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On fundamental forms and osculating bundles

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To the memory of Gianni Sacchiero

Abstract. *We define higher order fundamental forms and osculating spaces of projective algebraic varieties, using sheaves of principal parts. We show that the mth fundamental form can be viewed as the differential of the $(m-1)$th Gauss map, and explain why the vanishing of the mth fundamental form implies that the variety is contained in a general $(m-1)$th osculating space. Pointwise, the fundamental forms give linear systems on the projectivized tangent spaces. We show that, at each point, the Jacobian of the mth fundamental form is contained in the $(m-1)$th fundamental form. In the case of ruled varieties, we describe these linear systems. We discuss conditions for a surface to be ruled, in terms of the second fundamental form and the Fubini cubic.*

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1. Introduction

In classical differential geometry the *second fundamental form* of a surface in $\mathbb{R}^3$ at a smooth point is a quadratic form on the tangent space to the surface at that point. The starting point of this paper is the work by Griffiths and Harris published in 1979 [10]. Using Darboux frames, they defined this quadratic form pointwise for a complex analytic projective variety and showed that it could be viewed as the fiber of a map from the second symmetric product of the tangent bundle to the normal bundle [10, (1.18), p. 366]. They also defined higher fundamental forms pointwise, using Darboux frames, and gave a similar description of the corresponding maps of bundles. Our reading of their paper led us, more than thirty years ago, to define, in a purely algebraic way and without using frames, the higher fundamental forms of a quasi-projective variety. This definition was not published at that time, but appeared in lectures and in the Master thesis of Tegnander [24]. A similar definition was recently given by Ein and Niu in their paper [7], which made us revisit our old notes and expand them into the present paper, where we give various interpretations and properties of these fundamental forms and the linear systems associated with them. Several of our results can also be found, or have analogues, in papers by other authors, such as [4, 10, 14, 23].

The paper is organized as follows. In the next section we define fundamental forms of an algebraic variety X in projective space, using sheaves of principal parts that define the osculating spaces of the variety. We relate our definition to the definition by Altman and Kleiman [1] of the second fundamental form of a subsheaf and prove Theorem 2.4, which generalizes an observation by Perkinson [19]. We define higher order Gauss maps, and show that Proposition 2.3 and Theorem 2.4 imply that the mth fundamental form is equal to the differential of the $(m-1)$th Gauss map. We also explain why the vanishing of the mth fundamental form implies that X is contained in its $(m-1)$th osculating space at a (general) point.

In the third section we consider the interpretation of fundamental forms as linear systems on the projectivized tangent spaces. We prove in Theorem 3.2 that the Jacobian of the mth fundamental form is contained in the $(m-1)$th fundamental form. We illustrate our results by three examples; these are non-ruled surfaces in $\mathbb{P}^5$ such that the second order osculating spaces have dimension 4 (instead of the expected dimension 5). The second fundamental forms are pencils of quadrics in $\mathbb{P}^1$; in one case, these pencils have a base point, in the two other cases, they do not.

In the fourth section we study and describe the fundamental forms of projective ruled varieties $\pi\colon X = \mathbb{P}(\mathcal{E}) \to Y$. We use a result of Landsberg [16] to give a condition for a *surface* to be ruled, in terms of the second fundamental form and the Fubini cubic discussed in [10]. We can view ruled varieties as varieties in a Grassmann variety, and we show that the bundles of principal parts of $\mathcal{E}$ on Y are equal to the pushdowns of the bundles of principal parts of $\mathcal{O}_X(1)$ on X.

This work grew out of old notes by the authors. A part of these notes were based on writings by the second author in 1988–89, while she was a Science Scholar at the Bunting Institute of Radcliffe College,[1] and on her lecture “Espaces osculateurs, formes fondamentales et multiplicités des discriminants” given at École Normale Supérieure in Paris on November 26, 1992. In the 1992 Master thesis of Cathrine Tegnander this definition of fundamental forms is used [24, §4.1], and some of our surface examples are taken from her thesis.

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2. Fundamental forms

Let k be an algebraically closed field and V a k-vector space of dimension $N+1$. Suppose X is a smooth (but not necessarily proper), irreducible k-scheme, of dimension r, and that $f\colon X \to \mathbb{P}(V)$ is a morphism which is birational onto its image.

Set $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}(V)}(1)$, and let $\mathcal{P}^m_X(\mathcal{L})$ denote the sheaf of principal parts of order m of $\mathcal{L}$, for $m \geq 0$. Recall the definition of these sheaves: Consider the product $X \times X$, and let $\mathcal{I} \subset \mathcal{O}_{X\times X}$ denote the ideal sheaf of the diagonal. The mth infinitesimal neighborhood of the diagonal is the subscheme $X^{(m)} \subset X \times X$ defined by the ideal sheaf $\mathcal{I}^{m+1}$. Let $p_i\colon X^{(m)} \to X$, $i = 1, 2$ denote the projection maps. If $\mathcal{F}$ is a sheaf on X, then $\mathcal{P}^m_X(\mathcal{F}) := p_{1*}p_2^*\mathcal{F}$ is the sheaf of principal parts of $\mathcal{F}$ of order m. We refer to [11, §16.7] and [20, § 6, pp. 492–494] for further details.

Since X is smooth, of dimension r, and $\mathcal{L}$ is a line bundle, $\mathcal{P}^m_X(\mathcal{L})$ is locally free, with rank $\binom{r+m}{m}$, and there are exact sequences, for $m \geq 1$,

$$0 \to S^m\Omega^1_X \otimes \mathcal{L} \to \mathcal{P}^m_X(\mathcal{L}) \to \mathcal{P}^{m-1}_X(\mathcal{L}) \to 0.$$

Moreover, for each m, there is a natural map

$$H^0(X, \mathcal{L})_X := H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{P}^m_X(\mathcal{L}).$$

Note that $V = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$, so that

$$V_X := V \otimes \mathcal{O}_X = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))_X := H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \otimes \mathcal{O}_X.$$

We denote by

$$a^m\colon V_X \to \mathcal{P}^m_X(\mathcal{L})$$

the map obtained by composing the above natural map with the homomorphism

$$V_X = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))_X \to H^0(X, \mathcal{L})_X.$$

The maps a^m are locally just Taylor series expansion up to order m of the coordinate functions on X, with the variables being local coordinates on X. They are compatible with the surjections

$$\mathcal{P}^m_X(\mathcal{L}) \to \mathcal{P}^{m-i}_X(\mathcal{L}).$$

Now we set $\mathcal{K}_m = \operatorname{Ker}(a^m)$, and consider the maps ϕ_m defined by the commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_{m-1} & \longrightarrow & V_X & \xrightarrow{a^{m-1}} & \mathcal{P}^{m-1}_X(\mathcal{L}) & & \\
 & & \Big\downarrow{\scriptstyle \phi_m} & & \Big\downarrow{\scriptstyle a^m} & & \Big\| & & \\
0 & \longrightarrow & S^m\Omega^1_X \otimes \mathcal{L} & \longrightarrow & \mathcal{P}^m_X(\mathcal{L}) & \longrightarrow & \mathcal{P}^{m-1}_X(\mathcal{L}) & \longrightarrow & 0.
\end{array}$$

It follows that $\mathcal{K}_m \subseteq \mathcal{K}_{m-1}$ and that $\operatorname{Ker}(\phi_m) = \mathcal{K}_m$.

Definition 2.1. *The mth fundamental form of X (with respect to f) is the injective homomorphism induced by ϕ_m,*

$$\Phi_m \colon \mathcal{K}_{m-1}/\mathcal{K}_m \to S^m\Omega^1_X \otimes \mathcal{L}.$$

Let us first briefly compare this definition with the "classical" fundamental forms, as defined locally (see e.g. [10, (1.18), p. 366 and (1.46), p. 373] for $m = 2, 3$). For this, we may assume that f is an embedding. Let $T_X := (\Omega^1_X)^\vee$ denote the tangent bundle to X. If $m!$ is invertible in k, the natural map

$$S^m T_X \to (S^m\Omega^1_X)^\vee$$

is an isomorphism (see [2, Lemma (2.13), p. 21], [9, B. 3, p.476], and [18, p. 248]). Hence we obtain a map

$$S^m T_X \to \mathcal{K}^\vee_{m-1}/\mathcal{K}^\vee_m \otimes \mathcal{L}$$

by composing with $\Phi^\vee_m \otimes \mathrm{id}_{\mathcal{L}}$.

For $m = 2$ we get

$$\phi_2 \colon \mathcal{K}_1 = \mathcal{N}_{X/\mathbb{P}(V)} \otimes \mathcal{L} \to S^2\Omega^1_X \otimes \mathcal{L},$$

where $\mathcal{N}_{X/\mathbb{P}(V)}$ is the conormal sheaf of X in $\mathbb{P}(V)$, and hence a map $S^2T_X \to \mathcal{N}^\vee_{X/\mathbb{P}(V)}$, whose fibers are the classical second fundamental forms. We shall consider the linear systems induced by the fundamental forms in Section 3.

Altman and Kleiman [1, I.3, p. 10] gave a general definition of the second fundamental form of a subsheaf of a quasi-coherent sheaf on a scheme. We shall now recall their construction, in our situation.

Suppose $\mathcal{F}$ is a coherent sheaf on X and that

$$\alpha \colon V_X \to \mathcal{F}$$

is a homomorphism. Set $\mathcal{E} := \operatorname{Ker}(\alpha)$, $Z := X \times \mathbb{P}(V)$, $Y := \mathbb{P}(V_X/\mathcal{E}) \subseteq Z$, and let $\mathrm{pr}_1 \colon Z \to X$, $\mathrm{pr}_2 \colon Z \to \mathbb{P}(V)$, and $p \colon Y \to X$ denote the projection morphisms. By [2, Lemma (2.6), p. 17], there is a natural homomorphism

$$p^*\mathcal{E} \otimes \mathcal{O}_Y(-1) \to \mathcal{N}_{Y/Z},$$

where $\mathcal{N}_{Y/Z}$ denotes the conormal sheaf of Y in Z, which is an isomorphism if $\mathcal{F}$ is locally free. Now we compose this homomorphism with the homomorphisms

$$\mathcal{N}_{Y/Z} \to \Omega^1_Z|_Y = (\mathrm{pr}_1^*\Omega^1_X \oplus \mathrm{pr}_2^*\Omega^1_{\mathbb{P}(V)})|_Y \to \mathrm{pr}_1^*\Omega^1_X|_Y = p^*\Omega^1_X.$$

Thus we obtain a homomorphism

$$p^*\mathcal{E} \otimes \mathcal{O}_Y(-1) \to p^*\Omega^1_X,$$

hence also

$$p^*\mathcal{E} \to p^*\Omega^1_X \otimes \mathcal{O}_Y(1),$$

and, by adjunction and the projection formula (since Ω^1_X is locally free),

$$\mathcal{E} \to p_*(p^*\Omega^1_X \otimes \mathcal{O}_Y(1)) \cong \Omega^1_X \otimes p_*\mathcal{O}_Y(1).$$

Let $U \subseteq X$ be the open dense subset where α has constant rank. On U, $p_*\mathcal{O}_Y(1) \cong \operatorname{Im}(\alpha) = V_X/\mathcal{E}$, so, by composing with the inclusion $V_X/\mathcal{E} \to \mathcal{F}$, we finally obtain a homomorphism, defined on U, and denoted $F(\alpha)$,

$$F(\alpha)\colon \mathcal{E}|_U \to \Omega^1_U \otimes \mathcal{F}|_U.$$

Definition 2.2 ([1, I.3, p. 10]). *The second fundamental form of* $\mathcal{E} := \operatorname{Ker}(\alpha)$ *in* V_X *is the homomorphism*

$$F(\alpha)\colon \mathcal{E}|_U \to (\Omega^1_X \otimes \mathcal{F})|_U$$

constructed above.

By [1, Thm. (3.1), p. 11] (see also [8, B.5.8, p. 435]), the second fundamental form of the kernel of the surjection $a^0\colon V_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1)$ identifies this kernel with the sheaf $\Omega^1_{\mathbb{P}(V)}(1)$. Via this identification, we get $\mathcal{K}_0 = f^*\Omega^1_{\mathbb{P}(V)} \otimes \mathcal{L}$ and $\phi_1 = df \otimes \mathrm{id}_{\mathcal{L}}$. Moreover, if $U \subseteq X$ denotes the open dense subset such that $f|_U$ is an embedding, then on U, $\mathcal{K}_1 = \mathcal{N}_{f(X)/\mathbb{P}(V)}|_U \otimes \mathcal{L}$ and $\mathcal{K}_0/\mathcal{K}_1 \cong \Omega^1_X \otimes \mathcal{L}$, where $\mathcal{N}_{f(X)/\mathbb{P}(V)}$ denotes the conormal sheaf of $f(X)$ in $\mathbb{P}(V)$. Hence $\Phi_1|_U = \mathrm{id}$ is trivial.

As remarked by Perkinson [19, Remark 2.4, p. 3183], in this case the map ϕ_1, induced by the Taylor map $a^1\colon V_{\mathbb{P}(V)} \to \mathcal{P}^1_{\mathbb{P}(V)}(1)$, is equal to $-F(a^0)$. We shall generalize this in Theorem 2.4.

For a vector space V and a non-negative integer s, let $Grass_{s+1}(V)$ denote the Grassmann variety of s-dimensional linear subspaces of $\mathbb{P}(V)$. Note that if $V \to V'$ is a $(s+1)$-dimensional quotient of V, then $\mathbb{P}(V') \subset \mathbb{P}(V)$ is a s-dimensional linear subspace.

Proposition 2.3. *Assume* $\mathcal{F}$ *is locally free, with rank* $s+1$, *and that*

$$\alpha\colon V_X \to \mathcal{F}$$

is surjective. Set $\mathcal{E} := \operatorname{Ker}(\alpha)$, *and* $G := \operatorname{Grass}_{s+1}(V)$, *and let* $\psi\colon X \to G$ *denote the morphism corresponding to* α. *Then the second fundamental form of* $\mathcal{E}$,

$$F(\alpha)\colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{F},$$

induces

$$d\psi\colon \psi^*\Omega^1_G \to \Omega^1_X$$

via the isomorphism $\psi^*\Omega^1_G \cong \mathcal{E}\otimes\mathcal{F}^\vee$.

Proof. See [1, I.3, p. 10] and [8, B.5.8, p. 435]. □

We shall now show how the mth fundamental form of X (with respect to the map $f\colon X\to\mathbb{P}(V)$) is related to the second fundamental form of the kernel $\mathcal{K}_{m-1}$ of the homomorphism

$$a^{m-1}\colon V_X \to \mathcal{P}^{m-1}_X(\mathcal{L}).$$

We have the following result.

Theorem 2.4. *The second fundamental form of* $\mathcal{K}_{m-1}$,

$$F(a^{m-1})\colon \mathcal{K}_{m-1}\to \Omega^1_X\otimes\mathcal{P}^{m-1}_X(\mathcal{L}),$$

factors through the inclusion

$$\Omega^1_X\otimes S^{m-1}\Omega^1_X\otimes\mathcal{L}\hookrightarrow \Omega^1_X\otimes\mathcal{P}^{m-1}_X(\mathcal{L}),$$

and the induced homomorphism

$$\overline{\phi}_m\colon \mathcal{K}_{m-1}\to S^m\Omega^1_X\otimes\mathcal{L}$$

satisfies

$$\overline{\phi}_m = -m\phi_m,$$

where ϕ_m *is the map used to define the* m*th fundamental form of* X.

Proof. In order to show that $F(a^{m-1})$ factors as stated, we must show that the image of $F(a^{m-1})(\mathcal{K}_{m-1})$ is zero under the homomorphism

$$\Omega^1_X\otimes\mathcal{P}^{m-1}_X(\mathcal{L})\to\Omega^1_X\otimes\mathcal{P}^{m-2}_X(\mathcal{L}).$$

This can be checked locally around a point $x\in X$: Let u_1,...,u_r be local parameters for X at x (i.e., generators for the maximal ideal $\mathfrak{m}$ of the local ring of X at x), and let $x_0,\dots,x_N\in\mathcal{O}_{X,x}\cong\mathcal{O}_X(1)_x$ be the images of a basis $X_0,\dots,X_N$ for $V=H^0(\mathbb{P}(V),\mathcal{O}_{\mathbb{P}(V)}(1))$, so that the x_j are functions of $u_1,\dots,u_r$. A set of generators for the free $\mathcal{O}_{X,x}-$module

$$\mathcal{P}^{m-1}_X(\mathcal{L})_x\cong\mathcal{P}^{m-1}_{X,x}\cong\mathcal{O}_{X,x}\otimes\mathcal{O}_{X,x}/\mathfrak{m}^m_x$$

is then $\{du^I\}_{|I|\le m-1}$, where $I=(i_1,...,i_r)$ and $du^I=(du_1)^{i_1}\cdots(du_r)^{i_r}$. Set $\partial_i:=\partial/\partial u_i$ and $\partial^I:=\partial^{i_1}_{i_1}\cdots\partial^{i_r}_{i_r}$. The map a^{m-1}_x is given by

$$a^{m-1}_x(1\otimes X_j)=\sum_{|I|\le m-1} D_I x_j du^I$$

for $j = 0, \dots, N$, where $D_I := \frac{1}{i_1! \cdots i_r!}\partial^I$ is the Hasse differential operator "dual" to du^I given by the coefficients in Taylor series expansions. In particular, $D_I(du^J) = \delta_{IJ}$ (the Kronecker delta). (See [11, 16.11.2], and [18, p. 248].)

From [1, I.3, p. 10] it follows that the map $F(a^{m-1})_x$ is given by sending an element $g \in (\mathcal{K}_{m-1})_x$,

$$g = \sum_{j=0}^{N} g_j \otimes X_j \in \mathcal{O}_{X,x} \otimes V$$

to

$$\sum_{j=0}^{N} d(g_j) \otimes a_x^{m-1}(1 \otimes X_j) \in \Omega^1_{X,x} \otimes \mathcal{P}_X^{m-1}(\mathcal{L})_x$$

(here $d = d_x \colon \mathcal{O}_{X,x} \to \Omega^1_{X,x} \cong \oplus_{k=1}^r \mathcal{O}_{X,x} du_k$, so that $dg = \sum_{k=1}^r \partial_k(g) du_k$, where $\partial_k = D_{(0,\dots,1,\dots,0)}$ is differentiation with respect to u_k).

We need to show that the image of g in $\Omega^1_{X,x} \otimes \mathcal{P}_X^{m-2}(\mathcal{L})_x$ is zero. But this image is equal to

$$\begin{aligned}\sum_{j=0}^{N} d(g_j) \otimes a_x^{m-2}(1 \otimes x_j) &= \sum_{j=0}^{N} \sum_{k=1}^{r} \partial_k g_j du_k \otimes \sum_{|I| \le m-2} D_I x_j du^I \\ &= \sum_{k=1}^{r} du_k \otimes \sum_{j=0}^{N} \partial_k g_j \sum_{|I| \le m-2} D_I x_j du^I.\end{aligned}$$

Since $g \in (\mathcal{K}_{m-1})_x = \operatorname{Ker}(a_x^{m-1})$, we have

$$0 = a_x^{m-1}(\sum_{j=0}^{N} g_j \otimes x_j) = \sum_{j=0}^{N} g_j \sum_{|I| \le m-1} D_I x_j du^I.$$

Since the du^I's generate the free $\mathcal{O}_{X,x}$-module $\mathcal{P}_X^{m-1}(\mathcal{L})_x$, this implies that

$$\sum_{j=0}^{N} g_j D_I x_j = 0 \text{ for each } I \text{ with } |I| \le m-1. \tag{1}$$

Applying the differential operators ∂_k to these equations, we get, for $|I| \le m-1$,

$$0 = \partial_k(\sum_{j=0}^{N} g_j D_I x_j) = \sum_{j=0}^{N} (\partial_k g_j \cdot D_I x_j + g_j \cdot \partial_k D_I x_j)$$

so that

$$\sum_{j=0}^{N} \partial_k g_j D_I x_j = -\sum_{j=0}^{N} g_j \cdot \partial_k D_I x_j = -(i_k + 1) \sum_{j=0}^{N} g_j \cdot D_{I_k} x_j$$

where we have set $I_k := (i_1, ..., i_{k-1}, i_k + 1, i_{k+1}, ..., i_r)$ if $I = (i_1, ..., i_r)$.

From (1) it therefore follows that

$$\sum_{j=0}^{N} \partial_k g_j \cdot D_I x_j = 0 \text{ for } |I| \leq m-2, \tag{2}$$

and we also get

$$\sum_{j=0}^{N} \partial_k g_j \cdot D_I x_j = -(i_k+1) \sum_{j=0}^{N} g_j \cdot D_{I_k} x_j \text{ if } |I| = m-1. \tag{3}$$

Because of (1), $F(a^{m-1})$ factors as stated, and we shall use (2) to compare the induced map $\overline{\phi}_m$ with ϕ_m.

If $g \in (\mathcal{K}_{m-1})_x$ is as above, ϕ_m is given locally by

$$\begin{aligned}\phi_m(g) = a^m(\sum_{j=0}^{N} g_j \otimes X_j) &= \sum_{j=0}^{N} g_j \sum_{|I|\leq m} D_I x_j du^I \\ &= \sum_{j=0}^{N} g_j \sum_{|I|=m} D_I x_j du^I = \sum_{|I|=m} (\sum_{j=0}^{N} g_j D_I x_j) du^I.\end{aligned}$$

The map $\overline{\phi}_m$ is given by

$$\begin{aligned}\overline{\phi}_m(g) = \overline{\phi}_m(\sum_{j=0}^{N} g_j \otimes x_j) &= \sum_{j=0}^{N} dg_j \otimes a_x^{m-1}(1 \otimes x_j) \\ &= \sum_{j=0}^{N}\sum_{k=1}^{r} \partial_k g_j \sum_{|I|\leq m-1} D_I x_j du_k du^I = \sum_{|I|=m-1} \sum_{k=1}^{r}\sum_{j=0}^{N} \partial_k g_j \cdot D_I x_j du_k du^I \\ &= -\sum_{|I|=m-1} \sum_{j=0}^{N} g_j \sum_{k=1}^{r} (i_k+1) D_{I_k} x_j du_k du^I \\ &= -\sum_{j=0}^{N} g_j \sum_{|I|=m-1} \sum_{k=1}^{r} (i_k+1) D_{I_k} du_k du^I \\ &= -\sum_{j=0}^{N} g_j \sum_{|J|=m} (\sum_{k=1}^{r} j_k) D_J du^J = -m \sum_{|J|=m} \sum_{j=0}^{N} g_j \cdot D_J x_j du^J,\end{aligned}$$

where we used (2) and (3). This completes the proof of the theorem. □

Given $f \colon X \to \mathbb{P}(V)$ as before. Recall that the mth order osculating space of X at a point $x \in X$ is defined to be the subspace $\mathrm{Osc}^m_X(x) := \mathbb{P}(\mathrm{Im}\, a^m(x)) \subset$

$\mathbb{P}(V)$. We let $s(m)$ denote the dimension of $\mathrm{Osc}_X^m(x)$ for a general point x, i.e., the map

$$a^m \colon V_X \to \mathcal{P}_X^m(\mathcal{L})$$

has generic rank $s(m)+1$. Let $U_m \subseteq X$ be the open dense subset where a^m has this rank. Set $\mathcal{P}_m := \mathrm{Im}(a^m)$. On U_m the sheaf $\mathcal{P}_m$ is a $(s(m)+1)$-bundle, the mth *osculating bundle* of X. Hence there is a rational map, which is a morphism on U_m,

$$\psi_m \colon X \dashrightarrow \mathrm{Grass}_{s(m)+1}(V),$$

called the mth Gauss map, or mth associated map, of X (see [21, p. 336]).

We saw in Proposition 2.3 that the second fundamental form of the kernel $\mathcal{K}_{m-1}$ of

$$a^{m-1} \colon V_X \to \mathcal{P}_{m-1}$$

restricted to U_{m-1}, induces

$$d\psi_{m-1} \colon \mathcal{K}_{m-1} \otimes \mathcal{P}_{m-1}^\vee \to \Omega_X^1,$$

and hence a map (on U_{m-1})

$$\mathcal{K}_{m-1} \to \Omega_X^1 \otimes \mathcal{P}_{m-1} \hookrightarrow \Omega_X^1 \otimes \mathcal{P}_X^{m-1}(\mathcal{L}).$$

It follows from Theorem 2.4 that this map factors through $\Omega_X^1 \otimes S^{m-1}\Omega_X^1 \otimes \mathcal{L}$, and hence we get an induced map

$$\overline{\phi}_m \colon \mathcal{K}_{m-1} \to S^m\Omega_X^1 \otimes \mathcal{L}.$$

The equality

$$\overline{\phi}_m = -m\phi_m.$$

can be interpreted as a verification of the statement "the mth fundamental form of X is equal to the differential of the $(m-1)$th Gauss map" (cf. [10, (1.62), p. 379], [14, Remark, p. 307], [4, Thm. 1.18, p. 1202], and [7, Thm. 3.3]).

Remark 2.5. In [10, (1.52), p. 376] and in the introduction of [7], it is noted that the vanishing of the third fundamental form implies that X is contained in its second order osculating space at a general point of X. In fact, as stated in [7, Remark 2.14], this statement generalizes to higher order fundamental forms. A simple way to see this is as follows.

Consider the mth fundamental form $\Phi_m \colon \mathcal{K}_{m-1}/\mathcal{K}_m \to S^m\Omega_X^1 \otimes \mathcal{L}$. Assume Φ_m is zero on U_m, where $U_m \subseteq X$ denotes the open dense subset on which $\operatorname{rank} a^m$ is constant. Since Φ_m is (generically) injective, it follows that $\mathcal{K}_{m-1} = \mathcal{K}_m$ on U_m. Hence we get an equality of osculating bundles $\mathcal{P}_m|_{U_m} = \mathcal{P}_{m-1}|_{U_m}$. This means that "adding derivatives" does not make the mth osculating spaces bigger than the $(m-1)$th. But this can only happen if the $(m-1)$th osculating spaces are constant, which implies that X is contained in $\mathrm{Osc}_X^{m-1}(x)$, for (any) $x \in U_m$.

3. Geometric interpretation

Now we turn to the geometric interpretation of the fundamental forms as linear systems on the projectivized tangent spaces of X.

Let $x \in X$ be a point such that dim $\mathrm{Osc}_X^m(x) = s(m)$ and consider the linear subspace

$$\mathcal{K}_{m-1}(x) \subseteq V = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$$

whose elements correspond to hyperplanes containing $\mathrm{Osc}_X^{m-1}(x)$. At the point x, the map a^{m-1} is Taylor series expansion:

$$a^{m-1}(x)\colon V \to \mathcal{P}_X^{m-1}(\mathcal{L})(x) \cong \mathcal{O}_{X,x}/\mathfrak{m}_x^m.$$

Hence, if $h \in V$, then $h \in \mathcal{K}_{m-1}(x)$ iff $a^{m-1}(x)(h) \in \mathfrak{m}_x^m$. In particular, this shows that the hyperplane H defined by $h = 0$ is such that $x \in X \cap H$ is a point of multiplicity $\geq m$, and that this multiplicity is $> m$ iff $H \supseteq \mathrm{Osc}_X^m(x)$.

Consider the induced map

$$\begin{aligned}\Phi_m(x)\colon \mathcal{K}_{m-1}(x)/\mathcal{K}_m(x) \to &(S^m\Omega_X^1 \otimes \mathcal{L})(x) \cong \mathfrak{m}_x^m/\mathfrak{m}_x^{m+1}\\ &\cong H^0(PT(x), \mathcal{O}_{PT(x)}(m)),\end{aligned}$$

where we have set $PT(x) := \mathbb{P}(\mathfrak{m}_x/\mathfrak{m}_x^2)$, the projectivized tangent space to X at x. Thus $\Phi_m(x)$, the mth fundamental form of X at x, gives rise to a linear system of degree m and dimension $s(m) - s(m-1) - 1$ on $PT(x) \cong \mathbb{P}^{r-1}$. Denote this linear system by $|\Phi_m(x)|$. The geometric interpretation of the members of this linear system is as follows: Let H be the hyperplane defined by $h \in \mathcal{K}_{m-1}(x)$. Then there is an exact sequence of $\mathcal{O}_X$-modules

$$0 \to (h) \to \mathfrak{m}_{X,x} \to \mathfrak{m}_{X\cap H,x} \to 0$$

and a surjection

$$\mathfrak{m}_{X,x}^m/\mathfrak{m}_{X,x}^{m+1} \to \mathfrak{m}_{X\cap H,x}^m/\mathfrak{m}_{X\cap H,x}^{m+1}.$$

Then the inclusion

$$\begin{aligned}\mathrm{Proj}(\oplus_{m\geq 0}\mathfrak{m}_{X\cap H,x}^m/\mathfrak{m}_{X\cap H,x}^{m+1}) \hookrightarrow &\mathrm{Proj}(\oplus_{m\geq 0}\mathfrak{m}_{X,x}^m/\mathfrak{m}_{X,x}^{m+1})\\ &= \mathbb{P}(\mathfrak{m}_x/\mathfrak{m}_x^2) = PT(x)\end{aligned}$$

is given as the zeroes of $\Phi_m(x)(h) \in H^0(PT(x), \mathcal{O}_{PT(x)}(m))$.

We have thus shown the following:

Proposition 3.1. *Let $U_m \subseteq X$ be the open dense subset such that, for $x \in U_m$, dim $\mathrm{Osc}_X^i(x) = s(i)$ for $i = m-1, m$. Then the mth fundamental form*

$$\Phi_m\colon \mathcal{K}_{m-1}/\mathcal{K}_m \to S^m\Omega_X^1 \otimes \mathcal{L}$$

gives a family (over U_m) of linear systems $|\Phi_m(x)|$ on $PT(x) \cong \mathbb{P}^{r-1}$, of degree m and dimension $s(m) - s(m-1) - 1$. The members of $|\Phi_m(x)|$ are the projectivized tangent cones of $X \cap H$ at x, for $H \in \mathbb{P}(V^\vee)$, $H \supseteq \mathrm{Osc}_X^{m-1}(x)$, $H \not\supseteq \mathrm{Osc}_X^m(x)$.

The *Jacobian* of a linear system of divisors on a projective space is the linear system generated by the partial derivatives of the members of the original system. More generally, let X be a variety and assume $\mathcal{K} \subseteq S^m\Omega_X^1$ is a subsheaf. Define the *Jacobian* $J(\mathcal{K})$ of $\mathcal{K}$ to be the image in $S^{m-1}\Omega_X^1$ of $\mathcal{K} \otimes (\Omega_X^1)^\vee$ under the natural contraction map

$$S^m\Omega_X^1 \otimes (\Omega_X^1)^\vee \to S^{m-1}\Omega_X^1,$$

given locally by sending $v_1 \cdots v_m \otimes w^\vee$ to $\sum_{i=1}^m w^\vee(v_i) v_1 \cdots \widehat{v_i} \cdots v_m$ [9, (B.14), p. 476]. The next theorem says that the Jacobian of the linear systems associated to the mth fundamental form is contained in the linear systems associated to the $(m-1)$th fundamental form. This result was stated in [10, (1.47), p. 373], where an analytic proof was sketched in the case $m = 3$. See also [15, 4.2], [4, Thm. 1.12, p. 1199], [5, Cor. 3.5, p. 5143], and [7, Remark 2.14].

Theorem 3.2. *The Jacobian of the linear system $|\Phi_m(x)|$ is contained in the linear system $|\Phi_{m-1}(x)|$.*

Proof. The theorem follows from the next proposition. □

Proposition 3.3. *Set $\mathcal{K} := \Phi_m(\mathcal{K}_{m-1}/\mathcal{K}_m) \otimes \mathcal{L}^{-1} \subseteq S^m\Omega_X^1$. Then*

$$J(\mathcal{K}) \subseteq \Phi_{m-1}(\mathcal{K}_{m-2}/\mathcal{K}_{m-1}) \otimes \mathcal{L}^{-1} \subseteq S^{m-1}\Omega_X^1.$$

Proof. By restricting to an open subset of X, we may assume that the ranks of a^m and a^{m-1} are constant. It suffices to show (locally) that $J(\mathcal{K}) \subseteq \phi_{m-1}(\mathcal{K}_{m-2}) \otimes \mathcal{L}^{-1}$. We use the local description of ϕ_m given in the proof of Theorem 2.4. Let $g \in (\mathcal{K}_{m-1})_x$. Then $\phi_m(g) = \sum_{|I|=m}(\sum_{j=0}^N g_j D_I x_j) du^I$. The contraction map sends $\phi_m(g) \otimes \partial u_k$ to $\sum_{|I|=m-1}(\sum_{j=0}^N g_j D_{I^k} x_j) du^{I^k}$, where $I^k := (i_1, \ldots, i_k - 1, \ldots, i_r)$. Now $g \in (\mathcal{K}_{m-2})_x$, since $\mathcal{K}_{m-1} \subseteq \mathcal{K}_{m-2}$, hence $\sum_{|I^k|=m-1}(\sum_{j=0}^N g_j D_{I^k} x_j) du^{I^k} \in \phi_{m-1}(\mathcal{K}_{m-2})_x$. □

The geometrical interpretation of the fundamental forms gives of course a finer invariant for the osculating behavior of X than just the dimensions $s(m)$ of the osculating spaces. The simplest way to illustrate this, is to look at the case of surfaces with $s(2) = 4$.

Example 3.4. (Togliatti's Del Pezzo surface [25, p. 261], [23, Ex. 1, p. 248].) This is the (toric) surface $X \subset \mathbb{P}^5$ given by the rational parameterization $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$ where

$$f(x,y) = (1 : x : y : xy^2 : x^2y : x^2y^2).$$

It is the projection of the Del Pezzo surface of degree 6 in $\mathbb{P}^6$ from the common point of its second order osculating spaces (the construction can be generalized to higher degrees). In this case, the second fundamental forms are pencils with no base point.

To see this, we first find (local) equations for X: $G_1 := X_3 - X_1X_2^2 = 0$, $G_2 := X_4 - X_1^2X_2 = 0$, $G_3 := X_5 - X_1^2X_2^2 = 0$. We take $u_1 = x$ and $u_2 = y$. The map $\phi_2\colon \mathcal{K}_1 \to S^2\Omega^1_X \otimes \mathcal{L}$ is then given locally by the matrix product $\overline{A}^{(2)} \cdot K_1$, where $\overline{A}^{(2)}$ is the matrix obtained by taking the last three rows of the matrix of the map a^2:

$$A^{(2)} = \begin{pmatrix} 1 & x & y & xy^2 & x^2y & x^2y^2 \\ 0 & 1 & 0 & y^2 & 2xy & 2xy^2 \\ 0 & 0 & 1 & 2xy & x^2 & 2x^2y \\ 0 & 0 & 0 & 0 & y & y^2 \\ 0 & 0 & 0 & 2y & 2x & 4xy \\ 0 & 0 & 0 & x & 0 & x^2 \end{pmatrix}$$

and the matrix

$$K_1 = \begin{pmatrix} 2xy^2 & 2x^2y & 3x^2y^2 \\ -y^2 & -2xy & -2xy^2 \\ -2xy & -x^2 & -2x^2y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is obtained from the matrix

$$\begin{pmatrix} \partial G_1/\partial X_0 & \partial G_2/\partial X_0 & \partial G_3/\partial X_0 \\ \partial G_1/\partial X_1 & \partial G_2/\partial X_1 & \partial G_3/\partial X_1 \\ \vdots & \vdots & \vdots \\ \partial G_1/\partial X_5 & \partial G_2/\partial X_5 & \partial G_3/\partial X_5 \end{pmatrix}$$

by substituting $X_0 = 1$, $X_1 = x$, $X_2 = y$, $X_3 = xy^2$, $X_4 = x^2y$, $X_5 = x^2y^2$. (Alternatively, one can compute directly the kernel of the matrix $A^{(2)}$.) Hence the map ϕ_2 is given by the matrix

$$\overline{A}^{(2)} \cdot K_1 = \begin{pmatrix} 0 & y & y^2 \\ 2y & 2x & 4xy \\ x & 0 & x^2 \end{pmatrix}$$

Thus the image of ϕ_2 is generated by $2ydxdy + xdy^2$ and $ydx^2 + 2xdxdy$. This means that the linear system $|\Phi_2(x_0, y_0)|$ at the point $(1 : x_0 : y_0 : x_0y_0^2 : x_0^2y_0 : x_0^2y_0^2)$ is equal to $\langle 2y_0v_1v_2 + x_0v_2^2, y_0v_1^2 + 2x_0v_1v_2\rangle$, where $(v_1 : v_2)$ are coordinates on $PT(f(x_0, y_0)) \cong \mathbb{P}^1$. These linear systems have no base points.

To find the third fundamental forms, note that the kernel of $A^{(2)}$ is given by the column matrix $K_2 := (-x^2y^2, xy^2, x^2y, -x, -y, 1)^T$. Let

$$\overline{A}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2y \\ 0 & 0 & 0 & 1 & 0 & 2x \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

denote the matrix obtained by taking the last four rows of the matrix $A^{(3)}$. Then the map Φ_2 is given by the product $\overline{A}^{(3)} \cdot K_2 = (0, y, x, 0)^T$. Therefore we get the linear system $\Phi_3(x_0, y_0) = \langle y_0v_1^2v_2 + x_0v_1v_2^2\rangle$. We observe that the partial derivatives of the generator for $\Phi_3(x_0, y_0)$ are the generators for $\Phi_2(x_0, y_0)$, i.e., the Jacobian of $\Phi_3(x_0, y_0)$ is equal to $\Phi_2(x_0, y_0)$.

EXAMPLE 3.5. Any ruled, non-developable surface in $\mathbb{P}^N$, $N \geq 5$, has $s(2) = 4$, hence the linear systems corresponding to the second fundamental forms are 1-dimensional. It is easy to see that they have a base point, corresponding to the (direction of the) ruling (see Proposition 4.1). In [10, p. 377], the authors ask whether this only occurs for ruled surfaces. However, in their Appendix B, they assert that the answer is no. Indeed, Shifrin [23, p. 248] gave an explicit example of a non-ruled surface such that the second fundamental forms have a base point. A slightly modified version of this surface is given by the rational parametrization $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$, where

$$f(x, y) = (1 : x + y^2 : y : y^3 + 3xy : y^4 + 6xy^2 + 3x^2 : y^5 + 10xy^3 + 15x^2y),$$

which satisfies a "differential heat equation"

$$\partial^2 f/\partial y^2 = \partial f/\partial x.$$

This surface has $s(2) = 4$, it is not ruled, but the second fundamental forms have a base point, as was also shown in [24, pp. 49–51].

In fact, any surface in $\mathbb{P}^5$ of heat equation type has this property [23, Thm. (2.14), p. 237]. Suppose the surface is given by a parametrization $f(x, y)$, satisfying

$$\partial^2 f/\partial y^2 = \varphi(x, y)\partial f/\partial x,$$

for some function $\varphi(x, y)$. Let $A^{(1)}$ denote the matrix corresponding to the map a^1. Then with the notation of the previous example, $A^{(1)} \cdot K_1 = 0$, in particular $\partial f/\partial x \cdot K_1 = 0$. The last row of the matrix $\overline{A}^{(2)}$ is given by $\partial^2 f/\partial y^2$, so that

$\partial^2 f/\partial y^2 \cdot K_1 = 0$. Hence each linear system is generated by linear forms in v_1^2 and $v_1 v_2$ and therefore has a base point $(0:1)$.

In Shifrin's example, local equations for the surface X are $G_1 := X_3 - 3X_1X_2 + 2X_2^3 = 0$, $G_2 := X_4 + 2X_2^4 - 3X_1^2 = 0$, $G_3 := X_5 - 6X_2^5 + 20X_1X_2^3 - 15X_1^2X_2 = 0$. Computations as in Example 3.4 then give

$$\overline{A}^{(2)} \cdot K_1 = \begin{pmatrix} 0 & 3 & 15y \\ 3 & 12y & 30(x+y^2) \\ 0 & 0 & 0 \end{pmatrix},$$

so that $\Phi_2(f(x_0, y_0)) = \langle v_1^2, v_1v_2 \rangle$.

With notations as in the previous example, set

$$\overline{A}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 & 6 & 30y \\ 0 & 0 & 0 & 1 & 4y & 10x + 10y^2 \end{pmatrix}$$

To compute the product $\overline{A}^{(3)} \cdot K_2$, we only need to know the three last entries in K_2. We find that $K_2 = (*, *, *, -10x + 10y^2, -5y, 1)^T$, and hence we get $\overline{A}^{(3)} \cdot K_2 = (0, 15, 0, 0)^T$. Therefore $\Phi_3(f(x_0, y_0)) = \langle v_1^2 v_2 \rangle$, again confirming that the Jacobian of the third fundamental form is equal to the second fundamental form.

Example 3.6. There are also non-rational non-ruled surfaces in $\mathbb{P}^N$ such that $s(2) = 4$. An example is the Kummer surface Γ_2 provided by Dye [6, p. 1] (see also the discussion in [17]). Consider the intersection of three quadrics $\sum_{i=0}^5 b_i^j X_i^2 = 0$, $j = 0, 1, 2$, in $\mathbb{P}^5$, where the b_i are distinct (and nonzero) elements of the base field. It was shown in [24, pp. 53–56], that in this case the second fundamental forms are pencils without a base point. According to [23, Thm. (2.17), p. 239], this surface must be of "wave equation type".

4. Ruled varieties

Let Y be a smooth variety of dimension n and $g\colon Y \to \mathrm{Grass}_{e+1}(V)$ a morphism, where V is a vector space of dimension $N+1$. Let $V_Y \to \mathcal{E}$ denote the pullback via g of the tautological $(e+1)$-quotient on $\mathrm{Grass}_{e+1}(V)$. Set $X := \mathbb{P}(\mathcal{E})$, let $f\colon X \to \mathbb{P}(V)$ denote the induced morphism and $\pi\colon X \to Y$ the projection. Set $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Note that $\pi_*\mathcal{L} = \mathcal{E}$ [12, Ex. 8.4.(a), p. 253].

Let $x \in \mathbb{P}(\mathcal{E})$ be a general point and $U \subseteq Y$ an open subset, with $\pi(x) \in U$, such that $\pi^{-1}(U) \cong U \times \mathbb{P}^e$. Then we can find coordinates such that, around $x \in U \times \mathbb{A}^e$, the morphism f is parametrized by

$$f(u_1, \ldots, u_n, t_1, \ldots, t_e) = (1 : x_1(\underline{u}, \underline{t}) : \cdots : x_N(\underline{u}, \underline{t})),$$

where the x_i are linear in the t_j. Let again $\overline{A}^{(m)}$ denote the last $\binom{n+e+m-1}{m-1}$ rows of the matrix defining (locally) the homomorphism $a_X^m \colon V_X \to \mathcal{P}_X^m(\mathcal{L})$. Since the x_i are linear in the t_j, their partial derivatives of order ≥ 2 with respect to the t_j are 0. The first column is also 0, so in order to study the mth fundamental forms we can replace the matrix $\overline{A}^{(m)}$ by the $(\binom{m+n}{n} + e\binom{m-1+n}{n}) \times N$-matrix (we use the same name)

$$\overline{A}^{(m)} = \begin{pmatrix} \partial^m x_1/\partial u_1^m & \partial^m x_2/\partial u_1^m & \dots & \partial^m x_N/\partial u_1^m \\ \partial^m x_1/\partial u_1^{m-1}\partial u_2 & \partial^m x_2/\partial u_1^{m-1}\partial u_2 & \dots & \partial^m x_N/\partial u_1^{m-1}\partial u_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial^m x_1/\partial u_n^m & \partial^m x_2/\partial u_n^m & \dots & \partial^m x_N/\partial u_n^m \\ \partial^m x_1/\partial u_1^{m-1}\partial t_1 & \partial^m x_2/\partial u_1^{m-1}\partial t_1 & \dots & \partial^m x_N/\partial u_1^{m-1}\partial t_1 \\ \vdots & \vdots & \vdots & \vdots \\ \partial^m x_1/\partial u_n^{m-1}\partial t_e & \partial^m x_2/\partial u_n^{m-1}\partial t_e & \dots & \partial^m x_N/\partial u_n^{m-1}\partial t_e \end{pmatrix}.$$

PROPOSITION 4.1. *Let $f \colon X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}(V)$ be a ruled variety as above. Let $x \in X$ be a general point, let $L_x := \pi^{-1}(\pi(x))$ denote the ruling containing x, and let $PT_L(x) \subset PT(x)$ denote the projectivized tangent space to L_x at x. Then each member of the linear system $|\Phi_m(x)|$, for $m \geq 2$, contains $PT_L(x)$. In particular, if $n = 1$, $PT_L(x)$ is a fixed component of each member, and if $n \geq 2$ and $m \geq 3$, then each member is singular along $PT_L(x)$.*

Proof. Let $v_1, \dots, v_n, w_1, \dots, w_e$ denote homogeneous coordinates on the tangent space $PT(x) \cong \mathbb{P}^{n+e-1}$, where v_i corresponds to du_i and w_j to dt_j. The subspace $PT_L(x) \subset PT(x)$ is defined by $v_1 = \dots = v_n = 0$. It follows from the shape of $\overline{A}^{(m)}$ that $|\Phi_m(x)|$ consists of hypersurfaces defined by some linear combination of the monomials

$$v_1^m, v_1^{m-1}v_2, \dots, v_n^m, v_1^{m-1}w_1, \dots, v_1^{m-1}w_e, v_1^{m-2}v_2w_1, \dots, v_n^{m-1}w_e.$$

The first statement follows from this. So does the third, by taking partial derivatives. If $n = 1$, then $PT_L(x)$ has dimension $n + e - 1 - 1 = e - 1$, hence is a hyperplane in $PT(x)$. □

COROLLARY 4.2.

$$\dim |\Phi_m(x)| \leq \binom{n+m-1}{m} + e\binom{n+m-2}{m-1} - 1.$$

One can ask for a "converse" statement to Proposition 4.1, namely how can one characterize ruled varieties given their fundamental forms. Here we shall just consider the case of surfaces. To show that a surface in $\mathbb{P}^N$, $N \geq 5$, is ruled, it is necessary that the second fundamental forms are pencils with a base point. However, we have seen that this is not sufficient.

To show that a surface $X \subset \mathbb{P}^N$ is ruled, it suffices to show that a general projection $\overline{X} \subset \mathbb{P}^3$ is ruled. So we may assume $X \subset \mathbb{P}^3$. Consider a general point $x \in X$. We may choose coordinates such that $x = (1 : 0 : 0 : 0)$ and $X \cap \mathbb{A}^3$ has Monge form $(x_1, x_2, f(x_1, x_2))$, where $f(x_1, x_2) = f_2(x_1, x_2) + f_3(x_1, x_2) + f_4(x_1, x_2) + \cdots$, with $f_2(x_1, x_2) = x_1x_2$. Then the tangent plane to X at x is the plane $x_3 = 0$, and $x_1 = 0$ and $x_2 = 0$ are the principal tangents. Write $f_3(x_1, x_2) = ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3$. The intersection of the zero loci $f_2 = 0$ and $f_3 = 0$ in $PT(x)$ is given by $x_1x_2 = ax_1^3 + dx_2^3 = 0$, and hence is empty unless $a = 0$ or $d = 0$. Say $a = 0$, then the principal tangent $x_2 = x_3 = 0$ intersects the surface X in the scheme $k[x_1, x_2, x_3]/(x_3 - f, x_2, x_3) = k[x_1]/f(x_1, 0)$. Now $f(x_1, 0) = f_4(x_1, 0) + \cdots$, so that the tangent line intersects the surface with multiplicity at least 4. By [16, Thm. 1, p. 55], it follows that the tangent is contained in X. So if this happens at (almost) all points, then X is ruled. The form f_2 is the second fundamental form at x. The form f_3 is called the *Fubini cubic form* in [10, pp. 448–449] and it is studied and generalized by Ivey and Landsberg in [13, pp. 356–357]. It would be interesting to define this cubic form in terms of bundle maps and diagrams as we have done with the second fundamental form.

We have shown that a sufficient condition for a surface in $\mathbb{P}^3$ to be ruled is that the intersection of the second fundamental form and the Fubini cubic in the projectivized tangent space $PT(x)$ is non-empty, for almost all points $x \in X$. See also the discussion in [10, pp. 448–449] and in [23, pp. 235–236].

Let $\mathcal{P}^m_Y(\mathcal{E})$ denote the sheaf of principal parts of order m of $\mathcal{E}$, which is a bundle of rank $\binom{n+m}{m}(e+1)$. Set $a^m_Y(\mathcal{E})\colon V_Y \to \mathcal{P}^m_Y(\mathcal{E})$ equal to the natural homomorphism obtained by composing

$$V_Y = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))_Y \to H^0(X, \mathcal{L})_Y$$

with $H^0(X, \mathcal{L})_Y = H^0(Y, \pi_*\mathcal{L})_Y = H^0(Y, \mathcal{E})_Y \to \mathcal{P}^m_Y(\mathcal{E})$ [20, § 6, p. 492]. Note that $a^m_X\colon V_X \to \mathcal{P}^m_X(\mathcal{L})$ is the composition $\pi^*V_Y \to \pi^*\mathcal{E} = \pi^*\pi_*\mathcal{L} \to \mathcal{P}^m_X(\mathcal{L})$.

Proposition 4.3. *For each integer $m \geq 1$, we have a natural isomorphism of exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^m\Omega^1_Y \otimes \mathcal{E} & \longrightarrow & \mathcal{P}^m_Y(\mathcal{E}) & \longrightarrow & \mathcal{P}^{m-1}_Y(\mathcal{E}) & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha_m}\big\downarrow\simeq & & {\scriptstyle \beta_m}\big\downarrow\simeq & & {\scriptstyle \beta_{m-1}}\big\downarrow\simeq & & \\
0 & \longrightarrow & \pi_*(S^m\Omega^1_X \otimes \mathcal{L}) & \longrightarrow & \pi_*\mathcal{P}^m_X(\mathcal{L}) & \longrightarrow & \pi_*\mathcal{P}^{m-1}_X(\mathcal{L}) & \longrightarrow & 0
\end{array}$$

and the diagram

$$\begin{array}{ccc} V_Y & \xrightarrow{a^m_Y(\mathcal{E})} & \mathcal{P}^m_Y(\mathcal{E}) \\ \| & & \beta_m \big\downarrow \simeq \\ V_Y & \xrightarrow{\pi_* a^m_X} & \pi_*\mathcal{P}^m_X(\mathcal{L}) \end{array}$$

commutes.

Proof. Define β_m as the adjoint of the composition of the natural maps

$$\pi^*\mathcal{P}^m_Y(\mathcal{E}) \to \mathcal{P}^m_X(\pi^*\mathcal{E}) = \mathcal{P}^m_X(\pi^*\pi_*\mathcal{L}) \to \mathcal{P}^m_X(\mathcal{L}).$$

Thus the map α_m making the diagram commute is defined as the adjoint of

$$\pi^*(S^m\Omega^1_Y \otimes \mathcal{E}) = S^m\pi^*\Omega^1_Y \otimes \pi^*\pi_*\mathcal{L} \to S^m\Omega^1_X \otimes \mathcal{L}.$$

We want to show that the α_m's are isomorphisms.

Consider the exact sequence

$$0 \to \pi^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0,$$

which gives

$$0 \to S^m\pi^*\Omega^1_Y \to S^m\Omega^1_X \to G_m := S^m\Omega^1_X/S^m\pi^*\Omega^1_Y \to 0.$$

Then $S^m\Omega^1_X$ has a filtration [12, II, Ex. 5.16 (c)]

$$S^m\Omega^1_X = F^0 \supseteq F^1 \supseteq \dots \supseteq F^m \supseteq F^{m+1} = 0$$

such that

$$F^j/F^{j+1} \cong S^j\pi^*\Omega^1_Y \otimes S^{m-j}\Omega^1_{X/Y}.$$

Now consider the exact sequences

$$0 \to S^m\pi^*\Omega^1_Y \otimes \mathcal{L} \to S^m\Omega^1_X \otimes \mathcal{L} \to G_m \otimes \mathcal{L} \to 0$$

and the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \pi_*(S^m\pi^*\Omega^1_Y \otimes \mathcal{L}) & \longrightarrow & \pi_*(S^m\Omega^1_X \otimes \mathcal{L}) \\ & & \| & & \| \\ & & S^m\Omega^1_Y \otimes \mathcal{E} & \xrightarrow{\alpha_m} & \pi_*(S^m\Omega^1_X \otimes \mathcal{L}) \\ & & \| & & \| \\ & & \pi_*(F^m \otimes \mathcal{L}) & \longrightarrow & \pi_*(F^0 \otimes \mathcal{L}) \end{array}$$

In order to show that α_m is an isomorphism, it suffices to show that the maps

$$\pi_*(F^{j+1} \otimes \mathcal{L}) \to \pi_*(F^j \otimes \mathcal{L})$$

are isomorphisms for $j = 0, 1, ..., m-1$. But this will follow if we can show that

$$\pi_*(F^j/F^{j+1} \otimes \mathcal{L}) = 0$$

for $j = 0, 1, ..., m-1$. Now we have

$$\begin{aligned}\pi_*(F^j/F^{j+1} \otimes \mathcal{L}) &= \pi_*(S^j \pi^* \Omega^1_Y \otimes S^{m-j}\Omega^1_{X/Y} \otimes \mathcal{L}) \\ &= S^j\Omega^1_Y \otimes \pi_*(S^{m-j}\Omega^1_{X/Y} \otimes \mathcal{L}).\end{aligned}$$

It therefore suffices to show

$$\pi_*(S^{m-j}\Omega^1_{X/Y} \otimes \mathcal{L}) = 0,$$

for $j = 0, 1, ..., m-1$. Consider the base change map

$$\pi_*(S^{m-j}\Omega^1_{X/Y} \otimes \mathcal{L}) \otimes k(y) \to H^0(\pi^{-1}(y), S^{m-j}\Omega^1_{\pi^{-1}(y)} \otimes \mathcal{L}_y).$$

Since $\pi\colon X \to Y$ is a projective fiber bundle, with $\pi^{-1}(y) \cong \mathbb{P}^e$, the right hand side has constant dimension for $y \in Y$. By Grauert's theorem [12, Cor. 12.9, p. 288], it follows that $\pi_*(S^{m-j}\Omega^1_{X/Y} \otimes \mathcal{L})$ is locally free and that the base change map is an isomorphism. Hence it suffices to show that $H^0(\mathbb{P}^e, S^i\Omega^1_{\mathbb{P}^e} \otimes \mathcal{O}_{\mathbb{P}^e}(1)) = 0$ for $i = 1, \ldots, m$. But this holds by Bott's theorem [3, Prop. 14.4, p. 246]. □

We can also define fundamental forms for varieties in Grassmann varieties. Set $\mathcal{K}^{\mathcal{E}}_m := \operatorname{Ker} a^m_Y(\mathcal{E})$. We get a map $\phi_m(\mathcal{E})\colon \mathcal{K}^{\mathcal{E}}_{m-1} \to S^m\Omega^1_Y \otimes \mathcal{E}$, which induces

$$\Phi_m(\mathcal{E})\colon \mathcal{K}^{\mathcal{E}}_{m-1}/\mathcal{K}^{\mathcal{E}}_m \to S^m\Omega^1_Y \otimes \mathcal{E}.$$

Corollary 4.4. *With notations as above, we have*

$$\pi_*\Phi_m = \Phi_m(\mathcal{E}).$$

Proof. This is an immediate consequence of Proposition 4.3. □

EXAMPLE 4.5. (Rational normal scrolls [22].) Assume $Y = \mathbb{P}^1$ and $\mathcal{E} = \bigoplus_{i=0}^{e} \mathcal{O}_{\mathbb{P}^1}(d_i)$, with $0 < d_0 \leq \cdots \leq d_e$. Then $X = \mathbb{P}(\mathcal{E}) \subset \mathbb{P}(V)$, where $V = \bigoplus_{i=0}^{e} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_i))$ has dimension $\sum_{i=0}^{e}(d_i + 1)$, is a rational normal scroll of degree $d := \sum_{i=0}^{e} d_i$.

Assume $2 \leq m \leq d_0$. We have $\operatorname{rank} a_X^m = m(e+1) + 1$. The rank of $\mathcal{P}_Y^m(\mathcal{E})$ is $\binom{1+m}{m}(e+1) = (m+1)(e+1)$, and this is also the rank of $a_{\mathbb{P}^1}^m(\mathcal{E})$. Indeed, we have

$$\mathcal{P}_Y^m(\mathcal{E}) = \mathcal{P}_{\mathbb{P}^1}^m(\bigoplus_{i=0}^{e} \mathcal{O}_{\mathbb{P}^1}(d_i)) = \bigoplus_{i=0}^{e} \mathcal{P}_{\mathbb{P}^1}^m(d_i) = \bigoplus_{i=0}^{e} \mathcal{O}_{\mathbb{P}^1}(d_i - m)^{m+1},$$

and $a_{\mathbb{P}^1}^m(\mathcal{E}) = \bigoplus_{i=0}^{e} a_{\mathbb{P}^1}^m$, where each $a_{\mathbb{P}^1}^m$ has rank $m+1$.

We can parametrize an open subset of X (X is a toric variety) by the map $(\mathbb{C}^*)^{1+e} \to \mathbb{P}(V)$, given by

$$(t, s_1, \ldots, s_e) \mapsto (1 : t : \cdots : t^{d_0} : s_1 : s_1 t : \cdots : s_1 t^{d_1} : \cdots : s_e : s_e t : \cdots : \cdots : s_e t^{d_e}).$$

This gives

$$\overline{A}^{(m)} = \begin{pmatrix} M_{d_0}^m & s_1 M_{d_1}^m & \ldots & s_e M_{d_e}^m \\ 0 & M_{d_1}^{m-1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & M_{d_e}^{m-1} \end{pmatrix},$$

where $M_{d_i}^m$ denotes the $1 \times (d_i + 1)$-matrix $(0, \ldots, 0, 1, \binom{m+1}{m} t, \ldots, \binom{d_i}{m} t^{d_i - m})$. From this one can deduce, with the notations from the proof of Proposition 4.1 that the linear system $|\Phi_m(x)|$ is generated by $v^m, v^{m-1}w_1, \ldots, v^{m-1}w_e$, and hence that its dimension is e and its fixed component is given by $v = 0$.

It is not quite clear how to give a geometric interpretation of the fundamental forms for a variety in a Grassmann variety. In the case of a rational normal scroll, we have

$$\phi_m(\mathcal{E}) \colon \mathcal{K}_{m-1}^{\mathcal{E}} \to S^m \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{E} = \bigoplus_{i=0}^{e} S^m \Omega_{\mathbb{P}^1}^1 \otimes \mathcal{O}_{\mathbb{P}^1}(d_i).$$

Since $a_{\mathbb{P}^1}^m(\mathcal{E}) = \bigoplus_{i=0}^{e} a_{\mathbb{P}^1}^m$, we can view $\Phi_m(\mathcal{E})$ as giving $e+1$ linear systems of degree m in each $PT_Y(y)$, for $y \in Y = \mathbb{P}^1$. So it means that the linear system $\langle v^m, v^{m-1}w_1, \ldots, v^{m-1}w_e \rangle$ corresponds to $\langle v^m \rangle$ in each of $e+1$ copies of $PT_Y(y)$.

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