Rend. Istit. Mat. Univ. Trieste
Vol. 56 (2024), Art. No. 3, 10 pages
DOI: 10.13137/2464-8728/36463

Inequalities for the arithmetic mean of the first n prime numbers

Horst Alzer and Mehdi Hassani

Abstract. *Let A_n be the (unweighted) arithmetic mean of the first n prime numbers. We prove that for $n \geq 2$,*

$$A_n^{1+\frac{\alpha}{n\log(n)}} \leq A_{n+1} \leq A_n^{1+\frac{\beta}{n\log(n)}}$$

with the best possible constants $\alpha \approx 0.43525$ and $\beta \approx 1.22596$. The right-hand side improves a result given by Z.-W. Sun in 2013.

Keywords: Prime number, arithmetic mean, inequality, monotonicity.
MS Classification 2020: 11N05, 11B83.

1. Introduction and statement of the main results

I. In this paper, we study the unweighted arithmetic mean of the first n prime numbers, that is,

$$A_n = \frac{1}{n}\sum_{k=1}^{n} p_k,$$

where p_k denotes the k-th prime number. Several mathematicians presented interesting inequalities involving A_n. Mandl's conjecture states that

$$A_n < \frac{1}{2}p_n \quad (n \geq 9). \tag{1}$$

The following companion to (1) is due to Robin,

$$p_{[n/2]} \leq A_n \quad (n \geq 2). \tag{2}$$

Proofs for (1) and (2) were given by Dusart [3, Section 1.9]. He used the elegant integral formula

$$A_n = p_n - \frac{1}{n}\int_2^{p_n} \pi(x)dx$$

to settle (1). Here, $\pi(x)$ denotes the number of primes less than or equal to x.

In search of the order of the magnitude of A_n, Hassani [4] offered new upper and lower bounds for A_n,

$$\frac{1}{2}p_n - \frac{9}{4}n < A_n < \frac{1}{2}p_n - \frac{1}{12}n. \tag{3}$$

The left-hand side of (3) holds for $n \geq 2$. The right-hand side of (3) provides a refinement of (1). It is valid for $n \geq 10$.

A simple inequality involving A_n, A_{n+1} and p_{n+1} was published by Popoviciu:

$$p_{n+1} < \frac{A_{n+1}^{n+1}}{A_n^n} \quad (n \geq 1).$$

The following related result is due to Klamkin. If A_n^* denotes the arithmetic mean of $1/p_1, \ldots, 1/p_n$, then

$$1 < (n+1)\sqrt{A_{n+1}A_{n+1}^*} - n\sqrt{A_n A_n^*} \quad (n \geq 1);$$

see Bullen et al. [2, Section II.3.1].

Inequalities for the ratio of the arithmetic and geometric means of the first n primes were given by Hassani [4]. Axler [1] provided an asymptotic formula for A_n and Matomäki [5] studied the set of natural numbers n such that $A(n)$ is an integer.

II. Our work has been inspired by a remarkable paper published by Sun [8] in 2013. Motivated by the open Firoozbakht conjecture, which states that the sequence $(p_n^{1/n})_{n\geq 1}$ is strictly decreasing (see Ribenboim [6, p. 185]), he proved (among others) that $\left(A_n^{1/n}\right)_{n\geq 1}$ is strictly decreasing. This leads to

$$A_{n+1} < A_n^{1+1/n} \quad (n \geq 1). \tag{4}$$

Is it possible to improve (4)? More precisely, we ask: does there exist a real number $c < 1$ such that

$$A_{n+1} \leq A_n^{1+c/n} \tag{5}$$

is valid for $n \geq 1$? Numerous calculations led us to the conjecture that (5) holds with $c \approx 0.76257$. Here, we present the following refinement and converse of this result.

THEOREM 1.1. *For all integers $n \geq 2$, we have*

$$A_n^{1+\frac{\alpha}{n\log(n)}} \leq A_{n+1} \leq A_n^{1+\frac{\beta}{n\log(n)}} \tag{6}$$

with the best possible constants

$$\alpha = 2\log(2)\frac{\log(4/3)}{\log(5/2)} = 0.43524... \tag{7}$$

and

$$\beta = 18\log(3)\frac{\log(1161/1000)}{\log(100/9)} = 1.22596.... \tag{8}$$

Remark 1.2. The sign of equality holds on the left-hand side of (6) if and only if $n=2$, and on the right-hand side if and only if $n=9$.

An application of the second inequality in (6) leads to the following monotonicity results.

Corollary 1.3. *The sequences*

$$\big(A_n^{\frac{\sqrt{\log(n)}}{n}}\big)_{n\geq 5}, \quad \big(A_n^{\frac{1+\log(n)}{n}}\big)_{n\geq 5}, \quad \big(A_n^{\frac{1}{\sqrt{n}}}\big)_{n\geq 12}$$

are strictly decreasing. Moreover, for each integer $k\geq 1$ there exists a positive integer n_k such that the sequence

$$\big(A_n^{\frac{\log^k(n)}{n}}\big)_{n\geq n_k}$$

is strictly decreasing. In particular, we have

k	1	2	3	4	5
n_k	10	22	57	151	395

III. In the next section, we introduce some helpful notation and in Section 3, we collect seven lemmas. The proofs of Theorem 1.1 and Corollary 1.3 are presented in Section 4. We have used Maple 17 to verify the validity of three inequalities for a finite number of integers. The three computer programs are given in the supporting file "CAS-Supplement".

2. Notation

Throughout, α and β are the constants given in (7) and (8). Moreover, x and n denote a real number and a natural number, respectively. In order to prove Theorem 1.1 we need the following functions and constants.

$$v(x) = \frac{x^2}{2}\big[\log(x)+\log(\log(x))-1.4\big], \quad w(x) = \frac{x}{x+1}\Big[\frac{0.99x\log(x)}{v(x)}+1\Big],$$

$$g(x) = \frac{x^2}{2}\big[\log(x)+\log(\log(x))-1.5\big],$$

$$h(x) = g(x) + \frac{x^2}{2\log(x)}\big[\log(\log(x))-2.5\big],$$

$$T(x) = h(x)\Big[\frac{x+1}{x}\Big(\frac{h(x)}{x}\Big)^{\frac{\alpha}{x\log(x)}} - 1\Big],$$

$$B(x,c) = x\big[\log(x) + \log(\log(x)) - c\big],$$

$$u(x) = \log(x) + \log(\log(x)) - 1.5, \quad R(x) = g(x)\Big[\frac{x+1}{x}\Big(\frac{g(x)}{x}\Big)^{\frac{\beta}{x\log(x)}} - 1\Big],$$

$$n_0 = 305494, \quad c_0 = 1 + \frac{1}{n_0}, \quad c_1 = \frac{1379}{2500} + 2\log(2),$$

$$c_2 = 2c_0\Big(1 + \frac{c_1}{u(26n_0)}\Big) = 2.25..., \quad c_3 = \frac{c_2 - 1}{\beta} = 0.99999616...,$$

$$c_4 = 0.9999962, \quad c_5 = \frac{3c_4 - 1}{2}.$$

3. Lemmas

We use the notation introduced in Section 2. The following lemmas play an important role in the proof of our main result.

LEMMA 3.1. (i) *For $x \geq e$, we have*

$$\frac{\alpha}{\log(x)} \log\Big(\frac{v(x)}{x}\Big) < \frac{3}{4}. \tag{9}$$

(ii) *For $x \geq 61279$, we have*

$$\frac{3}{4} < x\log(w(x)). \tag{10}$$

Proof. (i) The function

$$\delta(x) = 2e^{0.7x} - x - \log(x) + 1.4$$

is convex on $[1, \infty)$ with $\delta'(1) = 0.8...$ and $\delta(1) = 4.4....$ It follows that δ is positive on $[1, \infty)$. Since $3/(4\alpha) = 1.72...$, we obtain for $x \geq e$,

$$x^{3/(4\alpha)} > x^{1.7} > x^{1.7} - \frac{x}{2}\delta(\log(x)) = \frac{v(x)}{x}.$$

This leads to (9).
(ii) Let

$$\mu(x) = x - (x+1)e^{3/(4x)} \quad \text{and} \quad \phi(x) = \frac{x}{x + \log(x) - 1.4}.$$

Then

$$(x+1)\Big[w(x) - e^{3/(4x)}\Big] = \mu(x) + \frac{99}{50}\phi(\log(x)) = \Lambda(x), \quad \text{say.} \tag{11}$$

We have

$$e^{-3/(4x)}\mu'(x) = \kappa(1/x),$$

where

$$\kappa(x) = -1 + \frac{3}{4}x + \frac{3}{4}x^2 + e^{-3x/4}.$$

The function κ is strictly convex on $[0,\infty)$ with $\kappa'(0) = \kappa(0) = 0$. It follows that κ is positive on $(0,\infty)$. Thus, $\mu'(x) > 0$ for $x > 0$. Moreover, we have for $x \geq \exp(12/5)$,

$$\frac{1}{25}\phi'(x) = \frac{\log(x) - 12/5}{(5x + 5\log(x) - 7)^2} \geq 0.$$

From (11) we obtain that Λ is increasing for $x \geq \exp(\exp(12/5)) = 61278.01\ldots$. This implies that for $x \geq 61279$,

$$\Lambda(x) \geq \Lambda(61279) = 0.065\ldots.$$

Applying (11) we conclude that (10) holds. □

LEMMA 3.2. *For $x \geq 61279$, we have*

$$T(x) < 0.99x\log(x). \tag{12}$$

Proof. Since

$$\frac{\log(\log(x)) - 2.5}{\log(x)} < \frac{1}{10} \quad (x > 1),$$

we obtain

$$h(x) < g(x) + \frac{x^2}{20} = v(x).$$

This implies

$$T(x) \leq v(x)\Big[\frac{x+1}{x}\Big(\frac{v(x)}{x}\Big)^{\frac{\alpha}{x\log(x)}} - 1\Big]. \tag{13}$$

Using (9) and (10) gives for $x \geq 61279$,

$$\frac{\alpha}{x\log(x)}\log\Big(\frac{v(x)}{x}\Big) < \log(w(x)),$$

which is equivalent to

$$v(x)\Big[\frac{x+1}{x}\Big(\frac{v(x)}{x}\Big)^{\frac{\alpha}{x\log(x)}} - 1\Big] < 0.99x\log(x). \tag{14}$$

Combining (13) and (14) yields (12). □

LEMMA 3.3. *For $x \geq 640$, we have*

$$0.99x\log(x) < B(x+1, 1.5). \tag{15}$$

Proof. Let

$$\theta(x) = \frac{B(x+1,1.5) - 0.99x\log(x)}{x+1} \quad \text{and} \quad \sigma(x) = x + 1 - 99\log(x).$$

Since σ is positive on $[640, \infty)$, we obtain

$$100(x+1)^2\log(x+1)\theta'(x) = \log(x+1)\sigma(x) + 100(x+1) > 0.$$

Thus

$$\theta(x) \geq \theta(640) = 0.44... \quad (x \geq 640).$$

This implies (15). □

LEMMA 3.4. *For $x \geq 13$, we have*

$$x + c_2 < (x+1)\Big(\frac{xu(x)}{2}\Big)^{\frac{\beta}{x\log(x)}}. \tag{16}$$

Proof. We define for $x \geq 13$,

$$q(x) = x^{1-c_4}u(x).$$

Since

$$x^{c_4}q'(x) = (1-c_4)\big(\log(x) + \log(\log(x))\big) + c_5 + \frac{1}{\log(x)} > 0$$

and $q(13) = 2.006...$, we conclude that $q(x) > 2$. This leads to

$$x^{c_4} < \frac{xu(x)}{2}. \tag{17}$$

Using

$$1 + t < e^t \quad (t \neq 0)$$

with $t = (c_2 - 1)/(x+1)$ gives

$$\Big(\frac{x+c_2}{x+1}\Big)^{\frac{(x+1)\log(x)}{\beta}} < x^{c_3}. \tag{18}$$

Since $c_3 < c_4$, we conclude from (17) and (18) that (16) holds. □

LEMMA 3.5. *For $n \geq 30$, we have*

$$B(n+1, 0.9484) < R(n). \tag{19}$$

Proof. We consider two cases.
Case 1. $n \geq 26n_0$. Since u is increasing, we obtain $u(n) \geq u(26n_0)$. It follows that

$$nu(n)\Big(\frac{n}{2} + c_0\Big) + c_0c_1n \leq nu(n)\Big(\frac{n}{2} + c_0\Big) + \frac{c_0c_1}{u(26n_0)}nu(n)$$
$$= \frac{nu(n)}{2}(n + c_2). \tag{20}$$

Combining (16) and (20) yields

$$\frac{nu(n)}{2}(n + 2c_0) + c_0c_1n < (n+1)\Big(\frac{nu(n)}{2}\Big)^{1+\frac{\beta}{n\log(n)}}. \tag{21}$$

Since $g(n) = n^2u(n)/2$, we obtain that (21) is equivalent to

$$c_0n\big(u(n) + c_1\big) < R(n). \tag{22}$$

We have

$$\begin{aligned} B(n+1, 0.9484) &< (n+1)\big(\log(2n) + \log(\log(n^2)) - 0.9484\big) \\ &= (n+1)\big(u(n) + c_1\big) \\ &\leq \Big(1 + \frac{1}{n_0}\Big)n\big(u(n) + c_1\big) \\ &= c_0n\big(u(n) + c_1\big). \end{aligned} \tag{23}$$

From (22) and (23) we conclude that (19) holds.
Case 2. $30 \leq n \leq 26n_0 - 1$. By using Maple 17 we obtain that (19) is valid for these finite numbers. □

The next lemma provides upper and lower bounds for the sum of the first n primes. These results are due to Axler [1] and Dusart [3, p. 51]. Let

$$S(n) = \sum_{k=1}^{n} p_k.$$

Lemma 3.6. (i) *For $n \geq 115149$, we have $S(n) \leq h(n)$.*
(ii) *For $n \geq 305494$, we have $g(n) \leq S(n)$.*

We conclude this section with two inequalities for p_n proved by Rosser and Schoenfeld [7] and Dusart [3, p. 32].

Lemma 3.7. (i) *For $n \geq 2$, we have*

$$B(n, 1.5) < p_n.$$

(ii) *For $n \geq 39017$, we have*

$$p_n \leq B(n, 0.9484).$$

4. Proofs of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. A short calculation gives that (6) is equivalent to

$$Z(n,\alpha) \le p_{n+1} \le Z(n,\beta) \tag{24}$$

with

$$Z(n,x) = S(n)\Big[\frac{n+1}{n}\Big(\frac{S(n)}{n}\Big)^{\frac{x}{n\log(n)}} - 1\Big].$$

First, we prove the left-hand side of (24).
Case 1. $n \ge 115149$. Applying Lemma 3.6 (i) gives

$$Z(n,\alpha) < T(n), \tag{25}$$

and from Lemma 3.2, Lemma 3.3 and Lemma 3.7 (i) we get

$$T(n) < 0.99n\log(n) < B(n+1,1.5) < p_{n+1}. \tag{26}$$

Combining (25) and (26) leads to the left-hand side of (24) with "$<$" instead of "$\le$".
Case 2. $2 \le n \le 115148$. If $n = 2$, then equality holds. We apply Maple 17 and obtain that if $3 \le n \le 115148$, then the left-hand side of (24) holds with "$<$" instead of "$\le$".

Now, we prove the right-hand side of (24).
Case 1. $n \ge 305494$. From Lemma 3.6 (ii), Lemma 3.5 and Lemma 3.7 (ii) we obtain

$$Z(n,\beta) \ge R(n) > B(n+1, 0.9484) \ge p_{n+1}.$$

This settles the right-hand side with "$<$" instead of "$\le$".
Case 2. $2 \le n \le 305493$. By direct computation, we find that for $n \in \{2,3,...,9\}$ we have $p_{n+1} \le Z(n,\beta)$ with equality if and only if $n = 9$. Next, we use Maple 17. We find that if $10 \le n \le 305493$, then the right-hand side of (24) holds with "$<$" instead of "$\le$". □

Proof of Corollary 1.3. The proofs of the monotonicity of the sequences presented in Corollary 1.3 are similar, so that we provide the details only for $\big(A_n^{\frac{\log^k(n)}{n}}\big)$, $k \in \{1,2,3,4,5\}$. Let

$$f_k(x) = \frac{\log^k(x)}{x}, \quad F_k(x) = \frac{f_k(x)}{f_k(x+1)} - \frac{\beta}{x\log(x)}$$

and

$$Q_k(x) = \log(x) - k - \frac{k}{\log(x+1)}\log\Big(1+\frac{1}{x}\Big)^x - \beta\frac{1+\log(x)}{\log(x+1)}\Big(\frac{\log(x+1)}{\log(x)}\Big)^{k+1}.$$

By differentiation, we obtain

$$-x^2 \log(x)\Big(\frac{\log(x+1)}{\log(x)}\Big)^k F_k'(x) = Q_k(x).$$

Since $Q_k(x)$ tends to ∞ as $x \to \infty$, there exists a number n_k such that $F_k'(x) < 0$ for $x \geq n_k$. Using

$$\lim_{x\to\infty} F_k(x) = 1$$

we conclude that $F_k(x) > 1$ for $x \geq n_k$. This leads to

$$1 + \frac{\beta}{x\log(x)} < \frac{f_k(x)}{f_k(x+1)}. \tag{27}$$

From (6) and (27) we get for $n \geq n_k$,

$$A_{n+1} \leq A_n^{1+\frac{\beta}{n\log(n)}} < A_n^{\frac{f_k(n)}{f_k(n+1)}},$$

which implies that $\big(A_n^{\frac{\log^k(n)}{n}}\big)_{n\geq n_k}$ is strictly decreasing. For $k \in \{1,2,3,4,5\}$ and $x \geq 1250$, we obtain

$$Q_k(x) \geq \log(1250) - 5 - \frac{5}{\log(1251)} - \beta\frac{1+\log(1250)}{\log(1251)}\Big(\frac{\log(1251)}{\log(1250)}\Big)^6 = 0.03....$$

It follows that $\big(A_n^{\frac{\log^k(n)}{n}}\big)_{n\geq 1250}$ is strictly decreasing. More precisely, by direct computation, we find that $\big(A_n^{\frac{\log^k(n)}{n}}\big)_{n\geq n_k}$ is strictly decreasing, where n_k is exactly as given in the table. □

Acknowledgement

We thank the referee for helpful comments.

References

[1] C. Axler, *On the sum of the first n prime numbers*, J. Théor. Nombres Bordeaux **31** (2019), 293–311.
[2] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and their inequalities*, D. Reidel Publishing Co., Dordrecht, 1988.
[3] P. Dusart, *Autour de la fonction qui compte le nombre de nombres premiers*, Ph.D. thesis, Université de Limoges, 1998.
[4] M. Hassani, *On the ratio of the arithmetic and geometric means of the prime numbers and the number* e, Int. J. Number Theory **9** (2013), 1593–1603.

[5] K. Matomäki, *A note on the sum of the first n primes*, Q. J. Math. **61** (2010), 109–115.
[6] P. Ribenboim, *The little book of bigger primes*, Springer, New York, 2004.
[7] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.
[8] Z.-W. Sun, *On a sequence involving sums of primes*, Bull. Aust. Math. Soc. **88** (2013), 197–205.

Authors' addresses:

Horst Alzer
Morsbacher Straße 10
51545 Waldbröl, Germany
E-mail: h.alzer@gmx.de

Mehdi Hassani
Department of Mathematics, Faculty of Science
University of Zanjan
University Blvd., 45371-38791, Zanjan, Iran
E-mail: mehdi.hassani@znu.ac.ir

Received March 28, 2024
Revised June 6, 2024
Accepted June 19, 2024