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The fundamental group of $SO(n)$ via quotients of braid groups

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Abstract. *Some topological properties of a Lie group can be deduced by studying a discrete group of homotopy classes of paths from the identity to elements of a finite subgroup of the given Lie group. In this way a "skeleton" of the universal cover is constructed in terms of generators and relations. We use this approach to describe an algebraic derivation of the well-known fact that the fundamental group of $SO(n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ when $n \geq 3$. The fundamental group of $SO(n)$ appears in our treatment as a subgroup of the center of a finite factor of the braid group B_n, obtained by imposing one additional relation and turns out to be a nontrivial central extension by $\mathbb{Z}/2\mathbb{Z}$ of the corresponding group of rotational symmetries of the hyperoctahedron in dimension n.*

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1. Introduction

An important property of the rotation groups $SO(n)$ when $n \geq 3$ is that they are not simply connected and their fundamental group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Other Lie groups, whose maximal compact subgroups are $SO(n)$, such as the connected components containing the identity of $SO(1,n)$, $GL(n,\mathbb{R})$, and $SL(n,\mathbb{R})$, share the same property. This specific topological feature especially for $SO(3)$ and the restricted Lorentz group $SO^+(1,3)$ has a far-reaching consequence for our physical world – the existence of precisely two fundamentally different classes of elementary particles – bosons and fermions. Bosons are particles with integer spin and their wave function transforms under a representation of the group $SO(3)$ (in non-relativistic quantum mechanics) or $SO^+(1,3)$ (in the relativistic case). On the other hand fermions have half-integer spin and their wave functions transform under representations of the universal covering groups $SU(2)$ and $SL(2,\mathbb{C})$, respectively. This possibility, discovered heuristically by Pauli and Dirac, can be traced to the fact that the components of the wave function in quantum mechanics do not have direct physical meaning. The careful analysis showed ([2, 5]) that the wave function has to transform properly only under transformations which are in a small neighborhood of the identity, which is essentially equivalent to looking at representations of the Lie algebra.

The standard proof that $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ when $n = 3$ uses substantially Lie theory. A two-to-one homomorphism $SU(2) \to SO(3)$ is exhibited, which is a local isomorphism of Lie groups. Since $SU(2)$ is topologically a sphere we obtain a double covering map sending any two antipodal points in $SU(2)$ to one point in $SO(3)$. The case $n > 3$ reduces to the above result by applying powerful techniques from homotopy theory, not particularly accessible to non-experts in algebraic topology.

One interesting observation is that there is a connection between the fundamental group of $SO(n)$ and the braid group with n strands B_n. The case $n = 3$ is fairly intuitive and was studied in [7] where an isomorphism was constructed between homotopy classes of closed paths in $SO(3)$ and a certain factor group of the group P_3 of pure braids with three strands. This factor group turns out to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The method of [7], however, has the disadvantage that it does not lend itself to a generalization to higher dimensions. (A geometric braid in $\mathbb{R}^n$ is always trivial when $n > 3$.) Still, the connection to braid groups persists in higher dimension, as we will see.

The idea of the present paper is to develop an algebraic approach to calculating $\pi_1(SO(n))$ exploiting substantially the fact that our manifold is a group. We study a certain discrete (in fact finite) group of homotopy classes of paths in $SO(n)$, starting at the identity and ending at points which are elements of some fixed finite subgroup of $SO(n)$. It turns out that it is convenient to use the finite group of rotational symmetries of the hyperoctahedron. Each homotopy class contains an element consisting of a chain of rotations by $\pi/2$ in different coordinate planes. These simple motions play the role of generators of our group. Certain closed paths obtained in this way remain in a small neighborhood of the identity (in an appropriate sense, explained later) and can be shown explicitly to be contractible. Thus, certain products among the generators must be set to the identity and we get a group defined by a set of generators and relations. Interestingly, the number of (independent) generators is $n - 1$ and the relations, apart from one of them, are exactly Artin's relations for the braid group B_n. In this way we obtain for each n a finite group, which is the quotient of B_n by the normal closure of the group generated by the additional relation. When $n = 3$ the order of the group turns out to be 48 and it is the so-called *binary octahedral group* which is a nontrivial extension by $\mathbb{Z}/2\mathbb{Z}$ of the group of rotational symmetries of the octahedron. We may think of the former as a double cover of the latter and this is a finite version of the double cover $SU(2) \to SO(3)$. The next groups in the series have orders 384, 3840, 46080, etc.; in fact the order is given by $2^n n!$. Note that these groups have the same orders as the Coxeter groups of all symmetries (including reflections as well as rotations) of the respective hyperoctahedra, but they are different. This is analogous to the relationship between $O(n)$ and $SO(n)$ on the one hand, and $Spin(n)$ and $SO(n)$ on the other.

It turns out that the subgroup of homotopy classes of closed paths, i.e. $\pi_1(SO(n))$, in each case is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and either coincides or lies in the center of the respective group. Factoring by it, the respective rotational hyperoctahedral groups are obtained. Thus we believe that as a side result we obtain interesting new presentations of all rotational hyperoctahedral groups and their double covers as factors of braid groups.

An open challenge is to extend the method described to calculating higher homotopy groups of $SO(n)$.

2. Groups of Homotopy Classes of Paths

We will consider paths in $SO(n)$ starting at the identity. In other words, we have continuous functions $R : [0,1] \to SO(n)$, subject to the restriction $R(0) = Id$. Because the target space is a group, there is a natural product of such paths, i.e. if R_1 and R_2 are paths of this type, we first translate R_1 by the constant element $R_2(1)$ to the path $R_1R_2(1)$. Then we concatenate R_2 with $R_1R_2(1)$. Thus, by R_1R_2 we mean the path:

$$(R_1R_2)(t) = \begin{cases} R_2(2t), & \text{if } t < \frac{1}{2}\,, \\ R_1(2t-1)R_2(1), & \text{if } t \geq \frac{1}{2}\,. \end{cases}$$

For each R we denote by R^{-1} the path given by

$$R^{-1}(t) = (R(t))^{-1}\ .$$

The set formed by continuous paths in $SO(n)$ obviously contains an identity, which is the constant path. The product in this set is not associative since $(R_1R_2)R_3$ and $R_1(R_2R_3)$ are different paths (due to the way we parametrize them), even though they trace the same curve. Also, R^{-1} is by no means the inverse of R. In fact, there are no inverses in this set. However, the set of homotopy classes of such paths (with fixed ends) is a group with respect to the induced operations. Further, we will use R to denote the homotopy class of R. Now R^{-1} is the inverse of R. Recalling how the universal covering space of a topological space is defined we see that the group we have defined is none other than the universal covering group of $SO(n)$. In the case $n = 3$ this is the group $SU(2)$; in general it is the group denoted by $Spin(n)$.

Our aim is of course to show that the subgroup corresponding to closed paths is $\mathbb{Z}_2$ or equivalently that the covering map, which is obtained by taking the end-point of a representative path, is two-to-one. Since the full group of homotopy classes of paths starting at the identity is uncountable and difficult to handle, the main idea of this paper is to study a suitable discrete, in fact finite, subgroup by limiting the end-points to be elements of the group of rotational symmetries of the hyperoctahedron in dimension n. In what follows we denote this group by G. (One could perhaps take any large enough finite subgroup of $SO(n)$, but using the hyperoctahedral group seems the simplest.)

The further study of G requires algebra and just a couple of relatively easy results from analysis and geometry. We state these here and leave the proofs for the appendix.

By a *generating path* R_{ij} we will mean a rotation from 0 to $\pi/2$ in the coordinate plane ij.

LEMMA 1. *Every homotopy class of paths in $SO(n)$, $n \geq 3$, starting at the identity and ending at an element of the rotational hyperoctahedral group contains a representative which is a product of generating paths.*

Each vertex of the hyperoctahedron lies on a coordinate axis—either in the positive or negative direction — and determines a closed half-space (of all points having the respective coordinate nonnegative or nonpositive) to which it belongs. Let us call *local closed paths* those closed paths in $SO(n)$ for which no vertex of the hyperoctahedron leaves the closed half-space to which it belongs initially.

LEMMA 2. *Local closed paths are contractible. A closed path consisting of generating paths is contractible if and only if the word representing it can be reduced to the identity by inserting expressions describing local closed paths.*

3. The case $n = 3$

We consider three-dimensional rotations separately since the essential algebraic properties of G are present already here and in a sense the higher-dimensional cases are a straightforward generalization. It is worth recalling some facts about the group of rotational symmetries of the octahedron in three-dimensions. The octahedron is one of the five regular convex polyhedra, known as Platonic solids. It has six vertices and eight faces which are identical equilateral triangles. The octahedron is the dual polyhedron of the cube and so they have the same symmetry group. The analogous statement is true in any dimension. We find it convenient to think of a spherical model of the octahedron — the edges connecting the vertices are parts of large circles on the unit sphere (Fig.1).

The octahedron has three types of rotational symmetries belonging to cyclic subgroups generated by elements of different orders – rotations by $\pi/2$ about the three coordinate axes (order 4); rotations by π about axes connecting the centers of six pairs of opposite edges (order 2); rotations by $2\pi/3$ about axes connecting the centers of four pairs of opposite faces (order 3). It is well-known that the group of rotational symmetries of the octahedron is faithfully

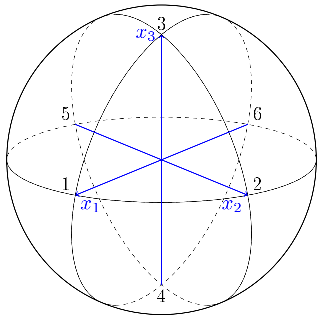

Figure 1: A spherical octahedron with its three axes of rotational symmetries of order 4.

represented as the group of permutations of the four pairs of opposite faces. Therefore, it is isomorphic to S_4. All symmetric groups can be realized as finite reflection groups and thus finite Coxeter groups of type A_n [6]. We have $S_n \cong A_{n-1}$ and in the case at hand the group is A_3 — the full symmetry group (including reflections) of the tetrahedron. The case $n = 3$ is an exception. When $n > 3$ the respective rotational hyperoctahedral group is not a Coxeter group. For any n however, it is a normal subgroup of index 2 of the respective full hyperoctahedral group which is a Coxeter group of type B_n (not to be confused with the braid group on n strands, for which the same notation is used).

The full hyperoctahedral group can also be considered as a *wreath product* $S_2 \wr S_n$. The wreath product in this special case is the semidirect product of $\Sigma = S_2 \times S_2 \times \cdots \times S_2$ with S_n, where S_n acts on the first factor by permuting its components. The n pairs of opposite vertices determine n mutually orthogonal (non-oriented) lines in $\mathbb{R}^n$. A rotation in the ijth plane by $\pi/2$ permutes the ith and jth lines. An arbitrary symmetry can be realized by an arbitrary permutation of the lines plus possible reflections with respect to the hyperplanes perpendicular to the n lines. This explains the structure of the full hyperoctahedral group as a wreath product. One advantage is that it is easy to see that the order of the group is $2^n n!$.

The rotational hyperoctahedral group forms a normal subgroup of index 2 in the full hyperoctahedral group. In terms of orthogonal matrices, this is the subgroup of matrices with unit determinant. Note that each rotation is a product of two reflections. Coming back to the rotational octahedral group (i.e. the case $n = 3$), we observe that it is generated by three elements of order four, denoted by r_1, r_2 and r_3 — rotations by $\pi/2$ about the three coordinate axes. Obviously, these three elements generate the respective cyclic groups of order four but they also generate all other rotational symmetries. For example $r_1r_2r_3$ has order two and is a rotation by π about an axes connecting two opposite edges, while r_1r_2 has order three and is a rotation by $2\pi/3$ about an axis connecting the centers of two opposing faces.

We consider the discrete group G generated by the generating paths $R_1 := R_{23}$, $R_2 := R_{31}$, and $R_3 := R_{12}$, treated as homotopy classes. (We have $R_i(1) = r_i,\ i = 1,2,3.$) Local closed paths built out of the generators R_i and their inverses are contractible by Lemma 2 and their corresponding words must be set to identity. Apart from trivial cases where an R_i is followed by its inverse, we have a family of paths for which each vertex either goes around the edges of a single triangular face of the octahedron or moves along an edge and comes back. Inspecting all possible ways in which such "triangular" closed paths can be built, we see that each one is represented by a word of four letters. Each word contains either R_i or R_i^{-1} for each i. No word contains twice a given letter but it may contain a letter together with its inverse. In that case this letter and its inverse conjugate one of the other two letters. Here is a list of all identities that follow:

$$
\begin{aligned}
1 &= R_3^{-1}R_1^{-1}R_2^{-1}R_1 = R_3R_1^{-1}R_2R_1 = R_2R_3^{-1}R_2^{-1}R_1 = R_2^{-1}R_3R_2R_1 \\
&= R_3^{-1}R_2^{-1}R_3R_1 = R_3R_2R_3^{-1}R_1 = R_2^{-1}R_1^{-1}R_3R_1 = R_2R_1^{-1}R_3^{-1}R_1 \\
&= R_1R_3R_1^{-1}R_2 = R_1^{-1}R_3^{-1}R_1R_2 = R_3R_1^{-1}R_3^{-1}R_2 = R_3^{-1}R_1R_3R_2 \\
&= R_3^{-1}R_2^{-1}R_1R_2 = R_3R_2^{-1}R_1^{-1}R_2 = R_1R_2^{-1}R_3R_2 = R_1^{-1}R_2^{-1}R_3^{-1}R_2 \\
&= R_1^{-1}R_2R_1R_3 = R_1R_2^{-1}R_1^{-1}R_3 = R_2R_3^{-1}R_1R_3 = R_2^{-1}R_3^{-1}R_1^{-1}R_3 \\
&= R_2^{-1}R_1^{-1}R_2R_3 = R_2R_1R_2^{-1}R_3 = R_1^{-1}R_3^{-1}R_2R_3 = R_1R_3^{-1}R_2^{-1}R_3.
\end{aligned} \tag{1}
$$

Very few of these 24 identities are actually independent. First, we notice that one of the generators, e.g. R_3 can be expressed as a combination of the other two and their inverses, in several different ways. For example, if we use the second identity in the fifth row and the first identity in the last row we get $R_3 = R_1R_2R_1^{-1} = R_2^{-1}R_1R_2$, from which follows

$$R_2R_1R_2 = R_1R_2R_1, \tag{2}$$

while if we use $R_3 = R_1^{-1}R_2^{-1}R_1 = R_1R_2R_1^{-1}$, and also $R_3 = R_2^{-1}R_1R_2 = R_2R_1^{-1}R_2^{-1}$, we obtain:

$$R_1^2 = R_2R_1^2R_2, \qquad R_2^2 = R_1R_2^2R_1. \tag{3}$$

All other identities are consequences of these three. Therefore, G is presented as a group generated by two generators and a set of relations, one of which (Equation 2) is precisely Artin's braid relation for the braid group with three

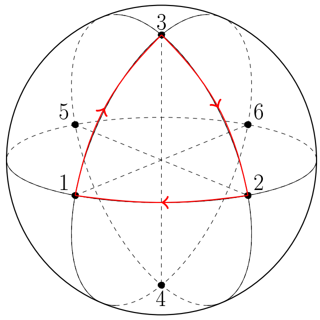

Figure 2: Visualization of the triangular closed path $R_3^{-1}R_1^{-1}R_2^{-1}R_1$ (the path traced by vertex 1).

strands B_3. In other words G is the quotient of B_3 by the normal closure of the subgroup generated by the additional relations (3).

Actually, only one of the identities (3) is independent:

LEMMA 3. *The second identity in (3) follows from the first one and Artin's braid relation (2).*

Proof. Using Artin's relation twice it is almost immediate that

$$R_2^2 = R_1R_2R_1^2R_2^{-1}R_1^{-1}.$$

Now using the first of the identities (3) inside the expression above we get

$$R_2^2 = R_1R_2R_1^2R_2^{-1}R_1^{-1} = R_1R_2R_2R_1^2R_2R_2^{-1}R_1^{-1} = R_1R_2^2R_1. \qquad \square$$

Note: As shown by Artin [1], the braid group B_n can be thought as a group of isotopy classes of geometric braids with n strands or as a group generated by $n-1$ generators satisfying what came to be called Artin's braid relations. When $n=3$ there is just one relation (2) between the two generators. The geometric picture has the advantage of being more intuitive and providing us with ways to see identities, which can then be shown algebraically. Thus, for example, the proof above has a geometric version which is easy to visualize (Fig.3). The same situation will be in place when we consider $n>3$. Our group G will be generated by $n-1$ generators satisfying the standard braid relations plus some additional ones. Using the geometric picture will allow us to arrive at conclusions which are difficult to see algebraically.

COROLLARY 1. *The group G has a presentation*

$$G = \left\langle R_1, R_2 \,|\, R_1R_2R_1 = R_2R_1R_2, R_1^2 = R_2R_1^2R_2 \right\rangle . \qquad (4)$$

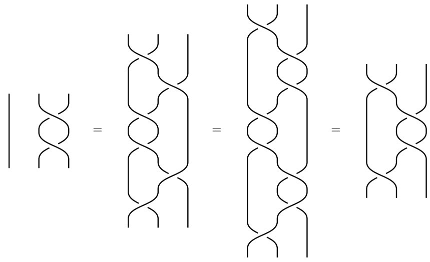

Figure 3: Geometric Proof of Lemma 3 that $R_2^2 = R_1 R_2^2 R_1$ follows from $R_1^2 = R_2 R_1^2 R_2$.

Note: We will see shortly that the group defined by (4) is the binary octahedral group $2O$, which is the universal (double) cover of the group of rotational symmetries of the octahedron. The latter is obtained from (4) by imposing the additional relation $R^4 = Id$ and is also known as the Von Dyck group (also "rotational triangle group") $D(2,3,4) := \langle x, y \,|\, x^2 = y^3 = (xy)^4 = \mathbf{1} \rangle$. This construction has a generalization, discussed in detail in [8]. Consider a two-dimensional surface with constant positive, zero, or negative curvature, tessellated by equilateral geodesic triangles. In the positive case we get a spherical model of the tetrahedron, octahedron or icosahedron, depending on the number n of triangles meeting at each vertex. When $n \geq 6$ we get a tessellation of the infinite Euclidean plane or hyperbolic plane, respectively. In all cases the group of orientation-preserving isometries preserving the tessellation is the Von Dyck group $D(2,3,n) := \langle x, y \,|\, x^2 = y^3 = (xy)^n = \mathbf{1} \rangle$. As shown in [8], instead of the standard generators for $D(2,3,n)$ of order 2 and 3, one can generate the group by rotations about two vertices of a given triangle. They always satisfy Artins's braid relation. In fact we have

$$D(2,3,n) = \langle R_1, R_2 \,|\, R_1 R_2 R_1 = R_2 R_1 R_2, R_1^2 = R_2 R_1^{n-2} R_2, R^n = Id \rangle \ . \quad (5)$$

By dropping the third relation in (5) we obtain a presentation of the universal cover of the respective Von Dyck group. Notice that the second relation still carries the information about the type of tessellation.

Proposition 1. *The order of R_1 and R_2 in G is eight.*

Proof. Using relations (3) we conclude that $R_1^2 = R_2 R_1^2 R_2 = R_2^2 R_1^2 R_2^2$ and similarly $R_2^2 = R_1^2 R_2^2 R_1^2$. Therefore

$$R_1^2 = R_2^2 R_1^2 (R_1^2 R_2^2 R_1^2) \Longrightarrow Id = R_2^2 R_1^4 R_2^2 \Longrightarrow R_1^4 = R_2^{-4}.$$

Note that the last equality above immediately implies that R_1^4 is central. Next, one can write

$$R_2^4 = R_2R_2^2R_2 = R_2R_1R_2^2R_1R_2 = R_2R_1R_2R_1R_2R_1 = R_1R_2R_1^2R_2R_1 = R_1^4.$$

Putting together the results in the last two equations we see that $R_1^4 = R_1^{-4}$ and similarly $R_2^4 = R_2^{-4}$, which imply $R_1^8 = R_2^8 = Id$.

From this we can tell that the order of R_1 and R_2 is at most 8. To conclude that it is exactly 8 (and not for example 4) is not a trivial problem. One way to do this is to perform the Todd – Coxeter algorithm (see e.g. Ken Brown's short description [3]). If the group defined by a finite set of generators and relations is finite, the algorithm will (in theory) close and stop and will produce the order of the group and a table for the action of the generators on all elements. We used the simple computer program graciously made available by Ken Brown on his site. The resulting table showed that the order of G is 48 and the central element $R_1^4 = R_2^4$ is not trivial. □

We know that taking the end-point of any element $R \in G$ gives a homomorphism from G onto the group of rotational symmetries of the octahedron and the latter can be considered as a subgroup of the symmetric group S_6 since it permutes the six vertices. So we have a homomorphism $\theta : G \to S_6$. To prove that the fundamental group of $SO(3)$ is $\mathbb{Z}_2$, we need to show that the kernel of the homomorphism described above is $\mathbb{Z}_2$, because the kernel consists of those elements of G which are classes of closed paths in $SO(3)$. In other words, these are the motions that bring the octahedron back to its original position. It will be helpful to come up with a way to list all elements of G as words in the generators R_1, R_2 and their inverses, i.e., find a *canonical form* for the elements of G.

Proposition 2. *Any $x \in G$ can be written uniquely in one of the three forms:*
1. $R_1^mR_2^n$ (32 elements),
2. $R_1^mR_2R_1$ (8 elements),
3. $R_1^mR_2^3R_1$ (8 elements),
where $m \in \{0, 1, 2, 3, ..., 7\}$ and $n \in \{0, 1, 2, 3\}$.

Proof. Any element of G can be obtained by multiplying the identity by a sequence of the generators and their inverses either on the right or on the left. We are using right multiplication. The idea is to show that when multiplying an element $x \in G$, which is written in the form 1, 2 or 3 by any of the two generators of G or their inverses, we get again an expression of these three types. Since $R_i^{-1} = R_i^7$ it is enough to check the above for positive powers.

The proof is a direct verification using Artin's braid relation and the identity $R_2^2R_1 = R_1^{-1}R_2^2$ (and the symmetric one with R_1 and R_2 interchanged). Thus, e.g., multiplying the first expression by R_1 we get $R_1^{m\pm1}R_2^n$, if n is even and either expression 2 or 3, if n is odd.

Similarly, multiplying expression 3 by R_1 we get

$$R_1^m R_2^3 R_1^2 = R_1^{m+2} R_2^{-3} = R_1^{m+2} R_2^5 = R_1^{m+6} R_2$$

while multiplying it by R_2 gives

$$R_1^m R_2^3 R_1 R_2 = R_1^m R_2^2 R_2 R_1 R_2 = R_1^m R_2^2 R_1 R_2 R_1 = R_1^{m-1} R_2^3 R_1 \ .$$

Uniqueness is proven by inspection. Let us show as an example that $R_1^m R_2^n \neq R_1^k R_2 R_1$. Indeed, the assumption that the two are equal leads to the following sequence of equivalent statements:

$$R_1^{m-k} R_2^n = R_2 R_1 \Rightarrow R_1^{m-k+1} R_2^n = R_1 R_2 R_1 = R_2 R_1 R_2 \Rightarrow$$
$$R_1^{m-k+1} R_2^{n-1} = R_2 R_1 \Rightarrow \cdots \Rightarrow R_1^{m-k+n} = R_2 R_1 \Rightarrow R_1^{m-k+n-1} = R_2.$$

The last identity is apparently wrong since no power of R_1 can be equal to R_2. □

PROPOSITION 3. *The kernel of the homomorphism $\theta : G \to S_6$ is isomorphic to $\mathbb{Z}_2$.*

Proof. The proof is straightforward and consists of checking how the images under θ of the elements of G permute the vertices. We skip the details. It turns out that the only nontrivial element mapped to the identity is R_1^4 and since $R_1^8 = Id$ we obtain

$$\pi_1(SO(3)) \cong \ker(\theta) = \{Id, R_1^4\} \cong \mathbb{Z}_2.$$ □

Since $R_1^4 = R_2^4$ is a central element of order two, the group generated by it, $\{Id, R_1^4\}$, belongs to the center $Z(G)$. It is not obvious a priori that there are no other central elements. We make the following observation: the center of the braid group B_n is known to be isomorphic to $\mathbb{Z}$. It is generated by a full twist of all n strands — $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ (using σ_i for the generators of B_n). Since G is obtained from B_n by imposing one more relation, any central element in B_n will be central in G, although there is no guarantee that we will obtain a nontrivial element in G in this way. In addition, there may be central elements in G which come from non-central elements in B_n. When $n = 3$ the above argument ensures that $(R_1R_2)^3$ is a central element in G. The calculation in Proposition 1 shows that

$$(R_1R_2)^3 = R_1R_2R_1R_2R_1R_2 = R_1^4 = R_2R_1R_2R_1R_2R_1 = R_2^4,$$

so we do not get any new central element, different from the one we have already found. When $n = 4$ however, the analogous calculation, using the braid relations plus the additional relations as in equation 3, gives

$$(R_1R_2R_3)^4 = R_1^2 R_3^{-2}.$$

This is another central element, different from $R_1^4 = R_2^4 = R_3^4$. Taking the product of the two we obtain a third central element $R_1^2 R_3^2$. Therefore when $n = 4$, $Z(G)$ contains (in fact coincides with) the product of two copies of $\mathbb{Z}_2$. This obviously generalizes to any even n — the element $R_1^2 R_3^2 \cdots R_{n-1}^2$ is central, as can be checked explicitly.

The difference between even and odd dimensions can be traced back to the difference between rotation groups in even and odd dimensions. As we shall see shortly, when we factor G by $\ker\theta$ we obtain the rotational hyperoctahedral group. In odd dimensions this has trivial center, which follows for example from Schur's lemma and the fact that the matrix $-\mathbf{1}$ is not a rotation. However, in even dimensions reflection of all axes, given by $-\mathbf{1}$, is a rotation and therefore the corresponding rotational hyperoctahedral group has a nontrivial center.

According to the general construction, we expect that when we factorize G by the kernel of the covering map, which is nothing but $\ker\theta$, we should obtain the rotational octahedral group. This can also be established directly. Denoting by G_1 the quotient, we have a presentation for it:

$$G_1 = \left\langle R_1, R_2 \,|\, R_1R_2R_1 = R_2R_1R_2, R_1^2 = R_2R_1^2R_2, R_1^4 = Id \right\rangle . \qquad (6)$$

The order of G_1 is 24. We can list its elements using the same expressions as in Proposition 2, except that now the two integers n and m run from 0 to 3. An easy calculation shows that G_1 is the symmetric group S_4, which on the other hand is the group of the octahedral rotational symmetries. Given the structure of G, we conclude that it is a nontrivial extension by $\mathbb{Z}_2$ of S_4 described by a non-split short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow G \longrightarrow S_4 \longrightarrow 1 .$$

The non-isomorphic central extensions by $\mathbb{Z}_2$ of S_4 are in one-to-one correspondence with the elements of the second cohomology group (for trivial group action) of S_4 with coefficients in $\mathbb{Z}_2$ and the latter is known to be isomorphic to the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The identity in cohomology corresponds to the trivial extension $\mathbb{Z}_2 \times S_4$. The other three elements of the cohomology group classify the three non-isomorphic nontrivial extensions, namely the binary octahedral group $2O$, the group $GL(2,3)$ of nonsingular 2×2 matrices over the field with three elements, and the group $SL(2,4)$ of 2×2 matrices with unit determinant over the ring of integers modulo 4.

Proposition 4. *G is isomorphic to the binary octahedral group $2O$.*

Note: By construction, we expect the group G to be a subgroup of $SU(2)$ and the covering map $G \to S_4$ to be a restriction of the covering map $SU(2) \to SO(3)$. The fact that G with the presentation (4) is isomorphic to $2O$ is mentioned in Section 6.5 of [4]. The GAP ID of G is [48, 28].

Proof. The binary octahedral group, being a subgroup of $SU(2)$, can be realized as a group of unit quaternions. In fact it consists of the 24 Hurwitz units

$$\left\{\pm 1, \pm i, \pm j, \pm k, \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)\right\}$$

and the following 24 additional elements:

$$\left\{\tfrac{1}{\sqrt{2}}(\pm 1 \pm i), \tfrac{1}{\sqrt{2}}(\pm 1 \pm j), \tfrac{1}{\sqrt{2}}(\pm 1 \pm k), \tfrac{1}{\sqrt{2}}(\pm i \pm j), \tfrac{1}{\sqrt{2}}(\pm i \pm k), \tfrac{1}{\sqrt{2}}(\pm j \pm k)\right\}$$

To prove the isomorphism it is enough to find two elements in $2O$ which generate the whole group and which satisfy the same relations as the defining relations of G. Obviously we are looking for elements of order 8. Let, e.g., $u_1 = \frac{1}{\sqrt{2}}(1-k), u_2 = \frac{1}{\sqrt{2}}(1-j) \in 2O$. It is a simple exercise in quaternion algebra to prove that $u_2u_1u_2 = u_1u_2u_1$. Similarly, we check that $u_1^2 = u_2u_1^2u_2$, or equivalently that $u_1^2u_2 = u_2^{-1}u_1^2$. Indeed, since $u_1^2 = -k$ and since k and j anticommute,

$$u_1^2u_2 = -k\tfrac{1}{\sqrt{2}}(1-j) = \tfrac{1}{\sqrt{2}}(1+j)(-k) = u_2^{-1}u_1^2\ .$$

A direct verification further shows that the whole $2O$ is generated by u_1 and u_2. □

The 24 Hurwitz units, when considered as points in $\mathbb{R}^4$, lie on the unit sphere S^3 and are the vertices of a regular 4-polytope — the 24-cell, one of the exceptional regular polytopes with symmetry — the Coxeter group F_4. The set is also a subgroup of $2O$ — the binary tetrahedral group, denoted as $2T$. The second set of 24 unit quaternions can be thought of as the vertices of a second 24-cell, obtained from the first one by a rotation, given by multiplication of all Hurwitz units by a fixed element, e.g. $\frac{1}{\sqrt{2}}(1+i)$. The convex hull of all 48 vertices is a 4-polytope, called disphenoidal 288-cell.

It may be instructive to consider the symmetric group S_4 with its presentation given by equation (6) and try to construct explicitly all non-isomorphic central extensions by $\mathbb{Z}_2$. This means that the three relations in the presentation of S_4 must now be satisfied up to a central element, belonging to the (multiplicative) cyclic group with two elements $\{1, -1\}$:

$$R_1R_2R_1 = aR_2R_1R_2, \quad R_1^2 = bR_2R_1^2R_2, \quad R_1^4 = c, \quad a, b, c \in \mathbb{Z}_2.$$

The element a in the first relation can be absorbed by replacing R_1 by aR_1, so Artin's braid relation remains unchanged. We are left with four choices for b and c:

1. The choice $b = c = 1$ leads to the trivial extension as a direct product $\mathbb{Z}_2 \times S_4$.
2. The choice $b = 1, c = -1$ leads to the already familiar group

$$G \cong 2O = \langle R_1, R_2 \mid R_1R_2R_1 = R_2R_1R_2, R_1^2 = R_2R_1^2R_2 \rangle .$$

3. The choice $b = c = -1$ leads to the group $GL(2,3)$ with presentation

$$GL(2,3) = \langle R_1, R_2 \mid R_1R_2R_1 = R_2R_1R_2, R_1^2 = R_2R_1^6R_2, R_2R_1^4 = R_1^4R_2 \rangle .$$

4. The choice $b = -1, c = 1$ leads to the group $SL(2,4)$ with presentation

$$SL(2,4) = \langle R_1, R_2, b \mid R_1R_2R_1 = R_2R_1R_2, R_1^2 = bR_2R_1^2R_2, R_1^4 = b^2 = 1 \rangle .$$

or, after simplifications, to

$$SL(2,4) = \langle R_1, R_2 \mid R_1R_2R_1 = R_2R_1R_2, R_1^4 = (R_1R_2)^6 = 1 \rangle .$$

The proofs of points 3 and 4 above repeat the logic of the proof of Proposition 4. For the group $GL(2,3)$ we can make the following identifications

$$R_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \in GL(2,3)$$

and then check that they generate the whole $GL(2,3)$ and satisfy the respective relations. Similarly, for $SL(2,4)$ we can set

$$R_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} \in SL(2,4).$$

Notice that there is an essential difference between the extensions described in 2 and 3, and the extension in 4. The extensions in 2 and 3 have presentations as the presentation of S_4 (equation 6) with the same set of generators and some relation removed. This is not the case with $SL(2,4)$ where, if we want to keep the form of the relations, we need three generators. Notice also that for $2O$ and $GL(2,3)$ the order of R_1 and R_2 becomes 8 (it is 4 in S_4), while in $SL(2,4)$ the order of R_1 and R_2 remains 4. (It may be worth pointing out that the properties of the braid group B_n ensure that in any of its factors the order of any two (standard) generators has to be the same.)

The groups $2O$ and $GL(2,3)$ are so-called *stem extensions* of S_4, i.e., the abelian group by which we extend is not only contained in the center of the extended group, but also in its commutator subgroup. One can check explicitly that for $2O$ and $GL(2,3)$ the corresponding central element $R_1^4 = R_2^4$ can be written as a commutator, while in the case $SL(2,4)$ the center is generated by $(R_1R_2)^3$ and it is not in the commutator subgroup. The Cayley graphs of $2O$ and $GL(2,3)$ look like (topological) double covers of the corresponding Cayley graph of S_4. For this reason stem extensions are called covering groups and are discrete versions of covering groups of Lie groups. It is intriguing that while $SO(3)$ has $SU(2)$ as its unique (double) cover, the finite subgroup S_4 has one additional double cover, namely $GL(2,3)$, which does not come from lifting $SO(3)$ to $SU(2)$.

Finally, we may notice that in the presentations of $2O$ and $GL(2,3)$ there is no condition imposed on the order of the central element. Its order comes out to be 2 automatically, which means that S_4 does not admit bigger stem extensions and the two groups considered are maximal stem extensions, i.e., *Schur extensions.*

4. Generalization to arbitrary n

The n-dimensional case is an easy generalization of the three-dimensional one. We consider products of generating paths R_{ij} in $SO(n)$ as defined in Section 2. If we take a closed path of the form $R_{ij}R_{kl}R_{ij}^{-1}R_{kl}^{-1}$, where all four indices are different, it is clear that this is contractible as these are rotations in two separate planes and at the level of homotopy classes the generators R_{ij} and R_{kl} will commute. When one of the indices coincides, we have a motion that takes place in a 3-dimensional subspace of $\mathbb{R}^n$ and we may use the algebraic relations we had in the previous section. In particular, if we consider the elements R_{ij}, R_{jk} and R_{ki}, we can express one, e.g. the third, in terms of the other two, just as we expressed $R_3 = R_{12}$ as the conjugation of $R_2 = R_{31}$ by $R_1 = R_{23}$. At this point, it seems convenient to choose a different notation, where $R_1 := R_{12}, R_2 := R_{23}, \ldots, R_{n-1} := R_{n-1\,n}$. With this notation we have $R_{31} = R_{13}^{-1} = R_1R_2R_1^{-1}$, $R_{41} = R_{13}R_{34}R_{13}^{-1} = R_1R_2^{-1}R_1^{-1}R_3R_1R_2R_1^{-1}$, etc. In this way, all R_{ij} are products of the $n-1$ generators R_i and their inverses. We have $R_iR_j = R_jR_i$ when $|i-j| \geq 2$. Further, since local closed paths that correspond to rotations in any 3-dimensional subspace must be set to identity when passing to homotopy classes, we have identities analogous to the ones in equation 1 for any two generators R_i and R_{i+1}. In particular, Artin's braid relation is satisfied for any $i = 1, \ldots, n-2$:

$$R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}.$$

The other relation must also hold:

$$R_iR_{i+1}^2R_i = R_{i+1}^2.$$

The fact that there are no additional relations follows from the observation that any contractible closed path in $SO(n)$ which is a product of generating paths can be written as a product of "triangular" local closed paths of the type described in Section 3 (see Appendix).

The properties of the braid group lead to some interesting restrictions on the type of additional relations that can be imposed on the generators. In particular, if $P(R_1, R_2) = 1$ is some relation involving the first two generators and their inverses, then it follows that $P(R_i, R_{i+1}) = 1$ (translation) and

$P(R_{i+1}, R_i) = 1$ (symmetry) will be satisfied automatically. These have a simple geometric explanation. For example, the first relation involves the first three strands of the braid, so if we want to prove the relation for R_2 and R_3, we flip the first strand over the next three, apply the relation and then flip back the former first strand to the first place. The symmetry property follows from the first and the fact that the braid group has an outer automorphism $R_i \to R_{n-i}$ which correspond to "looking at the same braid from behind." It is also clear that the order of all generators in the factor group will be the same. Therefore, for any n the group G generated by the generating paths, up to homotopy, has presentation

$$G = \langle R_1, \ldots, R_{n-1} \,|\, R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}; \\ R_i R_j = R_j R_i, |i-j| \geq 2; R_1^2 = R_2 R_1^2 R_2 \rangle . \quad (7)$$

The group G for arbitrary n has many features in common with the case $n = 3$. In particular, all R_i have order 8 and $R_1^4 = R_2^4 = \cdots = R_{n-1}^4$ is central. The Todd-Coxeter algorithm when run on the computer gives the following results for the order of G when $n = 3, 4, 5, 6$ respectively — 48, 384, 3840, 46080. In fact, we have $|G| = 2^n n!$, which will be shown next. As mentioned already in Section 3, the full hyperoctahedral group in dimension n, which is the Coxeter group B_n, has the same order, but G is a different group — it is a non-trivial double cover of the orientation-preserving subgroup of the full hyperoctahedral group.

A canonical form of the elements of G can be defined inductively as follows: Let $x_{(i)} \in G$ denote a word which contains no R_j with $j > i$. Then we will say that $x_{(i)}$ is in canonical form if it is written as $x_{(i)} = x_{(i-1)} y_{(i)}$ with $x_{(i-1)}$ being in canonical form and $y_{(i)}$ being an expression of one of the types:
1. $R_i^k,\ k \in \{0, 1, 2, 3\}$,
2. $R_i R_{i-1} \cdots R_{i-j},\ j \in \{1, \ldots, i-1\}$,
3. $R_i^3 R_{i-1} \cdots R_{i-j},\ j \in \{0, \ldots, i-1\}$,

As an example, let us list all elements of G in the case $n = 4$, by multiplying all canonical expressions containing R_1 and R_2 (see Proposition 2) with all the expressions as above, with $i = 3$. We have $R_1^m R_2^n R_3^k$ (128 elements), $R_1^m R_2^n R_3 R_2$ (32 elements), $R_1^m R_2^n R_3 R_2 R_1$ (32 elements), $R_1^m R_2^n R_3^3 R_2$ (32 elements), $R_1^m R_2^n R_3^3 R_2 R_1$ (32 elements), $R_1^m R_2 R_1 R_3^k$ (32 elements), $R_1^m R_2 R_1 R_3 R_2$ (8 elements), $R_1^m R_2 R_1 R_3 R_2 R_1$ (8 elements), $R_1^m R_2 R_1 R_3^3 R_2$ (8 elements), $R_1^m R_2 R_1 R_3^3 R_2 R_1$ (8 elements), $R_1^m R_2^3 R_1 R_3^k$ (32 elements), $R_1^m R_2^3 R_1 R_3 R_2$ (8 elements), $R_1^m R_2^3 R_1 R_3 R_2 R_1$ (8 elements), $R_1^m R_2^3 R_1 R_3^3 R_2$ (8 elements), $R_1^m R_2^3 R_1 R_3^3 R_2 R_1$ (8 elements). The following is a straightforward generalization of Proposition 2.

Proposition 5. *Any element of G can be written uniquely in the canonical form defined above. The number of elements is $2^n n!$.*

Proof. The idea is to show that by multiplying an element in canonical form on the right by any R_j, one gets another element that can be brought to a canonical form as well. There is nothing conceptually different from the proof of Proposition 2 and we skip the details. In order to calculate the order of G, we notice that the elements in canonical form of type $x_{(i)}$ are obtained by all possible products of all elements of type $x_{(i-1)}$ with the $4+2(i-1)=2(i+1)$ different expressions of type $y_{(i)}$. Starting with $i=2$ where we have $48=2^3 3!$ and remembering that i runs from 1 to $n-1$ we get

$$|G| = 2^3 \cdot 3! \cdot 2 \cdot 4 \cdot 2 \cdot 5 \cdots 2 \cdot n = 2^n n!. \qquad \square$$

The group G consists of homotopy classes of paths in $SO(n)$ starting at the identity and ending at an element of $SO(n)$ which is a rotational symmetry of the hyperoctahedron in n dimensions (also called n-orthoplex or n-cross polytope). In particular we have a homomorphism $\theta : G \to S_{2n}$ because we have a permutation of the $2n$ vertices of the hyperoctahedron (we take left action of the group on the set). Since the fundamental group of $SO(n)$ consists of the homotopy classes of closed paths, we investigate the kernel of θ.

THEOREM. $\pi_1(SO(n)) \cong ker\theta = \{Id, R_1^4\} \cong \mathbb{Z}_2$.

Proof. We choose to enumerate the $2n$ vertices of the hyperoctahedron as $\{1,-1,2,-2,\dots n,-n\}$ taking ± 1 to denote the two opposite vertices on the first axis, ± 2 on the second axis, etc. Then the element $\theta(R_i)$ permutes cyclically only the elements $\{i, i+1, -i, -(i+1)\}$ leaving the rest in place. Consider an element $x_{(n-1)} = x_{(n-2)} y_{(n-1)} \in G$ in canonical form. By looking at the three possible expressions for $y_{(n-1)}$ we see that the leftmost letter is R_{n-1} to some power, possibly preceded (on the right) by a sequence of R_is with decreasing i. Since in the word $y_{(n-1)}$ only $\theta(R_{n-1})$ moves n and $-n$, we have $\theta(y_{(n-1)})(n) \neq n$, unless $y_{(n-1)} = 1$. Now, because $x_{(n-2)}$ does not contain R_{n-1}, the number n is not the image of any number $k \neq n$ under the action of $\theta(x_{(n-2)})$. Therefore we have $\theta(x_{(n-1)})(n) \neq n$, unless $x_{(n-1)} = x_{(n-2)}$. Proceeding in this way we see that $x \in ker\,\theta$ if and only if $x = R_1^m$. Finally, as $\theta(R_1) = (1\ 2\ -1\ -2)$ (cyclic permutation of $\{1,2,-1,-2\}$; the rest fixed), the only possible cases are when $m=0,4$. Therefore, R_1^4 is the only nontrivial element of $ker\,\theta$ and it has order 2. $\square$

We obtained a series of finite groups G from the braid groups B_n by imposing one additional relation, namely $R_1^2 = R_2 R_1^2 R_2$. These groups are nontrivial double covers of the corresponding rotational hyperoctahedral groups and have order $2^n n!$. It is quite obvious that we obtain a second, nonisomorphic series of double covers if we impose the relations $R_1^2 = R_2 R_1^6 R_2$ and $R_1^4 R_2 = R_2 R_1^4$, instead. Note that for the second series we need the additional condition, which then implies that $R_1^4 = R_2^4 = \cdots = R_{n-1}^4$ is central. When $n=3$ the two groups are the two Schur extensions of the base group. This is perhaps the case also for arbitrary n.

5. Appendix

Proof of Lemmas 1 and 2. It is helpful to introduce a function, measuring the (square of a) "distance" between two points on $SO(n)$. Let $X,\ Y \in SO(n)$ be written as $n \times n$ orthogonal matrices and define

$$D(X, Y) := \mathrm{Tr}(Id - X^T Y)\ .$$

It is a simple exercise to show that this is a positive-definite symmetric function on $SO(n) \times SO(n)$. Notice that $(X^T Y)_{ii}$ is the cosine of the angle between the image of the standard coordinate basis vector $\mathbf{e}_i$ under the action of X and its image under the action of Y. The function D does not satisfy the triangle inequality and is not a true distance, but this causes no difficulties in our considerations.

Further we write the proof for $SO(3)$ for brevity. It is obvious that the same method works in general. Let $X \in SO(3)$ be written as a 3×3 orthogonal matrix:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix}.$$

The "distance" of X to the identity (writing just one argument) is :

$$D(X) = 3 - x_{11} - x_{22} - x_{33}.$$

Thus $D(X) \geq 0$ and $D(X) = 0$ implies $X = Id$. The gradient of $D(X)$ is a 9-dimensional vector field, which we choose to write as a 3×3 matrix. We have

$$\mathrm{grad}\, D(X) = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix}.$$

$SO(3)$ is a 3-dimensional submanifold of $\mathbb{R}^9$ consisting of the points satisfying six algebraic equations, ensuring that the 3×3 matrix X is orthogonal. In vector form these are equivalent to the statement that the (column) vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ form an orthonormal basis:

$$\mathbf{x}_1 \cdot \mathbf{x}_1 = 1,\ \mathbf{x}_2 \cdot \mathbf{x}_2 = 1,\ \mathbf{x}_3 \cdot \mathbf{x}_3 = 1,\ \mathbf{x}_1 \cdot \mathbf{x}_2 = 0,\ \mathbf{x}_1 \cdot \mathbf{x}_3 = 0,\ \mathbf{x}_2 \cdot \mathbf{x}_3 = 0.$$

The respective gradients of these six functions are

$$\begin{aligned} &\begin{pmatrix} \mathbf{x}_1 & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & \mathbf{x}_2 & \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x}_3 \end{pmatrix}, \\ &\begin{pmatrix} \mathbf{x}_2 & \mathbf{x}_1 & \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{x}_3 & \mathbf{0} & \mathbf{x}_1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & \mathbf{x}_3 & \mathbf{x}_2 \end{pmatrix}. \end{aligned}$$

It is well known and an easy exercise that for each $X \in SO(3)$ these six vectors are linearly independent and their orthogonal complement is precisely the tangent space of $SO(3)$. (This is in fact how it is shown that these six equations in $\mathbb{R}^9$ define indeed a three-dimensional submanifold.) We want to show that under our assumptions $\operatorname{grad} D(X)$ has a nonzero tangential component. Indeed, assuming the opposite implies that $\operatorname{grad} D(X)$ is in the span of the six vectors above, which leads leads to

$$\begin{aligned}\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} = a\begin{pmatrix}\mathbf{x}_1 & \mathbf{0} & \mathbf{0}\end{pmatrix} + b\begin{pmatrix}\mathbf{0} & \mathbf{x}_2 & \mathbf{0}\end{pmatrix} + c\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{x}_3\end{pmatrix}\\ + d\begin{pmatrix}\mathbf{x}_2 & \mathbf{x}_1 & \mathbf{0}\end{pmatrix} + e\begin{pmatrix}\mathbf{x}_3 & \mathbf{0} & \mathbf{x}_1\end{pmatrix} + f\begin{pmatrix}\mathbf{0} & \mathbf{x}_3 & \mathbf{x}_2\end{pmatrix},\end{aligned}$$

which in turn is equivalent to the three vector equations

$$\mathbf{e}_1 = a\mathbf{x}_1 + d\mathbf{x}_2 + e\mathbf{x}_3, \quad \mathbf{e}_2 = b\mathbf{x}_2 + d\mathbf{x}_1 + f\mathbf{x}_3, \quad \mathbf{e}_3 = c\mathbf{x}_3 + e\mathbf{x}_1 + f\mathbf{x}_2.$$

Taking the scalar product of the first equation with $\mathbf{x}_2$ yields $d = x_{12}$, while taking the scalar product of the second equation with $\mathbf{x}_1$ yields $d = x_{21}$. In a similar way we see that $x_{ij} = x_{ji}$ for any i, j. Therefore the matrix X must be symmetric and being also orthogonal its square is the identity. The eigenvalues can only be 1 and -1 but the latter is excluded by the assumption that under the transformation corresponding to X the coordinate axes do not leave the closed half-space they belong to initially.

Thus the tangential component of $-\operatorname{grad} D(X)$ defines a vector field on $SO(3)$ which will be nonzero for any $X = R(t)$, $X \neq Id$, where $R : [0, 1] \to SO(3)$ is a local closed path. This means that the flow along this vector field defines a homotopy from $R(t)$ to the identity of $SO(3)$. This proves the first part of Lemma 2.

Let us denote by $G' \subset SO(n)$ the respective rotational hyperoctahedral group in dimension n. Taking an arbitrary path $R : [0, 1] \to SO(n)$ with $R(0) = Id$ and $R(1) = r \in G'$, we want to construct another path, homotopic to the first one, which is a product of generating paths R_i. We can proceed as follows: If $t_1 \in [0, 1]$ is the smallest t for which the "distance" from $R(t)$ to some $r_1 \in G'$ becomes equal to the "distance" to Id, we take a product of generating paths $R^{(1)}_{k_1} \cdots R^{(1)}_1$ with $(R^{(1)}_{k_1} \cdots R^{(1)}_1)(0) = Id$ and $(R^{(1)}_{k_1} \cdots R^{(1)}_1)(t_1) = r_1$. Note that the transformation $(R^{(1)}_{k_1} \cdots R^{(1)}_1)(t)$ leaves every vertex of the hyperoctahedron in the closed half-space determined by it (as defined in Section 2) or, equivalently, the angle between the standard basis vector $\mathbf{e}_i$ and its image under $(R^{(1)}_{k_1} \cdots R^{(1)}_1)(t)$ does not exceed $\pi/2$ for any i and any t. Indeed, as we consider a continuous path in $SO(n)$ starting at the identity, we may think of the motion of the n points on the $(n-1)$–dimensional sphere (the images of the vectors $\mathbf{e}_i$ under $R(t)$. Initially all points remain in some respective adjacent $(n-1)$-cells (these are $(n-1)$–simplices (spherical)) forming the spherical hyperoctahedron. If the point i is to leave the closed half-space to which it initially belonged, it must reach the $(n-2)$-dimensional boundary opposite to $\mathbf{e}_i$ for some t. (At the same time there will be at least one more point i' belonging to a boundary of an $(n-1)$-cell since two points cannot belong to the interior of the same $(n-1)$–cell). But then it follows that there will be a vertex j the angular distance to which, from i is less than or equal to $\pi/2$. This implies that the element $r_{ij} \in G'$ giving rotation by $\pi/2$ in the ijth plane is not further to $R(t)$ than the "distance" between $R(t)$ and the identity. The above argument shows that even though the element r_1 above does not determine the product $R^{(1)}_{k_1} \cdots R^{(1)}_1$ uniquely, it is unique up to homotopy.

Next we take t_2 as the smallest $t \geq t_1$ for which the "distance" from $R(t)$ to some r_2 becomes equal to the "distance" to r_1. There is an element r'_2 in G', such that $r_2 = r'_2 r_1$ and r'_2 satisfies the same property as r_1 above. We take a product of generating paths $R^{(2)}_{k_2} \cdots R^{(2)}_1$ with $(R^{(2)}_{k_2} \cdots R^{(2)}_1)(t_1) = Id$ and $(R^{(2)}_{k_2} \cdots R^{(2)}_1)(t_2) = r'_2$. Proceeding in this way and taking the product of products of generating paths we produce a path (after renumbering) $R' := R_k \cdots R_1$ with $(R_k \cdots R_1)(0) = Id$ and $(R_k \cdots R_1)(1) = r$. The elements r_i which we have to use at each step may not be unique and we will have to make a choice but the end result will lead to homotopic paths. To show that the path we have constructed is homotopic to the original path R we can use the following trick — for each fixed t take $R''(t) := R'(t)R^{-1}(t)$. The path R'' is closed and is local by the construction of R'. Therefore it is homotopic to the identity by the first part of Lemma 2. With this Lemma 1 is proven.

Finally, we need to show that the expression corresponding to any contractible closed path consisting of generating paths can be reduced to the identity by inserting in it words giving local closed paths. More precisely the words that must be inserted give triangular closed paths as described in Section 3. This is essential as we must be sure that there are no additional relations other than the ones as in Equations 2 and 3. First we carry out the proof for local closed paths consisting of generating paths (these are contractible because of locality) by induction on the length of the word representing the path. A single-letter path R_{ij} cannot be closed. A two-letter path can only be closed if the word is $R_{ji}R_{ij}$. A three-letter path cannot be closed as can easily be seen considering all possibilities. For example if we want to try to close the path starting with $R_{23}R_{12}$ we have to add R_{31} on the left so that vertex 1 goes back to 1 but $R_{31}R_{23}R_{12}$ is not closed since 2 goes to -3 and 3 goes to -2. (We adopt enumeration of the $2n$ vertices with $\{1, 2, \ldots, n, -1, -2, \ldots, -n\}$ where $-i$ denotes the vertex opposite to i. The element R_{ij} moves vertex i to vertex j, vertex j to vertex $-i$ and leaves all other vertices in place.) Considering four-letter paths it is easy to check that they can be closed either if they consist of a product of two closed two-letter paths or if they involve only rotations in a three-dimensional subspace spanned by some three axes i, j and k. These are precisely what we called triangular closed paths in Section 3. In general notations it turns out that all such closed paths are words that are cyclic permutations of the following four expressions:

$$R_{kj}R_{ki}R_{jk}R_{ij}, \quad R_{jk}R_{ik}R_{kj}R_{ij}, \quad R_{ki}R_{jk}R_{ik}R_{ij}, \quad R_{ik}R_{kj}R_{ki}R_{ij}. \tag{8}$$

Notice that setting these expressions to one gives conjugation identities, e.g. $R_{kj}R_{ik}R_{jk} = R_{ij}$, etc. Suppose now that the statement we want to prove is valid for all words with length $2n$ and consider a word of $2n+2$ letters. Suppose that the first letter on the right is R_{ij}. We can move R_{ij} to the left across any R_{kl} with k, l different from i and j, as rotations in two such planes commute. If R_{ij} gets next to a letter R_{ji} we are allowed to cancel the two and obtain a word of length $2n$ and we are done. The possibility to obtain $R_{ij}R_{ij}$ is excluded by locality since anything to the right leaves vertex i invariant and R_{ij}^2 sends i to $-i$. There are four additional possibilities in which R_{ij} gets next to a letter with which it does not commute. These are $R_{jk}R_{ij}$, $R_{kj}R_{ij}$, $R_{ik}R_{ij}$ and $R_{ki}R_{ij}$. Using the identities following from setting the expressions in Equation (8) to one we replace the above combinations by products of two letters in which the index i appears only in the letter on the left. Here are the actual identities that can be used:

$$R_{jk}R_{ij} = R_{ik}R_{jk},\ R_{kj}R_{ij} = R_{ki}R_{kj},\ R_{ik}R_{ij} = R_{ij}R_{kj},\ R_{ki}R_{ij} = R_{ij}R_{jk}.$$

Next we continue moving the letter involving i to the left. Notice that the index i may now appear as the second index, so we may need to use the additional identities

$$R_{jk}R_{ji} = R_{ki}R_{jk},\ R_{kj}R_{ji} = R_{ik}R_{kj},\ R_{ik}R_{ji} = R_{ji}R_{jk},\ R_{ki}R_{ji} = R_{ji}R_{kj}.$$

Eventually the letter involving the index i will reach the left end of the word and there will be no letters involving i to its right. Such a path cannot be closed as it does not leave vertex i in place. Therefore the only possibility is that the letter involving i gets canceled in the process and the length of the word becomes $2n$.

Now consider a general closed path R consisting of generating paths and homotopic to the identity. In general the homotopy from the identity to R need not pass through paths consisting of generating paths but we can modify it so that it does. Indeed if $R(s)$ is the homotopy from the identity to R, i.e. $R(0) \equiv Id$, $R(1) = R$, for each s we can homotope $R(s)$ to the nearest closed path consisting of generating paths, as described earlier in this section. For s small enough the nearest will be Id. For some s_1 the construction will yield a path R' consisting of generating paths, such that R' is homotopic to $R(s_1)$. Furthermore the construction is such that R' is a local closed path as any vertex remains in some $(n-1)$-cell without leaving it. Then for some $s_2 > s_1$ we will get R'' which is nearest to $R(s_2)$. The path R'' is not local but $R'^{-1}R''$

(usual product of paths by concatenation) is local. Since the words giving R' and $R'^{-1}R''$ can be reduced to the identity by inserting words giving triangular closed paths, the same will be true for R''. Proceeding in this way, after a finite number of steps we show that the word for the closed path R has the same property.

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