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New inequalities for the confluent hypergeometric function

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Abstract. *In this paper, we obtain some new inequalities for the confluent hypergeometric function. These results, at least for large argument, improve the well-known inequalities of Y. L. Luke and B. C. Carlson.*

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1. Introduction and main results

The confluent hypergeometric function, which is defined as

$$ {}_1F_1(a;b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!} $$

for $b \neq 0, -1, ...$, has been studied in great detail from its mathematical point of view (see, for instance, [4, 12, 20]). In particular, the estimate of the confluent hypergeometric function ${}_1F_1(a;b;x)$, as well as for its particular cases, has been considered in several papers from different point of views (see [1, 2, 5, 8, 9, 10, 11, 13, 16, 18, 19, 21], and references therein). For instance, Luke [11] and Carlson [5] proved the following inequalities, respectively,

$$ {}_1F_1(a;b;x) < \frac{(b-1)e^x}{(b-a-1)(1+x)}, \qquad x > 0,\ b-1 > a > 0, \tag{1} $$

$$ {}_1F_1(a;b;x) < 1 - \frac{a}{b} + \frac{a}{b}e^x, \qquad x > 0,\ b > a > 0, \tag{2} $$

whereas in [8] it has been showed that

$$ |{}_1F_1(a;b;x)| \leq e^x, \tag{3} $$

where $a < 0$, $b > 1$, $x \geq 0$, and equality holds only when $x = 0$.

On the other hand, the estimate of the cumulative gamma distribution $F(x;\alpha,\lambda)$, which in terms of the confluent hypergeometric functions can be written as

$$F(x;\alpha,\lambda) = \frac{(\lambda x)^{\alpha} e^{-\lambda x}}{\alpha\Gamma(\alpha)} {}_1F_1(1;\alpha+1;\lambda x), \qquad x,\alpha,\lambda > 0,$$

has been studied, among others, in [9, 13, 16, 18, 19]. In particular, in [9] the sharper inequality than (2) has been obtained when $a = 1$, i.e.

$$ {}_1F_1(1;b;x) \le \min \begin{cases} 1 - \frac{1}{b} + \frac{1}{b}e^x - \frac{(b-1)x^2}{2b(b+1)}, \\ \frac{1}{\left(1-\frac{x}{b}\right)^+}, \end{cases} \qquad x > 0,\ b > 1. \tag{4}$$

As has been remarked in [8], from the asymptotics

$$ {}_1F_1(a;b;x) = \frac{\Gamma(b)}{\Gamma(a)} x^{a-b} e^x [1 + O(x^{-1})], \qquad x \to +\infty, \tag{5}$$

where a is not a negative integer or zero, it follows that e^x in (3) cannot be replaced by any e^{cx}, $0 \le c < 1$, when $x \to \infty$. In the case when a is a negative integer or zero, we have

$$\max_{x\ge 0} e^{-\frac{x}{2}} \left| {}_1F_1(a;b;x) \right| = 1, \qquad b > 1. \tag{6}$$

On the other hand, it is well known that, for $a < 0$ and $b > 0$, there are precisely $-\lfloor a \rfloor$ positive zeros of ${}_1F_1(a;b;x)$ (see [14, §13.9]). All these real zeros of ${}_1F_1(a;b;x)$, except at most one, lie in the oscillatory interval $\mathcal{I} = (0, x_+)$, where (see Appendix A)

$$x_+ = \begin{cases} 2(b-2a) & \text{if } 0 < b \le 2, \\ b - 2a + \sqrt{(b-2a)^2 + b(2-b)} & \text{if } b \ge 2. \end{cases}$$

When $x \in \left(0, \frac{(2b-1)(b-2a)}{b}\right)$, the following sharp inequality has been established in [8]:

$$\left| {}_1F_1(a;b;x) \right| \le e^{\frac{x}{2}},$$

where $a < 0$ and $b > 2$. Obviously, for large b we have $x_+ < \frac{(2b-1)(b-2a)}{b}$.

We note that, according to (5), inequalities (1), (2), (3) and (4) are not sharp when x becomes large. In this paper we obtain some new inequalities for the confluent hypergeometric function which, at least for large x, improve inequalities (1), (2), (3) and (4).

Here is our results.

THEOREM 1.1. (i) *For* $x > 0$, $a < 0$ *and* $\Re e(b) > 1$

$$x^{\Re e(b)-1} e^{-x-\frac{\pi}{2}|\Im m(b)|} \left|{}_1F_1(a;b;x)\right| \leq \frac{|\Gamma(b)|}{2\pi} B\left(\frac{1}{2}, \frac{\Re e(b)-1}{2}\right), \tag{7}$$

where $B(\cdot,\cdot)$ *is beta function.*

(ii) *For* $x > 0$, $a > 0$ *and* $\Re e(b) - a > 1$

$$x^{\Re e(b)-a-1} e^{-x-1-\frac{\pi}{2}|\Im m(b)|} \left|{}_1F_1(a;b;x)\right| \leq \frac{|\Gamma(b)|}{2\pi} B\left(\frac{1}{2}, \frac{\Re e(b)-a-1}{2}\right). \tag{8}$$

(iii) *For* $x > 0$, $2a < b$ *and* $b > 0$

$$x^{\frac{b}{2}} e^{-x} \left|{}_1F_1(a;b;x)\right| \leq \frac{\Gamma(b)}{2\Gamma(b-a)} \sqrt{2^{2a-b}(b-2a+1)\Gamma(b-2a)}. \tag{9}$$

Next, for some special cases of parameters and argument, we compare our results with (1), (2), (3) and (4).

(i) For $a < 0$, $b > 1$ and

$$x > \left[\frac{\Gamma(b)}{2\pi} B\left(\frac{1}{2}, \frac{b-1}{2}\right)\right]^{\frac{1}{b-1}},$$

the inequality (7) is sharper than the inequality (3).

(ii) When $a > 0$, $b > a + 2$ and x is large, the inequality (8) is sharper than the inequality (1).

(iii) For $a > 0$, $b - a > 1$ and

$$x > \left[\frac{\Gamma(b+1)}{2a\pi} B\left(\frac{1}{2}, \frac{b-a-1}{2}\right)\right]^{\frac{1}{b-a-1}},$$

the inequality (8) is sharper than the upper bound of (2).

(iv) At least when x is large and

- $a > 1$, $b > 2a$;
- $a > 0$, $b > 2a$;
- $a < 0$, $b > 0$;

the inequality (9) is sharper than the inequality (1), the upper bound of (2), and the inequality (3), respectively.

(v) At least when x is large and $a = 1$, $b > 2$ the inequalities (8) and (9) are sharper than the inequality (4).

REMARK 1.2. Using inequalities for Pochhammer symbol (see [7])

$$|(\alpha)_n| \le (|\alpha|)_n\,,$$
$$|(\alpha)_n| \ge \left(\cos\frac{\theta}{2}\right)^{n-1} (|\alpha|)_n\,, \qquad \theta = \arg(\alpha),\ |\theta| < \pi,\ n \in \mathbb{N}$$

we obtain

$$|{}_1F_1(a;b;x)| \le \cos\frac{\theta}{2}\,{}_1F_1\left(|a|;|b|;|x|\sec\frac{\theta}{2})\right), \qquad \theta = \arg(b), |\theta| < \pi.$$

With the help of this observation, we can obtain the complex extensions of inequalities (1), (2), (7), and (8) for $a > 0$.

2. Proof of Theorem 1.1

(i) The proof is based on the well-known inverse Laplace transform of ${}_1F_1$ (see [12, p. 60])

$${}_1F_1(a;b;x) = \frac{\Gamma(b)}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^t t^{-b}\left(1-\frac{x}{t}\right)^{-a} dt,$$

where $x > 0$, $\Re e(b) > 0$ and $\gamma > x$, which can be transformed to the integral over the real axis

$${}_1F_1(a;b;x) = \frac{\Gamma(b)}{2\pi}\int_{-\infty}^{+\infty} e^{\gamma+i\delta}(\gamma+i\delta)^{-b}\left(1-\frac{x}{\gamma+i\delta}\right)^{-a} d\delta. \qquad (10)$$

Taking absolute values in (10) we get

$$\begin{aligned} e^{-\gamma-\frac{\pi}{2}|\Im m(b)|}\,|{}_1F_1(a;b;x)| &\le \frac{|\Gamma(b)|}{2\pi}\int_{-\infty}^{+\infty}(\gamma^2+\delta^2)^{-\frac{\Re e(b)}{2}}d\delta \\ &= \frac{|\Gamma(b)|}{\pi}\int_{0}^{+\infty}(\gamma^2+\delta^2)^{-\frac{\Re e(b)}{2}}d\delta \\ &< \frac{|\Gamma(b)|}{\pi}\int_{0}^{+\infty}(x^2+\delta^2)^{-\frac{\Re e(b)}{2}}d\delta, \end{aligned}$$

where $a < 0$ and we implement inequalities

$$|z^c| = |z|^{\Re e(c)}e^{-\Im m(c)\cdot\arg(z)} \le |z|^{\Re e(c)}e^{\frac{\pi}{2}|\Im m(c)|}, \quad \Re e(z) > 0.$$

$$\left|1-\frac{x}{\gamma+i\delta}\right| < 1, \qquad x < 2\gamma.$$

Now, taking into account the integral (see [17, p. 295])

$$\int_0^{+\infty} (z^2+t^2)^{-\rho} dt = \frac{z^{1-2\rho}}{2} B\left(\frac{1}{2}, \rho - \frac{1}{2}\right),$$

where $\Re e(z) > 0$ and $\Re e(\rho) > \frac{1}{2}$, we have

$$x^{\Re e(b)-1} e^{-\gamma - \frac{\pi}{2}|\Im m(b)|} \left|{}_1F_1(a;b;x)\right| < \frac{|\Gamma(b)|}{2\pi} B\left(\frac{1}{2}, \frac{\Re e(b)-1}{2}\right).$$

If for any finite $x > 0$ we choose γ such that $\gamma = x + \epsilon$, $\epsilon > 0$, we get

$$x^{\Re e(b)-1} e^{-x - \frac{\pi}{2}|\Im m(b)|} \left|{}_1F_1(a;b;x)\right| < e^{\epsilon} \frac{|\Gamma(b)|}{2\pi} B\left(\frac{1}{2}, \frac{\Re e(b)-1}{2}\right).$$

Since the above inequality holds for any $\epsilon > 0$, we get the desired result (7) for any finite $x > 0$.

On the other hand, according to (5) and (6), we have for large $x > 0$

$$x^{\Re e(b)-1} e^{-x - \frac{\pi}{2}|\Im m(b)|} \left|{}_1F_1(a;b;x)\right| = o(1),$$

where $a < 0$ and $\Re e(b) > 1$. This completes the proof of case (i).

(ii) By making use of (10) and the fact that for $x < \gamma - 1$

$$|(\gamma - x) + i\delta| > 1,$$

the proof of (8) can be completed by following the proof of (7).

(iii) For the proof of (9), we shall use the Mellin–Barnes integral representation (see [15, §3.4.1])

$${}_1F_1(a;b;-x) = \frac{\Gamma(b)}{2\pi i \Gamma(a)} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(a+s)\Gamma(-s)x^s}{\Gamma(b+s)} ds,$$

where $\Re e(x) > 0$ and $0 > \gamma > -\Re e(a)$, which can be transformed to the integral over the real axis

$${}_1F_1(a;b;-x) = \frac{\Gamma(b)}{2\pi \Gamma(a)} \int_{-\infty}^{+\infty} \frac{\Gamma(a+\gamma+i\delta)\Gamma(-\gamma-i\delta)x^{\gamma+i\delta}}{\Gamma(b+\gamma+i\delta)} d\delta.$$

In particular, for $\gamma = -\frac{b}{2}$ we have

$${}_1F_1(a;b;-x) = \frac{\Gamma(b)}{2\pi \Gamma(a)} \int_{-\infty}^{+\infty} \frac{\Gamma\left(a - \frac{b}{2} + i\delta\right)\Gamma\left(\frac{b}{2} - i\delta\right)x^{-\frac{b}{2}+i\delta}}{\Gamma\left(\frac{b}{2}+i\delta\right)} d\delta.$$

Since

$$\overline{\Gamma(z)} = \Gamma(\overline{z}),$$

it follows that

$$|\Gamma(z)| = |\Gamma(\overline{z})|\,.$$

Thus, for $0 < b < 2a$ and $x > 0$

$$|{}_1F_1(a;b;-x)| \leq \frac{\Gamma(b)x^{-\frac{b}{2}}}{2\pi\Gamma(a)} \int_{-\infty}^{+\infty} \left|\Gamma\left(a - \frac{b}{2} + i\delta\right)\right| d\delta.$$

On the other hand, applying Ramanujan's remarkable integrals involving gamma functions, it is easily observed that (see Appendix B for details)

$$\int_{-\infty}^{+\infty} |\Gamma(x+iy)|\, dy \leq \pi\sqrt{\frac{\Gamma(2x+2)}{2^{2x+1}x}}. \tag{11}$$

Thus,

$$|{}_1F_1(a;b;-x)| \leq \frac{\Gamma(b)x^{-\frac{b}{2}}}{2\Gamma(a)} \sqrt{2^{b-2a}(2a-b+1)\Gamma(2a-b)}.$$

Finally, based on the Kummer transformation

$${}_1F_1(a;b;-x) = e^{-x}\,{}_1F_1(b-a;b;x),$$

we obtain, after some algebraic operations, the proof of inequality (9).

A. The oscillatory interval

A remarkable result established by Tricomi (see [22]) states that the positive zeros of ${}_1F_1(a;b;x)$, possibly with the exception of the largest, have the upper bound

$$x_+ = b - 2a + \sqrt{(b-2a)^2 + b(2-b)}.$$

Later on, this result has been proved by several authors by using different approaches (see, for instance, [6, 23], and references therein).

Here we improve the Tricomi result when $0 < b \leq 2$. For this, we need the following result due to Picone (see [23]):
Let p and q be two functions with first and second derivatives in some interval (x_1, x_2). Consider the self-adjoint linear second order differential equation

$$[p(x)y'(x)]' + q(x)y(x) = 0. \tag{12}$$

If p and q have different signs in (x_1, x_2), then the solutions y can vanish at most once in that interval.

It is straightforward to check that the function $y(x) = e^{-\frac{x}{2}}\,{}_1F_1(a;b;x)$ satisfies the following self-adjoint linear second order differential equation (see, for instance, [8])

$$(x^b y'(x))' + \frac{2b-4a-x}{4}x^{b-1}y(x) = 0,$$

which corresponds to equation (12) with

$$p(x) = x^b, \qquad q(x) = \frac{2b - 4a - x}{4} x^{b-1}.$$

Thus, for $a < 0$ and $b > 0$, the functions p and q have different signs in $(2b - 4a, \infty)$. Now, by Picone's result, we conclude that the function ${}_1F_1(a; b; x)$ has at most one zero on the interval $(2b - 4a, +\infty)$.

B. Proof of inequality (11)

By using different estimates of gamma function, next we present several inequalities for the integral

$$\int_{-\infty}^{+\infty} |\Gamma(x + iy)|\, dy.$$

(i) Taking into account Ramanujan's integrals involving gamma functions (see, for instance, [3])

$$\int_{-\infty}^{+\infty} \left| \frac{\Gamma(x + iy)}{\Gamma(z + 1 + iy)} \right|^2 dy = \frac{\sqrt{\pi}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right)\Gamma\left(z - x + \frac{1}{2}\right)}{\Gamma(z + 1)\Gamma\left(z + \frac{1}{2}\right)\Gamma(z - x + 1)}, \quad 0 < x < z + \frac{1}{2},$$

$$\int_{-\infty}^{+\infty} |\Gamma(x + iy)|^2\, dy = \pi 2^{1-2x}\Gamma(2x), \quad x > 0,$$

and using Cauchy–Schwarz inequality we have

$$\left(\int_{-\infty}^{+\infty} |\Gamma(x + iy)|\, dy\right)^2 \le \int_{-\infty}^{+\infty} |\Gamma(x + 1 + iy)|^2\, dy \times \int_{-\infty}^{+\infty} \left| \frac{\Gamma(x + iy)}{\Gamma(x + 1 + iy)} \right|^2 dy = \frac{\pi^2\Gamma(2x + 2)}{2^{2x+1}x},$$

which proves inequality (11).

(ii) Taking into account inequalities (see [7])

$$|\Gamma(x + iy)| < \frac{e^2\Gamma(x + 1)}{|x + iy|} e^{-\frac{\pi}{2}|y|}, \qquad x > 0, |y| \le \frac{2}{\pi},$$

$$|\Gamma(x + iy)| < 2^{\frac{1}{2}-x} \frac{e\sqrt{\Gamma(2x + 2)}}{|x + iy|} \left(\frac{\pi}{2}|y|\right)^{x+\frac{1}{2}} e^{-\frac{\pi}{2}|y|}, \qquad x > 0, |y| \ge \frac{2}{\pi},$$

it is straightforward to show that

$$\int_{|y|\le \frac{2}{\pi}} |\Gamma(x+iy)|\,dy < \frac{4e(e-1)}{\pi}\Gamma(x),$$

$$\int_{|y|\ge \frac{2}{\pi}} |\Gamma(x+iy)|\,dy < 2^{\frac{1}{2}-x}\frac{e\pi^2\sqrt{\Gamma(2x+2)}}{\sqrt{(\pi x)^2+4}}\Gamma\left(x+\frac{3}{2},1\right),$$

where $\Gamma(\cdot,\cdot)$ is incomplete gamma function.

As consequence,

$$\int_{-\infty}^{+\infty} |\Gamma(x+iy)|\,dy < \frac{4e(e-1)}{\pi}\,\Gamma(x) + \frac{e\pi^2\sqrt{\Gamma(2x+2)}}{2^{x-\frac{1}{2}}\sqrt{(\pi x)^2+4}}\,\Gamma\left(x+\frac{3}{2},1\right). \quad (13)$$

(iii) Using inequality (see [14, §5.6])

$$\begin{aligned}|\Gamma(x+iy)| &\le \sqrt{2\pi}(x^2+y^2)^{\frac{2x-1}{4}}e^{-\frac{\pi}{2}|y|}e^{\frac{1}{6|x+iy|}}, \qquad x>0,\\ &\le \sqrt{2\pi}(x^2+y^2)^{\frac{2x-1}{4}}e^{-\frac{\pi}{2}|y|}e^{\frac{1}{6x}}\end{aligned}$$

we get

$$\int_{-\infty}^{+\infty} |\Gamma(x+iy)|\,dy \le 2\sqrt{2\pi}e^{\frac{1}{6x}}\int_0^{+\infty}(x^2+y^2)^{\frac{2x-1}{4}}e^{-\frac{\pi}{2}|y|}$$

Now we can apply equation (see [14, (11.2.5)] and [17, p. 323])

$$\int_0^{+\infty}(x^2+y^2)^{\nu}e^{-\mu y}dy = \frac{\sqrt{\pi}}{2}\left(\frac{2x}{\mu}\right)^{\nu+\frac{1}{2}}\Gamma(\nu+1)\mathbf{K}_{\nu+\frac{1}{2}}(\mu x),$$

where $|\arg(x)|<\pi$, $\Re e(\mu)>0$ and $\mathbf{K}_\nu(\cdot)$ is Struve function of the second kind, to yield

$$\int_{-\infty}^{+\infty} |\Gamma(x+iy)|\,dy \le \pi\sqrt{2}e^{\frac{1}{6x}}\left(\frac{4x}{\pi}\right)^{\frac{2x+1}{4}}\Gamma\left(\frac{2x+3}{4}\right)\times\mathbf{K}_{\frac{2x+1}{4}}\left(\frac{\pi x}{2}\right). \quad (14)$$

In our opinion inequality (11) is to be preferred over inequalities (13) and (14), because the upper bound of (11) is given in terms of simplest functions.

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