin and the lines  $\tilde{S}^+$  and  $\tilde{S}^-$ , transversal to  $\Gamma$  (these distances are of order  $|\ln(\varpi)|^{-1}$ ). This parameter is needed just for the proofs of Theorems 4.2 and 4.3 and it does not appear in the statement of the results.

Then we choose  $\delta = \delta(\varpi, \varepsilon) \in ]0, \varpi[$ :  $\delta$  measures the displacement of the trajectories performing a loop with respect to  $\tilde{W}$ : in fact each trajectory appearing in Theorems 4.2 and 4.3 has distance  $O(d)$  from  $\tilde{W}$  when it crosses  $L^0$ , where  $0 < d \leq \delta$ . Finally we possibly reduce the size of  $\varepsilon_0$  so that the fixed point arguments described in §5.1 and §5.2 work.

Let us spend a few more words about the order in which these parameters need to be chosen. The parameter  $\varepsilon$  is assigned from (PS); we choose  $\beta_0 > 0$  and  $\varepsilon_0 > 0$  so that Lemma 3.6 holds for any  $0 < \varepsilon^{\sigma^{\text{fb}}/2} < \beta \leq \beta_0$  and any  $0 < \varepsilon \leq \varepsilon_0$ , and the maps appearing in Lemma 3.6, Theorems 4.2 and 4.3 are well defined.

Then we choose  $\mu > 0$ : in fact we have to choose a parameter  $\mu_1 > 0$  which is needed for the estimates of the trajectories going from  $L^0$  to  $\tilde{S}^+$  and from  $\tilde{S}^-$  to  $L^0$ , and a parameter  $\mu_2 > 0$  which is needed for the estimates of the trajectories going from  $\tilde{S}^+$  to  $L^{\text{in}}$  and from  $L^{\text{in}}$  to  $\tilde{S}^-$ . Then we set  $\mu_2 = 2\mu_1$  and  $\mu = c_\mu\mu_2$  where  $c_\mu \geq 7$  is a constant (depending only on  $\lambda_s^\pm, \lambda_u^\pm$ , see (83) below), and we need to choose  $\mu$  so that  $0 < \mu \leq \mu_0$ . So, the choice of  $\mu$  prescribes the values of  $\mu_1$  and  $\mu_2$ .

Then we choose  $\varpi = \varpi(\mu, \varepsilon)$ , nextly  $\delta = \delta(\varpi, \mu, \varepsilon)$ ; finally for any fixed  $0 < \mu \leq \mu_0$  we choose  $\varepsilon_0 = \varepsilon_0(\mu)$  so that our argument works for any  $0 < \varepsilon \leq \varepsilon_0$ . We think that in fact the estimates might be improved and it could be shown that  $\mu$  could be chosen as a linear function of  $\varepsilon$ , but this is beyond the interest of this paper, see Remark 5.29. In fact we believe that one can improve further the estimates if  $\mathbf{g}_x(t, 0, \varepsilon) \equiv \mathbf{0}$ .

REMARK 5.2. By Lemma 3.6 we know that if  $\vec{Q} \in A^{\text{fwd},+}(\tau)$ , then  $\vec{x}(t, \tau; \vec{Q})$  is confined in the region  $K_A^{\text{fwd},+}(t)$  for any  $0 < t < \tau_1(\vec{Q}, \tau)$ . To simplify the presentation and avoid cumbersome notation we will tacitly assume that the systems (PS+) and (31)<sub>+</sub> are well defined, through any smooth extension, even when  $\vec{x} \in \Omega^0 \cup \Omega^-$  is “sufficiently close” to  $\Omega^+$ . In fact, the choice of the extension does not affect our argument since we just focus on  $\Omega^0 \cup \Omega^+$ .

We will maintain the same kind of assumption when dealing with  $\vec{x}(t, \tau; \vec{Q})$  and  $\vec{Q} \in A^{\text{bwd},-}(\tau)$ .

Before starting the actual proof we recall some facts concerning exponential dichotomy. The roughness of exponential dichotomy (see [11, Proposition 1, §4] and also [9, Appendix]) yields that the linear systems

$$\dot{\vec{x}}(t) = \mathbf{F}_x^\pm(\vec{0}, t, \varepsilon)\vec{x}(t) \quad (31)_\pm$$

admit exponential dichotomy on the whole of  $\mathbb{R}$  with projections  $\mathbf{P}^{\pm, \varepsilon}$ , and exponents  $\lambda_s^\pm + k\varepsilon < 0 < \lambda_u^\pm - k\varepsilon$  for some  $k > 0$ .

Keeping in mind the definitions of  $\underline{\lambda}$  and  $\bar{\lambda}$  given in (10), we shall assume w.l.o.g. that  $k\varepsilon_0 < \frac{1}{4}\underline{\lambda}$ , so that

$$-2\bar{\lambda} < \lambda_s^\pm - k\varepsilon < \lambda_s^\pm + k\varepsilon < -\frac{1}{2}\underline{\lambda} < 0 < \frac{1}{2}\underline{\lambda} < \lambda_u^\pm - k\varepsilon < \lambda_u^\pm + k\varepsilon < 2\bar{\lambda}.$$

In this paper we use the version of exponential dichotomy introduced in [9, Appendix] in which we estimate the projections using actual solutions of (31)<sub>±</sub>.

In the whole section (unless otherwise stated)  $\varepsilon > 0$  is fixed: so from now on we leave this dependence unsaid.

Let us denote by  $\vec{w}_u^\pm(t)$  and by  $\vec{w}_s^\pm(t)$  the unique solutions of the linear

equations (31)<sub>±</sub> satisfying the following conditions

$$\begin{aligned} \|\bar{w}_u^\pm(0)\| = 1, \quad \lim_{t \rightarrow -\infty} \|\bar{w}_u^\pm(t)\| = 0, \quad \lim_{t \rightarrow -\infty} \left( \frac{\bar{w}_u^\pm(t)}{\|\bar{w}_u^\pm(t)\|} \right)^* \bar{v}_u^\pm > 0, \\ \|\bar{w}_s^\pm(0)\| = 1, \quad \lim_{t \rightarrow +\infty} \|\bar{w}_s^\pm(t)\| = 0, \quad \lim_{t \rightarrow +\infty} \left( \frac{\bar{w}_s^\pm(t)}{\|\bar{w}_s^\pm(t)\|} \right)^* \bar{v}_s^\pm > 0. \end{aligned} \quad (32)$$

Let us denote by  $\bar{v}_u^\pm(t)$  and by  $\bar{v}_s^\pm(t)$  the unit vectors  $\bar{v}_u^\pm(t) = \frac{\bar{w}_u^\pm(t)}{\|\bar{w}_u^\pm(t)\|}$  and  $\bar{v}_s^\pm(t) = \frac{\bar{w}_s^\pm(t)}{\|\bar{w}_s^\pm(t)\|}$ . Then, following [9, §6.2] we define

$$z_u^\pm(t, s) = \frac{\|\bar{w}_u^\pm(t)\|}{\|\bar{w}_u^\pm(s)\|}, \quad z_s^\pm(t, s) = \frac{\|\bar{w}_s^\pm(t)\|}{\|\bar{w}_s^\pm(s)\|}. \quad (33)$$

Note that for any  $t, r, s \in \mathbb{R}$  we have

$$z_s^\pm(t, s) = z_s^\pm(t, r)z_s^\pm(r, s), \quad z_u^\pm(t, s) = z_u^\pm(t, r)z_u^\pm(r, s).$$

Moreover, there are  $C > 0$  and  $k_1 \geq 1$  such that for any  $t, s \in \mathbb{R}$  we have

$$\begin{aligned} \|\bar{v}_u^\pm(t) - \bar{v}_u^\pm(s)\| \leq C\varepsilon, \quad \|\bar{v}_s^\pm(t) - \bar{v}_s^\pm(s)\| \leq C\varepsilon, \\ (k_1)^{-1} e^{\lambda_u^\pm(t-s) - k\varepsilon|t-s|} \leq z_u^\pm(t, s) \leq k_1 e^{\lambda_u^\pm(t-s) + k\varepsilon|t-s|}, \\ (k_1)^{-1} e^{\lambda_s^\pm(t-s) - k\varepsilon|t-s|} \leq z_s^\pm(t, s) \leq k_1 e^{\lambda_s^\pm(t-s) + k\varepsilon|t-s|}. \end{aligned} \quad (34)$$

So that, for any  $t < s$  we find

$$z_u^\pm(t, s) \leq k_1 e^{-\frac{1}{2}\lambda|t-s|}, \quad z_s^\pm(s, t) \leq k_1 e^{-\frac{1}{2}\lambda|t-s|}. \quad (35)$$

Denote by  $\mathbf{X}^{\pm, \varepsilon}(t)$  the fundamental solutions of (31)<sub>±</sub> satisfying  $\mathbf{X}^{\pm, \varepsilon}(0) = \mathbb{I}$ . We also need to define the shifted projections

$$\mathbf{P}^{\pm, \varepsilon}(\tau) := \mathbf{X}^{\pm, \varepsilon}(\tau) \mathbf{P}^{\pm, \varepsilon}(\mathbf{X}^{\pm, \varepsilon}(\tau))^{-1}.$$

Notice that both  $\|\mathbf{P}^{\pm, \varepsilon}(\tau)\|$  and  $\|\mathbb{I} - \mathbf{P}^{\pm, \varepsilon}(\tau)\|$  are bounded, so we can set

$$k_2 := 2 \sup\{\|\mathbf{P}^{\pm, 0}(\tau)\| \mid \tau \in \mathbb{R}\} \cup \{\|\mathbb{I} - \mathbf{P}^{\pm, 0}(\tau)\| \mid \tau \in \mathbb{R}\}$$

so that  $\max\{\|\mathbf{P}^{\pm, \varepsilon}(\tau)\|; \|\mathbb{I} - \mathbf{P}^{\pm, \varepsilon}(\tau)\|\} \leq k_2$  for any  $\varepsilon > 0$  small enough and any  $\tau \in \mathbb{R}$ . We need the following result.

LEMMA 5.3. *For any  $s, t \in \mathbb{R}$  and any  $\vec{\xi} \in \mathbb{R}^2$  we have the following.*

$$\begin{aligned} \|\mathbf{X}^{\pm, \varepsilon}(t) \mathbf{P}^{\pm, \varepsilon}(\mathbf{X}^{\pm, \varepsilon}(s))^{-1} \vec{\xi}\| &\leq k_2 z_s(t, s) \|\vec{\xi}\|, \\ \|\mathbf{X}^{\pm, \varepsilon}(t) (\mathbb{I} - \mathbf{P}^{\pm, \varepsilon})(\mathbf{X}^{\pm, \varepsilon}(s))^{-1} \vec{\xi}\| &\leq k_2 z_u(t, s) \|\vec{\xi}\| \end{aligned}$$

for some  $k_2 > 0$ .

*Proof.* Let  $\mathbf{A}$  be a linear operator, we denote by  $\mathcal{R}\mathbf{A}$  its range and by  $\mathcal{N}\mathbf{A}$  its kernel. Fix  $\tau \in \mathbb{R}$ , then for any  $\vec{\xi} \in \mathbb{R}^2$  we find  $\vec{\xi}_s^\pm(\tau) \in \text{span}[\vec{v}_s^\pm(\tau)] = \mathcal{R}\mathbf{P}^{\pm,\varepsilon}(\tau)$  and  $\vec{\xi}_u^\pm(\tau) \in \text{span}[\vec{v}_u^\pm(\tau)] = \mathcal{N}\mathbf{P}^{\pm,\varepsilon}(\tau)$  such that  $\vec{\xi} = \vec{\xi}_s^\pm(\tau) + \vec{\xi}_u^\pm(\tau)$ . Further  $\|\vec{\xi}_s^\pm(\tau)\| \leq k_2\|\vec{\xi}\|$  and  $\|\vec{\xi}_u^\pm(\tau)\| \leq k_2\|\vec{\xi}\|$ .

Next by the invariance of the projections of exponential dichotomy we have

$$\begin{aligned} \mathbf{X}^{\pm,\varepsilon}(t)\mathbf{P}^{\pm,\varepsilon}(\mathbf{X}^{\pm,\varepsilon}(s))^{-1} &= \mathbf{X}^{\pm,\varepsilon}(t)(\mathbf{X}^{\pm,\varepsilon}(s))^{-1}\mathbf{P}^{\pm,\varepsilon}(s), \\ \mathbf{X}^{\pm,\varepsilon}(t)[\mathbb{I} - \mathbf{P}^{\pm,\varepsilon}](\mathbf{X}^{\pm,\varepsilon}(s))^{-1} &= \mathbf{X}^{\pm,\varepsilon}(t)(\mathbf{X}^{\pm,\varepsilon}(s))^{-1}[\mathbb{I} - \mathbf{P}^{\pm,\varepsilon}(s)]. \end{aligned}$$

Hence for any  $\vec{\xi} \in \mathbb{R}^2$  we find

$$\begin{aligned} \|\mathbf{X}^{\pm,\varepsilon}(t)\mathbf{P}^{\pm,\varepsilon}(\mathbf{X}^{\pm,\varepsilon}(s))^{-1}\vec{\xi}\| &= \|\mathbf{X}^{\pm,\varepsilon}(t)(\mathbf{X}^{\pm,\varepsilon}(s))^{-1}\vec{\xi}_s^\pm(s)\| \\ &= z_s(t,s)\|\vec{\xi}_s^\pm(s)\| \leq k_2z_s(t,s)\|\vec{\xi}\|, \\ \|\mathbf{X}^{\pm,\varepsilon}(t)[\mathbb{I} - \mathbf{P}^{\pm,\varepsilon}](\mathbf{X}^{\pm,\varepsilon}(s))^{-1}\vec{\xi}\| &= \|\mathbf{X}^{\pm,\varepsilon}(t)(\mathbf{X}^{\pm,\varepsilon}(s))^{-1}\vec{\xi}_u^\pm(s)\| \\ &= z_u(t,s)\|\vec{\xi}_u^\pm(s)\| \leq k_2z_u(t,s)\|\vec{\xi}\|, \end{aligned}$$

so the lemma is proved.  $\square$

Using standard arguments from exponential dichotomy theory (see again [9, §6.2]), we get that there is  $c_k > 1$  such that

$$\begin{aligned} (c_k)^{-1}e^{-2\bar{\lambda}\theta_s} &< \|\vec{x}(\theta_s + \tau, \tau; \vec{P}_s(\tau))\| < c_k e^{-\frac{1}{2}\lambda\theta_s}, \\ (c_k)^{-1}e^{-2\bar{\lambda}|\theta_u|} &< \|\vec{x}(\theta_u + \tau, \tau; \vec{P}_u(\tau))\| < c_k e^{-\frac{1}{2}\lambda|\theta_u|} \end{aligned} \quad (36)$$

whenever  $\theta_u \leq 0 \leq \theta_s$ , for any  $\tau \in \mathbb{R}$ .

### 5.1. The loop: from $L^0$ to $S^+(\tau)$ and from $S^-(\tau)$ to $L^0$

In the whole section  $\varpi > 0$  is a small parameter and we choose  $\delta = \delta(\varpi) < \varpi$  according to Lemmas 5.4, 5.9, and then we let  $0 < d \leq \delta$ . We recall that  $\vec{P}_s(\tau)$  is the unique intersection between  $\vec{W}^s(\tau)$  and  $L^0$  while  $\vec{P}_u(\tau)$  is the unique intersection between  $\vec{W}^u(\tau)$  and  $L^0$ ; further  $\vec{Q}_s(d, \tau)$  and  $\vec{Q}_u(d, \tau)$  are as defined at the beginning of §4. We introduce the following notation

$$\begin{aligned} \vec{Q}^+(d, \tau) &= -d\vec{v}_u^+ + \vec{\pi}_s(\tau) \in S^+(\tau), \\ \vec{Q}^-(d, \tau) &= -d\vec{v}_s^- + \vec{\pi}_u(\tau) \in S^-(\tau). \end{aligned} \quad (37)$$

We focus firstly on the trajectory  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$  going from  $L^0$  to  $\vec{S}^+$ . The next lemma is a consequence of Lemma 3.6.



LEMMA 5.4. Let  $\varpi > 0$  be a small constant, then there is  $\delta = \delta(\varpi)$  such that for any  $0 < d \leq \delta$  and any  $\tau \in \mathbb{R}$  there are  $T_1^f(d, \tau) > 0$  and  $\vec{P}_f^+(d, \tau)$  such that  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau)) \in K_A^{fwd,+}(t)$  for any  $\tau < t \leq \tau + T_1^f(d, \tau)$  and it crosses transversely  $S^+(T_1^f(d, \tau) + \tau) \subset \tilde{S}^+$  at  $t = \tau + T_1^f(d, \tau)$  in  $\vec{P}_f^+(d, \tau) = \vec{x}(T_1^f(d, \tau) + \tau, \tau; \vec{Q}_s(d, \tau))$ .

Conversely, going backwards in  $t$ , there are  $T_4^b(d, \tau) > 0$  and  $\vec{P}_b^+(d, \tau)$  such that  $\vec{x}(t, \tau; \vec{Q}^+(d, \tau)) \in K_A^{bwd,+}(t)$  for any  $\tau - T_4^b(d, \tau) < t \leq \tau$  and it crosses transversely  $L^0$  at  $t = \tau - T_4^b(d, \tau)$  in  $\vec{P}_b^+(d, \tau) = \vec{x}(\tau - T_4^b(d, \tau), \tau; \vec{Q}^+(d, \tau))$ .

Now we proceed to estimate the functions  $T_1^f, \vec{P}_f^+, T_4^b, \vec{P}_b^+$  defined in Lemma 5.4. We denote by  $-\tilde{T}_4^b < 0 < \tilde{T}_1^f, \tilde{T}_4^b = \tilde{T}_1^f$  the values such that  $\vec{\gamma}(\tilde{T}_1^f) \in \tilde{S}^+$ .

REMARK 5.5. Observe that if  $\varepsilon = 0$  then  $T_1^f(0, \tau) \equiv \tilde{T}_1^f$  and  $T_4^b(0, \tau) \equiv \tilde{T}_4^b$ . From (36) we find

$$\frac{1}{2\lambda} \ln \left( \frac{1}{c_k \|\vec{\gamma}(\tilde{T}_1^f)\|} \right) \leq \tilde{T}_1^f \leq \frac{2}{\lambda} \ln \left( \frac{c_k}{\|\vec{\gamma}(\tilde{T}_1^f)\|} \right).$$

Using  $\|\vec{\gamma}(\tilde{T}_1^f)\| = \frac{c}{|\ln(\varpi)|}$  for some  $0 < c < 2$ , we derive

$$\begin{aligned} \frac{1}{2\lambda} \ln \left( \frac{|\ln(\varpi)|}{c_k} \right) &\leq \tilde{T}_1^f \leq \frac{2}{\lambda} \ln(c_k |\ln(\varpi)|), \\ \frac{1}{2\lambda} \ln \left( \frac{|\ln(\varpi)|}{c_k} \right) &\leq \tilde{T}_4^b \leq \frac{2}{\lambda} \ln(c_k |\ln(\varpi)|). \end{aligned} \quad (38)$$

Hence for any  $0 < \varepsilon \leq \varepsilon_0$  we have

$$\begin{aligned} \frac{1}{2\lambda} \ln \left( \frac{|\ln(\varpi)|}{2c_k} \right) &\leq T_1^f(0, \tau) \leq \frac{2}{\lambda} \ln(2c_k |\ln(\varpi)|), \\ \frac{1}{2\lambda} \ln \left( \frac{|\ln(\varpi)|}{2c_k} \right) &\leq T_4^b(0, \tau) \leq \frac{2}{\lambda} \ln(2c_k |\ln(\varpi)|). \end{aligned} \quad (39)$$

LEMMA 5.6. Assume **H** and let  $\varpi > 0$  be a small constant, then there is  $\delta = \delta(\varpi)$  such that for any  $0 < d \leq \delta$  and any  $\tau \in \mathbb{R}$  we have

$$|T_1^f(d, \tau) - T_1^f(0, \tau)| \leq 1, \quad (40)$$

$$|T_4^b(d, \tau) - T_4^b(0, \tau)| \leq 1. \quad (41)$$

*Proof.* Let us set  $\bar{T} = \frac{2}{\lambda} \ln(4c_k |\ln(\varpi)|)$ , using continuous dependence of (PS) on initial data (see Remark 3.7), we see that there is  $\delta = \delta(\varpi)$  such that

$$\|\vec{x}(\theta + \tau, \tau; \vec{Q}_s(d, \tau)) - \vec{x}(\theta + \tau, \tau; \vec{P}_s(\tau))\| \leq |\ln(\varpi)|^{-4} \quad (42)$$

for any  $0 \leq \theta \leq \bar{T}$  and for any  $0 \leq d \leq \delta$ . Since  $\vec{f}(\vec{x}(T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau)))$  is transversal to  $\tilde{S}^+$  and  $1 + T_1^f(0, \tau) + \tau < \bar{T}$ , using (39) we find

$$\begin{aligned} & \dot{\vec{x}}(T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau)) \\ &= \mathbf{f}_x^+(\vec{0})\vec{x}(T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau)) + o(\vec{x}(T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau))) + O(\varepsilon) \quad (43) \\ &= \frac{\lambda_s^+}{|\ln(\varpi)|} (\vec{v}_s^+ + O(|\ln(\varpi)|^{-\alpha}) + O(\varepsilon)). \end{aligned}$$

Further, a similar estimate holds when  $\theta \in [0, 1]$ , namely

$$\left\| \dot{\vec{x}}(\theta + T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau)) - \frac{\lambda_s^+ e^{\lambda_s^+ \theta}}{|\ln(\varpi)|} \vec{v}_s^+ \right\| = \frac{O(\varepsilon)}{|\ln(\varpi)|} + O\left(\frac{1}{|\ln(\varpi)|^{1+\alpha}}\right). \quad (44)$$

So, from (43) and (44) we find that

$$B(\vec{x}(1 + T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau)); 2|\ln(\varpi)|^{-2}) \cap \tilde{S}^+ = \emptyset.$$

Then using (42) we find

$$B(\vec{x}(1 + T_1^f(0, \tau) + \tau, \tau; \vec{Q}_s(d, \tau)); |\ln(\varpi)|^{-2}) \cap \tilde{S}^+ = \emptyset,$$

so we get  $T_1^f(d, \tau) < T_1^f(0, \tau) + 1$ . Using a specular argument we prove (40); the proof of (41) is analogous and it is omitted.  $\square$

Adapting slightly the proof of Lemmas 6.2 and 6.3 in [15], which are in fact based on Theorem 12.15 in [22] we obtain the following estimates of the fly time and displacement with respect to  $\tilde{W}^s$ , moving forward and backward in time.

**PROPOSITION 5.7.** *Assume  $\mathbf{H}$  and let  $\varpi > 0$  be a small constant, then there is  $\delta = \delta(\varpi)$  such that for any  $0 < d \leq \delta$  and any  $\tau \in \mathbb{R}$  we have*

$$\frac{\ln(|\ln(\varpi)|)}{4\lambda} < T_1^f(d, \tau) < \frac{4\ln(|\ln(\varpi)|)}{\lambda}. \quad (45)$$

Further  $\vec{P}_f^+(d, \tau) = -D_1^f(d, \tau)\vec{v}_u^+ + \vec{\pi}_s(\tau + T_1^f(d, \tau))$  where

$$0 < d|\ln(\varpi)|^{-\tilde{C}} \leq D_1^f(d, \tau) \leq d|\ln(\varpi)|^{\tilde{C}}. \quad (46)$$

Here and below  $\tilde{C} > 0$  is a constant independent of  $\varpi$ ,  $\delta$  and  $\varepsilon$ .

Respectively,

$$\frac{\ln(|\ln(\varpi)|)}{4\lambda} < T_4^b(d, \tau) < \frac{4\ln(|\ln(\varpi)|)}{\lambda},$$

and  $D_4^b(d, \tau) = \mathcal{D}(\vec{P}_b^+(d, \tau), \vec{P}_s(\tau - T_4^b(d, \tau)))$  verifies

$$0 < d|\ln(\varpi)|^{-\tilde{C}} \leq D_4^b(d, \tau) \leq d|\ln(\varpi)|^{\tilde{C}}.$$

*Proof.* The estimate (45) follows immediately from (40) and (39), and similarly for the estimates concerning  $T_4^b(d, \tau)$ .

Let us now estimate the displacement

$$D_1^f(d, \tau) = D_1^f(d, \tau, \varepsilon) = \frac{\partial D_1^f}{\partial d}(0, \tau, 0)d + o(d + \varepsilon) \quad (47)$$

since  $D_1^f(0, \tau, \varepsilon) = 0$ . Now we estimate  $\frac{\partial D_1^f}{\partial d}(0, \tau, 0)$ , so from now on in this proof we consider  $\varepsilon = 0$ . Let us set

$$M_f = \sup \left\{ \left\{ |\operatorname{tr} \mathbf{f}_x^-(\vec{x})| \mid \vec{x} \in B(\mathbf{\Gamma}^-, 1) \right\} \cup \left\{ |\operatorname{tr} \mathbf{f}_x^+(\vec{x})| \mid \vec{x} \in B(\mathbf{\Gamma}^+, 1) \right\} \right\},$$

where  $\mathbf{\Gamma}^\pm = \mathbf{\Gamma} \cap \Omega^\pm$ . In this proof we denote for the sake of brevity  $\vec{p}(\theta, \vec{P}) = \vec{x}(\theta + \tau, \tau; \vec{P}, \varepsilon = 0)$ . Notice that  $\vec{p}(\theta, \vec{\gamma}(0)) = \vec{\gamma}(\theta)$  and  $\dot{\vec{p}}(\theta, \vec{\gamma}(0)) = \vec{f}^+(\vec{\gamma}(\theta))$ , if  $\theta \geq 0$ . Adapting Theorem 12.15 in [22] we consider the variational equation

$$\dot{\vec{x}}(\theta) = \mathbf{f}_x^+(\vec{\gamma}(\theta))\vec{x}(\theta). \quad (48)$$

Following [22] we see that if  $\mathbf{X}(\theta)$  is the fundamental matrix of (48) then for any  $T > 0$  we get

$$\det \mathbf{X}(T) = \exp \left( \int_0^T \operatorname{tr} \mathbf{f}_x^+(\vec{\gamma}(\theta)) d\theta \right).$$

Further, recalling that  $\mathbf{f}_x^+(\vec{\gamma}(\theta))$  is close to  $\mathbf{f}_x^+(\vec{0})$  when  $\theta \rightarrow +\infty$  we find

$$\|\mathbf{X}(\tilde{T}_1^f)\| \leq e^{2\bar{\lambda}\tilde{T}_1^f} \leq |\ln(\varpi)|^{8\bar{\lambda}/\lambda}. \quad (49)$$

Next from (45) we get

$$|\ln(\varpi)|^{-4M_f/\lambda} \leq \det(\mathbf{X}(\tilde{T}_1^f)) \leq |\ln(\varpi)|^{4M_f/\lambda} \quad (50)$$

for any  $0 \leq d \leq \delta$ . Let now  $\vec{v}$  be the unit vector tangent to  $\Omega^0$  in  $\vec{\gamma}(0)$  aiming towards  $E^{\text{in}}$ : notice that  $\vec{v}$  is transversal to  $\dot{\vec{\gamma}}(0^+) = \vec{f}^+(\vec{\gamma}(0^+))$ , cf.  $\mathbf{K}$ .

Observe that by construction  $\vec{Q}_s(0, \tau) = \vec{P}_s(\tau) = \vec{\gamma}(0)$ ,  $\frac{\partial \vec{Q}_s(0, \tau)}{\partial d} = \vec{v}$ ; further,  $\frac{\partial}{\partial \vec{P}} \vec{p}(\theta, \vec{\gamma}(0)) = \mathbf{X}(\theta)$  is the fundamental matrix of (48).

Now we proceed to evaluate  $T_d'(\tau) := \frac{\partial}{\partial d} T_1^f(d, \tau)|_{d=0}$ . Note that, by construction,  $[\mathbf{P}^+ \vec{p}(T_1^f(d, \tau), \vec{Q}_s(d, \tau))]^* \vec{v}_s^+ \equiv |\ln(\varpi)|^{-1}$  for any  $0 < d \leq \delta$  and any  $\tau \in \mathbb{R}$ . Hence differentiating we find

$$\begin{aligned} T_d'(\tau) &= - \frac{\left[ \mathbf{P}^+ \frac{\partial}{\partial \vec{P}} \vec{p}(T_1^f(0, \tau), \vec{P}_s(\tau)) \frac{\partial \vec{Q}_s(0, \tau)}{\partial d} \right]^* \vec{v}_s^+}{\left[ \mathbf{P}^+ \dot{\vec{p}}(T_1^f(0, \tau), \vec{P}_s(\tau)) \right]^* \vec{v}_s^+} \\ &= - \frac{[\mathbf{P}^+ \mathbf{X}(\tilde{T}_1^f) \vec{v}]^* \vec{v}_s^+}{[\mathbf{P}^+ \vec{f}^+(\vec{\gamma}(\tilde{T}_1^f))]^* \vec{v}_s^+}. \end{aligned} \quad (51)$$

Now, observing that

$$\tilde{p}(T_1^f(0, \tau), \tilde{P}_s(\tau)) = \tilde{\gamma}(\tilde{T}_1^f) = \frac{\tilde{v}_s^+}{|\ln(\varpi)|} + o\left(\frac{1}{|\ln(\varpi)|}\right)$$

we get

$$\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f)) = \mathbf{f}_x^+(\tilde{0})\tilde{\gamma}(\tilde{T}_1^f) + o(\tilde{\gamma}(\tilde{T}_1^f)) = \frac{\lambda_s^+}{|\ln(\varpi)|}\tilde{v}_s^+ + o\left(\frac{1}{|\ln(\varpi)|}\right)$$

whence

$$\begin{aligned} [\mathbf{P}^+ \tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))]^* \tilde{v}_s^+ &= \frac{\lambda_s^+}{|\ln(\varpi)|} + o\left(\frac{1}{|\ln(\varpi)|}\right) \quad \text{and} \\ \frac{\lambda}{2|\ln(\varpi)|} &\leq \|\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))\| \leq \frac{2\bar{\lambda}}{|\ln(\varpi)|}. \end{aligned} \quad (52)$$

Recall that  $\max\{\|\mathbf{P}^+\|; \|\mathbb{I} - \mathbf{P}^-\|\} \leq k_2$ ; then, using (49) and (52) in (51), we find

$$|T_d'(\tau)| \leq \frac{k_2}{\lambda} |\ln(\varpi)|^{8\frac{\bar{\lambda}}{\lambda} + 1}. \quad (53)$$

To proceed in the computation, it is convenient (cf. p. 377–388 in [22]) to consider the matrix representation  $\mathbf{Y}(\tilde{T}_1^f)$  of  $\mathbf{X}(\tilde{T}_1^f)$  with respect to the bases  $\left\{\frac{\tilde{f}^+(\tilde{\gamma}(0))}{\|\tilde{f}^+(\tilde{\gamma}(0))\|}, \tilde{v}\right\}$  in the domain and  $\left\{\frac{\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))}{\|\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))\|}, \tilde{v}_u^+\right\}$  in the codomain, i.e., to set  $\mathbf{Y}(\tilde{T}_1^f) = \mathbf{B}\mathbf{X}(\tilde{T}_1^f)\mathbf{A}$  where

$$\mathbf{A} := \left(\frac{\tilde{f}^+(\tilde{\gamma}(0))}{\|\tilde{f}^+(\tilde{\gamma}(0))\|}, \tilde{v}\right), \quad \mathbf{B} := \left(\frac{\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))}{\|\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))\|}, \tilde{v}_u^+\right)^{-1}.$$

Since  $\dot{\tilde{p}}(\theta, \tilde{\gamma}(0))$  is a solution of the variational equation (48) we get

$$\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f)) = \dot{\tilde{p}}(\tilde{T}_1^f, \tilde{\gamma}(0)) = \mathbf{X}(\tilde{T}_1^f)\dot{\tilde{p}}(0, \tilde{\gamma}(0)) = \mathbf{X}(\tilde{T}_1^f)\tilde{f}^+(\tilde{\gamma}(0))$$

or equivalently

$$\left(\frac{\|\tilde{f}^+(\tilde{\gamma}(\tilde{T}_1^f))\|}{\|\tilde{f}^+(\tilde{\gamma}(0))\|}, 0\right)^* = \mathbf{Y}(\tilde{T}_1^f) \cdot (1, 0)^*.$$

We recall that

$$-D_1^f(d, \tau)\tilde{v}_u^+ = \tilde{p}(T_1^f(d, \tau), \tilde{Q}_s(d, \tau)) - \tilde{\pi}_s(T_1^f(d, \tau) + \tau).$$

Notice that  $\tilde{\pi}_s(\tau) \equiv \tilde{\gamma}(\tilde{T}_1^f) \in \tilde{S}^+$  for any  $\tau$  when  $\varepsilon = 0$ , so  $\frac{d\tilde{\pi}_s}{dt}(\tau) \equiv \tilde{0}$  if  $\varepsilon = 0$ .

Now, recalling that

$$\dot{p}(T_1^f(d, \tau), \vec{Q}_s(d, \tau)) = \vec{f}^+(p(T_1^f(d, \tau), \vec{Q}_s(d, \tau)))$$

we find

$$\begin{aligned} -\frac{\partial D_1^f}{\partial d}(0, \tau) \vec{v}_u^+ &= \frac{\partial}{\partial \vec{P}} \bar{p}(T_1^f(0, \tau), \vec{P}_s(\tau)) \frac{\partial \vec{Q}_s(0, \tau)}{\partial d} \\ + \dot{p}(T_1^f(0, \tau), \vec{P}_s(\tau)) \frac{\partial T_1^f(0, \tau)}{\partial d} &= \mathbf{X}(\tilde{T}_1^f) \vec{v} + \vec{f}^+(\vec{\gamma}(\tilde{T}_1^f)) T_d'(\tau). \end{aligned}$$

Thus,  $\mathbf{Y}(\tilde{T}_1^f)$  is the matrix

$$\mathbf{Y}(T_1^f(0, \tau)) = \begin{pmatrix} \frac{\|\vec{f}^+(\vec{\gamma}(\tilde{T}_1^f))\|}{\|\vec{f}^+(\vec{\gamma}(0))\|} & -\|\vec{f}^+(\vec{\gamma}(\tilde{T}_1^f))\| T_d'(\tau) \\ 0 & -\frac{\partial D_1^f}{\partial d}(0, \tau) \end{pmatrix}. \quad (54)$$

Now let  $K_A$  and  $K_B$  equal the determinants respectively of  $\mathbf{A}$  and  $\mathbf{B}$ : since their columns are made up by unit vectors which are transversal, then  $|K_A| > 0$  and  $|K_B| > 0$ . Hence

$$\det \mathbf{X}(\tilde{T}_1^f) = \frac{\det \mathbf{Y}(\tilde{T}_1^f)}{K_A K_B} = -\frac{\partial D_1^f}{\partial d}(0, \tau) \frac{\|\vec{f}^+(\vec{\gamma}(\tilde{T}_1^f))\|}{\|\vec{f}^+(\vec{\gamma}(0))\| K_A K_B}. \quad (55)$$

So from (50) and (52) we find  $0 < c_1 < c_2$  such that

$$c_1 |\ln(\varpi)|^{-\frac{4M_f}{\lambda} + 1} \leq \left| \frac{\partial D_1^f}{\partial d}(0, \tau) \right| \leq c_2 |\ln(\varpi)|^{\frac{4M_f}{\lambda} + 1}.$$

So, taking into account (47) and setting  $\tilde{C} = \frac{4M_f}{\lambda} + 2$  we conclude the proof of (46).

Notice that (45) is independent of  $\delta$  and  $\tau$  so the estimate of  $T_4^b(d, \tau)$  is obtained from (45). Then the proof concerning  $D_4^b(d, \tau)$  is obtained from (46) reversing the role of  $d$  and  $D_1^f$ .  $\square$

We emphasize that we have proved also the following result, cf. (53) which allows us to improve (40).

REMARK 5.8. Let the assumptions of Proposition 5.7 be satisfied. Then

$$|T_d'(\tau)| := \left| \frac{\partial}{\partial d} T_1^f(0, \tau) \right| \leq \frac{k_2}{\lambda} |\ln(\varpi)|^{8\frac{\bar{\lambda}}{\lambda} + 1};$$

so, for any  $0 < d \leq \delta \leq \varpi$ , we have

$$|T_f^1(d, \tau) - T_1^f(0, \tau)| \leq \frac{2k_2}{\lambda} |\ln(\delta)|^{8\frac{\bar{\lambda}}{\lambda} + 1} d \leq d\delta^{-\mu_1/2}.$$

Analogously let us consider the trajectory  $\vec{x}(t, \tau; \vec{Q}_u(d, \tau))$  backward from  $L^0$  to  $\tilde{S}^-$ ; using again Lemma 3.6 we obtain the following.

LEMMA 5.9. *Let  $\varpi > 0$  be a small constant, then there is  $\delta = \delta(\varpi)$  such that for any  $0 < d \leq \delta$  and any  $\tau \in \mathbb{R}$  there are  $T_1^b(d, \tau) > 0$  and  $\vec{P}_b^-(d, \tau)$  such that  $\vec{x}(t, \tau; \vec{Q}_u(d, \tau)) \in K_A^{bwd, -}(t)$  for any  $\tau - T_1^b(d, \tau) \leq t < \tau$  and it crosses transversely  $S^-(\tau - T_1^b(d, \tau)) \subset \tilde{S}^-$  at  $t = \tau - T_1^b(d, \tau)$  in  $\vec{P}_b^-(d, \tau) = \vec{x}(\tau - T_1^b(d, \tau), \tau; \vec{Q}_u(d, \tau))$ .*

*Conversely, going forward in  $t$ , there are  $T_4^f(d, \tau) > 0$  and  $\vec{P}_f^-(d, \tau)$  such that  $\vec{x}(t, \tau; \vec{Q}^-(d, \tau)) \in K_A^{fwd, -}(t)$  for any  $\tau \leq t < \tau + T_4^f(d, \tau)$  and it crosses transversely  $L^0$  at  $t = \tau + T_4^f(d, \tau)$  in  $\vec{P}_f^-(d, \tau) = \vec{x}(\tau + T_4^f(d, \tau), \tau; \vec{Q}^-(d, \tau))$ .*

Then, arguing as in Proposition 5.7 we find

PROPOSITION 5.10. *Assume **H** and let  $\varpi > 0$  be a small constant. Then there is  $\delta = \delta(\varpi)$  such that for any  $0 < d \leq \delta$  we have*

$$\frac{\ln(|\ln(\varpi)|)}{4\lambda} < T_1^b(d, \tau) < \frac{4\ln(|\ln(\varpi)|)}{\lambda},$$

for any  $\tau \in \mathbb{R}$ . Further  $\vec{P}_b^-(d, \tau) = -D_1^b(d, \tau)\vec{v}_s^- + \vec{\pi}_u(\tau - T_1^b(d, \tau))$  where

$$0 < d|\ln(\varpi)|^{-\tilde{C}} \leq D_1^b(d, \tau) \leq d|\ln(\varpi)|^{\tilde{C}},$$

and  $\tilde{C} > 0$  is as in Proposition 5.7. Respectively,

$$\frac{\ln(|\ln(\varpi)|)}{4\lambda} < T_4^f(d, \tau) < \frac{4\ln(|\ln(\varpi)|)}{\lambda},$$

and  $D_4^f(d, \tau) = \mathcal{D}(\vec{P}_f^-(d, \tau), \vec{P}_u(\tau + T_4^f(d, \tau)))$  verifies

$$0 < d|\ln(\varpi)|^{-\tilde{C}} \leq D_4^f(d, \tau) \leq d|\ln(\varpi)|^{\tilde{C}}.$$

The proof is analogous to the one of Proposition 5.7 and it is omitted.

From an analysis of the proof of Proposition 5.7 we get the following.

LEMMA 5.11. *Assume **H** and fix  $0 < \mu_1 < 1/2$  and  $\tau \in \mathbb{R}$ , then there are  $\varpi = \varpi(\mu_1)$  and  $\delta = \delta(\mu_1, \varpi)$  such that*

$$\|\vec{x}(\theta + \tau, \tau; \vec{Q}_s(d, \tau)) - \vec{x}(\theta + \tau, \tau; \vec{P}_s(\tau))\| \leq d\delta^{-\mu_1},$$

$$\|\vec{x}(\phi + \tau, \tau; \vec{Q}^+(d, \tau)) - \vec{x}(\phi + \tau, \tau; \vec{\pi}_s(\tau))\| \leq d\delta^{-\mu_1}$$

for any  $-T_4^b(d, \tau) \leq \phi \leq 0 \leq \theta \leq T_1^f(d, \tau)$ , and any  $0 \leq d \leq \delta$ ; further

$$\|\vec{x}(\phi + \tau, \tau; \vec{Q}_u(d, \tau)) - \vec{x}(\phi + \tau, \tau; \vec{P}_u(\tau))\| \leq d\delta^{-\mu_1},$$

$$\|\vec{x}(\theta + \tau, \tau; \vec{Q}^-(d, \tau)) - \vec{x}(\theta + \tau, \tau; \vec{\pi}_u(\tau))\| \leq d\delta^{-\mu_1}$$

for any  $-T_1^b(d, \tau) \leq \phi \leq 0 \leq \theta \leq T_4^f(d, \tau)$ , and any  $0 \leq d \leq \delta$ .

## 5.2. The loop: from $S^+(\tau)$ to $L^{\text{in}}$

In this section we consider the fly time and the displacement of trajectories travelling from  $S^+(\tau)$  to  $L^{\text{in}}$ , see Figure 6. In this case we linearize centering in the origin and we decompose the solution as a sum of three terms: a “stable” nonlinear component, i.e., a trajectory of the stable manifold, a linear “unstable” component, i.e., a trajectory of the linearization (31) of (PS) in the origin, and a remainder.

Let us recall that  $\vec{Q}^\pm(d, \tau) \in S^\pm(\tau)$  are defined in (37), see also Figure 6.

In the whole section we assume that  $0 < \delta < \varpi$  are suitably small parameters, see Remark 5.15 (in fact for the results in this section we do not need to find a function  $\delta = \delta(\varpi)$  small enough: we just need to assume  $\delta < \varpi$ ).

**LEMMA 5.12.** *Assume  $\mathbf{H}$  and fix  $\tau \in \mathbb{R}$ ; for any  $0 < d \leq \delta$  there are  $C^r$  functions  $T_2^f(d, \tau)$  and  $D_2^f(d, \tau)$  such that  $\vec{x}(\theta + \tau, \tau; \vec{Q}^+(d, \tau)) \in K_A^{fwd,+}(\theta + \tau)$  for any  $0 \leq \theta < T_2^f(d, \tau)$  and it crosses transversely  $\Omega^0$  at  $\theta = T_2^f(d, \tau)$  in a point such that  $\|\vec{x}(T_2^f(d, \tau) + \tau, \tau; \vec{Q}^+(d, \tau))\| = D_2^f(d, \tau)$ .*

*Proof.* It follows from the argument of Lemma 3.6, taking into account the mutual positions of  $\vec{v}_u^\pm, \vec{v}_s^\pm$ .  $\square$

**REMARK 5.13.** Note that the mapping  $\Psi^+ : (d, \tau) \mapsto (D_2^f(d, \tau), T_2^f(d, \tau) + \tau)$  is injective on  $]0, \delta] \times \mathbb{R}$  due to the reversibility of the flow.

Let us recall that  $\vec{y}_s(\theta) = \vec{x}(\theta + \tau, \tau; \vec{\pi}_s(\tau))$ ; following [9, §4.2] we can estimate  $\vec{y}_s$  as indicated in the next remark.

**REMARK 5.14.** There are  $0 < \bar{c}_s < c_s/2$ ,  $\tilde{a}_u^+(\theta)$  and  $\tilde{a}_s^+(\theta)$  such that

$$\vec{y}_s(\theta) = \tilde{a}_s^+(\theta) \frac{z_s^+(\theta + \tau, \tau)}{|\ln(\varpi)|} \vec{v}_s^+(\theta + \tau) + \tilde{a}_u^+(\theta) \left( \frac{z_s^+(\theta + \tau, \tau)}{|\ln(\varpi)|} \right)^{1+\alpha} \vec{v}_u^+(\theta + \tau) \quad (56)$$

and  $\bar{c}_s < \tilde{a}_s^+(\theta) < c_s/2$ ,  $|\tilde{a}_u^+(\theta)| < c_s/2$  for any  $\theta > 0$ .

Since  $\varepsilon > 0$  is fixed in this section we leave this dependence unsaid, so we write  $\mathbf{X}^+(\theta)$  for the fundamental matrix of (31)<sub>+</sub> and  $\vec{F}^+(\vec{x}, \tau)$  instead of  $\vec{F}^+(\vec{x}, \tau, \varepsilon)$ . Denote by

$$\vec{\ell}^+(\theta) := -\mathbf{X}^+(\theta + \tau)(\mathbf{X}^+(\tau))^{-1} d\vec{v}_u^+(\tau) = -dz_u^+(\theta + \tau, \tau) \vec{v}_u^+(\theta + \tau). \quad (57)$$

Notice that by definition  $\dot{\vec{\ell}}^+(\theta) = \mathbf{F}_x^+(\vec{0}, \theta + \tau) \vec{\ell}^+(\theta)$ ,  $\vec{\ell}^+(0) = -d\vec{v}_u^+(\tau)$ .

Now, using (56) and (57), and taking into account Remark 5.2 we expand  $\vec{x}(\theta + \tau, \tau; \vec{Q}^+(d, \tau))$  as

$$\vec{x}(\theta + \tau, \tau; \vec{Q}^+(d, \tau)) = \vec{y}_s(\theta) + \vec{\ell}^+(\theta) + \vec{h}^+(\theta), \quad (58)$$

where  $\theta \geq 0$  and  $\vec{h}^+(\theta)$  is a remainder.

We want to write a fixed-point equation for the remainder, in a suitable exponentially weighted space. Notice first that  $\vec{h}^+(\theta)$  satisfies

$$\vec{h}^+(0) = \vec{Q}^+(d, \tau) - \vec{\pi}_s(\tau) + d\vec{v}_u^+(\tau) = \vec{0}$$

and the equation

$$\begin{aligned} \dot{\vec{h}}^+(\theta) &= \dot{\vec{x}}(\theta + \tau, \tau; \vec{Q}^+(d, \tau)) - \dot{\vec{y}}_s(\theta) - \dot{\vec{\ell}}^+(\theta) \\ &= \vec{F}^+(\vec{y}_s(\theta) + \vec{\ell}^+(\theta) + \vec{h}^+(\theta), \theta + \tau) \\ &\quad - \vec{F}^+(\vec{y}_s(\theta), \theta + \tau) - \mathbf{F}_x^+(\vec{0}, \theta + \tau)\vec{\ell}^+(\theta) \\ &= \mathbf{F}_x^+(\vec{0}, \theta + \tau)\vec{h}^+(\theta) + \vec{R}_1^+(\theta, \vec{h}^+(\theta)) + \vec{R}_2^+(\theta, \vec{h}^+(\theta)) \end{aligned}$$

where the remainder terms  $\vec{R}_1^+(\theta, \vec{h}^+(\theta))$  and  $\vec{R}_2^+(\theta, \vec{h}^+(\theta))$  equal

$$\begin{aligned} \vec{R}_1^+(\theta, \vec{h}) &:= \left[ \mathbf{F}_x^+(\vec{y}_s(\theta), \theta + \tau) - \mathbf{F}_x^+(\vec{0}, \theta + \tau) \right] (\vec{\ell}^+(\theta) + \vec{h}), \\ \vec{R}_2^+(\theta, \vec{h}) &:= \vec{F}^+(\vec{y}_s(\theta) + \vec{\ell}^+(\theta) + \vec{h}, \theta + \tau) - \vec{F}^+(\vec{y}_s(\theta), \theta + \tau) \\ &\quad - \mathbf{F}_x^+(\vec{y}_s(\theta), \theta + \tau)(\vec{\ell}^+(\theta) + \vec{h}). \end{aligned}$$

Let us set

$$D_0 := \frac{1}{|\ln(\delta)|^2}, \quad M_0 := \frac{2|\ln(\delta)|}{\lambda}. \quad (59)$$

Let now  $M \geq M_0$  be fixed, and  $X^+ = C([0, M], \mathbb{R}^2)$  be the Banach space of continuous functions equipped with the norm

$$\|\vec{u}\|_{X^+} := \max_{\theta \in [0, M]} \frac{\|\vec{u}(\theta)\|}{z_u^+(\theta + \tau, M + \tau)}.$$

Fix  $0 < D_\ell \leq D_0$  and set  $d = D_\ell/z_u^+(M + \tau, \tau)$ , so that from (35) we find

$$d \leq \frac{D_0}{z_u^+(M_0 + \tau, \tau)} \leq \frac{k_1 e^{-\frac{1}{2}\lambda M_0}}{|\ln(\delta)|^2} = \frac{k_1}{|\ln(\delta)|^2} \delta \leq \delta$$

since we can assume  $\delta \leq e^{-\sqrt{k_1}}$ , see Remark 5.1.

From (56) and (57), we see that

$$\begin{aligned} \|\vec{y}_s(\theta)\| &\leq \frac{c_s}{|\ln(\varpi)|} z_s^+(\theta + \tau, \tau) \leq \frac{c_s k_1}{|\ln(\varpi)|} e^{-\frac{1}{2}\lambda\theta}, \\ \|\vec{\ell}^+(\theta)\| &= dz_u^+(\theta + \tau, \tau) = dz_u^+(M + \tau, \tau)z_u^+(\theta + \tau, M + \tau) \\ &= D_\ell z_u^+(\theta + \tau, M + \tau) \leq \frac{k_1}{|\ln(\delta)|^2} e^{-\frac{1}{2}\lambda(M-\theta)}. \end{aligned} \quad (60)$$



Moreover, assumption **H** implies

$$\begin{aligned}
 \|\vec{R}_1^+(\theta, \vec{h})\| &\leq N_\alpha \|\vec{y}_s(\theta)\|^\alpha \cdot \|\vec{\ell}^+(\theta) + \vec{h}\|, \\
 \|\vec{R}_2^+(\theta, \vec{h})\| &= \left\| \int_0^1 \left[ \mathbf{F}_x^+(\vec{y}_s(\theta) + \sigma[\vec{\ell}^+(\theta) + \vec{h}], \theta + \tau) \right. \right. \\
 &\quad \left. \left. - \mathbf{F}_x^+(\vec{y}_s(\theta), \theta + \tau) \right] (\vec{\ell}^+(\theta) + \vec{h}) d\sigma \right\| \leq N_\alpha \|\vec{\ell}^+(\theta) + \vec{h}\|^{1+\alpha}.
 \end{aligned} \tag{61}$$

On the other hand,

$$\vec{R}_1^+(\theta, \vec{h}_2) - \vec{R}_1^+(\theta, \vec{h}_1) = \left[ \mathbf{F}_x^+(\vec{y}_s(\theta), \theta + \tau) - \mathbf{F}_x^+(\vec{0}, \theta + \tau) \right] (\vec{h}_2 - \vec{h}_1).$$

Further

$$\begin{aligned}
 &\vec{R}_2^+(\theta, \vec{h}_2) - \vec{R}_2^+(\theta, \vec{h}_1) \\
 &= \vec{F}^+(\vec{y}_s(\theta) + \vec{\ell}^+(\theta) + \vec{h}_2, \theta + \tau) \\
 &\quad - \vec{F}^+(\vec{y}_s(\theta) + \vec{\ell}^+(\theta) + \vec{h}_1, \theta + \tau) - \mathbf{F}_x^+(\vec{y}_s(\theta), \theta + \tau)(\vec{h}_2 - \vec{h}_1) \\
 &= \int_0^1 \left[ \mathbf{F}_x^+(\vec{y}_s(\theta) + \vec{\ell}^+(\theta) + \vec{h}_1 + \sigma(\vec{h}_2 - \vec{h}_1), \theta + \tau) \right. \\
 &\quad \left. - \mathbf{F}_x^+(\vec{y}_s(\theta), \theta + \tau) \right] (\vec{h}_2 - \vec{h}_1) d\sigma
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \|\vec{R}_1^+(\theta, \vec{h}_2) - \vec{R}_1^+(\theta, \vec{h}_1)\| &\leq N_\alpha \|\vec{y}_s(\theta)\|^\alpha \|\vec{h}_2 - \vec{h}_1\|, \\
 \|\vec{R}_2^+(\theta, \vec{h}_2) - \vec{R}_2^+(\theta, \vec{h}_1)\| & \\
 &\leq N_\alpha \left( \|\vec{\ell}^+(\theta)\| + \max\{\|\vec{h}_1\|, \|\vec{h}_2\|\} \right)^\alpha \|\vec{h}_2 - \vec{h}_1\|.
 \end{aligned} \tag{62}$$

Let us define the operator  $\mathcal{F}^+ : X^+ \rightarrow X^+$  as

$$\begin{aligned}
 \mathcal{F}^+(\vec{u})(\theta) &= \int_0^\theta \mathbf{X}^+(\theta + \tau)(\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_1^+(s, \vec{u}(s)) + \vec{R}_2^+(s, \vec{u}(s)) \right) ds \\
 &= \int_0^\theta \mathbf{X}^+(\theta + \tau) \mathbf{P}^+(\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_1^+(s, \vec{u}(s)) + \vec{R}_2^+(s, \vec{u}(s)) \right) ds \\
 &\quad + \int_0^\theta \mathbf{X}^+(\theta + \tau)(\mathbb{I} - \mathbf{P}^+)(\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_1^+(s, \vec{u}(s)) + \vec{R}_2^+(s, \vec{u}(s)) \right) ds.
 \end{aligned}$$

Notice that  $\vec{h}^+(\theta)$  is a fixed point of  $\mathcal{F}^+$  if and only if the function  $\vec{x}(\theta + \tau, \tau; \vec{Q}^+(d, \tau))$  given by (58) solves (PS) and  $\vec{h}^+(0) = \vec{0}$ .

REMARK 5.15. In the next Lemma 5.16 we let

$$\begin{aligned} \ln(\varpi) &\leq -\max\{(2C)^{2/\alpha}; 4\}, \\ \text{where } C &= (2c_s^\alpha + 1)N_\alpha \frac{2k_2k_1^{2+2\alpha}(\alpha + 1)}{\lambda\alpha}. \end{aligned} \quad (63)$$

Observe that this is not an optimal constant: however, the relevant fact is that both  $C$  and  $\varpi$  are independent of  $d$ ,  $\delta$  and  $\varepsilon$ .

LEMMA 5.16. Assume  $\mathbf{H}$ , fix  $\varpi > 0$  as in (63) let  $\delta = \delta(\varpi) < \varpi$  be as in Lemma 5.4 and Remark 5.9. With the above notation, for any  $0 < d \leq \delta$ , the fixed point equation  $\mathcal{F}^+(\vec{u}) = \vec{u}$  admits a unique solution  $\vec{h}$  in  $\overline{B}_r^{X^+} := \{\vec{v} \in X^+ \mid \|\vec{v}\|_{X^+} \leq r\}$  where  $r = D_\ell |\ln(\varpi)|^{-\alpha/2}$ .

*Proof.* First we show that  $\mathcal{F}^+$  maps  $\overline{B}_r^{X^+}$  into itself. For this purpose, notice that if  $\vec{u} \in \overline{B}_r^{X^+}$  then  $\|\vec{u}(\theta)\| \leq rz_u^+(\theta + \tau, M + \tau)$  for any  $\theta \in [0, M]$ . Now, using

$$\|\vec{\ell}^+(\theta) + \vec{u}(\theta)\| \leq (D_\ell + r)z_u^+(\theta + \tau, M + \tau) \leq 2r |\ln(\varpi)|^{\alpha/2} z_u^+(\theta + \tau, M + \tau),$$

from (60) and (61) we get

$$\begin{aligned} \|\vec{R}_1^+(\theta, \vec{u}(\theta))\| &\leq 2N_\alpha (c_s)^\alpha r |\ln(\varpi)|^{-\alpha/2} [z_s^+(\theta + \tau, \tau)]^\alpha z_u^+(\theta + \tau, M + \tau), \\ \|\vec{R}_2^+(\theta, \vec{u}(\theta))\| &\leq N_\alpha [2r |\ln(\varpi)|^{\alpha/2} z_u^+(\theta + \tau, M + \tau)]^{1+\alpha}. \end{aligned}$$

Moreover, from (35) we get the following estimates:

$$\begin{aligned} &\int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_s^+(s + \tau, \tau)]^\alpha z_u^+(s + \tau, M + \tau) ds \\ &\leq k_1^{1+\alpha} z_u^+(\theta + \tau, M + \tau) \int_0^\theta z_u^+(s + \tau, \theta + \tau) ds \\ &\leq \frac{2(k_1)^{2+\alpha}}{\lambda} z_u^+(\theta + \tau, M + \tau), \end{aligned}$$

$$\begin{aligned} &\int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, M + \tau)]^{1+\alpha} ds \\ &= [z_u^+(\theta + \tau, M + \tau)]^{1+\alpha} \int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, \theta + \tau)]^{1+\alpha} ds \\ &\leq k_1^{2+2\alpha} z_u^+(\theta + \tau, M + \tau) \int_0^\theta e^{-\frac{1}{2}\lambda(2+\alpha)(\theta-s)} ds \leq \frac{2k_1^{2+2\alpha}}{\lambda(2+\alpha)} z_u^+(\theta + \tau, M + \tau), \end{aligned}$$

$$\begin{aligned}
& \int_0^\theta z_u^+(\theta + \tau, s + \tau) [z_s^+(s + \tau, \tau)]^\alpha z_u^+(s + \tau, M + \tau) ds \\
&= z_u^+(\theta + \tau, M + \tau) \int_0^\theta [z_s^+(s + \tau, \tau)]^\alpha ds \\
&\leq \frac{2(k_1)^\alpha}{\lambda\alpha} z_u^+(\theta + \tau, M + \tau),
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^\theta z_u^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, M + \tau)]^{1+\alpha} ds \\
&= z_u^+(\theta + \tau, M + \tau) \int_0^\theta [z_u^+(s + \tau, M + \tau)]^\alpha ds \\
&\leq \frac{2(k_1)^\alpha}{\lambda\alpha} z_u^+(\theta + \tau, M + \tau).
\end{aligned}$$

Using the above, we get

$$\begin{aligned}
& \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau) \mathbf{P}^+(\mathbf{X}^+(s + \tau))^{-1} \vec{R}_1^+(s, \vec{u}(s)) ds \right\| \\
&\leq 2N_\alpha(c_s)^\alpha k_2 r |\ln(\varpi)|^{-\alpha/2} \\
&\quad \cdot \int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_s^+(s + \tau, \tau)]^\alpha z_u^+(s + \tau, M + \tau) ds \\
&\leq 2N_\alpha(c_s)^\alpha k_2 \frac{2k_1^{2+\alpha}}{\lambda} r |\ln(\varpi)|^{-\alpha/2} z_u^+(\theta + \tau, M + \tau),
\end{aligned}$$

$$\begin{aligned}
& \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau) \mathbf{P}^+(\mathbf{X}^+(s + \tau))^{-1} \vec{R}_2^+(s, \vec{u}(s)) ds \right\| \\
&\leq N_\alpha k_2 [2r |\ln(\varpi)|^{\alpha/2}]^{1+\alpha} \\
&\quad \cdot \int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, M + \tau)]^{1+\alpha} ds \\
&\leq N_\alpha k_2 \frac{2k_1^{2+2\alpha}}{\lambda(2+\alpha)} [2r |\ln(\varpi)|^{\alpha/2}]^{1+\alpha} z_u^+(\theta + \tau, M + \tau),
\end{aligned}$$

$$\begin{aligned}
& \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau)(\mathbb{I} - \mathbf{P}^+)(\mathbf{X}^+(s + \tau))^{-1} \vec{R}_1^+(s, \vec{u}(s)) ds \right\| \\
& \leq 2N_\alpha (c_s)^\alpha k_2 r |\ln(\varpi)|^{-\alpha/2} \\
& \quad \cdot \int_0^\theta z_u^+(\theta + \tau, s + \tau) [z_s^+(s + \tau, \tau)]^\alpha z_u^+(s + \tau, M + \tau) ds \\
& \leq 2N_\alpha (c_s)^\alpha k_2 \frac{2k_1^\alpha}{\lambda\alpha} r |\ln(\varpi)|^{-\alpha/2} z_u^+(\theta + \tau, M + \tau),
\end{aligned}$$

and

$$\begin{aligned}
& \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau)(\mathbb{I} - \mathbf{P}^+)(\mathbf{X}^+(s + \tau))^{-1} \vec{R}_2^+(s, \vec{u}(s)) ds \right\| \\
& \leq N_\alpha k_2 [2r |\ln(\varpi)|^{\alpha/2}]^{1+\alpha} \int_0^\theta z_u^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, M + \tau)]^{1+\alpha} ds \\
& \leq N_\alpha k_2 \frac{2k_1^\alpha}{\lambda\alpha} [2r |\ln(\varpi)|^{\alpha/2}]^{1+\alpha} z_u^+(\theta + \tau, M + \tau).
\end{aligned}$$

Since  $r |\ln(\varpi)|^{\alpha/2} = D_\ell \leq D_0 = |\ln \delta|^{-2} < |\ln(\varpi)|^{-2}$ , we have

$$\begin{aligned}
[2r |\ln(\varpi)|^{\alpha/2}]^{1+\alpha} &= r |\ln(\varpi)|^{-\alpha/2} 2^{1+\alpha} [r |\ln(\varpi)|^{\alpha/2} \cdot |\ln(\varpi)|]^\alpha \\
&\leq r |\ln(\varpi)|^{-\alpha/2} 2^{1+\alpha} |\ln(\varpi)|^{-\alpha} \ll r |\ln(\varpi)|^{-\alpha/2};
\end{aligned} \tag{64}$$

further

$$2r |\ln(\varpi)|^{\alpha/2} = 2D_\ell \leq 2|\ln(\varpi)|^{-2} \leq |\ln(\varpi)|^{-1/2}, \tag{65}$$

so in particular  $r |\ln(\varpi)|^{\alpha/2} < 1/2$ . Hence from the above estimates and Remark 5.15 it follows that

$$\|\mathcal{F}^+(\vec{u})\|_{X^+} \leq C |\ln(\varpi)|^{-\alpha/2} r \leq \frac{r}{2}.$$

Next, we show that  $\mathcal{F}^+$  is a contraction. First, by (60) and (62) we have the estimates:

$$\begin{aligned}
& \|\vec{R}_1^+(\theta, \vec{u}_1(\theta)) - \vec{R}_1^+(\theta, \vec{u}_2(\theta))\| \\
& \leq N_\alpha (c_s)^\alpha |\ln(\varpi)|^{-\alpha/2} [z_s^+(\theta + \tau, \tau)]^\alpha \|\vec{u}_1(\theta) - \vec{u}_2(\theta)\|, \\
& \|\vec{R}_2^+(\theta, \vec{u}_1(\theta)) - \vec{R}_2^+(\theta, \vec{u}_2(\theta))\| \\
& \leq N_\alpha [2r |\ln(\varpi)|^{\alpha/2}]^\alpha [z_u^+(\theta + \tau, M + \tau)]^\alpha \|\vec{u}_1(\theta) - \vec{u}_2(\theta)\|
\end{aligned}$$

for any  $\vec{u}_1, \vec{u}_2 \in \overline{B}_r^{X^+}$ . Consequently, using

$$\begin{aligned}
\|\vec{u}_1(s) - \vec{u}_2(s)\| &= z_u^+(s + \tau, M + \tau) \frac{\|\vec{u}_1(s) - \vec{u}_2(s)\|}{z_u^+(s + \tau, M + \tau)} \\
&\leq z_u^+(s + \tau, M + \tau) \|\vec{u}_1 - \vec{u}_2\|_{X^+}
\end{aligned}$$

for all  $s \in [0, M]$ , we derive

$$\begin{aligned} & \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau) \mathbf{P}^+(\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_1^+(s, \vec{u}_1(s)) ds - \vec{R}_1^+(s, \vec{u}_2(s)) \right) ds \right\| \\ & \leq \frac{k_2 N_\alpha (c_s)^\alpha}{|\ln(\varpi)|^{\alpha/2}} \int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_s^+(s + \tau, \tau)]^\alpha \|\vec{u}_1(s) - \vec{u}_2(s)\| ds \\ & \leq \frac{2k_1^{2+\alpha} k_2 N_\alpha (c_s)^\alpha}{\lambda} \|\vec{u}_1 - \vec{u}_2\|_{X+z_u^+(\theta + \tau, M + \tau)}, \end{aligned}$$

$$\begin{aligned} & \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau) \mathbf{P}^+(\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_2^+(s, \vec{u}_1(s)) ds - \vec{R}_2^+(s, \vec{u}_2(s)) \right) ds \right\| \\ & \leq k_2 N_\alpha [2r |\ln(\varpi)|^{\alpha/2}]^\alpha \\ & \quad \cdot \int_0^\theta z_s^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, M + \tau)]^\alpha \|\vec{u}_1(s) - \vec{u}_2(s)\| ds \\ & \leq \frac{2k_1^{2+2\alpha}}{\lambda(2 + \alpha)} k_2 N_\alpha [2r |\ln(\varpi)|^{\alpha/2}]^\alpha \|\vec{u}_1 - \vec{u}_2\|_{X+z_u^+(\theta + \tau, M + \tau)}, \end{aligned}$$

$$\begin{aligned} & \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau) (\mathbb{I} - \mathbf{P}^+) (\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_1^+(s, \vec{u}_1(s)) ds - \vec{R}_1^+(s, \vec{u}_2(s)) \right) ds \right\| \\ & \leq \frac{k_2 N_\alpha (c_s)^\alpha}{|\ln(\varpi)|^{\alpha/2}} \int_0^\theta z_u^+(\theta + \tau, s + \tau) [z_s^+(s + \tau, \tau)]^\alpha \|\vec{u}_1(s) - \vec{u}_2(s)\| ds \\ & \leq \frac{2k_1^\alpha k_2 N_\alpha (c_s)^\alpha}{\lambda \alpha} \|\vec{u}_1 - \vec{u}_2\|_{X+z_u^+(\theta + \tau, M + \tau)}, \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_0^\theta \mathbf{X}^+(\theta + \tau) (\mathbb{I} - \mathbf{P}^+) (\mathbf{X}^+(s + \tau))^{-1} \left( \vec{R}_2^+(s, \vec{u}_1(s)) ds - \vec{R}_2^+(s, \vec{u}_2(s)) \right) ds \right\| \\ & \leq k_2 N_\alpha [2r |\ln(\varpi)|^{\alpha/2}]^\alpha \\ & \quad \cdot \int_0^\theta z_u^+(\theta + \tau, s + \tau) [z_u^+(s + \tau, M + \tau)]^\alpha \|\vec{u}_1(s) - \vec{u}_2(s)\| ds \\ & \leq \frac{2k_1^\alpha}{\lambda \alpha} k_2 N_\alpha [2r |\ln(\varpi)|^{\alpha/2}]^\alpha \|\vec{u}_1 - \vec{u}_2\|_{X+z_u^+(\theta + \tau, M + \tau)}. \end{aligned}$$

Now using (65) and the previous four estimates, recalling Remark 5.15 and (65) we find

$$\|\mathcal{F}^+(\vec{u}_1) - \mathcal{F}^+(\vec{u}_2)\|_{X^+} \leq \frac{C}{|\ln(\varpi)|^{\alpha/2}} \|\vec{u}_1 - \vec{u}_2\|_{X^+} \leq \frac{1}{2} \|\vec{u}_1 - \vec{u}_2\|_{X^+}.$$

The proof is finished by the Banach fixed point theorem.  $\square$

REMARK 5.17. By fixing  $\delta > 0$ , the values of  $M_0$  and  $D_0$  are given. Then for any fixed  $M \geq M_0$ ,  $\vec{h} \in \overline{B}_r^{X^+}$  if and only if  $\|\vec{h}(\theta)\| \leq rz_u^+(\theta + \tau, M + \tau)$  for all  $\theta \in [0, M]$ . Here one may consider a sufficiently smooth extension of (PS+) beyond  $\Omega^0$  if needed. On the other hand, if  $0 < T < M_0$ , taking  $M \geq T + \frac{2}{\lambda} \ln k_1$  yields that if  $\vec{h} \in \overline{B}_r^{X^+}$  then

$$\begin{aligned} \|\vec{h}(\theta)\| &\leq rz_u^+(\theta + \tau, T + \tau)z_u^+(T + \tau, M + \tau) \\ &\leq rz_u^+(\theta + \tau, T + \tau)k_1 e^{-\frac{1}{2}\lambda(M-T)} \leq rz_u^+(\theta + \tau, T + \tau) \end{aligned}$$

for any  $\theta \in [0, T]$ .

Now, the trajectory  $\vec{x}(\theta + \tau, \tau; \vec{Q}^+(d, \tau))$  crosses the curve  $\Omega^0$  when its projection on the “stable” and on the “unstable” linear spaces are comparable. This idea allows us to estimate the crossing time and displacement: see Lemma 5.19 below.

Observe that there are  $K^\pm > 0$  such that the unit vector

$$\frac{\vec{v}_s^+ - K^+ \vec{v}_u^+}{\|\vec{v}_s^+ - K^+ \vec{v}_u^+\|} = \frac{\vec{v}_u^- - K^- \vec{v}_s^-}{\|\vec{v}_u^- - K^- \vec{v}_s^-\|} \quad (66)$$

is tangent to  $\Omega^0$  in the origin and aims towards  $E^{\text{in}}$ .

Hence the vector  $\rho(\vec{v}_s^+ - 2K^+ \vec{v}_u^+)$  points towards  $\Omega^-$  and  $\rho(\vec{v}_s^+ - \frac{1}{2}K^- \vec{v}_u^+)$  points towards  $\Omega^+$  for  $\rho > 0$  small enough, see Figure 2. Fix  $\delta < \varpi$  and let  $0 < d < \delta$ ; then

$$\begin{aligned} \rho(\vec{v}_s^+(\theta) - 2K^+ \vec{v}_u^+(\theta)) &\in \Omega^-, \\ \rho\left(\vec{v}_s^+(\theta) - \frac{1}{2}K^+ \vec{v}_u^+(\theta)\right) &\in \Omega^+, \quad \text{for any } \theta \in \mathbb{R}, \end{aligned}$$

for  $\rho > 0$  small enough. Hence there is  $\tilde{K}^+ = \tilde{K}^+(d, \tau) \in [\frac{1}{2}K^+, 2K^+]$  such that  $\vec{x}(T_2^f + \tau, \tau; \vec{Q}^+(d, \tau)) \in \Omega^0$ , where  $T_2^f = T_2^f(d, \tau)$ , verifies

$$\vec{x}(T_2^f + \tau, \tau; \vec{Q}^+(d, \tau)) \in \text{span}\{\vec{v}_s^+ - \tilde{K}^+ \vec{v}_u^+\},$$

i.e., there is  $\tilde{\rho}^+ = \tilde{\rho}^+(T_2^f + \tau, d) > 0$  such that

$$\begin{aligned} \vec{\ell}^+(T_2^f) + \vec{y}_s(T_2^f) + \vec{h}^+(T_2^f) \\ = \tilde{\rho}^+ \{\vec{v}_s^+(T_2^f + \tau) - \tilde{K}^+ \vec{v}_u^+(T_2^f + \tau)\}. \end{aligned} \quad (67)$$

REMARK 5.18. In the next Proposition 5.19 we fix  $\mu_0$  such that (14) holds, then for  $\mu_2 \in ]0, \mu_0/2]$  we let  $\varpi = \varpi(\mu_2)$  be such that the following estimates hold true:

$$|\ln(\varpi)| > K^+ c_s k_1^2, \quad \frac{k_1^2 |\ln(\varpi)|}{K^+ c_s} < \varpi^{-\mu_2/4}. \quad (68)$$

PROPOSITION 5.19. *Assume  $\mathbf{H}$ ; let  $\mu_2 \in ]0, \mu_0/2]$  where  $\mu_0$  satisfies (14) and choose  $\varpi = \varpi(\mu_2)$  satisfying both (63) and (68). Then there is  $\delta = \delta(\mu_2, \varpi) > 0$  such that for any  $0 < d \leq \delta$  and any  $\tau \in \mathbb{R}$  we find*

$$\Sigma_+^{fwd} \left(1 - \frac{\mu_2}{3}\right) |\ln(d)| \leq T_2^f(d, \tau) \leq \Sigma_+^{fwd} \left(1 + \frac{\mu_2}{3}\right) |\ln(d)| \quad (69)$$

and

$$d^{\sigma_+^{fwd} \left(1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)} \leq D_2^f(d, \tau) \leq d^{\sigma_+^{fwd} \left(1 - \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)} \quad (70)$$

where

$$D_2^f(d, \tau) := \|\vec{x}(T_2^f(d, \tau) + \tau, \tau; \vec{Q}^+(d, \tau))\|. \quad (71)$$

*Proof.* Let us fix  $\tau \in \mathbb{R}$ ; we choose  $\varpi$  and  $\delta < \varpi$  so that Lemma 5.16 holds and we set  $M = T_2^f(d, \tau)$ , so that for any  $0 < d < \delta$  we find  $D_\ell(d, \tau) := dz_u^+(T_2^f(d, \tau) + \tau, \tau)$ . In this proof we write  $\vec{Q}^+$ ,  $T_2^f$  and  $D^+$  for  $T_2^f(d, \tau)$ ,  $\vec{Q}^+(d, \tau)$  and  $D_2^f(d, \tau)$  to deal with less cumbersome notation. Notice that  $T_2^f \geq M_0$  (cf. (59)) so that the expansion (58) and Lemma 5.16 hold for  $0 < t \leq T_2^f$ . Let  $h_u^+(T_2^f)$  and  $h_s^+(T_2^f)$  be such that

$$\vec{h}^+(T_2^f) = h_u^+(T_2^f) \vec{v}_u^+(T_2^f + \tau) + h_s^+(T_2^f) \vec{v}_s^+(T_2^f + \tau).$$

Using (56) and (57), from (67) we see that

$$\begin{aligned} & -\frac{1}{\tilde{K}^+(T_2^f + \tau)} \left[ -dz_u^+(T_2^f + \tau, \tau) + \tilde{a}_u^+(T_2^f) \left( \frac{z_s^+(T_2^f + \tau, \tau)}{|\ln(\varpi)|} \right)^{1+\alpha} + h_u^+(T_2^f) \right] \\ & = \tilde{\rho}^+(T_2^f + \tau, d) = \tilde{a}_s^+(T_2^f) \frac{z_s^+(T_2^f + \tau, \tau)}{|\ln(\varpi)|} + h_s^+(T_2^f). \end{aligned} \quad (72)$$

We want to compare the highest order terms of (72). First note that

$\tilde{a}_u^+(T_2^f) \left( \frac{z_s^+(T_2^f + \tau, \tau)}{|\ln(\varpi)|} \right)^{1+\alpha}$  is negligible with respect to  $\tilde{a}_s^+(T_2^f) \frac{z_s^+(T_2^f + \tau, \tau)}{|\ln(\varpi)|}$ . Moreover,  $|h_s^+(T_2^f)| = \|\mathbf{P}^{+, \varepsilon}(T_2^f + \tau) \vec{h}^+(T_2^f)\| \leq k_2 \|\vec{h}^+(T_2^f)\|$  and  $|h_u^+(T_2^f)| = \|(\mathbb{I} - \mathbf{P}^{+, \varepsilon}(T_2^f + \tau)) \vec{h}^+(T_2^f)\| \leq k_2 \|\vec{h}^+(T_2^f)\|$ , and using Lemma 5.16 we get

$$\|\vec{h}^+(T_2^f)\| \leq \frac{D_\ell z_u^+(T_2^f + \tau, T_2^f + \tau)}{|\ln(\varpi)|^{\alpha/2}} = \frac{D_\ell}{|\ln(\varpi)|^{\alpha/2}} \ll D_\ell.$$

Hence also  $h_s^+(T_2^f)$  and  $h_u^+(T_2^f)$  are negligible with respect to  $D_\ell = dz_u^+(T_2^f + \tau, \tau)$ .

Thus the highest order terms in (72) are  $\frac{D_\ell}{\tilde{K}^+(T_2^f + \tau)}$  and  $\tilde{a}_s^+(T_2^f) \frac{z_s^+(T_2^f + \tau, \tau)}{|\ln(\varpi)|}$ .

So, picking up the highest order terms and neglecting the others, we can find a constant  $\tilde{C}_K \in \left[ \frac{K^+ \tilde{c}_s}{2}, K^+ c_s \right]$  (cf. (56)) such that

$$D_\ell(d, \tau) = dz_u^+(T_2^f + \tau, \tau) = \tilde{C}_K \frac{z_s^+(T_2^f + \tau, \tau)}{|\ln(\varpi)|}. \quad (73)$$

Now using

$$\frac{1}{k_1^2} e^{(\lambda_u^+ + |\lambda_s^+| - 2k\varepsilon)T_2^f} \leq \frac{z_u^+(T_2^f + \tau, \tau)}{z_s^+(T_2^f + \tau, \tau)} = \frac{\tilde{C}_K}{d|\ln(\varpi)|} \leq k_1^2 e^{(\lambda_u^+ + |\lambda_s^+| + 2k\varepsilon)T_2^f}$$

we get

$$\begin{aligned} \frac{\ln\left(\frac{\tilde{C}_K}{k_1^2 d |\ln \varpi|}\right)}{\lambda_u^+ + |\lambda_s^+| + 2k\varepsilon} &\leq T_2^f \leq \frac{\ln\left(\frac{\tilde{C}_K k_1^2}{d |\ln(\varpi)|}\right)}{\lambda_u^+ + |\lambda_s^+| - 2k\varepsilon}, \\ \ln\left(\frac{\tilde{C}_K k_1^2}{d |\ln(\varpi)|}\right) &= |\ln(d)| + \ln(\tilde{C}_K k_1^2) - \ln(|\ln(\varpi)|) < |\ln(d)| \end{aligned} \quad (74)$$

by the first inequality in (68); further from the second inequality in (68) we get

$$\ln\left(\frac{\tilde{C}_K}{k_1^2 d |\ln \varpi|}\right) \geq -\ln(d\varpi^{-\mu_2/4}) \geq \left(1 - \frac{\mu_2}{4}\right) |\ln(d)|.$$

So plugging these inequalities in (74) we get

$$\begin{aligned} \frac{1 - \mu_2/3}{\lambda_u^+ + |\lambda_s^+|} |\ln(d)| &\leq \frac{(1 - \mu_2/4)(1 - c_1\varepsilon)}{\lambda_u^+ + |\lambda_s^+|} |\ln(d)| \\ &\leq \frac{1 - \mu_2/4}{\lambda_u^+ + |\lambda_s^+| + 2k\varepsilon} |\ln(d)| \leq T_2^f \leq \frac{1}{\lambda_u^+ + |\lambda_s^+| - 2k\varepsilon} |\ln(d)| \\ &\leq \frac{1 + c_1\varepsilon}{\lambda_u^+ + |\lambda_s^+|} |\ln(d)| \leq \frac{1 + \mu_2/3}{\lambda_u^+ + |\lambda_s^+|} |\ln(d)| \end{aligned} \quad (75)$$

where  $c_1 = 4k/(|\lambda_s^+| + \lambda_u^+)$ ; so (69) is proved.

Then plugging (69) in (73) we find

$$\begin{aligned} D_\ell(d, \tau) &\leq k_1 d \left( e^{T_2^f} \right)^{\lambda_u^+ + k\varepsilon} \leq k_1 d^{1 - \left( \frac{1 + \mu_2/3}{\lambda_u^+ + |\lambda_s^+|} \right) (\lambda_u^+ + k\varepsilon)} \\ &= k_1 d^{\sigma_+^{\text{fwd}} - \left( \frac{\mu_2 \lambda_u^+}{3(\lambda_u^+ + |\lambda_s^+|)} \right) - \left( \frac{k\varepsilon(1 + \mu_2/3)}{\lambda_u^+ + |\lambda_s^+|} \right)} \\ &\leq d^{\sigma_+^{\text{fwd}} - \frac{\mu_2 \lambda_u^+}{2(\lambda_u^+ + |\lambda_s^+|)}} = d^{\sigma_+^{\text{fwd}} \left( 1 - \frac{\mu_2 \lambda_u^+}{2|\lambda_s^+|} \right)}. \end{aligned}$$



Similarly, plugging again (69) in (73),

$$\begin{aligned}
 D_\ell(d, \tau) &\geq \frac{d}{k_1} \left( e^{T_2^f} \right)^{\lambda_u^+ - k\varepsilon} \geq \frac{d^{1 - \left( \frac{1 - \mu_2/3}{\lambda_u^+ + |\lambda_s^+|} \right) (\lambda_u^+ - k\varepsilon)}}{k_1} \\
 &= \frac{d^{\sigma_+^{\text{fwd}} + \left( \frac{\lambda_u^+ \mu_2}{3(\lambda_u^+ + |\lambda_s^+|)} \right) + \left( \frac{k\varepsilon(1 - \mu_2/3)}{\lambda_u^+ + |\lambda_s^+|} \right)}}{k_1} \\
 &\geq d^{\sigma_+^{\text{fwd}} + \frac{\mu_2 \lambda_u^+}{2(\lambda_u^+ + |\lambda_s^+|)}} = d^{\sigma_+^{\text{fwd}} \left( 1 + \frac{\mu_2 \lambda_u^+}{2|\lambda_s^+|} \right)}.
 \end{aligned}$$

Summing up we have shown that

$$d^{\sigma_+^{\text{fwd}} \left( 1 + \frac{\mu_2 \lambda_u^+}{2|\lambda_s^+|} \right)} \leq D_\ell(d, \tau) \leq d^{\sigma_+^{\text{fwd}} \left( 1 - \frac{\mu_2 \lambda_u^+}{2|\lambda_s^+|} \right)}. \quad (76)$$

Now, again from (72) we see that there are  $\bar{c}_2 > \bar{c}_1 > 0$  (independent of  $d$  and  $\tau$ ) such that

$$\bar{c}_1 D_\ell(d, \tau) \leq \tilde{\rho}^+(T_2^f + \tau, d) \leq \bar{c}_2 D_\ell(d, \tau);$$

further from (67) and (71) we see that there are  $\bar{c}_4 > \bar{c}_3 > 0$  such that

$$\bar{c}_3 \tilde{\rho}^+(T_2^f + \tau, d) \leq D_2^f(d, \tau) \leq \bar{c}_4 \tilde{\rho}^+(T_2^f + \tau, d).$$

So, these last two inequalities imply that there are  $C > c > 0$ , independent of  $d, \tau, \varepsilon$  and  $\mu_2$ , such that

$$cD_\ell(d, \tau) \leq D_2^f(d, \tau) \leq CD_\ell(d, \tau). \quad (77)$$

Now assuming that  $\delta^{\frac{\mu_2 \lambda_u^+}{2|\lambda_s^+|} \sigma_+^{\text{fwd}}} \leq \min\{C^{-1}, c\}$ , the proof of (70) follows from (76) and (77). This completes the proof of Proposition 5.19.  $\square$

For future reference it is convenient to state the following remark concerning estimate (77).

REMARK 5.20. There are  $C > c > 0$  independent of  $d, \tau, \varepsilon$  and  $\mu_2$  such that

$$cD_\ell(d, \tau) \leq D_2^f(d, \tau) \leq CD_\ell(d, \tau).$$

Now we want to prove a result which is, to some extent, the converse of Proposition 5.19. Let us set  $\delta_1 = \delta^{\sigma_+^{\text{fwd}} \left( 1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|} \right)}$  and consider the set  $L^{\text{in}}(\delta_1)$ . For any  $0 < D < \delta_1$  we denote by  $\vec{R}(D)$  the unique point in  $L^{\text{in}}(\delta_1)$  such that  $\|\vec{R}(D)\| = D$ .

Then, adapting slightly Lemma 3.6, we prove the following.

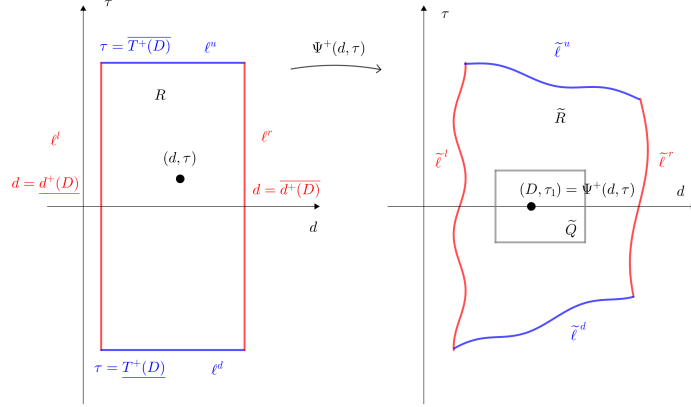


Figure 7: An explanation of the proof of Proposition 5.22.

REMARK 5.21. Fix  $\tau_1 \in \mathbb{R}$ ,  $0 < D < \delta_1 = \delta^{\sigma_{+}^{\text{fwd}}}\left(1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)$  and follow the trajectory  $\vec{x}(t, \tau_1; \vec{R}(D))$  backward in time. Then there are  $D_3^b(D, \tau_1) > 0$ ,  $T_3^b(D, \tau_1) > 0$  and  $\tau_0^+(D, \tau_1) = \tau_1 - T_3^b(D, \tau_1)$  such that  $\vec{x}(t, \tau_1; \vec{R}(D)) \in K_A^{\text{bwd},+}(t)$  for any  $\tau_0^+(D, \tau_1) \leq t < \tau_1$  and it crosses transversely  $\hat{S}^+(\tau_0^+(D, \tau_1))$  in the point  $\tilde{Q}^+(D_3^b(D, \tau_1), \tau_0^+(D, \tau_1))$ , where  $\tilde{Q}^+(\cdot, \cdot)$  is as defined in (37).

Using the uniformity in  $\tau$  of the estimates in (5.19) we show the following.

PROPOSITION 5.22. Assume **H**; let  $\mu_2 \in ]0, \mu_0/2]$  where  $\mu_0$  satisfies (14) and choose  $\varpi = \varpi(\mu_2)$  satisfying both (63) and (68). Then there is  $\delta = \delta(\mu_2, \varpi) > 0$  such that for any  $0 < D < \delta_1 = \delta^{\sigma_{+}^{\text{fwd}}}\left(1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)$  and any  $\tau_1 \in \mathbb{R}$  the functions  $T_3^b(D, \tau_1)$  and  $D_3^b(D, \tau_1)$  are well defined,  $C^r$  and one-to-one. Further

$$\begin{aligned} \frac{1}{|\lambda_s^+|} \left[ 1 - \mu_2 \left( 2 \frac{\lambda_u^+}{|\lambda_s^+|} + \frac{1}{2} \right) \right] |\ln(D)| \\ \leq T_3^b(D, \tau_1) \leq \frac{1}{|\lambda_s^+|} \left[ 1 + \mu_2 \left( 2 \frac{\lambda_u^+}{|\lambda_s^+|} + \frac{1}{2} \right) \right] |\ln(D)|, \\ D^{\frac{1}{\sigma_{+}^{\text{fwd}}}\left(1 + 2 \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)} \leq D_3^b(D, \tau_1) \leq D^{\frac{1}{\sigma_{+}^{\text{fwd}}}\left(1 - 2 \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)}. \end{aligned}$$

In particular  $\lim_{D \rightarrow 0} D_3^b(D, \tau_1) = 0$  uniformly in  $\tau_1 \in \mathbb{R}$ .

*Proof.* The proof is easier to follow while keeping in mind Figure 7. From

Remark 5.21 we see that the functions  $T_3^b(D, \tau_1)$  and  $D_3^b(D, \tau_1)$  are well defined; further they are  $C^r$  and one-to-one due to the smoothness of the flow, see Remark 3.8.

Let us consider the following function, as in Remark 5.13:

$$\Psi^+(d, \tau) : ]0, \delta] \times \mathbb{R} \rightarrow ]0, 1] \times \mathbb{R}, \quad \Psi^+(d, \tau) = (D_2^f(d, \tau), T_2^f(d, \tau) + \tau).$$

• We claim that  $(]0, \delta_1] \times \mathbb{R}) \subset \Psi^+(]0, \delta] \times \mathbb{R})$ .

Let  $(D, \tau_1) \in (]0, \delta_1] \times \mathbb{R})$ : we need to show that there is  $(d, \tau) \in (]0, \delta] \times \mathbb{R})$  such that  $\Psi^+(d, \tau) = (D, \tau_1)$ . We denote by  $d_0 = D^{1/\sigma_+^{\text{fwd}}}$ ; then we set

$$\begin{aligned} \underline{d}^+(D) &= D^{\frac{1}{\sigma_+^{\text{fwd}}}} \left( 1 + \frac{3\mu_2 \lambda_u^+}{|\lambda_s^+|} \right) = d_0^{1 + \frac{3\mu_2 \lambda_u^+}{|\lambda_s^+|}}, \\ \overline{d}^+(D) &= D^{\frac{1}{\sigma_+^{\text{fwd}}}} \left( 1 - \frac{3\mu_2 \lambda_u^+}{|\lambda_s^+|} \right) = d_0^{1 - \frac{3\mu_2 \lambda_u^+}{|\lambda_s^+|}}, \\ \underline{T}^+(D) &= \tau_1 - \left[ \Sigma_+^{\text{fwd}} \left( 1 + \frac{\mu_2}{3} \right) \left( 1 + \frac{3\mu_2 \lambda_u^+}{|\lambda_s^+|} \right) + 2\sigma_+^{\text{fwd}} \right] |\ln(d_0)|, \\ \overline{T}^+(D) &= \tau_1 - \left[ \Sigma_+^{\text{fwd}} \left( 1 - \frac{\mu_2}{3} \right) \left( 1 - \frac{3\mu_2 \lambda_u^+}{|\lambda_s^+|} \right) - 2\sigma_+^{\text{fwd}} \right] |\ln(d_0)|. \end{aligned}$$

Then consider the rectangle

$$\mathcal{R} = ([\underline{d}^+(D), \overline{d}^+(D)] \times [\underline{T}^+(D), \overline{T}^+(D)]) \subset ]0, 1/10] \times \mathbb{R},$$

and denote by  $\partial\mathcal{R}$  its border. Further set

$$\begin{aligned} \ell^l &= \{\underline{d}^+(D)\} \times [\underline{T}^+(D), \overline{T}^+(D)], \\ \ell^r &= \{\overline{d}^+(D)\} \times [\underline{T}^+(D), \overline{T}^+(D)], \\ \ell^d &= [\underline{d}^+(D), \overline{d}^+(D)] \times \{\underline{T}^+(D)\}, \\ \ell^u &= [\underline{d}^+(D), \overline{d}^+(D)] \times \{\overline{T}^+(D)\}, \end{aligned}$$

so that  $\partial\mathcal{R} = (\ell^l \cup \ell^d \cup \ell^r \cup \ell^u)$ .

Now observe that the restriction  $\tilde{\Psi}^+$  of  $\Psi^+$  defined by  $\tilde{\Psi}^+ : \mathcal{R} \rightarrow \tilde{\mathcal{R}}$  where  $\tilde{\mathcal{R}} = \Psi^+(\mathcal{R})$ , is a  $C^r$  diffeomorphism, see Remark 3.8. Hence, from Brouwer's domain invariance theorem (cf. [33, Proposition 16.9]), we see that  $\tilde{\mathcal{R}}$  is a compact and simply connected set and its border  $\partial\tilde{\mathcal{R}}$  satisfies  $\partial\tilde{\mathcal{R}} = \Psi(\partial\mathcal{R})$ .

Let  $\tilde{\ell}^l = \Psi(\ell^l)$  and similarly for  $\tilde{\ell}^r, \tilde{\ell}^d, \tilde{\ell}^u$ : all these sets are compact images of curves. Let

$$\tilde{\mathcal{Q}} = \left[ D^{1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}}, D^{1 - \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}} \right] \times [\tau_1 - |\ln(D)|, \tau_1 + |\ln(D)|]$$

and denote by  $\partial\tilde{Q}$  its border: obviously  $(D, \tau_1) \in \tilde{Q}$ .

Let  $\tau', \tau'' \in [\underline{T}^+(D), \overline{T}^+(D)]$  and consider

$$(\underline{d}^+(D), \tau') \in \ell^l \text{ and } (\overline{d}^+(D), \tau'') \in \ell^r ;$$

from Proposition 5.19 we find

$$\begin{aligned} D_2^f(\underline{d}^+(D), \tau') &\leq [\underline{d}^+(D)]^{\sigma_+^{\text{fwd}}\left(1 - \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right)} = D^{\left(1 - \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right)} \left(1 + 3\mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right) \\ &= D^{1 + 2\mu_2 \frac{\lambda_u^+}{|\lambda_s^+|} - 3\mu_2^2 \frac{(\lambda_u^+)^2}{(\lambda_s^+)^2}} < D^{1 + \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}}, \end{aligned}$$

$$\begin{aligned} D_2^f(\overline{d}^+(D), \tau'') &\geq [\overline{d}^+(D)]^{\sigma_+^{\text{fwd}}\left(1 + \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right)} = D^{\left(1 + \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right)} \left(1 - 3\mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right) \\ &= D^{1 - 2\mu_2 \frac{\lambda_u^+}{|\lambda_s^+|} - 3\mu_2^2 \frac{(\lambda_u^+)^2}{(\lambda_s^+)^2}} > D^{1 - \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}}. \end{aligned}$$

Whence  $\tilde{\ell}^l \cap \tilde{Q} = \emptyset = \tilde{\ell}^r \cap \tilde{Q}$ .

Similarly observe that for any  $d', d'' \in [\underline{d}^+(D), \overline{d}^+(D)]$  (so that  $(d', \underline{T}^+(D)) \in \ell^d$  and  $(d'', \overline{T}^+(D)) \in \ell^u$ ), from Proposition 5.19 we find

$$\begin{aligned} T_2^f(d', \underline{T}^+(D)) + \underline{T}^+(D) &\leq \underline{T}^+(D) + \Sigma_+^{\text{fwd}}\left(1 + \frac{\mu_2}{3}\right) |\ln(\underline{d}^+(D))| \\ &= \tau_1 - 2\sigma_+^{\text{fwd}} |\ln(d_0)| < \tau_1 - \sigma_+^{\text{fwd}} |\ln(d_0)| = \tau_1 - |\ln(D)|, \end{aligned}$$

$$\begin{aligned} T_2^f(d'', \overline{T}^+(D)) + \overline{T}^+(D) &\geq \overline{T}^+(D) + \Sigma_+^{\text{fwd}}\left(1 - \frac{\mu_2}{3}\right) |\ln(\overline{d}^+(D))| \\ &= \tau_1 + 2\sigma_+^{\text{fwd}} |\ln(d_0)| > \tau_1 + \sigma_+^{\text{fwd}} |\ln(d_0)| = \tau_1 + |\ln(D)|. \end{aligned}$$

Whence  $\tilde{\ell}^d \cap \tilde{Q} = \emptyset = \tilde{\ell}^u \cap \tilde{Q}$ .

Summing up we have shown that  $(\tilde{Q} \cap \partial\tilde{\mathcal{R}}) = \emptyset$ ; in fact we easily see that  $\tilde{Q}$  is contained in the bounded set enclosed by  $\partial\tilde{\mathcal{R}} = \Psi(\partial\mathcal{R})$ . So we see that there is  $(d, \tau) \in \mathcal{R}$  such that  $\Psi^+(d, \tau) = (D, \tau_1)$  and the claim is proved, by the arbitrariness of  $(D, \tau_1) \in ]0, \delta_1] \times \mathbb{R}$ .

Now from the claim we see that for any  $(D, \tau_1) \in ]0, \delta_1] \times \mathbb{R}$  there is a unique couple  $(d, \tau_0) \in ]0, \delta] \times \mathbb{R}$  such that  $(D, \tau_1) = \Psi^+(d, \tau_0)$ , since  $\Psi^+$  is defined through the flow of (PS).

Hence from Proposition 5.19 we get

$$d^{\sigma_+^{\text{fwd}}\left(1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)} \leq D \leq d^{\sigma_+^{\text{fwd}}\left(1 - \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)}$$

from which we get

$$\begin{aligned} D^{\frac{1}{\sigma_+^{\text{fwd}}}} \left(1 + \frac{2\mu_2 \lambda_u^+}{|\lambda_s^+|}\right) &\leq D^{\frac{1}{\sigma_+^{\text{fwd}}}} \left(1 - \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)^{-1} \\ &\leq d \leq D^{\frac{1}{\sigma_+^{\text{fwd}}}} \left(1 + \frac{\mu_2 \lambda_u^+}{|\lambda_s^+|}\right)^{-1} \leq D^{\frac{1}{\sigma_+^{\text{fwd}}}} \left(1 - \frac{2\mu_2 \lambda_u^+}{|\lambda_s^+|}\right). \end{aligned}$$

So the assertion concerning  $D$  is proved.

Again from Proposition 5.19 we find

$$\Sigma_+^{\text{fwd}} \left(1 - \frac{\mu_2}{3}\right) |\ln(d)| \leq \tau_1 - \tau = T_3^b(D, \tau_1) = T_2^f(d, \tau) \leq \Sigma_+^{\text{fwd}} \left(1 + \frac{\mu_2}{3}\right) |\ln(d)|.$$

So we conclude

$$\begin{aligned} T_3^b(D, \tau_1) &\leq \frac{\Sigma_+^{\text{fwd}}}{\sigma_+^{\text{fwd}}} \left(1 + \frac{\mu_2}{3}\right) \left(1 + 2\mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right) |\ln(D)| \\ &\leq \frac{1}{|\lambda_s^+|} \left[1 + \mu_2 \left(2 \frac{\lambda_u^+}{|\lambda_s^+|} + \frac{1}{2}\right)\right] |\ln(D)|, \\ T_3^b(D, \tau_1) &\geq \frac{\Sigma_+^{\text{fwd}}}{\sigma_+^{\text{fwd}}} \left(1 - \frac{\mu_2}{3}\right) \left(1 - 2\mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right) |\ln(D)| \\ &\geq \frac{1}{|\lambda_s^+|} \left[1 - \mu_2 \left(2 \frac{\lambda_u^+}{|\lambda_s^+|} + \frac{1}{2}\right)\right] |\ln(D)|. \end{aligned}$$

Then the assertion concerning  $T_3^b(D, \tau_1)$  is proved.  $\square$

LEMMA 5.23. Assume  $\mathbf{H}$  and let  $\mu_2$ ,  $\varpi$  and  $\delta = \delta(\mu_2, \varpi)$  be as in Proposition 5.19. Let  $0 < d \leq \delta$ ,  $\tau_0 \in \mathbb{R}$ , and consider the trajectory

$$\vec{x}(t, \tau_0; \vec{Q}^+(d, \tau_0)) \equiv x(t, \tau_1; \vec{R}(D)) \quad \tau_0 \leq t \leq \tau_1$$

where  $D = D_2^f(d, \tau_0)$ ,  $\vec{R}(D)$  is the point of  $L^{in}$  such that  $\|\vec{R}(D)\| = D \in ]0, \delta_1[$  and  $\tau_1 = \tau_0 + T_2^f(d, \tau_0)$ . Then

$$\begin{aligned} \|\vec{x}(t, \tau_0; \vec{Q}^+(d, \tau_0)) - \vec{y}_s(t - \tau_0)\| \\ \leq \frac{2}{c} d^{\sigma_+^{\text{fwd}}} \left(1 - \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right) z_u^+(t, \tau_1) \leq \frac{2k_1}{c} d^{\sigma_+^{\text{fwd}}} \left(1 - \mu_2 \frac{\lambda_u^+}{|\lambda_s^+|}\right) e^{-\frac{1}{2}\lambda(\tau_1 - t)} \end{aligned}$$

whenever  $\tau_0 \leq t \leq \tau_1$ , where  $c > 0$  is as in Remark 5.20.

Analogously let  $0 < D \leq \delta_1$ ,  $\tau_1 \in \mathbb{R}$ , and consider the trajectory

$$\vec{x}(t, \tau_0; \vec{Q}^+(d, \tau_0)) \equiv x(t, \tau_1; \vec{R}(D)) \quad \tau_0 \leq t \leq \tau_1$$

where  $d = D_3^b(D, \tau_1)$  and  $\tau_0 = \tau_1 - T_3^b(D, \tau_1)$ . Then

$$\|x(t, \tau_1; \vec{R}(D)) - \vec{y}_s(t - \tau_0)\| \leq \frac{2}{c} D z_u^+(t, \tau_1) \leq \frac{2k_1}{c} D e^{-\frac{1}{2}\lambda(\tau_1 - t)}$$

whenever  $\tau_0 \leq t \leq \tau_1$ .

*Proof.* Let us decompose  $\vec{x}(t, \tau_0; \vec{Q}^+(d, \tau))$  as in (58), and set  $\tau_1 = \tau_0 + T_2^f(d, \tau_0)$ . From Lemma 5.16 and Remark 5.20 we find

$$\|\vec{h}^+(\theta)\| \leq \|\vec{h}^+\|_{X+z_u^+}(t, \tau_1) \leq \frac{k_1}{c} |\ln(\varpi)|^{-\alpha/2} D e^{-\frac{1}{2}\lambda(\tau_1 - t)}$$

where  $t = \theta + \tau_0$ . Further from (60) and Remark 5.20

$$\|\vec{\ell}^+(\theta)\| \leq D_\ell z_u^+(t, \tau_1) \leq \frac{k_1}{c} D e^{-\frac{1}{2}\lambda(\tau_1 - t)}.$$

Therefore

$$\|\vec{x}(\theta + \tau_0, \tau_0; \vec{Q}^+(d, \tau)) - \vec{y}_s(\theta)\| \leq 2 \frac{k_1}{c} e^{-\frac{1}{2}\lambda(T_2^f(d, \tau_0) - \theta)} D. \quad (78)$$

Whence, using Proposition 5.19, we conclude the first part of the proof; the second part of the proof is analogous and it is omitted.  $\square$

### 5.3. The loop: from $S^-(\tau)$ to $L^{\text{in}}$

Let  $\tilde{S}^-$  and  $S^-(\tau)$  be as in (29) and (30); let us choose  $\vec{Q}^-(d, \tau) = -d\vec{v}_s^- + \vec{\pi}_u(\tau) \in S^-(\tau)$ . We aim to follow the trajectory  $\vec{x}(t, \tau; \vec{Q}^-(d, \tau))$  from  $\tilde{S}^-$  backwards to  $\Omega^0$ .

The next lemma is analogous to Lemma 5.12 and again it follows from the argument of Lemma 3.6, taking into account the mutual positions of  $\vec{v}_u^\pm, \vec{v}_s^\pm$ .

**LEMMA 5.24.** *Assume  $\mathbf{H}$  and fix  $\tau \in \mathbb{R}$ . For any  $0 < d \leq \delta$  there are  $C^r$  functions  $T_2^b(d, \tau)$  and  $D_2^b(d, \tau)$  such that  $\vec{x}(\theta + \tau, \tau; \vec{Q}^-(d, \tau)) \in K_A^{bwd, -}$  for any  $-T_2^b(d, \tau) < \theta \leq 0$  and it crosses transversely  $\Omega^0$  at  $\theta = -T_2^b(d, \tau)$  in a point such that  $\|\vec{x}(-T_2^b(d, \tau) + \tau, \tau; \vec{Q}^-(d, \tau))\| = D_2^b(d, \tau)$ .*

In order to get estimates of  $T_2^b(d, \tau)$  and  $D_2^b(d, \tau)$ , we perform an inversion of time argument.

Let us denote by  $\underline{f}^\pm(\vec{x}) = -\vec{f}^\pm(\vec{x})$  and  $\underline{g}(t, \vec{x}, \varepsilon) = -\vec{g}(-t, \vec{x}, \varepsilon)$ , and notice that if  $\vec{x}(t)$  is a solution of (PS), then  $\underline{\vec{x}}(t) = \vec{x}(-t)$  is a solution of

$$\dot{\underline{\vec{x}}} = \underline{f}^\pm(\underline{\vec{x}}) + \varepsilon \underline{g}(t, \underline{\vec{x}}, \varepsilon), \quad \underline{\vec{x}} \in \Omega^\pm. \quad (79)$$

Further,  $\frac{\partial \vec{f}^+}{\partial \mathbf{x}}(\vec{0})$  has eigenvalues  $-\lambda_u^+ < 0 < -\lambda_s^+$  with eigenvectors  $\vec{v}_u^+$  and  $\vec{v}_s^+$  respectively, while  $\frac{\partial \vec{f}^-}{\partial \mathbf{x}}(\vec{0})$  has eigenvalues  $-\lambda_u^- < 0 < -\lambda_s^-$  with eigenvectors  $\vec{v}_u^-$  and  $\vec{v}_s^-$  respectively.

So applying Proposition 5.19 to (79) and then going back to the original system we obtain the following.

**PROPOSITION 5.25.** *Assume  $\mathbf{H}$ ; let  $\mu_2 \in ]0, \mu_0/2]$  where  $\mu_0$  satisfies (14) and choose  $\varpi = \varpi(\mu_2)$  satisfying both (63) and (68). Then there is  $\delta = \delta(\mu_2, \varpi) > 0$  such that for any  $0 < d < \delta$  and any  $\tau \in \mathbb{R}$  we find*

$$\Sigma_-^{bwd} \left(1 - \frac{\mu_2}{3}\right) |\ln(d)| \leq T_2^b(d, \tau) \leq \Sigma_-^{bwd} \left(1 + \frac{\mu_2}{3}\right) |\ln(d)| \quad (80)$$

and

$$d^{\sigma_-^{bwd} \left(1 + \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)} \leq D_2^b(d, \tau) \leq d^{\sigma_-^{bwd} \left(1 - \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)} \quad (81)$$

where

$$D_2^b(d, \tau) := \|\vec{x}(\tau - T_2^b(d, \tau), \tau; Q^-(d, \tau))\|. \quad (82)$$

Similarly applying Remark 5.21 and Proposition 5.22 to (79) and then going back to the original system we get the following.

**REMARK 5.26.** Fix  $\tau_{-1} \in \mathbb{R}$ ,  $0 < D < \delta_{-1} = \delta^{\sigma_-^{bwd} \left(1 + \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)}$  and follow the trajectory  $\vec{x}(t, \tau_{-1}; \vec{R}(D))$  forward in time. Then there are  $D_3^f(D, \tau_{-1}) > 0$ ,  $T_3^f(D, \tau_{-1}) > 0$  and  $\tau_0^-(D, \tau_{-1}) = \tau_{-1} + T_3^f(D, \tau_{-1})$  such that  $\vec{x}(t, \tau_{-1}; \vec{R}(D)) \in K_A^{\text{fwd}, -}(t)$  for any  $\tau_{-1} < t \leq \tau_0^-(D, \tau_{-1})$  and it crosses transversely  $\vec{S}^-(\tau_0^-(D, \tau_{-1}))$  in the point  $\vec{Q}^-(D_3^f(D, \tau_{-1}), \tau_0^-(D, \tau_{-1}))$ , where  $\vec{Q}^-(\cdot, \cdot)$  is as defined in (37).

**PROPOSITION 5.27.** *Assume  $\mathbf{H}$ ; let  $\mu_2 \in ]0, \mu_0/2]$  where  $\mu_0$  satisfies (14) and choose  $\varpi = \varpi(\mu_2)$  satisfying both (63) and (68). Then there is  $\delta = \delta(\mu_2, \varpi) > 0$  such that for any  $0 < D < \delta_{-1} = \delta^{\sigma_-^{bwd} \left(1 + \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)}$  and any  $\tau_{-1} \in \mathbb{R}$  the functions  $T_3^f(D, \tau_{-1})$  and  $D_3^f(D, \tau_{-1})$  are well defined,  $C^r$  and one-to-one. Further*

$$\begin{aligned} \frac{1}{\lambda_u} \left[1 - \mu_2 \left(2 \frac{|\lambda_s^-|}{\lambda_u} + \frac{1}{2}\right)\right] |\ln(D)| \\ \leq T_3^f(D, \tau_{-1}) \leq \frac{1}{\lambda_u} \left[1 + \mu_2 \left(2 \frac{|\lambda_s^-|}{\lambda_u} + \frac{1}{2}\right)\right] |\ln(D)|, \\ D^{\frac{1}{\sigma_-^{bwd} \left(1 + 2 \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)}} \leq D_3^f(D, \tau_{-1}) \leq D^{\frac{1}{\sigma_-^{bwd} \left(1 - 2 \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)}}. \end{aligned}$$

In particular  $\lim_{D \rightarrow 0} D_3^f(D, \tau_{-1}) = 0$  uniformly in  $\tau_{-1} \in \mathbb{R}$ .

Let us recall that  $\vec{y}_u(\theta) = \vec{x}(\theta + \tau, \tau; \vec{\pi}_u(\tau))$ , see (29). We conclude the section with the following result, obtained applying the inversion of time argument to Lemma 5.23.

LEMMA 5.28. *Assume  $\mathbf{H}$  and let  $\mu_2$ ,  $\varpi$  and  $\delta = \delta(\mu_2, \varpi)$  be as in Proposition 5.25. Let  $0 < d \leq \delta$ ,  $\tau_0 \in \mathbb{R}$ , and consider the trajectory*

$$\vec{x}(t, \tau_0; \vec{Q}^-(d, \tau_0)) \equiv x(t, \tau_{-1}; \vec{R}(D)) \quad \tau_{-1} \leq t \leq \tau_0$$

where  $D = D_2^b(d, \tau_0)$ ,  $\vec{R}(D)$  is the point of  $L^{in}$  such that  $\|\vec{R}(D)\| = D \in ]0, \delta_{-1}]$

where  $\delta_{-1} = \delta^{\sigma_{bud} \left(1 + \frac{\mu_2 |\lambda_s^-|}{\lambda_u}\right)}$  and  $\tau_{-1} = \tau_0 - T_2^b(d, \tau_0)$ . Then

$$\begin{aligned} \|\vec{x}(t, \tau_0; \vec{Q}^-(d, \tau_0)) - \vec{y}_u(t - \tau_0)\| &\leq \frac{2}{c} d^{\sigma_{bud} \left(1 - \mu_2 \frac{|\lambda_s^-|}{\lambda_u}\right)} z_s^-(t, \tau_{-1}) \\ &\leq \frac{2k_1}{c} d^{\sigma_{bud} \left(1 - \mu_2 \frac{|\lambda_s^-|}{\lambda_u}\right)} e^{-\frac{1}{2}\lambda(t - \tau_{-1})} \end{aligned}$$

whenever  $\tau_{-1} \leq t \leq \tau_0$ .

Analogously let  $0 < D \leq \delta_{-1}$ ,  $\tau_{-1} \in \mathbb{R}$ , and consider the trajectory

$$\vec{x}(t, \tau_0; \vec{Q}^-(d, \tau_0)) \equiv x(t, \tau_{-1}; \vec{R}(D)) \quad \tau_{-1} \leq t \leq \tau_0$$

where  $d = D_3^f(D, \tau_{-1})$  and  $\tau_0 = \tau_{-1} + T_3^f(D, \tau_{-1})$ . Then

$$\|\vec{x}(t, \tau_{-1}; \vec{R}(D)) - \vec{y}_u(t - \tau_0)\| \leq \frac{2}{c} D z_s^-(t, \tau_{-1}) \leq \frac{2k_1}{c} D e^{-\frac{1}{2}\lambda(t - \tau_{-1})}$$

whenever  $\tau_{-1} \leq t \leq \tau_0$ .

#### 5.4. Final step: estimate of time and space displacement

Now we are ready to complete the proofs of Theorems 4.2, 4.3.

We start with Theorem 4.2: the proof of the result follows immediately from combining Propositions 5.7, 5.19, 5.27, 5.10 in forward time and Propositions 5.10, 5.25, 5.22, 5.7 in backward time.

However we proceed with the lengthy and tedious computation of the estimate of  $\mu_0$ , the upper bound in the errors performed to evaluate time and space displacement. In some sense, the only relevant fact is that the value of  $\mu_0$  (that is the upper bound of  $\mu$ ) depends only on the eigenvalues  $\lambda_s^\pm$ ,  $\lambda_u^\pm$ .

We make use of the following estimates

$$\frac{\lambda}{2\lambda} \leq \underline{\sigma} \leq \bar{\sigma} \leq \frac{\bar{\lambda}}{2\lambda}, \quad \frac{\lambda}{\bar{\lambda}} \leq \underline{\Sigma} \leq \bar{\Sigma} \leq \frac{\bar{\lambda}}{\lambda^2}.$$



*Proof of Theorem 4.2.* Let us set

$$c_\mu = c_T + c_d + \frac{1}{6} \quad (83)$$

where

$$c'_d = \frac{\bar{\lambda}}{\lambda}, \quad c_d = 7\frac{\bar{\lambda}^3}{\lambda^3}, \quad c_T = \frac{1}{6} + \frac{1}{2\lambda}. \quad (84)$$

Notice that  $c_\mu > c_d \geq 7$ . Assume that  $0 < 2\mu_1 = \mu_2$  and let  $0 < \mu = c_\mu \mu_2 < \mu_0$ , where  $\mu_0$  is as in (14). We can choose  $0 < d \leq \delta \leq \varpi$  small enough so that

$$d^{\mu_1} \leq \delta^{\mu_1} \leq \varpi^{\mu_1} \leq |\ln(\varpi)|^{-\tilde{C}}$$

where  $\tilde{C}$  is a constant as in Proposition 5.7.

Combining Propositions 5.7, 5.19 we find the following estimates of the space displacement

$$\begin{aligned} \|\mathcal{P}_{1/2}(d, \tau)\| &= D_2^f(D_1^f(d, \tau), \tau + T_1^f(d, \tau)), \\ d^{\sigma_+^{\text{fwd}} + \tilde{A}} &\leq \|\mathcal{P}_{1/2}(d, \tau)\| \leq d^{\sigma_+^{\text{fwd}} + \tilde{B}} \end{aligned} \quad (85)$$

where, using  $\frac{\lambda_u^+}{|\lambda_s^+|} \sigma_+^{\text{fwd}} \leq \frac{\bar{\lambda}}{2\lambda}$ , we find

$$\tilde{B} = (1 - \mu_1) \cdot \sigma_+^{\text{fwd}} \left(1 - \frac{\lambda_u^+}{|\lambda_s^+|} \mu_2\right) - \sigma_+^{\text{fwd}} \geq -\frac{\bar{\lambda}}{2\lambda} (\mu_1 + \mu_2) \geq -c'_d \mu_2 \geq -\mu, \quad (86)$$

$$\begin{aligned} \tilde{A} &= (1 + \mu_1) \cdot \sigma_+^{\text{fwd}} \left(1 + \frac{\lambda_u^+}{|\lambda_s^+|} \mu_2\right) - \sigma_+^{\text{fwd}} \leq \frac{\bar{\lambda}}{2\lambda} (\mu_1 + \mu_2 + \mu_1 \mu_2) \\ &\leq \frac{\bar{\lambda}}{2\lambda} (2\mu_1 + \mu_2) = c'_d \mu_2 \leq \mu. \end{aligned} \quad (87)$$

So we have shown that

$$d^{\sigma_+^{\text{fwd}} + \mu} \leq \|\mathcal{P}_{1/2}(d, \tau)\| \leq d^{\sigma_+^{\text{fwd}} - \mu}.$$

Let us set

$$\tau_0^f(d, \tau) = \tau, \quad D_0^f(d, \tau) = d, \quad \tau_i^f = T_i^f(D_{i-1}^f(d, \tau), \tau_{i-1}^f(d, \tau)) + \tau_{i-1}^f(d, \tau)$$

for  $i = 1, 2, 3, 4$ .

Now combining Propositions 5.7, 5.10, 5.19 and 5.27 we find the following

$$\begin{aligned} \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) \\ = D_4^f(D_3^f(D_2^f(D_1^f(d, \tau), \tau_1^f(d, \tau)), \tau_2^f(d, \tau)), \tau_3^f(d, \tau)), \end{aligned}$$

$$\begin{aligned} \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) &= D_4^f(D_3^f(\|\mathcal{P}_{1/2}(d, \tau)\|, \tau_f^2(d, \tau)), \tau_3^f(d, \tau)), \\ d^{\sigma^{\text{fwd}+A}} &\leq \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) \leq d^{\sigma^{\text{fwd}+B}} \end{aligned}$$

where, using (85) and  $\frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} \leq \frac{\bar{\lambda}^2}{\underline{\lambda}^2}$ , we find

$$\begin{aligned} B &= (1 - \mu_1) \cdot (\sigma_+^{\text{fwd}} - c'_d \mu_2) \cdot \frac{1}{\sigma_-^{\text{bwd}}} \left(1 - 2\mu_2 \frac{|\lambda_s^-|}{\lambda_u^-}\right) - \frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} \\ &\geq -\mu_1 \frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} - \mu_2 \frac{c'_d}{\sigma_-^{\text{bwd}}} - 2\mu_2 \frac{\sigma_+^{\text{fwd}} |\lambda_s^-|}{\sigma_-^{\text{bwd}} \lambda_u^-} \\ &\geq -\mu_2 \left( \frac{1}{2} \frac{\bar{\lambda}^2}{\underline{\lambda}^2} + \frac{2c'_d \bar{\lambda}}{\underline{\lambda}} + \frac{2\bar{\lambda}^3}{\underline{\lambda}^3} \right) \geq -\mu_2 \left(4 + \frac{1}{2}\right) \frac{\bar{\lambda}^3}{\underline{\lambda}^3} > -c_d \mu_2 \geq -\mu, \end{aligned} \quad (88)$$

$$\begin{aligned} A &= (1 + \mu_1) \cdot (\sigma_+^{\text{fwd}} + c'_d \mu_2) \cdot \frac{1}{\sigma_-^{\text{bwd}}} \left(1 + 2\mu_2 \frac{|\lambda_s^-|}{\lambda_u^-}\right) - \frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} \\ &= \frac{\mu_1}{\sigma_-^{\text{bwd}}} (\sigma_+^{\text{fwd}} + c'_d \mu_2) \left(1 + 2\mu_2 \frac{|\lambda_s^-|}{\lambda_u^-}\right) + \frac{\mu_2 c'_d}{\sigma_-^{\text{bwd}}} \left(1 + 2\mu_2 \frac{|\lambda_s^-|}{\lambda_u^-}\right) + 2\mu_2 \frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} \frac{|\lambda_s^-|}{\lambda_u^-} \\ &= \mu_1 \frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} \left(1 + 2\mu_2 \frac{|\lambda_s^-|}{\lambda_u^-}\right) + c'_d \frac{\mu_2}{\sigma_-^{\text{bwd}}} \left(1 + 2\mu_2 \frac{|\lambda_s^-|}{\lambda_u^-}\right) (1 + \mu_1) + 2\mu_2 \frac{\sigma_+^{\text{fwd}}}{\sigma_-^{\text{bwd}}} \frac{|\lambda_s^-|}{\lambda_u^-} \\ &\leq \mu_1 \frac{\bar{\lambda}^2}{\underline{\lambda}^2} \left(1 + \frac{1}{2} \frac{\bar{\lambda}}{\underline{\lambda}}\right) + c'_d \mu_2 \frac{2\bar{\lambda}}{\underline{\lambda}} \left(1 + \frac{1}{2} \frac{\bar{\lambda}}{\underline{\lambda}}\right) \left(1 + \frac{1}{8}\right) + 2\mu_2 \frac{\bar{\lambda}^3}{\underline{\lambda}^3} \\ &\leq \mu_2 \frac{\bar{\lambda}^3}{\underline{\lambda}^3} \left(\frac{3}{2} + 2 \cdot \frac{3}{2} \cdot \frac{9}{8} + 2\right) < 7\mu_2 \frac{\bar{\lambda}^3}{\underline{\lambda}^3} = c_d \mu_2 \leq \mu \end{aligned} \quad (89)$$

using  $\mu_2 \leq \frac{1}{4}$ . So, we have shown the first estimate of (21), i.e.

$$d^{\sigma^{\text{fwd}+\mu}} \leq \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) \leq d^{\sigma^{\text{fwd}-\mu}}. \quad (90)$$

Now we evaluate the space displacement backward in time: for this purpose we simply need to apply the previous analysis to the modified system (79), i.e., to perform an inversion of time argument, analogous to the one detailed in §5.3. This way from (85) and (90) we get respectively

$$d^{\sigma^{\text{bwd}+\mu}} \leq \|\mathcal{P}_{-1/2}(d, \tau)\| \leq d^{\sigma^{\text{bwd}-\mu}},$$

$$d^{\sigma^{\text{bwd}+\mu}} \leq \mathcal{D}(\mathcal{P}_{-1}(d, \tau), \vec{P}_s(\mathcal{T}_{-1}(d, \tau))) \leq d^{\sigma^{\text{bwd}-\mu}}.$$

Now we consider the time displacement; again we start by following the trajectory forward in time. Let us set

$$\tau'_a + \tau''_a \leq \mathcal{T}_{\frac{1}{2}}(d, \tau) - \tau \leq \tau'_b + \tau''_b,$$

$$\tau'_a + \tau''_a + \tau'''_a + \tau_a^{iv} \leq \mathcal{I}_1(d, \tau) - \tau \leq \tau'_b + \tau''_b + \tau'''_b + \tau_b^{iv}$$

where  $\tau'_a$  and  $\tau'_b$  are estimates from below and from above of the time spent by  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$  to travel from  $L^0$  to  $\tilde{S}^+$ ,  $\tau''_a$  and  $\tau''_b$  are estimates of the time spent to travel from  $\tilde{S}^+$  to  $L^{\text{in}}$ ,  $\tau'''_a$  and  $\tau'''_b$  are estimates of the time spent to travel from  $L^{\text{in}}$  to  $\tilde{S}^-$ ,  $\tau_a^{iv}$  and  $\tau_b^{iv}$  are estimates of the time spent to travel from  $\tilde{S}^-$  to  $L^0$ .

From Proposition 5.7, possibly choosing a smaller  $\varpi > 0$  we see that

$$0 < \tau'_a \leq \tau'_b \leq \frac{4}{\lambda} \ln(|\ln(\varpi)|) \leq \frac{4}{\lambda} \ln(\varpi^{-\frac{\mu_1 \lambda}{12}}) \leq \frac{\mu_1}{3} |\ln(\delta)|. \quad (91)$$

Analogously, from Proposition 5.10 we see that

$$0 < \tau_a^{iv} \leq \tau_b^{iv} \leq \frac{\mu_1}{3} |\ln(\delta)|.$$

Then combining Propositions 5.7 and 5.19 we find

$$\begin{aligned} \tau''_a &\geq \Sigma_+^{\text{fwd}} \left(1 - \frac{\mu_2}{3}\right) (1 - \mu_1) |\ln(d)| \\ &\geq \Sigma_+^{\text{fwd}} \left(1 - \frac{5\mu_2}{6}\right) |\ln(d)| \geq \Sigma_+^{\text{fwd}} (1 - \mu_2) |\ln(d)|, \end{aligned}$$

$$\begin{aligned} \tau''_b &\leq \Sigma_+^{\text{fwd}} \left(1 + \frac{\mu_2}{3}\right) (1 + \mu_1) |\ln(d)| \\ &= \Sigma_+^{\text{fwd}} \left(1 + \frac{5\mu_2 + \mu_2^2}{6}\right) |\ln(d)| \leq \Sigma_+^{\text{fwd}} (1 + \mu_2) |\ln(d)|. \end{aligned}$$

Summing up we have found

$$\begin{aligned} \mathcal{I}_{\frac{1}{2}}(d, \tau) - \tau &\leq \left(\Sigma_+^{\text{fwd}} + \frac{\mu_1}{3} + \Sigma_+^{\text{fwd}} \mu_2\right) |\ln(d)| \\ &\leq \left[\Sigma_+^{\text{fwd}} + \left(\frac{1}{6} + \frac{1}{2\lambda}\right) \mu_2\right] |\ln(d)| \\ &= (\Sigma_+^{\text{fwd}} + c_T \mu_2) |\ln(d)| \leq (\Sigma_+^{\text{fwd}} + \mu) |\ln(d)|, \\ \mathcal{I}_{\frac{1}{2}}(d, \tau) - \tau &\geq (\Sigma_+^{\text{fwd}} - \Sigma_+^{\text{fwd}} \mu_2) |\ln(d)| \geq \left(\Sigma_+^{\text{fwd}} - \frac{\mu_2}{2\lambda}\right) |\ln(d)| \\ &\geq (\Sigma_+^{\text{fwd}} - c_T \mu_2) |\ln(d)| \geq (\Sigma_+^{\text{fwd}} - \mu) |\ln(d)|. \end{aligned}$$

Thus we find

$$(\Sigma_+^{\text{fwd}} - \mu) |\ln(d)| \leq \mathcal{I}_{\frac{1}{2}}(d, \tau) - \tau \leq (\Sigma_+^{\text{fwd}} + \mu) |\ln(d)|. \quad (92)$$

Finally combining (85), (87), (86) and Proposition 5.27 we find

$$\begin{aligned}
\tau_b''' &\leq \frac{1}{\lambda_u^-} \left[ 1 + \mu_2 \left( 2 \frac{|\lambda_s^-|}{\lambda_u^-} + \frac{1}{2} \right) \right] |\ln(d^{\sigma_+^{\text{fwd}} + \bar{A}})| \\
&\leq \frac{1}{\lambda_u^-} \left[ 1 + \mu_2 \left( 2 \frac{\bar{\lambda}}{\lambda} + \frac{1}{2} \right) \right] (\sigma_+^{\text{fwd}} + c_d' \mu_2) |\ln(d)| \\
&\leq \left[ \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-} + \frac{\mu_2}{\lambda_u^-} \left( \sigma_+^{\text{fwd}} \frac{5\bar{\lambda}}{2\lambda} + c_d' \left( 1 + \mu_2 \frac{5\bar{\lambda}}{2\lambda} \right) \right) \right] |\ln(d)| \\
&\leq \left[ \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-} + \frac{\mu_2}{\lambda_u^-} \left( \frac{5\bar{\lambda}^2}{4\lambda^2} + \frac{13\bar{\lambda}^2}{8\lambda^2} \right) \right] |\ln(d)| \\
&< \left( \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-} + 3 \frac{\mu_2 \bar{\lambda}^2}{\lambda_u^- \lambda^2} \right) |\ln(d)| \leq \left( \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-} + c_d \mu_2 \right) |\ln(d)|.
\end{aligned}$$

Analogously we find

$$\tau_a''' \geq \left( \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-} - c_d \mu_2 \right) |\ln(d)|.$$

Observe now that  $\Sigma^{\text{fwd}} = \Sigma_+^{\text{fwd}} + \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-}$ , then, summing up all the estimates, we find

$$\begin{aligned}
\mathcal{F}_1(d, \tau) - \tau &\leq \tau_b' + \tau_b'' + \tau_b''' + \tau_b^{iv} \\
&\leq (\Sigma_+^{\text{fwd}} + c_T \mu_2) |\ln(d)| + \left( \frac{\sigma_+^{\text{fwd}}}{\lambda_u^-} + c_d \mu_2 \right) |\ln(d)| + \frac{\mu_1}{3} |\ln(\delta)| \\
&\leq \left[ \Sigma^{\text{fwd}} + \left( c_T + c_d + \frac{1}{6} \right) \mu_2 \right] |\ln(d)| = (\Sigma^{\text{fwd}} + c_\mu \mu_2) |\ln(d)| \\
&\leq (\Sigma^{\text{fwd}} + \mu) |\ln(d)|.
\end{aligned}$$

Analogously, using the fact that  $\mathcal{F}_1(d, \tau) - \tau \geq \tau_a' + \tau_a'' + \tau_a''' + \tau_a^{iv}$  and the previous estimates, we find

$$(\Sigma^{\text{fwd}} - \mu) |\ln(d)| \leq \mathcal{F}_1(d, \tau) - \tau \leq (\Sigma^{\text{fwd}} + \mu) |\ln(d)|. \quad (93)$$

Then, applying again an inversion of time argument, from (92) and (93) respectively we find

$$(\Sigma_-^{\text{bwd}} - \mu) |\ln(d)| \leq \tau - \mathcal{F}_{-1/2}(d, \tau) \leq (\Sigma_-^{\text{bwd}} + \mu) |\ln(d)|,$$

$$(\Sigma^{\text{bwd}} - \mu) |\ln(d)| \leq \tau - \mathcal{F}_{-1}(d, \tau) \leq (\Sigma^{\text{bwd}} + \mu) |\ln(d)|$$

and the proof is completed.  $\square$

REMARK 5.29. Modifying slightly the proof of Proposition 5.19 we might show that we can find a constant  $K_2^\mu > 1$  (independent of all the small parameters) such that we can choose  $\mu_2 = K_2^\mu \varepsilon$ ; then we could replace (69) and (70) by the following

$$\begin{aligned} \Sigma_+^{\text{fwd}} (1 - K_2^\mu \varepsilon) |\ln(d)| - |\ln(\varpi)| &\leq T_2^f(d, \tau) \leq \Sigma_+^{\text{fwd}} (1 + K_2^\mu \varepsilon) |\ln(d)| \\ |\ln(\varpi)|^{-2} d^{\sigma_+^{\text{fwd}}(1+K_2^\mu \varepsilon)} &\leq D_2^f(d, \tau) \leq |\ln(\varpi)|^2 d^{\sigma_+^{\text{fwd}}(1-K_2^\mu \varepsilon)}. \end{aligned} \quad (94)$$

Then, we conjecture that with a proper choice of a fixed  $\varpi = \varpi_0 > 0$  satisfying (63), eventually we would obtain, as we said in Remark 5.1,

$$\begin{aligned} C^{-1} d^{\sigma^{\text{fwd}} + K^\mu \varepsilon} &\leq \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) \leq C d^{\sigma^{\text{fwd}} + K^\mu \varepsilon}, \\ (\Sigma^{\text{fwd}} - K^\mu \varepsilon) |\ln(d)| - C &\leq (\mathcal{T}_1(d, \tau) - \tau) \leq (\Sigma^{\text{fwd}} + K^\mu \varepsilon) |\ln(d)| + C \end{aligned} \quad (95)$$

for suitable  $K^\mu > K_2^\mu > 0$  and  $C > 1$ . Further we think that if  $\mathbf{g}_x(t, \vec{0}, \varepsilon) = \mathbf{0}$ , so that for the exponential dichotomy estimates in (34) we can use  $\lambda_s^\pm$  and  $\lambda_u^\pm$  as exponents, we can even set  $K^\mu = 0$  in (95).

Now we prepare the proof of Theorem 4.3.

Notice that in Theorem 4.3 we need the distance between  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$  and  $\vec{x}(t, \tau; \vec{P}_s(\tau))$  to be small but in fact in §5.2 we measure the distance between  $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$  and  $\vec{x}(t, T_1^f(d, \tau) + \tau; \vec{\pi}_s(T_1^f(d, \tau) + \tau))$ . We need Lemmas 5.30 and 5.31 below to measure this mismatch.

With the notation of Lemma 5.4 we set

$$\vec{\ell}_s(d, \tau) := \vec{P}_f^+(d, \tau) - \vec{x}(T_1^f(d, \tau) + \tau, \tau; \vec{P}_s(\tau)),$$

$$\vec{v}_s(d, \tau) := \vec{\pi}_s(T_1^f(d, \tau) + \tau) - \vec{x}(T_1^f(d, \tau) + \tau, \tau; \vec{P}_s(\tau));$$

then we have the following.

LEMMA 5.30. *Let the assumptions of Proposition 5.7 be satisfied; recalling that  $0 < d \leq \delta \leq \varpi$  we find*

$$\begin{aligned} \left\| \frac{\partial}{\partial d} \vec{\ell}_s(d, \tau) \Big|_{d=0} \right\| &\leq |\ln(\varpi)|^{8\bar{\lambda}/\Delta}, \\ \|\vec{\ell}_s(d, \tau)\| &\leq 2 |\ln(\varpi)|^{8\bar{\lambda}/\Delta} d \leq 2 |\ln(\delta)|^{8\bar{\lambda}/\Delta} d \leq d \delta^{-\mu_1/2}. \end{aligned} \quad (96)$$

Further

$$\begin{aligned} \left\| \frac{\partial}{\partial d} \vec{v}_s(d, \tau) \Big|_{d=0} \right\| &\leq \frac{2k_2 \bar{\lambda}}{\lambda} |\ln(\varpi)|^{8\bar{\lambda}/\Delta}, \\ \|\vec{v}_s(d, \tau)\| &\leq \frac{4k_2 \bar{\lambda}}{\lambda} |\ln(\varpi)|^{8\bar{\lambda}/\Delta} d \leq \frac{4k_2 \bar{\lambda}}{\lambda} |\ln(\delta)|^{8\bar{\lambda}/\Delta} d \leq d \delta^{-\mu_1/2}. \end{aligned} \quad (97)$$

*Proof.* Notice that  $\vec{x}(T_1^f(0, \tau) + \tau, \tau; \vec{P}_s(\tau)) \in [\tilde{W}^s(T_1^f(0, \tau) + \tau) \cap \tilde{S}^+]$  and  $\tilde{W}^s(T_1^f(0, \tau) + \tau) \cap \tilde{S}^+ = \{\vec{\pi}_s(T_1^f(0, \tau) + \tau)\} = \{\vec{P}_f^+(0, \tau)\}$  for all  $\varepsilon$ . So we use an expansion as in (47) and in the rest of the proof we consider  $\varepsilon = 0$ . Hence  $\vec{\ell}_s(0, \tau) = \vec{0} = \vec{v}_s(0, \tau)$ .

Differentiating  $\vec{\ell}_s(d, \tau)$  with respect to  $d$  we get

$$\begin{aligned} \frac{\partial}{\partial d} \vec{\ell}_s(d, \tau)|_{d=0} &= \frac{\partial \vec{x}}{\partial \vec{P}}(T_1^f(d, \tau) + \tau, \tau; \vec{Q}_s(d, \tau))|_{d=0} \frac{\partial \vec{Q}_s(0, \tau)}{\partial d} \\ &+ [\dot{\vec{x}}(T_1^f(d, \tau) + \tau, \tau; \vec{Q}_s(d, \tau)) - \dot{\vec{x}}(T_1^f(d, \tau) + \tau, \tau; \vec{P}_s(\tau))]|_{d=0} T_d'(\tau) \\ &= \mathbf{X}(T_1^f(0, \tau)) \vec{v}, \end{aligned} \quad (98)$$

where  $\mathbf{X}(t)$  is the fundamental matrix of (48). So the first inequality in (96) follows from (49); then the second inequality in (96) follows immediately.

Now notice that if  $\varepsilon = 0$ , then  $\vec{\pi}_s(\tau) \equiv \vec{\pi}_{s, \Gamma} = \mathbf{\Gamma} \cap \tilde{S}^+$  for any  $\tau \in \mathbb{R}$ , so that  $\frac{\partial}{\partial \tau} \vec{\pi}_s(\tau) \equiv \vec{0}$ . Hence

$$\frac{\partial}{\partial d} \vec{v}_s(0, \tau) = -T_d'(\tau) \vec{f}^+(\vec{\gamma}(\vec{T}_1^f)), \quad (99)$$

then the first estimate in (97) follows from (52) and (53). Finally the second inequality in (97) follows immediately from the first and the fact that  $\|\vec{v}_s(0, \tau)\| = 0$ .  $\square$

We denote by  $\tau_1^f = \tau_1^f(d, \tau) = \tau + T_1^f(d, \tau)$  and  $\vec{\pi}_{s,1}(\tau_1^f) = \vec{x}(\tau_1^f, \tau; \vec{P}_s(\tau))$ . Notice that  $\vec{\pi}_{s,1}(\tau_1^f) \in \tilde{W}^s(\tau_1^f)$  but it is not necessarily in  $\tilde{S}^+$  and recall that by construction  $\{\vec{\pi}_s(\tau_1^f)\} = \tilde{W}^s(\tau_1^f) \cap \tilde{S}^+$ , cf. (27). Let us set

$$\vec{v}_s(\theta; d, \tau) := \vec{x}(\theta + \tau_1^f, \tau_1^f; \vec{\pi}_{s,1}(\tau_1^f)) - \vec{x}(\theta + \tau_1^f, \tau_1^f; \vec{\pi}_s(\tau_1^f)),$$

and recall that  $\vec{f}^\pm$  and  $\vec{g}$  are  $C^r$  and  $\alpha = \min\{r - 1; 1\}$ . Then we have the following estimate.

LEMMA 5.31. *Let  $0 < d \leq \delta$ , then*

$$\|\vec{v}_s(\theta; d, \tau)\| \leq d^{1-\mu_1} e^{\frac{\lambda_s^+ \theta}{1+\alpha/2}} \quad (100)$$

for any  $\theta \geq 0$ .

*Proof.* Observe that both  $\vec{x}(t, \tau_1^f; \vec{\pi}_{s,1}(\tau_1^f))$  and  $\vec{x}(t, \tau_1^f; \vec{\pi}_s(\tau_1^f))$  belong to  $\tilde{W}^s(t)$  for any  $t \geq \tau_1^f$ , and set  $\vec{y}_s(\theta) = \vec{x}(\theta + \tau_1^f, \tau_1^f; \vec{\pi}_{s,1}(\tau_1^f))$  for short. From Lemma 5.30 we know that

$$\|\vec{v}_s(0; d, \tau)\| = \|\vec{v}_s(d, \tau)\| = \|\vec{\pi}_{s,1}(\tau_1^f) - \vec{\pi}_s(\tau_1^f)\| \leq d^{1-\mu_1/2}. \quad (101)$$

To prove (100) we use exponential dichotomy and a fixed point argument in an exponentially weighted space. Notice that  $\vec{v}_s(\theta; d, \tau)$  is a solution of

$$\dot{\vec{h}}(\theta) = \mathbf{F}_x^+(\theta + \tau_1^f, \vec{y}_s(\theta), \varepsilon)\vec{h}(\theta) + \vec{N}(\theta + \tau_1^f, \vec{h}(\theta), \varepsilon) \quad (102)$$

where

$$\begin{aligned} \vec{N}(\theta + \tau_1^f, \vec{h}(\theta), \varepsilon) &= \vec{F}^+(\theta + \tau_1^f, \vec{y}_s(\theta) + \vec{h}(\theta), \varepsilon) - \vec{F}^+(\theta + \tau_1^f, \vec{y}_s(\theta), \varepsilon) \\ &\quad - \mathbf{F}_x^+(\theta + \tau_1^f, \vec{y}_s(\theta), \varepsilon)\vec{h}(\theta). \end{aligned}$$

Recall that  $\vec{F}^+ \in C^r$ , so there is  $C_\alpha > 0$  independent of  $\varepsilon, t$ , such that

$$\begin{aligned} \|\vec{N}(t, \vec{h}, \varepsilon)\| &\leq C_\alpha \|\vec{h}\|^{1+\alpha}, \\ \|\vec{N}(t, \vec{h}_1, \varepsilon) - \vec{N}(t, \vec{h}_2, \varepsilon)\| &\leq C_\alpha [\max\{\|\vec{h}_1\|, \|\vec{h}_2\|\}]^\alpha \|\vec{h}_1 - \vec{h}_2\| \end{aligned}$$

(cf.  $\vec{R}_2^+(\theta, \vec{h})$  in (61) and (62)).

Observe further, that  $\|\vec{y}_s(\theta) - \vec{\gamma}(\tilde{T}_1^f + \theta)\| = O(\varepsilon)$  uniformly for  $\theta \geq 0$ , so

$$\dot{\vec{h}}(\theta) = \mathbf{F}_x^+(\theta + \tau_1^f, \vec{y}_s(\theta), \varepsilon)\vec{h}(\theta) \quad (103)$$

admits exponential dichotomy in  $[0, +\infty[$ , and the exponents are  $\varepsilon$ -close to the respective ones of  $\dot{\vec{x}} = \mathbf{f}_x^+(\vec{\gamma}(\tilde{T}_1^f + \theta))\vec{x}$ , i.e., to  $\lambda_s^+, \lambda_u^+$ . Further the optimal dichotomy constant  $\bar{k}_1$  is  $|\ln(\varpi)|^{-1}$  close to  $k_1$  since  $\mathbf{f}_x^+(\vec{\gamma}(\tilde{T}_1^f + \theta))$  is  $|\ln(\varpi)|^{-1}$  close to  $\mathbf{f}_x^+(\vec{0})$ , so we can assume  $\bar{k}_1 = 2k_1$ .

So let  $\mathbf{X}_h(\theta)$  and  $\mathbf{P}_h$  be the fundamental matrix and the projection (of the exponential dichotomy) of (103). We can assume that

$$\begin{aligned} \|\mathbf{X}_h(t)\mathbf{P}_h[\mathbf{X}_h(s)]^{-1}\vec{\xi}\| &\leq 2k_1 e^{\frac{\lambda_s^+}{1+\alpha/2}(t-s)} & \text{if } t > s > 0, \\ \|\mathbf{X}_h(t)[\mathbb{I} - \mathbf{P}_h][\mathbf{X}_h(s)]^{-1}\vec{\xi}\| &\leq 2k_1 e^{\frac{\lambda_u^+}{1+\alpha/2}(t-s)} & \text{if } s > t > 0. \end{aligned} \quad (104)$$

Further, since  $\vec{v}_s(\theta; d, \tau)$  converges to 0 exponentially, for  $\theta \rightarrow +\infty$ , we see that it is on the stable manifold of (102). In particular  $\vec{v}_s(\theta; d, \tau) e^{\frac{|\lambda_s^+|}{1+\alpha/2}\theta}$  is bounded for  $\theta \geq 0$ .

In fact  $\vec{v}_s(\theta; d, \tau)$  is the unique bounded solution of (102) such that

$$\mathbf{P}_h \dot{\vec{h}}(0) = \mathbf{P}_h [\vec{\pi}_{s,1}(\tau_1^f) - \vec{\pi}_s(\tau_1^f)] =: \vec{c}_0. \quad (105)$$

So let  $\mathcal{H}$  be the space of all continuous functions from  $[0, +\infty[$  to  $\mathbb{R}^2$  endowed with the norm  $\|\vec{y}\|_{\mathcal{H}} = \sup_{\theta \geq 0} \{\|\vec{y}(\theta)\| \exp[\frac{|\lambda_s^+|}{1+\alpha/2}\theta]\}$ .

Let us consider the operator  $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$  defined by

$$\begin{aligned} \mathcal{T}(\vec{y})(\theta) &= \mathbf{X}_h(\theta) \mathbf{P}_h \vec{c}_0 + \int_0^\theta \mathbf{X}_h(\theta) \mathbf{P}_h [\mathbf{X}_h(s)]^{-1} \vec{N}(s + \tau_1^f, \vec{y}(s), \varepsilon) ds \\ &\quad - \int_\theta^{+\infty} \mathbf{X}_h(\theta) [\mathbb{I} - \mathbf{P}_h] [\mathbf{X}_h(s)]^{-1} \vec{N}(s + \tau_1^f, \vec{y}(s), \varepsilon) ds. \end{aligned} \quad (106)$$

Notice that  $\vec{y}(\theta)$  is a fixed point of  $\mathcal{T}$  if and only if it is a bounded solution of (102), (105). We aim to show that  $\mathcal{T}$  maps the ball of radius  $R = 6k_1 d^{1-\mu_1/2}$  of  $\mathcal{H}$  in itself and that it is a contraction. Then it follows that  $\vec{v}_s(\theta; d, \tau)$  is the unique fixed point of  $\mathcal{H}$  in this ball, and that  $\|\vec{v}_s(\theta; d, \tau)\|_{\mathcal{H}} \leq R$ .

In fact from (101) and (104) we find

$$\|\mathbf{X}_h(\theta) \mathbf{P}_h \vec{c}_0\| \leq 2k_1 d^{1-\mu_1/2} e^{\frac{\lambda_s^+}{1+\alpha/2}\theta} = \frac{R}{3} e^{\frac{\lambda_s^+}{1+\alpha/2}\theta}. \quad (107)$$

Further, possibly choosing a smaller  $\delta > d$ , and consequently a smaller  $R$ , we can assume  $R \leq \left(\frac{\alpha|\lambda_s^+|}{6k_1 C_\alpha(1+\alpha/2)}\right)^{1/\alpha}$ , so that whenever  $\|\vec{y}\|_{\mathcal{H}} \leq R$  we find

$$\begin{aligned} &\left\| \int_0^\theta \mathbf{X}_h(\theta) \mathbf{P}_h [\mathbf{X}_h(s)]^{-1} \vec{N}(s + \tau_1^f, \vec{y}(s), \varepsilon) ds \right\| \\ &\leq 2k_1 C_\alpha R^{1+\alpha} \int_0^\theta e^{\frac{\lambda_s^+}{1+\alpha/2}(\theta-s)} \cdot e^{\frac{\lambda_s^+}{1+\alpha/2}(1+\alpha)s} ds \\ &\leq \frac{2k_1 C_\alpha (1+\alpha/2)}{\alpha|\lambda_s^+|} R^{1+\alpha} e^{\frac{\lambda_s^+}{1+\alpha/2}\theta} \leq \frac{R}{3} e^{\frac{\lambda_s^+}{1+\alpha/2}\theta}; \end{aligned} \quad (108)$$

and finally

$$\begin{aligned} &\left\| \int_\theta^{+\infty} \mathbf{X}_h(\theta) [\mathbb{I} - \mathbf{P}_h] [\mathbf{X}_h(s)]^{-1} \vec{N}(s + \tau_1^f, \vec{y}(s), \varepsilon) ds \right\| \\ &\leq 2k_1 C_\alpha R^{1+\alpha} \int_\theta^{+\infty} e^{\frac{\lambda_s^+}{1+\alpha/2}(\theta-s)} \cdot e^{\frac{\lambda_s^+}{1+\alpha/2}(1+\alpha)s} ds \\ &\leq \frac{2k_1 C_\alpha (1+\alpha/2)}{\lambda_u^+ + (1+\alpha)|\lambda_s^+|} R^{1+\alpha} e^{\frac{\lambda_s^+}{1+\alpha/2}(1+\alpha)\theta} \leq \frac{R}{3} e^{\frac{\lambda_s^+}{1+\alpha/2}\theta}. \end{aligned} \quad (109)$$

So from (107), (108) and (109) we conclude that  $\mathcal{T}$  maps the ball of radius  $R$  in itself. With a similar argument we show that  $\mathcal{T}$  is a contraction in  $R$  and we conclude that  $\|\vec{v}_s(\theta; d, \tau)\|_{\mathcal{H}} \leq R$  so

$$\|\vec{v}_s(\theta; d, \tau)\|_{\mathcal{H}} \leq 6k_1 d^{1-\mu_1/2} \leq d^{1-\mu_1}$$

if  $0 < d \leq \delta$  is small enough; thus (100) is proved.  $\square$



Analogously, let us set

$$\begin{aligned}\vec{\ell}_u(d, \tau) &:= \vec{P}_b^-(d, \tau) - \vec{x}(\tau - T_1^b(d, \tau), \tau; \vec{P}_u(\tau)), \\ \vec{v}_u(d, \tau) &:= \vec{\pi}_u(\tau - T_1^b(d, \tau)) - \vec{x}(\tau - T_1^b(d, \tau), \tau; \vec{P}_u(\tau));\end{aligned}$$

then, as in Lemma 5.30, we have the following.

LEMMA 5.32. *Let the assumptions of Proposition 5.10 be satisfied, recalling that  $0 < d \leq \delta \leq \varpi$  we find*

$$\begin{aligned}\left\| \frac{\partial}{\partial d} \vec{\ell}_u(d, \tau) \Big|_{d=0} \right\| &\leq |\ln(\varpi)|^{8\bar{\lambda}/\lambda}, \\ \|\vec{\ell}_u(d, \tau)\| &\leq 2|\ln(\varpi)|^{8\bar{\lambda}/\lambda} d \leq 2|\ln(\delta)|^{8\bar{\lambda}/\lambda} d \leq d\delta^{-\mu_1/2}.\end{aligned}\tag{110}$$

Further

$$\begin{aligned}\left\| \frac{\partial}{\partial d} \vec{v}_u(d, \tau) \Big|_{d=0} \right\| &\leq |\ln(\varpi)|^{8\bar{\lambda}/\lambda}, \\ \|\vec{v}_u(0, \tau)\| &\leq 4k_2 \frac{\bar{\lambda}}{\lambda} |\ln(\varpi)|^{8\bar{\lambda}/\lambda} d \leq 4k_2 \frac{\bar{\lambda}}{\lambda} |\ln(\delta)|^{8\bar{\lambda}/\lambda} d \leq d\delta^{-\mu_1/2}.\end{aligned}\tag{111}$$

We denote by  $\tau_1^b = \tau_1^b(d, \tau) = \tau - T_1^b(d, \tau)$  and  $\vec{\pi}_{u,1}(\tau_1^b) = \vec{x}(\tau_1^b, \tau; \vec{P}_u(\tau))$ . Again, note that  $\vec{\pi}_{u,1}(\tau_1^b) \in \tilde{W}^u(\tau_1^b)$  but it is not necessarily in  $\tilde{S}^-$  while  $\{\vec{\pi}_u(\tau_1^b)\} = \tilde{W}^u(\tau_1^b) \cap \tilde{S}^-$  by construction, cf. (29). Let us set

$$\vec{v}_u(\theta; d, \tau) := \vec{x}(\theta + \tau_1^b, \tau_1^b; \vec{\pi}_{u,1}(\tau_1^b)) - \vec{x}(\theta + \tau_1^b, \tau_1^b; \vec{\pi}_u(\tau_1^b)).$$

Then, as in Lemma 5.31 we have the following estimate.

LEMMA 5.33. *Let  $0 < d \leq \delta$  then*

$$\|\vec{v}_u(\theta; d, \tau)\| \leq d^{1-\mu_1} e^{\frac{\lambda_- \theta}{1+\alpha/2}}\tag{112}$$

for any  $\theta \leq 0$ .

*Proof of Theorem 4.3.* Let us fix  $\mu_2 = 2\mu_1$ ,  $\mu = c_\mu \mu_2 \leq \mu_0$ ; then fix  $\varpi$  and  $\delta$  so that Lemmas 5.11, 5.23 and 5.32 hold true, and finally let  $0 < d \leq \delta$  and  $\tau \in \mathbb{R}$ . We start from (23); we denote by  $\tau_1^f = \tau_1^f(d, \tau) = \tau + T_1^f(d, \tau)$ ,  $\vec{\pi}_{s,1}(\tau_1^f) = \vec{x}(\tau_1^f, \tau; \vec{P}_s(\tau))$ . Notice that  $\vec{\pi}_s(\tau_1^f) = \tilde{W}^s(\tau) \cap \tilde{S}^+$ , see (27), while  $\vec{\pi}_{s,1}(\tau_1^f) \in \tilde{W}^s(\tau_1^f)$ , but it might not be in  $\tilde{S}^+$ . Let us set, as in §5.4,  $\vec{y}_s(\theta) := \vec{x}(\theta + \tau_1^f, \tau_1^f; \vec{\pi}_s(\tau_1^f))$  and

$$\vec{y}_{s,1}(\theta) := \vec{x}(\theta + \tau_1^f, \tau_1^f; \vec{\pi}_{s,1}(\tau_1^f)) \equiv \vec{x}(\theta + \tau_1^f, \tau; \vec{P}_s(\tau)).$$

From Lemmas 5.32 and 5.23, setting  $t = \theta + \tau_1^f$  we see that

$$\begin{aligned} & \|\vec{x}(t, \tau; \vec{Q}_s(d, \tau)) - \vec{x}(t, \tau; \vec{P}_s(\tau))\| \\ & \leq \|\vec{x}(\theta + \tau_1^f, \tau; \vec{Q}_s(d, \tau)) - \vec{y}_s(\theta)\| + \|\vec{y}_{s,1}(\theta) - \vec{y}_s(\theta)\| \\ & \leq \frac{2k_1}{c} d^{\sigma_+^{\text{fwd}}} \left(1 - \mu_2 \frac{\lambda_+}{|\lambda_+|}\right) + d^{1-\mu_1} \leq d^{\sigma_+^{\text{fwd}} - \mu} \end{aligned} \quad (113)$$

for any  $\tau_1^f \leq t \leq \mathcal{T}_{\frac{1}{2}}(d, \tau)$ , since  $\sigma_+^{\text{fwd}} < 1$  and  $\mu = c_\mu \mu_2$  (see the proof of Theorem 4.2).

Further, from Lemma 5.11 we find

$$\|\vec{x}(t, \tau; \vec{Q}_s(d, \tau)) - \vec{x}(t, \tau; \vec{P}_s(\tau))\| \leq d\delta^{-\mu_1} \leq d^{\sigma_+^{\text{fwd}} - \mu}$$

for any  $\tau \leq t \leq \tau_1^f$ . So (23) is proved.

The proofs of (24), (25) and (26) are similar and are omitted.  $\square$

## 6. Some remarks and future developments

In this section we spend a few more words concerning some projects we wish to develop using the results of Theorems 4.2 and 4.3, as we said in the Introduction.

Firstly, it is well known that, if  $\lambda_s + \lambda_u < 0$  and  $\varepsilon = 0$ , in the framework of Scenario 1 the set  $\mathbf{\Gamma}$  of (1) is asymptotically stable from inside (stable from outside in Scenario 2). We think that the results of this paper could be of use in exploring the possibility to construct an integral manifold which will be asymptotically stable from inside, when we have just one zero of the Melnikov function  $\mathcal{M}(\tau)$ , see (9). Notice that the presence of chaotic phenomena prevents the possibility of asymptotic stability if we have infinitely many zeros of  $\mathcal{M}(\tau)$ .

Secondly, we plan to use Theorems 4.2 and 4.3 to study the possibility to establish a sub-harmonic Melnikov theory. More precisely we think that if the function  $\mathcal{M}(\tau)$  has a unique zero at  $\tau = \tau_0$  then we should find two different type of results: if  $\mathcal{M}'(\tau_0) > 0$  then we expect that the homoclinic trajectory  $\vec{x}_b(t, \varepsilon)$  bifurcating from  $\vec{\gamma}(t)$  will be unique, while if  $\mathcal{M}'(\tau_0) < 0$  then we think there should be a monotone decreasing sequence of  $\varepsilon_k > 0$  such that if  $0 < \varepsilon \leq \varepsilon_k$  then there are  $k$  homoclinic trajectories, each of them performing exactly  $j$  loops (i.e., passing  $O(\varepsilon)$  close to  $\vec{\gamma}(0)$  exactly  $j$  times), where  $j = 1, \dots, k$ . In fact we expect that this will be the case if  $\mathcal{M}(\tau)$  does not converge to 0 when  $\tau$  converges either to  $+\infty$  or to  $-\infty$ , while some extra conditions on the speed of convergence to 0 is needed if  $\mathcal{M}(\tau) \rightarrow 0$ . This dynamical result is suggested also by the bubble-tower phenomena appearing in quasi-linear elliptic equations, see e.g. [10, 12, 13].

Thirdly, we believe that Theorems 4.2 and 4.3 could be used to find *safe regions* in systems characterized by Melnikov chaos, i.e., chaos generated by the non-autonomous perturbation of an autonomous homoclinic trajectory. More precisely we aim to find subsets of initial conditions close to  $\Gamma$ , which will remain far from the chaotic set for a long, possibly infinite time. Roughly speaking we think we can use these ideas to give some conditions sufficient to answer the question:

*How far should I stay from  $\Gamma$  to be sure not to finish in a chaotic pattern?*

We think this kind of result should be quite useful from an engineering point of view.

Finally, as we said in the Introduction, Melnikov theory has been extended to the context of piecewise smooth systems (PS) assuming that  $\vec{0} \in \Omega^0$  in [8], where we proved that the classical Melnikov condition is enough to ensure the persistence to perturbation of the homoclinic trajectory. However quite unexpectedly, it does not guarantee the persistence of chaos if  $\vec{g}$  is periodic in  $t$ , as it happens in a smooth setting, see e.g. [29] but also in a non-smooth setting if the origin does not belong to the discontinuity surface  $\Omega^0$ , see e.g. [2–4]. In fact in [15] we have found a big class of counterexamples to the presence of chaos: roughly speaking a geometrical obstruction forbids chaotic phenomena whenever we have sliding close to the origin.

In a forthcoming paper we plan to show that, if such a geometrical obstruction is removed then the usual Melnikov conditions guarantee chaos as in the smooth setting, and Theorems 4.2 and 4.3 of the present paper are essential in this project. We also think that these kinds of results should give us a better insight of what happens close to  $\Gamma$  in a smooth setting, possibly allowing us to give some results concerning the size and the position of the Cantor-like set giving rise to chaos.

## A. Sketch of the proofs of Lemmas 3.3 and 3.5

This section is devoted to the construction of the curves  $Z^{\text{fwd,in}}$ ,  $Z^{\text{fwd,out}}$ ,  $Z^{\text{bwd,in}}$ ,  $Z^{\text{bwd,out}}$ , i.e., to the proofs of Lemmas 3.3 and 3.5, which are adaption of the argument in [15, §6.2], see in particular [15, Lemmas 6.4, 6.7] and [15, §6.2.2]. We invite the reader to compare the references with Figures 8, 9.

In the whole section we assume the hypotheses of Lemmas 3.3 and 3.5 without further mentioning. Following [15, §6.2.2] we denote by  $\vec{f}^{\perp,\pm}(\vec{x})$  the  $\mathbb{R}^2$ -valued function such that  $\langle \vec{f}^{\perp,\pm}(\vec{x}), \vec{f}^{\pm}(\vec{x}) \rangle = 0$ ,  $\|\vec{f}^{\perp,\pm}(\vec{x})\| = \|\vec{f}^{\pm}(\vec{x})\|$ , i.e.  $\vec{f}^{\perp,\pm}(\vec{x})$  is the  $C^r$  function in  $\Omega^{\pm}$  obtained by rotating  $\vec{f}^{\pm}(\vec{x})$  by  $\pi/2$ . We choose the orientation in such a way that  $\vec{\gamma}(t) + c\vec{f}^{\perp,\pm}(\vec{\gamma}(t)) \in E^{\text{in}}$  if  $c > 0$  is small enough.

Then we set

$$\mathcal{K} := 2 \max\left\{ \sup\{\|\vec{g}(t, \vec{x}, 0)\|/\|\vec{f}^+(\vec{x})\| \mid \vec{x} \in B(\Gamma, 1) \cap (\Omega^+ \cup \Omega^0), t \in \mathbb{R}\}, \right. \\ \left. \sup\{\|\vec{g}(t, \vec{x}, 0)\|/\|\vec{f}^-(\vec{x})\| \mid \vec{x} \in B(\Gamma, 1) \cap (\Omega^- \cup \Omega^0), t \in \mathbb{R}\} \right\}.$$

Observe that  $\mathcal{K} > 0$  is bounded, see [15, Page 1457]. Then we denote by

$$\begin{aligned} \vec{f}_a^\pm(\vec{x}) &= \vec{f}^\pm(\vec{x}) + \varepsilon \mathcal{K} \vec{f}^{\pm, \pm}(\vec{x}), \\ \vec{f}_b^\pm(\vec{x}) &= \vec{f}^\pm(\vec{x}) - \varepsilon \mathcal{K} \vec{f}^{\pm, \pm}(\vec{x}). \end{aligned} \quad (114)$$

The curves  $Z^{\text{fwd}, \text{in}}$ ,  $Z^{\text{fwd}, \text{out}}$ ,  $Z^{\text{bwd}, \text{in}}$ ,  $Z^{\text{bwd}, \text{out}}$  are in fact obtained as orbits of trajectories of the autonomous systems  $\dot{\vec{x}} = \vec{f}_a^\pm(\vec{x})$  and  $\dot{\vec{x}} = \vec{f}_b^\pm(\vec{x})$ . From now on we denote with the subscript  $a$  quantities referred to the former system and with the subscript  $b$  quantities referred to the latter system. In particular we denote by  $\vec{y}_a(t, \vec{P})$  and  $\vec{y}_b(t, \vec{P})$ , resp., the trajectory of  $\dot{\vec{y}} = \vec{f}_a^\pm(\vec{y})$  and of  $\dot{\vec{y}} = \vec{f}_b^\pm(\vec{y})$  leaving from  $\vec{P}$  at  $t = 0$ . Notice that  $\vec{f}_a^\pm(\vec{x})$  on  $\Gamma$  aims towards  $E^{\text{in}}$  while  $\vec{f}_b^\pm(\vec{x})$  on  $\Gamma$  aims towards  $E^{\text{out}}$ .

Note that the origin is still a saddle for both the systems so it admits unstable and stable leaves  $W_a^u, W_b^u, W_a^s, W_b^s$ . Due to the discontinuity  $W_a^u$  (and similarly for  $W_b^u, W_a^s, W_b^s$ ) has a corner in the origin and it may be split on a component  $W_a^{u, -}$  entering  $\Omega^-$ , the one we are interested in, and another, say  $W_a^{u, +}$  entering  $\Omega^+$ .

If we follow  $W_a^u, W_a^s, W_b^u, W_b^s$  from the origin towards  $L^0$ , from a continuity argument, we see that they intersect  $L^0$  transversely the first time in points denoted respectively by  $\vec{\zeta}_a^u, \vec{\zeta}_a^s, \vec{\zeta}_b^u, \vec{\zeta}_b^s$ .

Further from [15, Lemma 6.9] we get the following.

REMARK A.1. There are positive constants  $c_a^u, c_b^u, c_a^s, c_b^s$  such that

$$\begin{aligned} \vec{\zeta}_a^u &= \vec{\gamma}(0) + (c_a^u \varepsilon + o(\varepsilon)) \vec{w}, & \vec{\zeta}_b^u &= \vec{\gamma}(0) - (c_b^u \varepsilon + o(\varepsilon)) \vec{w}, \\ \vec{\zeta}_a^s &= \vec{\gamma}(0) - (c_a^s \varepsilon + o(\varepsilon)) \vec{w}, & \vec{\zeta}_b^s &= \vec{\gamma}(0) + (c_b^s \varepsilon + o(\varepsilon)) \vec{w} \end{aligned}$$

where  $\vec{w}$  is a normalized vector, tangent to  $\Omega^0$  in  $\vec{\gamma}(0)$  and oriented so that  $\vec{\gamma}(0) + c\vec{w} \in E^{\text{in}}$  for  $c > 0$  small enough.

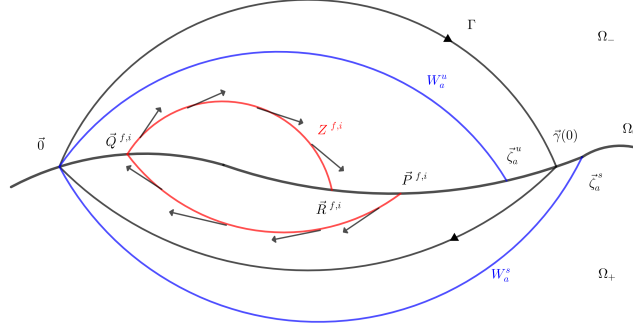
Let us start by a sketch of the proof of Lemma 3.3.

Let  $\vec{P}^{\text{fwd}, \text{in}}(\text{D})$  be the unique point such that

$$\vec{P}^{\text{fwd}, \text{in}}(\text{D}) \in L^0, \quad \|\vec{P}^{\text{fwd}, \text{in}}(\text{D}) - \vec{\zeta}_a^s\| = \text{D} > 0, \quad \langle \vec{P}^{\text{fwd}, \text{in}}(\text{D}) - \vec{\zeta}_a^s, \vec{w} \rangle > 0$$

where  $\text{D} > 0$  is small enough. The following result is the key ingredient to construct the curve  $Z^{\text{fwd}, \text{in}}$  (see Figure 8).

LEMMA A.2. *There is  $\Delta > 0$  such that for any  $0 < D \leq \Delta$  there are  $0 < T_a^{u, 1}(D) < T_a^{u, 2}(D)$  such that  $\vec{y}_a(t, \vec{P}^{\text{fwd}, \text{in}}(D)) \in (\Omega^+ \cap E^{\text{in}})$  for any  $0 < t <$*

Figure 8:  $Z^{\text{fwd,in}}$  constructed via Lemma A.2.

$T_a^{u,1}(D)$ ,  $\vec{y}_a(t, \vec{P}^{\text{fwd,in}}(D)) \in (\Omega^- \cap E^{\text{in}})$  for any  $T_a^{u,1}(D) < t < T_a^{u,2}(D)$ , and it crosses transversely  $L^{\text{in}}$  at  $t = T_a^{u,1}(D)$  in  $\vec{Q}^{\text{fwd,in}}(D) := \vec{y}_a(T_a^{u,1}(D), \vec{P}^{\text{fwd,in}}(D))$  and  $L^0$  at  $t = T_a^{u,2}(D)$  in  $\vec{R}^{\text{fwd,in}}(D) := \vec{y}_a(T_a^{u,2}(D), \vec{P}^{\text{fwd,in}}(D))$ . Further, for any  $0 < \mu \leq \mu_0$ , any  $0 < D \leq \Delta$  and  $0 < \varepsilon \leq \varepsilon_0$  we find

$$\begin{aligned} D^{\sigma^{\text{fwd}}+\mu} &\leq \|\vec{Q}^{\text{fwd,in}}(D)\| \leq D^{\sigma^{\text{fwd}}-\mu}, \\ D^{\sigma^{\text{fwd}}+\mu} &\leq \|\vec{R}^{\text{fwd,in}}(D) - \vec{\zeta}_a^u\| \leq D^{\sigma^{\text{fwd}}-\mu}. \end{aligned} \quad (115)$$

The proof of this lemma is obtained by adapting the argument in [15, §6.2.2] which derives from the estimates in [15, Lemma 6.4].

REMARK A.3. Note that from Remark A.1 and Lemma 115 we find

$$D^{\sigma^{\text{fwd}}+\mu} - \bar{c}\varepsilon \leq \|\vec{R}^{\text{fwd,in}}(D) - \vec{\gamma}(0)\| \leq D^{\sigma^{\text{fwd}}-\mu} + \bar{c}\varepsilon \quad (116)$$

where  $\bar{c} := 2 \max\{c_a^u, c_b^u, c_a^s, c_b^s\}$ .

Let  $\beta \geq \varepsilon^{\sigma^{\text{fb}}/2}$  be as in (13) and recall that  $\sigma^{\text{fb}} < 1$ ; it is easy to check that there is  $\bar{D}^{\text{f},i} \in [\beta - 2c_a^s\varepsilon, \beta - \frac{c_a^s}{2}\varepsilon]$  such that  $\vec{P}^{\text{fwd,in}} := \vec{P}^{\text{fwd,in}}(\bar{D}^{\text{f},i})$  satisfies  $\|\vec{P}^{\text{fwd,in}} - \vec{\gamma}(0)\| = \beta$  and  $\vec{P}^{\text{fwd,in}} \in E^{\text{in}}$ .

Now the curve  $Z^{\text{fwd,in}}$  is defined as follows

$$Z^{\text{fwd,in}} := \{\vec{y}_a(t, \vec{P}^{\text{fwd,in}}) \mid 0 \leq t \leq T_a^{u,2}(\bar{D}^{\text{f},i})\},$$

and we set  $\vec{Q}^{\text{fwd,in}} := \vec{Q}^{\text{fwd,in}}(\bar{D}^{\text{f},i})$ ,  $\vec{R}^{\text{fwd,in}} := \vec{R}^{\text{fwd,in}}(\bar{D}^{\text{f},i})$ .

In fact by construction the flow of (PS) on  $Z^{\text{fwd,in}}$  points towards the exterior of the bounded set enclosed by  $Z^{\text{fwd,in}}$  and the segment between  $\vec{P}^{\text{fwd,in}}$  and

$\vec{R}^{\text{fwd},\text{in}}$ , see Figure 8. Now the estimates concerning  $\vec{P}^{\text{fwd},\text{in}}$ ,  $\vec{Q}^{\text{fwd},\text{in}}$ ,  $\vec{R}^{\text{fwd},\text{in}}$  in Lemma 3.3 follow from Lemma A.2 and Remark A.3.

Let  $\vec{P}^{\text{fwd},\text{out}}(D)$  be the unique point such that

$$\vec{P}^{\text{fwd},\text{out}}(D) \in L^0, \quad \|\vec{P}^{\text{fwd},\text{out}}(D) - \vec{\zeta}_b^u\| = D > 0, \quad \langle \vec{P}^{\text{fwd},\text{out}}(D) - \vec{\zeta}_b^u, \vec{w} \rangle < 0$$

where  $D > 0$  is small enough. Rephrasing the argument of [15, §6.2.2] which is based on [15, Lemma 6.7], we get the following result needed to construct  $Z^{\text{fwd},\text{out}}$  (see Figure 9):

LEMMA A.4. *There is  $\Delta > 0$  such that for any  $0 < D < \Delta$  there are  $\tau_b^{u,-1}(D) < 0 < \tau_b^{u,1}(D)$  such that  $\vec{y}_b(t, \vec{P}^{\text{fwd},\text{out}}(D)) \in (\Omega^- \cap E^{\text{out}})$  for any  $\tau_b^{u,-1}(D) \leq t < 0$ ,  $\vec{y}_b(t, \vec{P}^{\text{fwd},\text{out}}(D)) \in (\Omega^+ \cap E^{\text{out}})$  for any  $0 < t \leq \tau_b^{u,1}(D)$ , and it crosses transversely  $L^{-,\text{out}}$  at  $t = \tau_b^{u,-1}(D)$  in  $\vec{O}^{\text{fwd},\text{out}}(D) := \vec{y}_b(\tau_b^{u,-1}(D), \vec{P}^{\text{fwd},\text{out}}(D))$  and  $L^{+,\text{out}}$  at  $t = \tau_b^{u,1}(D)$  in  $\vec{Q}^{\text{fwd},\text{out}}(D) := \vec{y}_b(\tau_b^{u,1}(D), \vec{P}^{\text{fwd},\text{out}}(D))$ . Further, for any  $0 < \mu \leq \mu_0$ , any  $0 < D \leq \Delta$  and  $0 < \varepsilon \leq \varepsilon_0$  we find*

$$\begin{aligned} D^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{O}^{\text{fwd},\text{out}}(D)\| \leq D^{\sigma^{\text{bwd}}-\mu}, \\ [D + (c_b^s + c_b^u)\varepsilon]^{\sigma_+^{\text{fwd}}+\mu} &\leq \|\vec{Q}^{\text{fwd},\text{out}}(D)\| \leq [D + (c_b^s + c_b^u)\varepsilon]^{\sigma_+^{\text{fwd}}-\mu}. \end{aligned} \quad (117)$$

Again, there is  $\bar{D}^{\text{f},\text{o}} \in [\beta - 2c_b^u\varepsilon, \beta - \frac{c_b^u}{2}\varepsilon]$  such that  $\vec{P}^{\text{fwd},\text{out}} := \vec{P}^{\text{fwd},\text{out}}(\bar{D}^{\text{f},\text{o}})$  satisfies  $\|\vec{P}^{\text{fwd},\text{out}} - \vec{\gamma}(0)\| = \beta$  and  $\vec{P}^{\text{fwd},\text{out}} \in E^{\text{out}}$ .

Then the curve  $Z^{\text{fwd},\text{out}}$  is defined as follows

$$Z^{\text{fwd},\text{out}} := \{\vec{y}_b(t, \vec{P}^{\text{fwd},\text{out}}) \mid \tau_b^{u,-1}(\bar{D}^{\text{f},\text{o}}) \leq t \leq \tau_b^{u,1}(\bar{D}^{\text{f},\text{o}})\},$$

and we set  $\vec{O}^{\text{fwd},\text{out}} := \vec{O}^{\text{fwd},\text{out}}(\bar{D}^{\text{f},\text{o}})$ ,  $\vec{Q}^{\text{fwd},\text{out}} := \vec{Q}^{\text{fwd},\text{out}}(\bar{D}^{\text{f},\text{o}})$ .

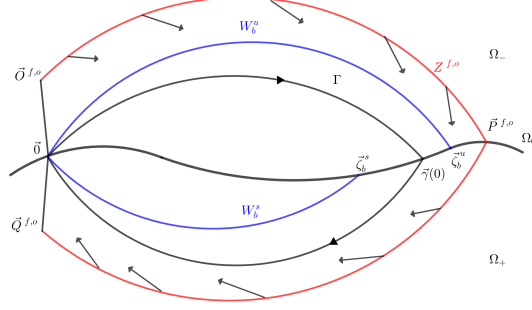
In fact by construction the flow of (PS) on  $Z^{\text{fwd},\text{out}}$  points towards the interior of the bounded set enclosed by  $Z^{\text{fwd},\text{out}}$  and the segments  $L^{-,\text{out}}$  between  $\vec{O}$  and  $\vec{O}^{\text{fwd},\text{out}}$ , and  $L^{+,\text{out}}$  between  $\vec{O}$  and  $\vec{Q}^{\text{fwd},\text{out}}$ . Further the estimates concerning  $\vec{O}^{\text{fwd},\text{out}}$ ,  $\vec{P}^{\text{fwd},\text{out}}$ ,  $\vec{Q}^{\text{fwd},\text{out}}$  in Lemma 3.3 follow from Lemma A.4. So Lemma 3.3 is proved.

The construction of the curve  $Z^{\text{bwd},\text{in}}$  (see Figure 10) is analogous to  $Z^{\text{fwd},\text{in}}$ . Namely, let  $\vec{P}^{\text{bwd},\text{in}}(D)$  be the unique point such that

$$\vec{P}^{\text{bwd},\text{in}}(D) \in L^0, \quad \|\vec{P}^{\text{bwd},\text{in}}(D) - \vec{\zeta}_b^u\| = D > 0, \quad \langle \vec{P}^{\text{bwd},\text{in}}(D) - \vec{\zeta}_b^u, \vec{w} \rangle > 0.$$

Reasoning as in Lemma A.2 we get the following.

LEMMA A.5. *There is  $\Delta > 0$  such that for any  $0 < D \leq \Delta$  there are  $T_b^{s,-2}(D) < T_b^{s,-1}(D) < 0$  such that  $\vec{y}_b(t, \vec{P}^{\text{bwd},\text{in}}(D)) \in (\Omega^- \cap E^{\text{in}})$  for any  $T_b^{s,-1}(D) < t <$*

Figure 9:  $Z^{\text{fwd,out}}$  constructed in Lemma A.4.

$0, \vec{y}_b(t, \vec{P}^{\text{bwd,in}}(D)) \in (\Omega^+ \cap E^{\text{in}})$  for any  $T_b^{s,-2}(D) < t < T_b^{s,-1}(D)$ . Let us set

$$\begin{aligned} \vec{Q}^{\text{bwd,in}}(D) &:= \vec{y}_b(T_b^{s,-1}(D), \vec{P}^{\text{bwd,in}}(D)) \quad \text{and} \\ \vec{R}^{\text{bwd,in}}(D) &:= \vec{y}_b(T_b^{s,-2}(D), \vec{P}^{\text{bwd,in}}(D)). \end{aligned}$$

Then, for any  $0 < \mu \leq \mu_0$ , for any  $0 < D \leq \Delta$  and  $0 < \varepsilon \leq \varepsilon_0$  we find

$$\begin{aligned} D^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{Q}^{\text{bwd,in}}(D)\| \leq D^{\sigma^{\text{bwd}}-\mu}, \\ D^{\sigma^{\text{bwd}}+\mu} &\leq \|\vec{R}^{\text{bwd,in}}(D) - \vec{\zeta}_b^s\| \leq D^{\sigma^{\text{bwd}}-\mu}. \end{aligned} \quad (118)$$

REMARK A.6. Analogously to Remark A.3 we see that

$$D^{\sigma^{\text{bwd}}+\mu} - \bar{c}\varepsilon \leq \|\vec{R}^{\text{bwd,in}}(D) - \vec{\gamma}(0)\| \leq D^{\sigma^{\text{bwd}}-\mu} + \bar{c}\varepsilon.$$

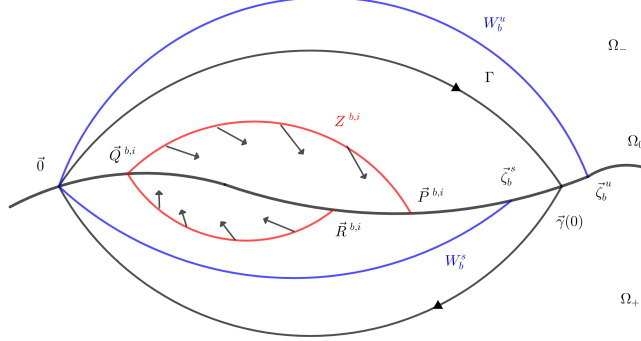
Again we see that there is  $\bar{D}^{\text{b,i}} \in [\beta + \frac{c_b^u}{2}\varepsilon, \beta + 2c_b^u\varepsilon]$  such that  $\vec{P}^{\text{bwd,in}} := \vec{P}^{\text{bwd,in}}(\bar{D}^{\text{b,i}})$  satisfies  $\|\vec{P}^{\text{bwd,in}} - \vec{\gamma}(0)\| = \beta$  and  $\vec{P}^{\text{bwd,in}} \in E^{\text{in}}$ .

So the curve  $Z^{\text{bwd,in}}$  is defined as follows

$$Z^{\text{bwd,in}} := \{\vec{y}_b(t, \vec{P}^{\text{bwd,in}}) \mid T_b^{s,-2}(\bar{D}^{\text{b,i}}) \leq t \leq 0\},$$

and we set  $\vec{Q}^{\text{bwd,in}} := \vec{Q}^{\text{bwd,in}}(\bar{D}^{\text{b,i}})$  and  $\vec{R}^{\text{bwd,in}} := \vec{R}^{\text{bwd,in}}(\bar{D}^{\text{b,i}})$ .

In fact by construction the flow of (PS) on  $Z^{\text{bwd,in}}$  aims toward the interior of the bounded set enclosed by  $Z^{\text{bwd,in}}$  and the segment between  $\vec{P}^{\text{bwd,in}}$  and  $\vec{R}^{\text{bwd,in}}$ . Further the estimates concerning  $\vec{P}^{\text{bwd,in}}$ ,  $\vec{Q}^{\text{bwd,in}}$ ,  $\vec{R}^{\text{bwd,in}}$  in Lemma 3.5 follow from Lemma A.5 and Remark A.6.

Figure 10:  $Z^{\text{bwd,in}}$  constructed in Lemma A.5.

Now, let  $\vec{P}^{\text{bwd,out}}(D)$  be the unique point such that

$$\vec{P}^{\text{bwd,out}}(D) \in L^0, \quad \|\vec{P}^{\text{bwd,out}}(D) - \vec{\zeta}_a^s\| = D > 0, \quad \langle \vec{P}^{\text{bwd,out}}(D) - \vec{\zeta}_a^s, \vec{w} \rangle < 0$$

where  $D > 0$  is small enough. Reasoning as in Lemma A.4 we get  $Z^{\text{bwd,out}}$  (see Figure 11):

LEMMA A.7. *There is  $\Delta > 0$  such that for any  $0 < D \leq \Delta$  there are  $\tau_a^{s,-1}(D) < 0 < \tau_a^{s,1}(D)$  such that  $\vec{y}_a(t, \vec{P}^{\text{bwd,out}}(D)) \in (\Omega^- \cap E^{\text{out}})$  for any  $\tau_a^{s,-1}(D) \leq t < 0$ ,  $\vec{y}_a(t, \vec{P}^{\text{bwd,out}}(D)) \in (\Omega^+ \cap E^{\text{out}})$  for any  $0 < t \leq \tau_a^{s,1}(D)$ , and it crosses transversely  $L^{-,\text{out}}$  at  $t = \tau_a^{s,-1}(D)$  in  $\vec{O}^{\text{bwd,out}}(D) := \vec{y}_a(\tau_a^{s,-1}(D), \vec{P}^{\text{bwd,out}}(D))$  and  $L^{+,\text{out}}$  at  $t = \tau_a^{s,1}(D)$  in  $\vec{Q}^{\text{bwd,out}}(D) := \vec{y}_a(\tau_a^{s,1}(D), \vec{P}^{\text{bwd,out}}(D))$ . Further, for any  $0 < \mu \leq \mu_0$ , any  $0 < D \leq \Delta$  and  $0 < \varepsilon \leq \varepsilon_0$  we find*

$$\begin{aligned} [D + (c_a^s + c_a^u)\varepsilon]^{\sigma_-^{\text{bwd}} + \mu} &\leq \|\vec{O}^{\text{bwd,out}}(D)\| \leq [D + (c_a^s + c_a^u)\varepsilon]^{\sigma_-^{\text{bwd}} - \mu}, \\ D^{\sigma_+^{\text{fwd}} + \mu} &\leq \|\vec{Q}^{\text{bwd,out}}(D)\| \leq D^{\sigma_+^{\text{fwd}} - \mu}. \end{aligned} \quad (119)$$

Again, there is  $\bar{D}^{\text{b},0} \in [\beta - 2c_a^s\varepsilon, \beta - \frac{c_a^s}{2}\varepsilon]$  such that  $\vec{P}^{\text{bwd,out}} := \vec{P}^{\text{bwd,out}}(\bar{D}^{\text{b},0})$  satisfies  $\|\vec{P}^{\text{bwd,out}} - \vec{\gamma}(0)\| = \beta$  and  $\vec{P}^{\text{bwd,in}} \in E^{\text{out}}$ .

Then the curve  $Z^{\text{bwd,out}}$  is defined as follows

$$Z^{\text{bwd,out}} := \{\vec{y}_a(t, \vec{P}^{\text{bwd,out}}(\bar{D}^{\text{b},0}) \mid \tau_a^{s,-1}(\bar{D}^{\text{b},0}) \leq t \leq \tau_a^{s,1}(\bar{D}^{\text{b},0})\},$$

and we set  $\vec{O}^{\text{bwd,out}} := \vec{O}^{\text{bwd,out}}(\bar{D}^{\text{b},0})$ ,  $\vec{Q}^{\text{bwd,out}} := \vec{Q}^{\text{bwd,out}}(\bar{D}^{\text{b},0})$ .

The flow of (PS) on  $Z^{\text{bwd,out}}$  aims towards the exterior of the bounded set enclosed by  $Z^{\text{bwd,out}}$  and the segments  $L^{-,\text{out}}$  between  $\vec{0}$  and  $\vec{O}^{\text{bwd,out}}$ ,



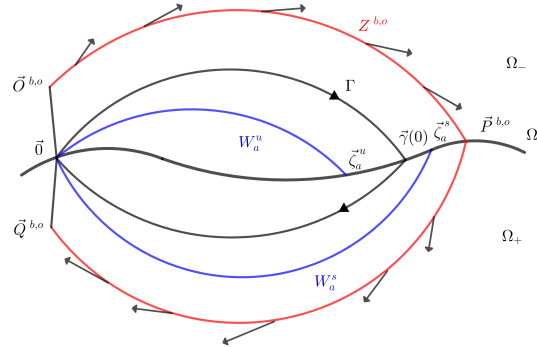


Figure 11:  $Z^{\text{bwd,out}}$  constructed in Lemma A.7.

and  $L^{+,out}$  between  $\vec{0}$  and  $\vec{Q}^{\text{bwd,out}}$ , see Figure 11. Moreover, the estimates concerning  $\vec{O}^{\text{bwd,out}}$ ,  $\vec{P}^{\text{bwd,out}}$ ,  $\vec{Q}^{\text{bwd,out}}$  follow from Lemma A.7. So Lemma 3.5 is proved.

### Acknowledgement

A. Calamai and M. Franca were partially supported by G.N.A.M.P.A. - INdAM (Italy) and by PRIN 2022 - Progetti di Ricerca di rilevante Interesse Nazionale, *Nonlinear differential problems with applications to real phenomena* (Grant Number: 2022ZXZTN2). M. Pospíšil was partially supported by the Slovak Research and Development Agency under the Contract no. APVV-23-0039, and by the Grants VEGA 1/0084/23 and VEGA-SAV 2/0062/24.

### REFERENCES

- [1] S. BANERJEE AND G. VERGHESE, *Nonlinear Phenomena in Power Electronics: Attractors, Bifurcations, Chaos and Nonlinear Control*, Wiley-IEEE Press, New York, 2001.
- [2] F. BATTELLI AND M. FEČKAN, *An example of chaotic behaviour in presence of a sliding homoclinic orbit*, Ann. Mat. Pura Appl. **189** (2010) no. 4, 615–642.
- [3] F. BATTELLI AND M. FEČKAN, *On the chaotic behaviour of discontinuous systems*, J. Dynam. Differential Equations **23** (2011) no. 3, 495–540.
- [4] F. BATTELLI AND M. FEČKAN, *Nonsmooth homoclinic orbits, Melnikov functions and chaos in discontinuous systems*, Phys. D **241** (2012), no. 22, 1962–1975.

- [5] M.D. BERNARDO, L. GAROFALO, AND F. VASCA, *Bifurcations in piecewise-smooth feedback systems*, Int. J. Control **75** (2002), 1243–1259.
- [6] G. BORG, *A condition for the existence of orbitally stable solutions of dynamical systems*, Kungl. Tekn. Högsk. Handl. Stockholm, **153** (1960), 1–12.
- [7] B. BROGLIATO, *Nonsmooth Mechanics*, Springer, London, 1999.
- [8] A. CALAMAI AND M. FRANCA, *Melnikov methods and homoclinic orbits in discontinuous systems*, J. Dynam. Differential Equations **25** (2013) no. 3, 733–764.
- [9] A. CALAMAI, J. DIBLÍK, M. FRANCA, AND M. POSPÍŠIL, *On the position of chaotic trajectories*, J. Dynam. Differential Equations **29** (2017) no. 4, 1423–1458.
- [10] C.C. CHEN AND C.S. LIN: *Blowing up with infinite energy of conformal metrics on  $S^n$* , Comm. Partial Differential Equations **24** (1999), 785–799.
- [11] W.A. COPPEL, *Dichotomies in Stability Theory*, Lecture Notes in Math., vol. 629, Springer, New York, 1978.
- [12] F. DALBONO, M. FRANCA, AND A. SFECCI, *Multiplicity of ground states for the scalar curvature equation*, Ann. Mat. Pura Appl. **199** (2020), No. 1, 273–298.
- [13] F. DALBONO, M. FRANCA, AND A. SFECCI, *Multiplicity of ground states for the scalar curvature equation without reciprocal symmetry*, J. Dynam. Differential Equations **34** (2022), 701–720.
- [14] L. DIECI, C. ELIA AND L. LOPEZ, *A Filippov sliding vector field on an attracting co-dimension 2 discontinuity surface, and a limited loss-of-attractivity analysis*, J. Differential Equations **254** (2013) no. 4, 1800–1832.
- [15] M. FRANCA AND M. POSPÍŠIL, *New global bifurcation diagrams for piecewise smooth systems: Transversality of homoclinic points does not imply chaos*, J. Differential Equations **266** (2019), 1429–1461.
- [16] M. FRANCA AND A. SFECCI, *On a diffusion model with absorption and production*, Nonlinear Anal. Real World Appl. **34** (2017), 41–60.
- [17] M. GARCIA, A. CHATTERJEE, A. RUINA, AND M. COLEMAN, *The simplest walking model: stability, complexity and scaling*, ASME. J. Biomech. Eng. **120** (1998), 281–288.
- [18] P. GIESL, *Necessary conditions for a limit cycle and its basin of attraction*, Nonlinear Analysis **56** (5) (2004), 643–677.
- [19] P. GIESL, *The basin of attraction of periodic orbits in nonsmooth differential equations*, Z. Angew. Math. Mech., **85** (2) (2005), 89–104.
- [20] P. GIESL AND M. RASMUSSEN, *Borg’s criterion for almost periodic differential equations*, Nonlinear Analysis **69** 11 (2008), 3722–3733.
- [21] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.
- [22] J. HALE AND H. KOÇAK, *Dynamics and Bifurcations*, Volume 3, Springer, New York, 1991.
- [23] P. HARTMAN AND C. OLECH, *On global asymptotic stability of solutions of differential equations*, Trans. Amer. Math. Soc. **104** (1962) 154–178.
- [24] R. JOHNSON, *Concerning a theorem of Sell*, J. Differential Equations **30** (1978), 324–339.
- [25] R. JOHNSON, X.B. PAN, AND Y.F. YI, *The Melnikov method and elliptic equations with critical exponent*, Indiana Univ. Math. J. **43** (1994), no. 3, 1045–1077.

- [26] G.A. LEONOV, I.M. BURKIN, AND A.I. SHEPELYAVYI, *Frequency Methods in Oscillation Theory*, in: Series in Mathematics and Its Applications, Vol. 357, Kluwer, Dordrecht/Boston/London, 1996.
- [27] S. LI, C. SHEN, W. ZHANG, AND Y. HAO, *The Melnikov method of heteroclinic orbits for a class of planar hybrid piecewise-smooth systems and application*, *Nonlinear Dynam.* **85** (2016), no. 2, 1091–1104.
- [28] V.K. MELNIKOV *On the stability of the center for time-periodic perturbations*, *Trans. Mosc. Math. Soc.* **12** (1963), 1–56.
- [29] K.J. PALMER, *Exponential dichotomies and transversal homoclinic points*, *J. Differential Equations* **55** (1984), no. 2, 225–256.
- [30] E. PLATHE AND S. KJØGLUM, *Analysis and genetic properties of gene regulatory networks with graded response functions*, *Phys. D* **201** (2005), 150–176.
- [31] L.P. SHILNIKOV, A.L. SHILNIKOV, D.V. TURAEV, AND L.O. CHUA, *Methods of Qualitative Theory in Nonlinear Dynamics, Part II*, World Scientific Publishing Company, Singapore, 2001.
- [32] S. WIGGINS, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, New York, 2003.
- [33] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications, I: Fixed Point Theorems*, Springer, New York, 1986.

Authors' addresses:

Alessandro Calamai  
 Dipartimento di Ingegneria Civile, Edile e Architettura  
 Università Politecnica delle Marche  
 Via Brecce Bianche 1  
 60131 Ancona, Italy  
 E-mail: [a.calamai@staff.univpm.it](mailto:a.calamai@staff.univpm.it)

Matteo Franca  
 Dipartimento di Scienze Matematiche  
 Università di Bologna  
 Piazza di Porta S. Donato, 5  
 40126 Bologna, Italy  
 E-mail: [matteo.franca4@unibo.it](mailto:matteo.franca4@unibo.it)

Michal Pospíšil  
 Department of Mathematical Analysis and Numerical Mathematics  
 Faculty of Mathematics, Physics and Informatics  
 Comenius University in Bratislava  
 Mlynská dolina, 842 48 Bratislava, Slovakia  
 Mathematical Institute  
 Slovak Academy of Sciences  
 Štefánikova 49, 814 73  
 Bratislava, Slovakia. E-mail: [michal.pospisil@fmph.uniba.sk](mailto:michal.pospisil@fmph.uniba.sk)

Received September 9, 2024  
 Accepted January 14, 2025