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Existence of positive solution for a nonlinear problem with mixed conditions

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Abstract. *In this work, we prove the existence of a positive solution to the second-order nonlinear problem $u'' + f(t, u, u') = 0$ with mixed boundary conditions, where f is an L^p-Carathéodory function satisfying certain properties. Three boundary conditions are analysed. Furthermore, we also prove the existence of a positive solution to the problem $u'' + b(t)g(u) = 0$, where $b(t)$ is an L^1 function and $g(u)$ is a continuous function. The proofs of the results are based on the Mawhin's coincidence degree.*

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1. Introduction

In some recent works by Guglielmo Feltrin and Fabio Zanolin, the existence of positive solutions for nonlinear problems with boundary conditions is studied. For example, in [4], they prove the existence of a positive solution for the second-order nonlinear equation

$$u'' + f(t, u, u') = 0, \ 0 < t < T$$

with Neumann or periodic boundary conditions. To obtain the result, they use the Mawhin coincidence degree to guarantee the existence of at least one solution. Then they apply a weak maximum principle to prove that the solution found is non-negative. Lastly, they use a strong maximum principle to prove that the obtained solution is positive.

In this work, we extend the result obtained by Guglielmo Feltrin and Fabio Zanolin to other boundary conditions, as it will be seen later. It will be applied the same techniques used by them, namely the Mawhin's coincidence degree and a maximum principle. It is worth mentioning that the maximum principle used here is slightly different from the one used by them. There is a specific section dedicated to its proof.

Assuming that $f : [0,T] \times [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a L^p-Carathéodory function, in this work, it will be studied the second-order nonlinear problem with boundary conditions

$$\begin{cases} u'' + f(t,u,u') = 0, \ 0 < t < T \\ \mathcal{B}(u) = (0,0), \end{cases} \tag{$\mathcal{P}$}$$

where the linear operator $\mathcal{B} : C^1([0,T],\mathbb{R}) \to \mathbb{R}^2$ represents the boundary conditions, which can be

$$\begin{aligned} &\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0)), \\ &\mathcal{B}(u) = (u'(T) - u'(0), u(0)) \qquad \text{or} \\ &\mathcal{B}(u) = (u(T) - u(0), u'(0)). \end{aligned}$$

A *solution* to problem ($\mathcal{P}$) is a function $u : [0,T] \to \mathbb{R}$, of class C^1 such that $u'(t)$ is absolutely continuous and $u(t)$ satisfies ($\mathcal{P}$) for almost every $t \in [0,T]$. We are interested in positive solutions of ($\mathcal{P}$), i.e., solutions u such that $u(t) > 0$ for all $t \in [0,T]$. However, when the problem is studied with the boundary condition $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$, we already know in advance that u vanishes at $t = 0$. In this case, the sought-after solution will be such that $u(t) > 0$ for all $t \in (0,T]$. After studying problem ($\mathcal{P}$), we present an application by proving the existence of a positive solution to the problem

$$\begin{cases} u'' + b(t)g(u) = 0, \ 0 < t < T \\ u(0) = u(T) \text{ and } u'(0) = 0 \end{cases}, \tag{$\mathcal{E}$}$$

where $g : [0,+\infty) \to [0,+\infty)$ is a continuous function such that

$$(g_1) \qquad g(0) = 0, \quad g(s) > 0 \text{ for } s > 0,$$

and the weight coefficient $b : [0,T] \to \mathbb{R}$ is a L^1 function that changes sign on $[0,T]$.

This work is organized as follows. In Section 2, some basic facts are recalled about Mawhin's coincidence degree. In this way, this section serves as a guide for the approach used in the study of the problems in the subsequent sections. In Section 3, is presented the main notations used in the study the problem ($\mathcal{P}$). Section 4 is used to carefully address the problems. Despite the similarity in the constructions of the problems, treating them separately and carefully allows us to better understand the differences generated by the alteration of the boundary conditions. In Section 5, we apply the result presented in Section 4 to the specific problem ($\mathcal{E}$). At the end, it is included an appendix where it is presented a maximum principle and a priori bounds result.

2. Mawhin's coincidence degree

Consider the real Banach spaces E and F, and the linear Fredholm mapping $L : \operatorname{dom} L \subseteq E \to F$ of index zero. Let $\ker L = L^{-1}(0)$ denote the kernel or nullspace of L, and $\operatorname{Im} L \subseteq F$ denote the range or image of L. We choose linear continuous projections $P : E \to \ker L$ and $Q : F \to \operatorname{coker} L \subseteq F$ with $\operatorname{coker} L \cong F/\operatorname{Im} L$ being the complementary subspace of $\operatorname{Im} L$ in F. The linear subspace $\ker P \subseteq E$ is the complementary subspace of $\ker L$ in E. This gives us the direct sum decomposition

$$E = \ker L \oplus \ker P \text{ and } Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

We define the right inverse of L as $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$, which satisfies $LK_P(w) = w$ for each $w \in \operatorname{Im} L$. As L is a Fredholm mapping of index zero, we have that $\operatorname{Im} L$ is a closed subspace of F, and $\ker L$ and $\operatorname{coker} L$ are finite dimensional vector spaces of the same dimension. Now fix an orientation on these spaces and consider a linear (orientation-preserving) isomorphism $J : \operatorname{coker} L \to \ker L$.

Now, let $N : E \to F$ be a possibly nonlinear operator, and consider the coincidence equation

$$Lu = Nu, \quad u \in \operatorname{dom} L. \tag{1}$$

According to [9], the coincidence equation is equivalent to the fixed point problem

$$u = \Phi(u) := Pu + JQNu + K_P(Id - Q)Nu, \quad u \in E, \tag{2}$$

Mawhin's coincidence degree theory is a powerful tool for solving equation (1) when L is not invertible. To apply this theory, it is needed to make some structural assumptions on the possibly nonlinear operator N. Specifically, it is assumed that N is L-completely continuous, meaning that N is continuous and that for every bounded set $B \subseteq E$, both $QN(B)$ and $K_P(Id - Q)N(B)$ are relatively compact sets.

Suppose we have an open and bounded set $\Omega \subseteq E$ such that $Lu \neq Nu$ for all $u \in \operatorname{dom} L \cap \partial\Omega$. In this case, we can define the coincidence degree of L and N in Ω as

$$D_L(L - N, \Omega) := \deg(Id - \Phi, \Omega, 0),$$

where "deg" denotes the Leray-Schauder degree. Here, Φ is the operator defined in equation (2). We denote the (finite dimensional) Brouwer degree by d_B.

Remarkably, the coincidence degree of L and N in Ω is independent of the choice of projectors P and Q, and is also independent of the choice of linear isomorphism J, provided that it is fixed an orientation on $\ker L$ and $\operatorname{coker} L$ and only consider orientation-preserving isomorphisms for J.

Mawhin's coincidence degree has several properties that provide us with information about the solutions of equation (1) in Ω. For example, we have additivity and homotopy invariance. An interesting property states that if $D_L(L-N,\Omega)\neq 0$, then (1) has at least one solution in Ω.

3. Notations and preliminaries

In this section, we will detail the notations used in the following sections as well as some preliminary hypotheses.

For a fixed T, let $E := C^1([0,T],\mathbb{R})$ equipped with the norm

$$\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$$

and let $F := L^1([0,T],\mathbb{R})$ equipped with the norm L^1, denoted by $\|\cdot\|_{L^1}$.

We define $L : \operatorname{dom} L \to F$ by

$$(Lu)(t) := -u''(t), \quad t\in[0,T],$$

where $\operatorname{dom} L = \{u\in E : u' \text{ is absolutely continuous and } \mathcal{B}(u) = (0,0)\}$.

Firstly, let $f : [0,T]\times[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ be an L^p-Carathéodory function, for some $1\le p\le\infty$, satisfying the following conditions:

(f_1) $f(t,0,\xi) = 0$, for almost every $t\in[0,T]$ and for every $\xi\in\mathbb{R}$;

(f_2) there exist a nonnegative function $k\in L^1[0,T]$ and a constant $\rho>0$ such that

$$|f(t,s,\xi)|\le k(t)(|s|+|\xi|),$$

for almost every $t\in[0,T]$, for every $0\le s\le\rho$, and $|\xi|\le\rho$;

(f_3) suppose that $f(t,s,\xi)$ satisfies a kind of Bernstein-Nagumo condition in order to have $|u'(t)|$ bounded whenever $u(t)$ is bounded.

For each $\eta>0$, there exists a continuous function

$$\phi = \phi_\eta : [0,+\infty)\to[0,+\infty), \text{ with } \int^\infty \frac{\xi^{\frac{p-1}{p}}}{\phi(\xi)}d\xi = \infty,$$

and a function $\psi = \psi_\eta \in L^p([0,T],[0,+\infty))$ such that

$$|f(t,s,\xi)|\le\psi(t)\phi(|\xi|), \text{ for almost every } t\in[0,T], \forall\, s\in[0,\eta], \forall\,\xi\in\mathbb{R}.$$

For technical reasons, when dealing with Nagumo functions $\phi(\xi)$ as above, we always assume that

$$\liminf_{\xi\to+\infty}\phi(\xi)>0.$$

This avoids the possibility of pathological examples as can be seen in [2, p. 46-47] and does not affect our application.

As the first step of our strategy, we extend the function f to a Carathéodory function $\tilde{f}$ defined on $[0,T] \times \mathbb{R}^2$, by

$$\tilde{f}(t,s,\xi) = \begin{cases} f(t,s,\xi), & \text{if } s \geq 0 \\ -s, & \text{if } s \leq 0 \end{cases}.$$

Now we introduce a nonlinear operator $N : E \to F$, the Nemytskii operator induced by $\tilde{f}$, that is,

$$(Nu)(t) := \tilde{f}(t, u(t), u'(t)),\ t \in [0,T].$$

The fact that we choose an extension of f that takes positive values for $s < 0$ is important for the application of the maximum principle, as we will see later.

4. The existence result

The objective of this section is to present the theorem that guarantees the existence of a positive solution to the problem $(\mathcal{P})$. To do this, we will first analyse the problem for each of the boundary conditions. And finally, we will present the theorem of existence of positive solution that encompasses all three cases.

4.1. First case: $u'(0) = u'(T) = u(0)$

Here, we will transform the problem

$$\begin{cases} u'' + f(t,u,u') = 0, & 0 < t < T \\ u'(0) = u'(T) = u(0). \end{cases} \tag{3}$$

into an equivalent operator equation, so that we can later use the degree theory. To do so, we will define the domain of the operator L, its kernel and image. With this, we will construct the projections P and Q. After this step, we will be ready to state the theorem of existence of positive solution.

A solution to problem (3) is a function $u : [0,T] \to \mathbb{R}$ of class C^1 such that u' is absolutely continuous and $u(t)$ satisfies (3) for almost every $t \in [0,T]$. In this section, we are interested in solutions u of (3) with $u(t) > 0$ for all $t \in [0,T]$.

For this problem, let us consider the operator L defined in section 3, where $\operatorname{dom} L \subseteq E$ is the vector subspace

$$\operatorname{dom} L = \{u \in E : u' \text{ is absolutely continuous and } u'(0) = u'(T) = u(0)\}.$$

In these conditions, the kernel of the operator L is given by

$$\ker L = \{u \in E : u(t) = at + a,\ a \in \mathbb{R}\},$$

which can be identified with the set of real numbers, i.e., $\ker L \equiv \mathbb{R}$. In fact, let $u \in \ker L$. That is, $u''(t) = 0$ for all $t \in [0, T]$. Thus, $u(t) = at + b$, with $a, b \in \mathbb{R}$. Hence, $u(0) = b$ and $u'(0) = u'(T) = a$. Since $u(0) = u'(0) = u'(T)$, we have $a = b$. Therefore, $u \in \{u \in E : u(t) = at + a,\ a \in \mathbb{R}\}$, i.e., $\ker L \subseteq \{u \in E : u(t) = at + a,\ a \in \mathbb{R}\}$. On the other hand, let $u \in E$ such that $u(t) = at + a$ with $a \in \mathbb{R}$. It is clear that $u'(t)$ is absolutely continuous. Moreover, $u(0) = u'(0) = u'(T) = a$. Therefore, $u \in \operatorname{dom} L$. And obviously, $u''(t) = 0$ for all $x \in [0, T]$. Hence, $\{u \in E : u(t) = at + a,\ a \in \mathbb{R}\} \subseteq \ker L$. We conclude that $\ker L = \{u \in E : u(t) = at + a,\ a \in \mathbb{R}\}$.

Furthermore, the image of the operator L is given by

$$\operatorname{im} L = \left\{ w \in F : \int_0^T w(t)dt = 0 \right\}.$$

Indeed, let $w \in \operatorname{im} L$. Then, $w = -u''$ for some $u \in \operatorname{dom} L$. Thus,

$$\int_0^T w(t)dt = -\int_0^T u''(t)dt = -u'(T) + u'(0) = 0.$$

On the other hand, let $w \in Z$ such that $\int_0^T w(t)dt = 0$. Define, for $s \in [0, T]$,

$$v(s) := -\int_0^s w(t)dt \ \text{ and } \ u(s) := \int_0^s v(t)dt.$$

By the fundamental theorem of calculus, we have, for $t \in [0, T]$,

$$u'(s) = v(s) \ \text{ and } \ -u''(s) = -v'(s) = w(s).$$

It is clear that u' is absolutely continuous and, moreover,

$$u'(0) = v(0) = 0,\ u'(T) = v(T) = 0 \ \text{ and } \ u(0) = 0.$$

Thus, $u \in \operatorname{dom} L$ and $Lu = w$. We conclude that

$$\operatorname{im} L = \left\{ w \in F : \int_0^T w(t)dt = 0 \right\}.$$

Observe that L is a Fredholm operator of index zero. In fact, L is a bounded linear operator with $\dim(\ker L) = 1$. It remains to prove that $\dim(\operatorname{coker} L) = 1$. To this end, note that $w \in F$ can be written as

$$w(t) = w_1(t) + w_2(t),$$

where

$$w_1(t) = w(t) - \frac{1}{T}\int_0^T w(\xi)\,d\xi \quad \text{and} \quad w_2(t) = \frac{1}{T}\int_0^T w(\xi)\,d\xi.$$

We have $w_1(t) \in \operatorname{im} L$ and $w_2(t) \in \operatorname{coker} L$. Since $w_2(t)$ is constant, we conclude that $\dim(\operatorname{coker} L) = 1$. Thus, the index of L is

$$\dim(\ker L) - \dim(\operatorname{coker} L) = 0.$$

At this point, we can define the projections $P : E \to \ker L$ and $Q : F \to \operatorname{coker} L$ as follows:

$$(Pu)(t) = \frac{u(T) - u(0)}{T}t + \frac{u(T) - u(0)}{T} \text{ and } (Qw)(t) = \frac{1}{T}\int_0^T w(t)dt.$$

Since $\dim(\ker L) = \dim(\operatorname{coker} L)$, we have $\operatorname{coker} L \equiv \mathbb{R}$. Moreover, $\ker P$ is given by C^1 functions such that $u(0) = u(T)$ and the linear operator $K_p : \operatorname{im} L \to \operatorname{dom} L \cap \ker P$, which is the right inverse of L, associates to each $w \in L^1([0, T], \mathbb{R})$ with $\displaystyle\int_0^T w(t)dt = 0$, an unique solution $u(t)$ of

$$u'' + w(t) = 0, \;\; u'(0) = u'(T) = u(0) = u(T).$$

In this scenario, u is a solution of the equation

$$Lu = Nu, \;\; u \in \operatorname{dom} L, \tag{4}$$

if, and only if, it is a solution of the problem

$$\begin{cases} u'' + \tilde{f}(t, u, u') = 0, \;\; 0 < t < T \\ u'(0) = u'(T) = u(0). \end{cases} \tag{5}$$

Furthermore, from the definition of $\tilde{f}$ for $s < 0$ and the conditions (f_1) and (f_2), it is easy to verify, using the maximum principle, that if $u \not\equiv 0$, then $u(t)$ is strictly positive and hence is a (positive) solution of problem (3).

4.2. Second case: $u'(0) = u'(T)$ and $u(0) = 0$

For this boundary condition, we will follow the same outline as in the previous case. We will transform the problem

$$\begin{cases} u'' + f(t, u, u') = 0, \;\; 0 < t < T \\ u'(0) = u'(T) \text{ and } u(0) = 0. \end{cases} \tag{6}$$

into an equivalent operator equation, so that we can later use the degree theory.

A solution to problem (6) is a function $u : [0, T] \to \mathbb{R}$ of class C^1 such that u' is absolutely continuous and $u(t)$ satisfies (6) for almost every $t \in [0, T]$. As already noted, in this section, we are interested in solutions u of (6) with $u(t) > 0$ for all $t \in\ (0, T]$.

For this problem, let us consider the operator L defined in section 3, where $\operatorname{dom} L \subseteq E$ is the vector subspace

$$\operatorname{dom}L = \{u \in E : u' \text{ is absolutely continuous and } u'(0) = u'(T) \text{ and } u(0) = 0\}.$$

In these conditions, the kernel of the operator L is given by

$$\ker L = \{u \in E : u(t) = at,\ a \in \mathbb{R}\},$$

which can be identified with the set of real numbers, i.e, $\ker L \equiv \mathbb{R}$. In fact, let $u \in \ker L$. That is, $u''(t) = 0$ for all $t \in [0, T]$. Thus, $u(t) = at + b$ with $a, b \in \mathbb{R}$. Therefore, $u(0) = b$ and $u'(0) = u'(T) = a$. Since $u(0) = 0$, we have $b = 0$. Hence, $u \in \{u \in E : u(t) = at, , a \in \mathbb{R}\}$, i.e., $\ker L \subseteq \{u \in E : u(t) = at, a \in \mathbb{R}\}$. On the other hand, let $u \in E$ such that $u(t) = at$ with $a \in \mathbb{R}$. It is clear that $u'(t)$ is absolutely continuous. Moreover, $u(0) = 0$ and $u'(0) = u'(T) = a$. Therefore, $u \in \operatorname{dom} L$. And obviously, $u''(t) = 0$ for all $t \in [0, T]$. Hence, $\{u \in E : u(t) = at, , a \in \mathbb{R}\} \subseteq \ker L$. We conclude that $\ker L = \{u \in E : u(t) = at, , a \in \mathbb{R}\}$. Furthermore, the image set of the operator L is given by

$$\operatorname{im} L = \left\{ w \in F : \int_0^T w(t)dt = 0 \right\}.$$

In fact, let $w \in \operatorname{im} L$. Then, $w = -u''$ for some $u \in \operatorname{dom} L$. Thus,

$$\int_0^T w(t)dx = -\int_0^T u''(t)dt = -u'(T) + u'(0) = 0.$$

On the other hand, let $w \in F$ such that $\displaystyle\int_0^T w(t)dt = 0$. Define, for $s \in [0, T]$,

$$v(s) := -\int_0^s w(t)dt \ \text{ e } \ u(s) := \int_0^s v(t)dx.$$

By the fundamental theorem of calculus, we have, for $s \in [0, T]$,

$$u'(s) = v(s) \ \text{ e } \ -u''(s) = -v'(s) = w(s).$$

Clearly, u' is absolutely continuous and, furthermore,

$$u'(0) = v(0) = 0,\ u'(T) = v(T) = 0 \ \text{ e } \ u(0) = 0.$$

Thus, $u \in \operatorname{dom} L$ and $Lu = w$. We conclude that

$$\operatorname{im} L = \left\{ w \in F : \int_0^T w(t)dt = 0 \right\}.$$

At this point, we can define the projections $P : E \to \ker L$ and $Q : F \to \operatorname{coker} L$ as follows:

$$(Pu)(t) = u'(0)t \text{ and } (Qw)(t) = \frac{1}{T}\int_0^T w(t)dt.$$

Similarly to the previous case, L is a Fredholm operator of index zero, and therefore $\dim(\ker L) = \dim(\operatorname{coker} L)$, so that $\operatorname{coker} L \equiv \mathbb{R}$. Moreover, $\ker P$ consists of C^1 functions such that $u'(0) = 0$, and the linear operator $K_p : \operatorname{im} L \to \operatorname{dom} \cap \ker P$ associates to each $w \in L^1([0,T], \mathbb{R})$ with $\displaystyle\int_0^T w(t)dt = 0$, a unique solution $u(t)$ of

$$u'' + w(t) = 0, \;\; u'(0) = u'(T) = u(0) = 0.$$

In this scenario, u is a solution of the equation

$$Lu = Nu, \;\; u \in \operatorname{dom} L, \tag{7}$$

if, and only if, it is a solution of the problem

$$\begin{cases} u'' + \tilde{f}(t, u, u') = 0, \;\; 0 < t < T \\ u'(0) = u'(T) \text{ and } u(0) = 0. \end{cases} \tag{8}$$

In addition, from the definition of $\tilde{f}$ for $s < 0$ and conditions (f_1) and (f_2), it is easy to verify, using the maximum principle, that if $u \not\equiv 0$, then $u(t)$ is strictly positive and thus is a (positive) solution of problem (6).

4.3. Third case: $u(0) = u(T)$ and $u'(0) = 0$

Again, we will follow the same structure as in the two previous cases. We will transform the problem

$$\begin{cases} u'' + f(t, u, u') = 0, \;\; 0 < t < T \\ u(0) = u(T) \text{ and } u'(0) = 0. \end{cases} \tag{9}$$

into an equivalent operator equation, so that we can later use the degree theory.

A solution to problem (9) is a function $u : [0,T] \to \mathbb{R}$ of class C^1 such that u' is absolutely continuous and $u(t)$ satisfies (9) for almost every $t \in [0,T]$.

In this section, we are interested in solutions u of (9) with $u(t) > 0$ for all $t \in [0, T]$.

For this problem, let us consider the operator L defined in section 3, where $\operatorname{dom} L \subseteq E$ is the vector subspace

$$\operatorname{dom} L = \{u \in E : u' \text{ is absolutely continuous and } u(0) = u(T) \text{ and } u'(0) = 0\}.$$

In these conditions, the kernel of the operator L is given by

$$\ker L = \{u \in E : u(t) = b,\, b \in \mathbb{R}\},$$

which can be identified with the set of real numbers, i.e., $\ker L \equiv \mathbb{R}$. In fact, let $u \in \ker L$. $u''(t) = 0$ for all $t \in [0, T]$. Thus, $u(t) = at + b$, with $a, b \in \mathbb{R}$. Hence, $u(0) = b$ e $u'(0) = a$. Since $u'(0) = 0$, we have $a = 0$. Therefore, $u \in \{u \in E : u(t) = b,\, b \in \mathbb{R}\}$, i.e, $\ker L \subseteq \{u \in E : u(t) = b,\, b \in \mathbb{R}\}$. On the other hand, let $u \in E$ such that $u(t) = b$ with $b \in \mathbb{R}$. It is clear that $u'(t)$ is absolutely continuous. Moreover, $u(0) = u(T) = b$ and $u'(0) = 0$. So, $u \in \operatorname{dom} L$. Therefore, $u \in \operatorname{dom} L$. And obviously, $u''(t) = 0$ for all $t \in [0, T]$. Hence, $\{u \in E : u(t) = b,\, b \in \mathbb{R}\} \subseteq \ker L$. We conclude that $\ker L = \{u \in X : u(t) = b,\, b \in \mathbb{R}\}$.

Furthermore, the image of the operator L is given by

$$\operatorname{im} L = \left\{ w \in F : \int_0^T \left(\int_0^s w(t)dt \right) ds = 0 \right\}.$$

Indeed, let $w \in \operatorname{im} L$. Then, $w = -u''$ for some $u \in \operatorname{dom} L$. Thus,

$$\begin{aligned} \int_0^T \left(\int_0^s w(t)dt \right) ds = - \int_0^T \left(\int_0^s u''(t)dt \right) ds \\ = - \int_0^T u'(s)ds = -u(T) + u(0) = 0. \end{aligned}$$

On the other hand, let $w \in F$ such that $\displaystyle\int_0^T w(t)dt = 0$. Define, for $t \in [0, T]$,

$$u(t) := u(0) - \int_0^t \left(\int_0^x w(s)ds \right) dx.$$

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By the fundamental theorem of calculus, we have, for $t \in [0, T]$,

$$u'(t) = - \int_0^t w(s)ds \text{ and } - u''(t) = w(t).$$

It is clear that u' is absolutely continuous and, moreover,

$$u'(0) = 0 \text{ and } u(0) = u(T).$$

Thus, $u \in \operatorname{dom} L$ and $Lu = w$. We conclude that

$$\operatorname{im} L = \left\{ w \in F : \int_0^T \left(\int_0^s w(t)dt \right) ds = 0 \right\}.$$

At this point we can define the projections $P : E \to \ker L$ e $Q : F \to \operatorname{coker} L$ as follows:

$$(Pu)(t) = \frac{1}{T} \int_0^T u(t)dt \ \text{ and } \ (Qw)(t) = \frac{2}{T^2} \int_0^T \left(\int_0^s w(t)dt \right) ds.$$

As in the two previous cases, here L is also a Fredholm operator of index zero. Thus, $\dim(\ker L) = \dim(\operatorname{coker} L)$, so $\operatorname{coker} L \equiv \mathbb{R}$. Moreover, $\ker P$ is given by C^1 functions with mean value zero and the linear operator $K_p : \operatorname{im} L \to \operatorname{dom} \cap \ker P$, which is the right inverse of L, associates to each $w \in L^1([0,T], \mathbb{R})$ with $\displaystyle\int_0^T \left(\int_0^s w(t)dt \right) ds = 0$, a unique solution $u(t)$ of

$$u'' + w(t) = 0, \ \ u(0) = u(T), \ \ u'(0) = 0 \text{ and } \int_0^T u(t)dt = 0.$$

In this scenario, u is a solution of the equation

$$Lu = Nu, \ \ u \in \operatorname{dom} L, \tag{10}$$

if, and only if, it is a solution of the problem

$$\begin{cases} u'' + \tilde{f}(t, u, u') = 0, \ \ 0 < t < T \\ u(0) = u(T) \text{ and } u'(0) = 0. \end{cases} \tag{11}$$

Furthermore, from the definition of $\tilde{f}$ for $s < 0$ and the conditions (f_1) and (f_2), it is easy to verify, using the maximum principle, that if $u \not\equiv 0$, then $u(t)$ is strictly positive and hence is a (positive) solution of problem (9).

4.4. Main result

In this section, we present the main result of this work, in which the existence of a positive solution for problem $(\mathcal{P})$ is proven. Before presenting the main theorem and its proof, we establish a technical framework that will serve as the foundation for our arguments.

Considering $\vartheta \in (0,1]$, the equation

$$Lu = \vartheta Nu, \ u \in \operatorname{dom} L, \tag{12}$$

is equivalent to problem

$$\begin{cases} u'' + \vartheta \tilde{f}(t,u,u') = 0, & 0 < t < T \\ \mathcal{B}(u) = (0,0). \end{cases} \tag{13}$$

Fixing $d > 0$, the condition (f_3) takes on an important role. Let $\phi = \phi_d : [0,+\infty) \to [0,+\infty)$ and $\psi = \psi_d \in L^p([0,T])$ be such that $|f(t,s,\xi)| \leq \psi(t)\phi(|\xi|)$, for almost every $t \in [0,T]$, for all $s \in [0,d]$ and $\xi \in \mathbb{R}$. Applying the Nagumo's lemma [2, § 4.4, Proposition 4.7], there is a constant $M = M_d > d$ such that, for some $\vartheta \in (0,1]$, any solution of (12) or, equivalently, any non-negative solution of

$$\begin{cases} u'' + \vartheta f(t,u,u') = 0, & 0 < t < T \\ \mathcal{B}(u) = (0,0), \end{cases} \tag{14}$$

satisfying $\|u\|_\infty \leq d$ is such that $\|u'\|_\infty < M_d$. This way, we can define the open and bounded set $\Omega_d \subseteq E$ by

$$\Omega_d := \{u \in E : \|u\|_\infty < d, \|u'\|_\infty < M_d\}. \tag{15}$$

When the boundary condition is $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))$, we have $\Omega_d \cap \ker L = \left(-\frac{d}{1+T}, \frac{d}{1+T}\right)$.

Let $u \in \partial\Omega_d \cap \ker L = \left\{u \in E : u(t) = a + at, |a| = \frac{d}{1+T}\right\}$. In this case,

$$-JQNu = -\frac{1}{T}\int_0^T \tilde{f}(t, a+at, a)dt.$$

By definition of $\tilde{f}$ we have

$$h(a) := -\frac{1}{T}\int_0^T \tilde{f}(t,a+at,a)dt = \begin{cases} -\dfrac{1}{T}\displaystyle\int_0^T f(t,a+at,a)dt, & \text{if } a > 0 \\ a + \dfrac{aT}{2}, & \text{if } a \leq 0\,. \end{cases}$$

Similarly, when the boundary condition is $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$ we obtain $\Omega_d \cap \ker L = \left(-\frac{d}{T}, \frac{d}{T}\right)$.

For $u \in \partial\Omega_d \cap \ker L = \left\{u \in E : u(t) = at, |a| = \frac{d}{T}\right\}$,

$$-JQNu = -\frac{1}{T}\int_0^T \tilde{f}(t, at, a)dt.$$

In this case,

$$h(a) := -\frac{1}{T}\int_0^T \tilde{f}(t, at, a)dt = \begin{cases} -\dfrac{1}{T}\displaystyle\int_0^T f(t, at, a)dt, & \text{if } a > 0 \\ \dfrac{aT}{2}, & \text{if } a \leq 0\,. \end{cases}$$

Following the same idea, when the boundary condition is $\mathcal{B}(u) = (u(T) - u(0), u'(0))$, we have $\Omega_d \cap \ker L = (-d, d)$.

For $u \in \partial\Omega_d \cap \ker L = \{u \in E : u(t) = a, |a| = d\}$,

$$-JQNu = -\frac{2}{T^2}\int_0^T \left(\int_0^s \tilde{f}(t, a, 0)dt\right) ds.$$

In this case,

$$\begin{aligned} h(a) := -\frac{2}{T^2}\int_0^T \left(\int_0^s \tilde{f}(t, a, 0)dt\right) ds \\ = \begin{cases} -\dfrac{2}{T^2}\displaystyle\int_0^T \left(\int_0^s f(t, a, 0)dt\right) ds, & \text{if } a > 0 \\ a, & \text{if } a \leq 0\,. \end{cases} \end{aligned}$$

This technical framework, together with the finite-dimensional reduction of Mawhin's coincidence degree [4, Lemma 2.1], will play an important role in the proof of the main theorem, which will be presented next. The proof of the theorem below follows the ideas used in [4, Theorem 2.1].

THEOREM 4.1. *Assume* (f_1), (f_2), *and* (f_3), *and suppose that there exist two constants* $r, R > 0$, *with* $r \neq R$, *such that the following hypotheses are true.*

(H_1) *The following condition is satisfied.*

$$JQNu < 0,$$

with $u \in \partial\Omega_r \cap \ker L$ *and* $u(t) > 0$ *in* $[0, T]$. *Moreover, any solution* $u(t)$ *of the problem (14) for* $0 < \vartheta \leq 1$, *such that* $u(t) > 0$ *in* $[0, T]$, *satisfies* $\|u\|_\infty \neq r$.

(H_2) *There exist a non-negative function* $v \in L^p([0, T], \mathbb{R})$ *with* $v \not\equiv 0$ *and a constant* $\alpha_0 > 0$, *such that every solution* $u(t) \geq 0$ *of the problem*

$$\begin{cases} u'' + f(t, u, u') + \alpha v(t) = 0, \;\; 0 < t < T \\ \mathcal{B}(u) = (0, 0), \end{cases} \tag{16}$$

for $\alpha \in [0, \alpha_0]$, *satisfies* $\|u\|_\infty \neq R$.

(H_3) *There are no solutions* $u(t)$ *of* (16) *for* $\alpha = \alpha_0$ *with* $0 \leq u(t) \leq R$*, for every* $t \in [0, T]$*.*

Then the problem ($\mathcal{P}$) *has at least one positive solution* $u(t)$ *with*

$$\min\{r, R\} < \max_{t \in [0,T]} u(t) < \max\{r, R\}.$$

Proof. As we have seen before, the choice of spaces E, $\operatorname{dom} L$, F and operators $L : u \mapsto -u''$ and N imply the equivalence between

$$Lu = Nu, \;\; u \in \operatorname{dom} L, \tag{17}$$

and

$$\begin{cases} u'' + \tilde{f}(t, u, u') = 0, \;\; 0 < t < T \\ \mathcal{B}(u) = (0, 0). \end{cases} \tag{18}$$

It is also we know that L is a Fredholm operator of index zero and that N is L- completely continuos.

Let us consider case $0 < r < R$ and, by (15), take the open and bounded set

$$\Omega_r := \{u \in E : \|u\|_\infty < r, \|u'\|_\infty < M_r\} \subseteq E.$$

By condition (H_1) we have

$$Lu \neq \vartheta N u, \;\; \forall u \in \operatorname{dom} L \cap \partial\Omega_r, \forall \vartheta \in \; (0, 1].$$

Regardless of the boundary condition used, by the first condition in (H_1), we have $h\,(-r) < 0 < h\,(r)$. In this way, we obtain

$$\deg_B\,(h, \Omega_r \cap \ker L, 0) = 1.$$

By the finite-dimensional reduction of the Mawhin's coincidence degree [4, Lemma 2.1], we conclude that

$$\begin{aligned} D_L(L - N, \Omega_r) = \deg_B\,&(-JQN|_{\ker L}, \Omega_r \cap \ker L, 0) \\ &= \deg_B\,(h, \Omega_r \cap \ker L, 0) = 1. \end{aligned} \tag{19}$$

Now let's analyze hypothesis (H_2) by studying equation

$$Lu = Nu + \alpha v, \;\; u \in \operatorname{dom} L, \tag{20}$$

for some $\alpha \geq 0$. This equation is equivalent to the problem

$$\begin{cases} u'' + \tilde{f}(t, u, u') + \alpha v(t) = 0 \\ \mathcal{B}(u) = (0, 0). \end{cases} \tag{21}$$

Taking any solution u of (20) for some $\alpha \geq 0$, the way we defined the extension $\tilde{f}$ allows us to apply the maximum principle and conclude that $u(t) \geq 0$ for all $t \in [0, T]$. Thus u is a solution of (16).

Once again, we will use condition (f_3). Let $\phi = \phi_R : [0, +\infty) \to [0, +\infty)$ and $\psi \in L^p([0, T], \mathbb{R})$ such that $|f(t, s, \xi)| \leq \psi(t)\phi(|\xi|)$, for almost every $t \in [0, T]$, for all $s \in [0, R]$ and for all $\xi \in \mathbb{R}$. Taking $\alpha \in [0, \alpha_0]$, we obtain

$$|f(t, s, \xi) + \alpha v(t)| \leq \psi(t)\phi(|\xi|) + \alpha_0 v(t) \leq \tilde{\phi}(t)\tilde{\psi}(|\xi|)$$

for almost every $t \in [0, T]$, for all $s \in [0, R]$ and for all $\xi \in \mathbb{R}$, where

$$\tilde{\psi}(t) = \psi(t) + \alpha_0 v(t) \text{ and } \tilde{\psi}(|\xi|) = \psi(|\xi|) + 1.$$

Note also that

$$\tilde{\psi} \in L^p([0, T]) \text{ and } \int^{\infty} \frac{\xi^{\frac{p-1}{p}}}{\tilde{\phi}(|\xi|)} d\xi = \infty.$$

Similarly to what we did earlier, we use Nagumo's lemma There is a positive constant $M = M_R > M_r$ such that any solution of (21), or equivalently, any non-negative solution of (16) satisfying $\|u\|_\infty \leq R$ satisfies $\|u'\|_\infty < M_R$. Thus we can define the open and bounded set $\Omega_R \subseteq E$ by

$$\Omega_R = \{u \in E : \|u\|_\infty < R, \|u'\|_\infty < M_R\}.$$

The condition (H_2) implies that

$$Lu \neq Nu + \alpha v, \ \forall u \in \operatorname{dom} L \cap \partial\Omega_R, \ \forall \alpha \in [0, \alpha_0].$$

In addition, the condition (H_3) implies that

$$Lu \neq Nu + \alpha_0 v, \ \forall u \in \operatorname{dom} L \cap \Omega_R.$$

Due to the homotopy invariance of the Mawhin coincidence degree, we have

$$D_L(L - N, \Omega_R) = 0. \tag{22}$$

Using (19), (22) and the additivity of the degree, it follows that

$$D_L(L - N, \Omega_R \setminus \overline{\Omega_r}) = -1.$$

This guarantees the existence of a nontrivial solution $\tilde{u}$ of (17) with $\tilde{u} \in \Omega_R \setminus \overline{\Omega_r}$. As $\tilde{u}$ is a nontrivial solution of (18), by the (strong) maximum principle, it follows that $\tilde{u}(t)$ is a solution of $\mathcal{P}$ with $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))$ and $\tilde{u}(t) > 0$ for all $t \in [0, T]$.

In the case where $0 < R < r$, we proceed analogously. Regarding the previous case, the only relevant change is the following. First, we fix a constant

$M = M_R > 0$ and obtain (22) for the set Ω_R. Then, we repeat the first part of the proof above, fix a constant $M_r > M_R$ and obtain (19) for the set Ω_r. Now we have

$$D_L(L - N, \Omega_r \setminus \overline{\Omega_R}) = 1.$$

This guarantees the existence of a nontrivial solution $\tilde{u}$ of (17) with $\tilde{u} \in \Omega_r \setminus \overline{\Omega_R}$ and we conclude, as above, that $\tilde{u}(t) > 0$ for all $t \in [0, T]$ (by the strong maximum principle). □

5. An application of Theorem 4.1

In this section, we give an application of Theorem 4.1 to the existence of positive solutions for problem

$$\begin{cases} u'' + b(t)g(u) = 0, \ 0 < t < T \\ u(0) = u(T) \text{ and } u'(0) = 0, \end{cases} \tag{$\mathcal{E}$}$$

where $g : [0, +\infty) \to [0, +\infty)$ is a continuous function such that

(g_1) $g(0) = 0, \quad g(s) > 0$ for $s > 0$;

(g_2) $\displaystyle\lim_{s \to 0^+} \frac{g(s)}{s} = 0.$

The weight coefficient $b : [0, T] \to \mathbb{R}$ is a L^1-function such that

(b_1) there exists $\delta > 0$ such that $b(t)$ is essentially negative on $[0, \delta]$ and also on $[T - \delta, T]$;

(b_2) there exist $m \geq 1$ intervals $I_1, \ldots, I_m$, closed and pairwise disjoint, such that

$$b(t) \geq 0, \text{ for a.e. } t \in I_i, \text{ with } b(t) \not\equiv 0 \text{ on } I_i \quad (i = 1, \ldots, m);$$

$$b(t) \leq 0, \text{ for a.e. } t \in [0, T] \setminus \bigcup_{i=1}^{m} I_i;$$

(b_3)
$$\int_0^s b(t)dt < 0 \text{ for all } 0 < s < T.$$

Let $\lambda_1^i, i = 1, \ldots, m$, be the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda b(t)\varphi = 0, \quad \varphi|_{\partial I_i} = 0.$$

From the assumptions on $b(t)$ in I_i it clearly follows that $\lambda_1^i > 0$ for each $i = 1, \ldots, m$.

A function $g : [0, +\infty) \to [0, +\infty)$ satisfying (g_1) is *regularly oscillating* at zero if

$$\lim_{\substack{s \to 0^+ \\ \omega \to 1}} \frac{g(\omega s)}{g(s)} = 1.$$

Now we are ready to present the following result. A similar result is found in [4]. However, here the hypotheses on $b(t)$ are substantially different. Additionally, a mixed boundary condition is considered. Nevertheless, as we will see below, a strategy similar to the one used in [4, Theorem 3.1] works in the proof of the theorem.

THEOREM 5.1. *Let $g(s)$ and $b(t)$ be as above. Suppose also that $g(s)$ is regularly oscillating at zero and satisfies*

$$g_\infty := \liminf_{s \to +\infty} \frac{g(s)}{s} > \max_{i=1,\dots,m} \lambda_1^i.$$

Then problem $(\mathcal{E})$ has at least one positive solution.

Proof. We define

$$f(t, s, \xi) = f(t, \xi) := b(t)g(s),$$

and observe that f is L^1 - Carathéodory. Now we prove the hypothesis (f_1), (f_2) and (f_3):

- (f_1) follows from $g(0) = 0$;
- (f_2) is an consequence of the fact that $g(s)/s$ is bounded on a right neighbourhood of $s = 0$ and $b \in L^1([0, T])$. We can that there exists $\rho > 0$ such that
$$|b(t)g(s)| \leq |b(t)||s|$$
for almost every $t \in [0, T]$ and for every $0 \leq s \leq \rho$.
- to prove the Nagumo condition (f_3) we can take $p = 1$, $\phi(\xi) \equiv 1$ and $\psi(t) = |b(t)| \max_{0 \leq s \leq \eta} g(s)$.

Let's start by verifying the validity of hypothesis (H_1). To verify the first part of (H_1), note that

$$\int_0^T \left(\int_0^s f(t, k, 0)dt \right) ds = \int_0^T \left(\int_0^s b(t)g(k)dt \right) ds = g(k) \int_0^T \left(\int_0^s b(t)dt \right) ds.$$

By the hypotheses (g_1) and (b_3), we have

$$\int_0^T \left(\int_0^s f(t, k, 0)dt \right) ds < 0, \quad \forall k > 0. \tag{23}$$

Now, let's verify the second part of (H_1). To do this, consider the following claim.

Claim 5.1. *There exists $r_0 > 0$ such that for all $0 < r \leq r_0$ and for all $\vartheta \in (0,1]$ there are no solutions $u(t)$ of (14) such that $u(t) > 0$ on $[0,T]$ and $\|u\|_\infty = r$.*

Verification of Claim. Consider a sequence $(r_n)_n$ of positive real numbers such that $r_n \to 0$. By contradiction, suppose that for $n \in \mathbb{N}$, there exist $\vartheta_n \in (0,1]$ and $u_n(t)$ positive solution of

$$u'' + \vartheta_n b(t) g(u) = 0, \quad u(0) = u(T) \text{ and } u'(0) = 0, \tag{24}$$

with $\|u_n\|_\infty = r_n$.

Using Rolle's theorem, there exists $t_n \in (0,T)$ such that $u_n'(t_n) = 0$. Integrating (24), we obtain

$$0 = -\int_0^{t_n} u_n''(t)dt = \vartheta_n \int_0^{t_n} b(t) g\left(u_n(t)\right) dt.$$

Then

$$\int_0^{t_n} b(t) g\left(u_n(t)\right) dt = 0. \tag{25}$$

Now, we define

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_\infty}.$$

Dividing (24) by $\|u_n\|_\infty$, we have

$$v_n''(t) + \vartheta_n b(t) \frac{g\left(u_n(t)\right)}{u_n(t)} v_n(t) = 0. \tag{26}$$

By integration of (26), we obtain

$$v_n'(t) = v_n'(0) - \vartheta_n \int_0^t b(\xi) \frac{g\left(u_n(\xi)\right)}{u_n(\xi)} v_n(\xi) d\xi,$$

hence,

$$\|v_n'\|_\infty \leq \int_0^T |b(t)| \frac{g\left(u_n(t)\right)}{u_n(t)} dt. \tag{27}$$

Recalling that $b \in L^1([0,T])$, by (g_2) and the dominated convergence theorem, we conclude that

$$\lim_{n\to\infty} v_n'(t) = 0, \qquad \text{uniformly on } [0,T]. \tag{28}$$

Note that, for each n, $\|v_n\|_\infty = 1$, therefore there exists $s_n \in [0,T]$ such that $v_n(s_n) = 1$ for all $n \in \mathbb{N}$. Since

$$v_n(t) = v_n\left(s_n\right) - \int_{s_n}^t v_n'(\xi) d\xi,$$

follow that

$$\lim_{n\to\infty} v_n(t) = 1, \qquad \text{uniformly on } [0,T]. \tag{29}$$

By (25), fix $n \in \mathbb{N}$ such that

$$\int_0^{t_n} b(t) g\left(u_n(t)\right) dt = 0.$$

Thus, we can write

$$0 = \int_0^{t_n} b(t) g\left(u_n(t)\right) dt = \int_0^{t_n} \left(b(t) g\left(r_n\right) + b(t)\left[g\left(r_n v_n(t)\right) - g\left(r_n\right)\right]\right) dt.$$

At this point, let us define, for each n, $\omega_n := v_n\left(t_n\right)$, for a suitable choice of $t_n \in [0,T]$. Note that, by (29), $\lim\limits_{n\to\infty} \omega_n(t_n) = 1$.

Since $g\left(r_n\right) > 0$, then

$$-\int_0^{t_n} b(t) dt = \int_0^{t_n} b(t) \frac{g\left(r_n v_n(t)\right) - g\left(r_n\right)}{g\left(r_n\right)} dt.$$

Consequently, by (b_3),

$$0 < -\int_0^{t_n} b(t) dt \leq \|b\|_{L^1} \max_{t\in[0,T]} \left|\frac{g\left(r_n v_n(t)\right)}{g\left(r_n\right)} - 1\right| = \|b\|_{L^1} \left|\frac{g\left(r_n \omega_n\right)}{g\left(r_n\right)} - 1\right|.$$

Moreover, since g is regularly oscillating at zero, we have

$$\lim_{n\to\infty} \frac{g\left(r_n \omega_n(t_n)\right)}{g\left(r_n\right)} = 1.$$

Therefore, we obtain

$$0 < \lim_{n\to\infty} \int_0^{t_n} b(t) dt < 0, \tag{30}$$

that is a contradiction. Thus, we have proved the claim. And with what was proved in (23), we conclude that the hypothesis (H_1) is satisfied for all $r \in (0, r_0]$.

Remark 5.2. To ensure that inequality (5.8) holds, it is necessary to guarantee that the sequence $(t_n)_n$ converges to a positive value. For this purpose, we use property (b_1), which states that there exists $\delta > 0$ such that $b(t) < 0$ for almost every $t \in [0, \delta]$. It follows that

$$\int_0^{\delta} b(t)\, dt < 0.$$

Now, let us consider a solution $u(t)$ of

$$u'' + \vartheta b(t)g(u) = 0, \quad u(0) = u(T) \text{ and } u'(0) = 0,$$

with $\vartheta \in (0, 1]$. Thus, $u''(t) > 0$ for almost every $t \in [0, \delta]$. Therefore, $u(t)$ is convex on $[0, \delta]$. Given $s \in (0, \delta]$, we have

$$u'(s) = -\int_0^\delta b(\xi)g(u(\xi))\, d\xi > 0.$$

We conclude that u is increasing on $[0, \delta]$, and hence it has a relative maximum $\hat{t} \geq \delta$. This ensures that we can take the sequence $(t_n)_n$ such that $t_n \geq \delta$ for all $n \in \mathbb{N}$.

Continuing with the proof, let's verify hypothesis (H_2).

Initially, we define $W := \bigcup_{i=1}^{m} I_i$ and let $v : [a, b] \to \mathbb{R}$ be the characteristic function of the set W. Note that $v \in L^1([0, T])$ with $v(t) \geq 0$ in W and $v(t) = 0$ in $[0, T] \setminus W$.

Since $b(t) \geq 0$ on each interval I_i, we have

$$b(t)g(s) \geq 0, \text{ a.e. } t \in I_i, \forall s \geq 0$$

and

$$\liminf_{s \to +\infty} \frac{b(t)g(s)}{s} \geq b(t)g_\infty, \text{ uniformly a.e } t \in I_i.$$

Due to the second condition in hypothesis (g_2), the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda g_\infty b(t)\varphi = 0, \quad \varphi|_{\partial I_i} = 0,$$

is strictly less than 1 , for all $i = 1, \ldots, m$. Thus, we can apply Lemma 6.2 in the Appendix and conclude that, for each $i = 1, \cdots m$, if $k : [0, T] \times [0 + \infty) \to \mathbb{R}$ is a Carathéodory function with

$$k(t, s) \geq h(t, s), \text{ a.e } t \in I_i, \forall s \geq 0,$$

then there exists a constant $R_{I_i} > 0$ such that every solution $u(t) \geq 0$ of the problem

$$u'' + k(t, s) = 0, \quad u(0) = u(T) \text{ and } u'(0) = 0 \tag{31}$$

satisfies $\max_{t \in I_i} u(t) < R_{I_i}$.

Therefore, consider a constant $R > r_0$, where r_0 was obtained in the first part of the proof, such that

$$R \geq \max_{i=1,\ldots,m} R_{I_i} \tag{32}$$

and fix $\alpha_0 > 0$ such that

$$\alpha_0 > \frac{\|b\|_{L^1} \max_{0\leq s\leq R} g(s)}{\|v\|_{L^1}}. \tag{33}$$

Take $\alpha \in [0, \alpha_0]$ and, setting $k(t,s) = b(t)g(s) + \alpha v(t)$, consider the problem

$$u'' + b(t)g(s) + \alpha v(t) = 0, \quad u(0) = u(T) \text{ and } u'(0) = 0, \tag{34}$$

which is equivalent to problem (16). Note that $b(t)g(s) + \alpha v(t) \geq b(t)g(s)$ a.e. $t \in W$ and for all $s \geq 0$. By applying Lemma 6.2 on each interval I_i, any solution $u(t) \geq 0$ of problem (34) satisfies $\max_{t\in W} u(t) < R$. observe that the solutions of (34) are convex on the intervals of $[0,T] \setminus W$. Hence,

$$\max_{t\in[0,T]} u(t) = \max_{t\in A} u(t).$$

As a result, we have that $\|u\|_\infty < R$. Thus, we conclude the proof of (H_2).

To prove hypothesis (H_3) we need to verify that for $\alpha = \alpha_0$, defined in (33), there are no solutions $u(t)$ of (16) with $0 \leq u(t) \leq R$ on $[0,T]$. Indeed, if u is a solution of

$$u'' + b(t)g(u) + \alpha v(x) = 0, \quad u(0) = u(T) \text{ and } u'(0) = 0, \tag{35}$$

with $0 \leq u(t) \leq R$, then, applying the hypothesis (b_1), there exists $\delta > 0$ such that

$$u''(t) = -b(t)g(u(t)) - \alpha v(t) > 0 \;\text{ a.e }\; t \in [T-\delta, T].$$

By a convexity argument, we have that $u'(T) > 0$.

Now, integrating (35) on $[0,T]$, we obtain

$$u'(T) + \alpha\|v\|_{L^1} = \alpha \int_0^T v(t)dt \leq \int_0^T |b(t)|g(u(t))dt \leq \|b\|_{L^1} \max_{0\leq s\leq R} g(s).$$

Due to the choice of α_0 we reach a contradiction, which proves hypothesis (H_3). Thus, we have proved hypotheses (H_1), (H_2) and (H_3). By applying Theorem 4.1, we conclude the proof. □

6. Appendix

The results presented in this section play an important role in the proofs of Theorem 4.1 and Theorem 5.1. Here, we will see two results related to problem

$$\begin{cases} u'' + h(t,u) = 0 \\ \mathcal{B}(u) = (0,0), \end{cases} \tag{36}$$

where $h : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a L^1 - Carathéodory function and the linear operator $\mathcal{B} : C^1([0,T],\mathbb{R}) \to \mathbb{R}^2$ represents the boundary conditions, which can be

$$\begin{aligned} &\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))\,, \\ &\mathcal{B}(u) = (u'(T) - u'(0), u(0)) \qquad\qquad \text{or} \\ &\mathcal{B}(u) = (u(T) - u(0), u'(0))\;. \end{aligned}$$

In the following lemma, we present the maximum principle. This lemma is divided into two parts: part (i) is the weak maximum principle, and part (ii) is the strong maximum principle. These parts ensure, respectively, the non-negativity and positivity of the solutions to the problem (36).

LEMMA 6.1 (Maximum principle). *Let $h : [0,T]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function.*

(i) *If $h(t,s) > 0$, almost every $t \in [0,T]$ and for all $s < 0$, then any solution of (36) is non-negative on $[0,T]$.*

(ii) *If $h(t,0) \equiv 0$ and there exists $q \in L^1([0,T],[0,+\infty))$ such that*

$$\limsup_{s\to 0^+} \frac{|h(t,s)|}{s} \le q(t),$$

uniformly for almost every $t \in [0,T]$, then any non-trivial non-negative solution of (36) satisfies:

- *$u(t) > 0$ for every $t \in (0,T]$ if $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$;*
- *$u(t) > 0$ for every $t \in [0,T]$ if $\mathcal{B}(u) = (u(T) - u(0), u'(0))$;*
- *$u(t) > 0$ for every $t \in [0,T]$ if $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))$.*

Proof. (i) Suppose there is a solution $u(t)$ of (36) and $\hat{t} \in [0,T]$ such that $u(\hat{t}) < 0$. Let $(t_1,t_2) \subseteq (0,T)$ be the maximal open interval containing $\hat{t}$ such that $u(t) < 0$ for all $t \in (t_1,t_2)$. We will analyse four cases, namely:

(a) $0 < t_1 < t_2 < T$:
By hypothesis, $u''(t) = -h(t,u(t)) < 0$ for almost every $t \in (t_1,t_2)$. This means that $u(t)$ is a concave function on $[t_1,t_2]$. Moreover, $u(t_1) = u(t_2) = 0$. Therefore, we have $u(t) \ge 0$ for all $t \in (t_1,t_2)$, which contradicts the assumption that $u(\hat{t}) < 0$. Thus, this case is not possible.

(b) $t_1 = 0$ and $t_2 = T$:
By hypothesis, we know that $u''(t) = -h(t,u(t)) < 0$ for almost every $t \in [0,T]$. This guarantees that $u'(t)$ is strictly decreasing in

$[0, T]$. Thus, we have $u'(T) < u'(0)$ which contradicts the boundary conditions in the cases $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))$ and $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$.

For the case where $\mathcal{B}(u) = (u(T) - u(0), u'(0))$, we use the facts that $u'(t)$ is strictly decreasing in $[0, T]$ and that $u'(0) = 0$ to conclude that $u'(t) < 0$ for all $t \in (0, T]$. Therefore, $u(t)$ is decreasing in $[0, T]$ implying that $u(T) < u(0)$, which contradicts the boundary condition. Hence, this case is also not possible.

(c) $t_1 = 0$ and $t_2 < T$:

By hypothesis, we know that $u''(t) = -h(t, u(t)) < 0$ for almost all $t \in [0, t_2]$. Therefore,

$$0 > \int_0^{t_2} u''(t)dt = u'(t_2) - u'(0). \tag{37}$$

Thus, we have $u'(t_2) < u'(0)$. When the boundary condition is $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))$, we have $u'(t_2) < u'(0) = u(0) \leq 0$. This is a contradiction since $u'(t_2) \geq 0$. In the case where the boundary condition is $\mathcal{B}(u) = (u(T) - u(0), u'(0))$, we have $u'(t_2) < u'(0) = 0$, again contradicting the fact that $u'(t_2) \geq 0$. We still need to look at the condition $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$. But in this case, $u(0) = u(t_2) = 0$, which can be solved with the same argument used in case (a).

(d) $0 < t_1$ and $t_2 = T$:

We start with the condition $\mathcal{B}(u) = (u(T) - u(0), u'(0)))$. Note that $u(T) \leq 0$. The possibility $u(T) = 0$ can be excluded by the same strategy used in part (a). Now, if $u(T)$ is negative, by the boundary condition, $u(0)$ is also negative. Suppose $\bar{t}_2 = \sup\{t : u \text{ is negative in } [0, t)\}$ and use the same argument as in part (c) for the interval $[0, \bar{t}_2]$.

We move to the condition $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$. Note that $u(t_1) \leq 0$. First, assume that $u'(T) \geq 0$. Since $u''(t) = -h(t, u(t)) < 0$ for almost every $t \in [0, t_2]$, it follows that

$$0 > \int_{t_1}^{T} u''(t)dt = u'(T) - u'(t_1), \tag{38}$$

which implies $u'(t_1) > u'(T) \geq 0$. This leads to a contradiction. If $u'(T)$ is negative, then by the boundary condition, we have $u(0) = 0$ and $u'(0) < 0$. Therefore, there is $s \in (0, T)$ such that $u(t) < 0$ in $(0, s)$. Suppose

$$\bar{s} = \sup\left\{s : u(t) \text{ é negativa em } (0, s)\right\}.$$

Note that $\overline{s} \leq t_1 < T$, so we go back to case (c).

Finally, let us consider the case $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0)))$. Assuming $u'(T) \geq 0$ and using (38) we conclude that $u'(t_1) > u'(T) \geq 0$, which leads to a contradiction. Now, if $u'(T) < 0$, then by the boundary condition, we have $u'(T) = u'(0) = u(0) < 0$. Therefore, there exists $s \in (0, T)$ such that $u(t) < 0$ in $(0, s)$. Suppose

$$\overline{s} = \sup \{s : u(t) \text{ is negative in }]0, s[\} .$$

Note that $\overline{s} \leq t_1 < T$, so we go back to case (c).

(ii) By contradiction, suppose that there exists a solution $u(t) \geq 0$ of (36) and $t^* \in [0, T]$ such that $u(t^*) = 0$. It is clear that if $t^* \in (0, T)$, then $u'(t^*) = 0$. Let us examine what happens to $u'(t^*)$ in the cases $t^* = 0$ or $t^* = T$ for each boundary condition:

- $\mathcal{B}(u) = (u'(T) - u'(0), u(0))$: In this case, we cannot make any assertions about $t^* = 0$. But if $t^* = T$, since $u'(0) = u'(T)$, $u(T) = 0$ and $u(t) \geq 0$ in $[0, T]$, we conclude that $u'(T) = 0$.
- $\mathcal{B}(u) = (u(T) - u(0), u'(0))$: Here, $u'(0) = 0$. Therefore, we know what happens when $t^* = 0$. But we cannot make any assertions for $t^* = T$.
- $\mathcal{B}(u) = (u'(T) - u(0), u'(0) - u(0))$: Assuming $t^* = 0$, by the boundary conditions, we have $u'(0) = u(0) = 0$. Setting $t^* = T$, then $u(T) = 0$. Since $u(t) \geq 0$ for all $t \in [0, T]$, we have $u'(T) \leq 0$. By the boundary condition, $u(0) = u'(T) \leq 0$. But $u(t) \geq 0$ in $t \in [0, T]$, so $u(0) = 0$, i.e., $u'(T) = 0$.

Now we will prove the following claim:

Claim 6.1. *There exist $\varepsilon > 0$ such that $u(t) = 0$ for all $x \in [t^* - \varepsilon, t^* + \varepsilon]$.*

From the hypothesis, we know that there is $\delta > 0$ such that

$$|h(t, s)| \leq q_1(t)s, \text{ for almost every } t \in [0, T], \text{ for all } s \in [0, \delta],$$

where $q_1(t) := q(t) + 1$. Using the continuity of $u(t)$, we fix $\varepsilon > 0$ such that $0 \leq u(t) \leq \delta$, for all $t \in [t^* - \varepsilon, t^* + \varepsilon]$. We will use $\|(\xi_1, \xi_2)\| = |\xi_1| + |\xi_2|$ as the norm for $\mathbb{R}^2$. For all $t \in (t^*, t^* + \varepsilon]$ we have

$$\begin{aligned} 0 \le \|(u(t), u'(t))\| &= |u(t)| + |u'(t)| \\ &= u(t^*) + \int_{t^*}^{t} u'(\xi) d\xi + \left| u'(t^*) + \int_{t^*}^{t} -h(\xi, u(\xi)) d\xi \right| \\ &\le \int_{t^*}^{t} |u'(\xi)| d\xi + \int_{t^*}^{t} |h(\xi, u(\xi))| d\xi \\ &\le \int_{t^*}^{t} \left[q_1(\xi)|u(\xi)| + |u'(\xi)| \right] d\xi \\ &\le \int_{t^*}^{t} (q_1(\xi) + 1)(|u(\xi)| + |u'(\xi)|) d\xi. \end{aligned}$$

Using the Gronwall inequality, we obtain

$$0 \le u(t) \le \|(u(t), u'(t))\| = 0, \ \forall t \in (t^*, t^* + \varepsilon].$$

With a similar calculation, we can prove $u(t) = 0$ for all $t \in [t^* - \varepsilon, t^*)$. Thus, the assertion is proved.

Finally, we need to show that Claim 6.1 leads to a contradiction. We will see this in the following claim:

Claim 6.2. *The Claim 6.1 implies that* $u \equiv 0$ *in* $[0, T]$*, which is a contradiction. Indeed, suppose* $t^* \in (0, T)$*. Fix* $\varepsilon > 0$ *such that* $J := [t^* - \varepsilon, t^* + \varepsilon] \subseteq (0, T)$ *e* $u \equiv 0$ *in* J*. Consider* $E = \{t \in [0, T] : u \equiv 0 \text{ in } [t^*, t] \text{ or } u \equiv 0 \text{ in } [t, t^*]\}$*. We want to show that* $E = [0, T]$*. If* $E \subsetneq [0, T]$*, then* $\sup E < T$ *or* $\inf E > 0$*. Let* $b = \sup E$ *and assume* $b < T$*. I claim that* $b \in E$*. In fact, since* $b = \sup E$*, there is a sequence* $(t_n)_n \subset E$ *with* $t_n \to b$*. As* u *is continuous,* $\lim_{n\to\infty} u(t_n) = u(b) = 0$*, since* $u(t_n) = 0$ *for all* n*. Note that* $b \in E$*, otherwise there would exist* $y \in (t^*, b)$ *with* $u(y) \neq 0$*. However, as* $t_n \to b$*, there is* n *such that* $t_n > y$*, which is absurd, since* $u \equiv 0$ *in* $[t^*, t_n]$*. Now we know that* $b \in (t^*, T)$*. By Claim 6.1, there exists* $\varepsilon > 0$ *such that* u *is zero in* $[b - \varepsilon, b + \varepsilon]$*. But then* $[t^*, b + \varepsilon] \subseteq E$*, which is absurd, since* $b = \sup E$*. Hence* $b = T$*.*

□

The second result of this section provides a priori bounds for non-negative solutions on the intervals where $h(t, s)$ is non-negative. The proof of this result can be found in [4, Lemma 6.2]. It is worth noting that in the proof, the boundary conditions are not used, hence this is a result that holds independently of the boundary condition considered. This lemma is used for verifying hypothesis (H_2) in Theorem 5.1.

LEMMA 6.2. *Let* $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a* L^1*-Carathéodory function. Suppose there exists a closed interval* $J \subseteq [0,T]$ *such that*

$$h(t,s) \geq 0, \qquad a.e.\ t \in J, \forall s \geq 0;$$

and there is a measurable function $q_\infty \in L^1(J, \mathbb{R}^+)$ *with* $q_\infty \not\equiv 0$*, such that*

$$\liminf_{s \to +\infty} \frac{h(t,s)}{s} \geq q_\infty(t), \qquad \text{uniformly a.e. } t \in J.$$

Let μ_J *be the first positive eigenvalue of the eigenvalue problem*

$$\varphi'' + \lambda q_\infty(t)\varphi = 0, \quad \varphi|_{\partial J} = 0,$$

and suppose that $\mu_J < 1$*. Then there exists* $R_J > 0$ *such that for each Carathéodory function* $k : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ *with*

$$k(t,s) \geq h(t,s), \qquad a.e.\ t \in J, \forall s \geq 0,$$

every solution $u(t) \geq 0$ *of the boundary value problem*

$$u'' + k(t,u) = 0, \quad \mathcal{B}(u) = (0,0),$$

satisfies $\max_{t \in J} u(t) < R_J$*.*

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