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Three general uniqueness theorems for BVPs of ODEs with applications

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Abstract. *We prove three uniqueness theorems for BVPs of ODEs whose unusual assumptions allow to prove existence results for BVPs related to various types of linear and nonlinear boundary conditions.*

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1. Introduction

This paper is devoted to the proof of three general uniqueness theorems for BVPs of ODEs and of a variety of consequent existence results.

The first theorem is obtained from that of R. Caccioppoli in [7] about diffeomorphisms between two Banach spaces. It provides sufficient conditions for the solvability of nonlinear BVPs when their linearizations have only the trivial solution. It is related to first-order systems of ODEs and applies to linear as well as to nonlinear boundary conditions (which, in particular, may be distinct from equation to equation in the same system).

The second theorem has been inspired by the early work of W.V. Petryshyn. In [12], Petryshyn developed a Spectral Theory for some non-symmetric linear operators in a given Hilbert space by introducing a new inner product making symmetric the starting operators (with the goal to obtain general iterative methods for the approximate calculation of eigenvalues). Petryshyn's results have been applied to some peculiar linear BVPs for ODEs in [5, 11, 12]. The lemma in §3 is a simple variant of those ideas and is the key for the proof of Theorem 3.2. Theorem 3.2 is related to higher-order scalar equations subjected to linear boundary conditions. Its assumption on the nonlinearity in the given ODE is centered on sign conditions and not on norms.

The third theorem fits into the context of the pioneering work of S. Ahmad and A.C. Lazer. Papers [10] and [1] inspired a variety of similar results as can be seen from the bibliographical references of [2], [3] and the introduction of [8]. Theorem 4.1 generalizes Lazer's theorem in [10]. It is related to higher-order systems with the peculiarity that the involved boundary conditions may change from equation to equation in the same system [in other words, following the title of [4], we have "meshed" spectra together with "meshed" boundary conditions].

The corollaries and examples show some of the various possibilities offered by the above theorems to have existence results for distinct BVPs.

At the beginning of each section below there is a short list of standing notations.

2. First-order systems with nonlinear boundary conditions

We shall use freely the following customary notations:

- $\mathcal{L}(X, Y)$ is the Banach space of bounded linear operators $X \to Y$;
- $\mathbb{R}^{N\times N}$ is the Banach space of real $N \times N$-matrices.

THEOREM 2.1. *Let K be a weakly sequentially compact, convex subset of $L^1([a,b], \mathbb{R}^{N\times N})$ and let $\mathcal{K}$ be a compact, convex subset of $\mathcal{L}\big(C^1([a,b],\mathbb{R}^N), \mathbb{R}^N\big)$ such that the linear BVP*

$$\begin{cases} u' = A(t) \cdot u \\ L\,u = 0 \end{cases}$$

has only the trivial solution whenever $A \in K$ and $L \in \mathcal{K}$.

If $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ and $B : C^1([a,b], \mathbb{R}^N) \to \mathbb{R}^N$ are continuously differentiable such that f_x is uniformly bounded, B is bounded on bounded sets and further

$$f_x\big(\cdot, u(\cdot)\big) \in K \quad \text{and} \quad B'(u) \in \mathcal{K} \qquad \big(u \in C^1([a,b], \mathbb{R}^N)\big),$$

then the functional BVP

$$\begin{cases} u' = f(t,u) \\ B(u) = r \end{cases}$$

has a unique solution for every $r \in \mathbb{R}^N$ and it depends C^1-continuously on r.

Proof. To simplify notations we set

$$C^0 := C^0([a,b], \mathbb{R}^N) \qquad \text{and} \qquad C^1 := C^1([a,b], \mathbb{R}^N).$$

A function u satisfies the given functional BVP if and only if

$$u(t) = u(a) + B(u) - r + \int_a^t f\big(s, u(s)\big)\, ds.$$

Define $F : C^1 \to C^1$ by

$$F(u)(t) := u(t) - u(a) - B(u) - \int_a^t f\bigl(s, u(s)\bigr)\, ds. \tag{1}$$

If we show that F is a diffeomorphism, then the equation $F(u) = -r$ will have a unique solution which will depend C^1-continuously on r (looking at "$-r$" as a constant map). To achieve this goal we shall use Caccioppoli's theorem in [7], so that we need to prove that F is a proper map which is continuously differentiable and whose derivatives are all invertible linear operators.

We claim that the derivative $F'(u)$ exists and is defined by

$$\bigl(F'(u)\bigr)\, v(t) := v(t) - v(a) - B'(u)\, v - \int_a^t f_x\bigl(s, u(s)\bigr)\, v(s)\, ds.$$

In fact, given $\varepsilon > 0$ then for $\|u - u_0\|_{C^1}$ small enough we have

$$\begin{aligned}
&\bigl\|F(u) - F(u_0) - F'(u_0)\cdot(u-u_0)\bigr\|_\infty \\
&\qquad \leqslant \bigl\|B(u) - B(u_0) - B'(u_0)\cdot(u-u_0)\bigr\|_\infty \\
&\qquad\qquad + (b-a)\,\bigl\|f_x\bigl(\cdot, u(\cdot)\bigr) - f_x\bigl(\cdot, u_0(\cdot)\bigr)\bigr\|_\infty \bigl\|u - u_0\bigr\|_\infty \\
&\qquad \text{[by the mean value theorem]} \\
&\qquad \leqslant \varepsilon\, \|u - u_0\|_{C^1} \\
&\qquad \text{[in view of the definition of derivative and the uniform continuity of } f_x \\
&\qquad\ \text{on bounded sets, \textit{provided that } } \|u-u_0\|_{C^1} \text{ \textit{is sufficiently small}]}
\end{aligned}$$

and similarly

$$\left\|\frac{d}{dt}\Bigl\{F(u) - F(u_0) - F'(u_0)\cdot(u-u_0)\Bigr\}\right\|_\infty \leqslant \varepsilon\, \bigl\|u - u_0\bigr\|_{C^1}$$

provided that $\|u - u_0\|_{C^1}$ is sufficiently small. Thus our claim holds and consequently F is continuously differentiable.

The identity $\bigl(F'(u)\bigr)\, v = 0$ means that v solves the BVP

$$\begin{cases} v' = f_x\bigl(t, u(t)\bigr)\, v \\ \bigl(B'(u)\bigr)\, v = 0\,. \end{cases}$$

Therefore $v \equiv 0$ by the hypotheses of the theorem. Consequently $\ker\bigl(F'(u)\bigr) = \bigl\{0\bigr\}$, hence $F'(u)$ is invertible (being the identity minus a compact linear operator).

To prove that F is proper, assume $F(u_n) = z_n$ with $(z_n)_n$ relatively compact in C^1 and let us show the existence of a convergent subsequence of $(u_n)_n$ in C^1.

In view of (1), it suffices to prove that $(u_n)_n$ is bounded in C^1 [because then each of the sequences made by the $u_n(a)$'s, u_n's and $\int_a^{\cdot} f\big(s, u_n(s)\big)\,ds$'s are relatively compact]. Passing to a subsequence if necessary, assume $z_n \to z_\infty$ in C^1. For contradiction, suppose that $\|u_n\|_{C^1} \to \infty$ (passing to a subsequence if necessary). The identity $F(u_n) = z_n$ means that u_n solves the BVP

$$\begin{cases} u_n' = f(t, u_n) + z_n' \\ B(u_n) = -z_n(a) \end{cases}$$

that can be rewritten equivalently as

$$\begin{cases} u_n'(t) = f(t,0) + \displaystyle\int_0^1 f_x\big(t, \xi\, u_n(t)\big) \cdot u_n(t)\, d\xi + z_n'(t) \\ B(0) + \displaystyle\int_0^1 B'(\xi\, u_n) \cdot u_n\, d\xi = -z_n(a)\,. \end{cases}$$

Setting $v_n := u_n / \|u_n\|_{C^1}$ we have

$$\begin{cases} v_n'(t) = \dfrac{f(t,0)}{\|u_n\|_{C^1}} + \displaystyle\int_0^1 f_x\big(t, \xi\, u_n(t)\big)\, d\xi \cdot v_n(t) + \dfrac{z_n'(t)}{\|u_n\|_{C^1}} \\ \dfrac{B(0)}{\|u_n\|_{C^1}} + \displaystyle\int_0^1 B'(\xi\, u_n)\, d\xi \cdot v_n = \dfrac{-z_n(a)}{\|u_n\|_{C^1}}\,. \end{cases}$$

Let C be a weakly sequentially compact and convex subset of a Banach space and let $g : [0,1] \to C$ be continuous. The Riemann sums related to the integral of g are members of C. Consequently $\int_0^1 g(\xi)\, d\xi \in C$. Therefore

$$\int_0^1 f_x\big(t, \xi\, u_n(t)\big)\, d\xi \in K \qquad \text{and} \qquad \int_0^1 B'(\xi\, u_n)\, d\xi \in \mathcal{K}$$

for every n. In addition, we have

$$\begin{aligned} \|v_n(t) - v_n(s)\| \leqslant & \int_s^t \frac{\|f(\zeta,0)\|}{\|u_n\|_{C^1}}\, d\zeta \\ & + \int_s^t d\zeta \int_0^1 \big\|f_x\big(\zeta, \xi\, u_n(\zeta)\big)\big\|\, \|v_n(\zeta)\|\, d\xi + \int_s^t \frac{\|z_n'(\zeta)\|}{\|u_n\|_{C^1}}\, d\zeta \end{aligned}$$

which implies that $(v_n)_n$ is equicontinuous since f_x is uniformly bounded. By Ascoli Theorem, it follows that $(v_n)_n$ is relatively compact in C^0. Summing up, we deduce the existence of $n_k \uparrow \infty$, $A_\infty \in K$, $L_\infty \in \mathcal{K}$ and $v_\infty \in C^0$ such that

$$\int_0^1 f_x\big(t,\xi\, u_{n_k}(t)\big)\, d\xi \rightharpoonup A_\infty \quad \text{in} \quad L^1([a,b],\mathbb{R}^{N\times N}),$$
$$\int_0^1 B'(\xi\, u_{n_k})\, d\xi \to L_\infty \quad \text{in} \quad \mathcal{L}\big(C^0,\mathbb{R}^N\big),$$
$$v_{n_k} \to v_\infty \quad \text{in} \quad C^0.$$

Then taking limits in

$$\begin{aligned} v_{n_k}(t) = v_{n_k}(a) &+ \frac{B(0)}{\|u_{n_k}\|_{C^1}} \\ &+ \int_0^1 B'(\xi\, u_{n_k})\cdot v_{n_k}\, d\xi + \frac{z'_{n_k}(a)}{\|u_{n_k}\|_{C^1}} + \int_a^t \frac{f(\zeta,0)}{\|u_n\|_{C^1}}\, d\zeta \\ &\quad + \int_a^t d\zeta \int_0^1 f_x\big(\zeta,\xi\, u_{n_k}(\zeta)\big)\cdot v_{n_k}(\zeta)\, d\xi + \int_a^t \frac{z'_{n_k}(\zeta)}{\|u_{n_k}\|_{C^1}}\, d\zeta \end{aligned}$$

yields

$$v_\infty(t) = v_\infty(a) + L_\infty\, v_\infty + \int_a^t A_\infty \cdot v_\infty\, d\zeta$$

[as continuous bilinear operators transform those sequences which converge both weakly in the first components and strongly in the others, into weakly convergent sequences (and weak and strong convergence are equal in $\mathbb{R}^N$)] so that v_∞ solves the BVP

$$\begin{cases} v'_\infty = A_\infty \cdot v_\infty \\ L_\infty\, v_\infty = 0\,. \end{cases}$$

We claim that $v_\infty \neq 0$. For, if $v_\infty = 0$, then $\|v'_{n_k}\|_\infty \to 1$, which is impossible because

$$\|v'_{n_k}\|_\infty \leqslant \frac{\|f(\cdot,0)\|_\infty}{\|u_{n_k}\|_{C^1}} + \|f_x\|_\infty\, \|v_{n_k}\|_\infty + \frac{\|z'_{n_k}\|_\infty}{\|u_{n_k}\|_{C^1}} \to 0.$$

Now we are done, since $v_\infty \neq 0$ contradicts $A_\infty \in K$ and $L_\infty \in \mathcal{K}$. □

Corollary 2.2. *Let K be a weakly sequentially compact, convex subset of $L^1([a,b],\mathbb{R}^{N\times N})$ and let $L \in \mathcal{L}\big(C^1([a,b],\mathbb{R}^N),\mathbb{R}^N\big)$ be such that the BVP*

$$\begin{cases} u' = A(t)\cdot u \\ L\, u = 0 \end{cases}$$

has only the trivial solution whenever $A \in K$.

If $f : [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$ is continuously differentiable with f_x uniformly bounded and

$$u \in C^1([a,b],\mathbb{R}^N) \qquad \Rightarrow \qquad f_x\big(\cdot, u(\cdot)\big) \in K,$$

then the functional BVP

$$\begin{cases} u' = f(t,u) \\ L\,u = r \end{cases}$$

has a unique solution for every $r \in \mathbb{R}^N$.

Proof. This is the special case $\mathcal{K} := \big\{L\big\}$ of the above theorem. □

The following example represents an application of Theorem 2.1 to a problem with nonlinear boundary conditions.

Example 2.3. The BVP

$$\begin{cases} -u'' = f(t,u) \\ u(0) = 0, \quad u(\pi) = g\big(u(\pi)\big) \end{cases}$$

has a unique solution when f, $g : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable and the two conditions

(i) there exist constants μ, ν such that

- either $-\infty < \mu \leqslant f_x \leqslant \nu < 0$,
- or there exists $n \geqslant 1$ such that $n^2 < \mu \leqslant f_x \leqslant \nu < (n+1)^2$,

(ii) g' is uniformly bounded and there exists $\gamma > 0$ such that

$$1 - g' \geqslant \gamma,$$

are fulfilled.

Proof. With $F : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ and $B : C^1([0,\pi],\mathbb{R}^2) \to \mathbb{R}^2$ defined by

$$F(t,z) := \big(z_2,\, -f(t,z_1)\big) \qquad \text{and} \qquad B(w) := \Big(w_1(0), w_2(\pi) - g\big(w_2(\pi)\big)\Big),$$

the given BVP is equivalent to the BVP

$$\begin{cases} w' = F(t,w) \\ B(w) = 0 \end{cases} \tag{2}$$

related to a planar system. The maps F and B are continuously differentiable with

$$\begin{aligned} & F_z(t,z)\cdot y = \big(y_2,\ -f_x(t,z_1)\,y_1\big) \\ \text{and}\qquad & B'(w)\cdot v := \Big(v_1(0), v_2(\pi) - g'\big(w_2(\pi)\big)\,v_2(\pi)\Big). \end{aligned}$$

Therefore for every $w \in C^1([a,b],\mathbb{R}^2)$ we have

$$\begin{aligned} F_z\big(\cdot, w(\cdot)\big) &\in \left\{ \begin{pmatrix} 0 & 1 \\ -h & 0 \end{pmatrix} :\ \ h : [0,\pi] \to [\mu,\nu] \text{ measurable} \right\} =: K, \\ B'(w) &\in \Big\{ (P_1, P_2 - \alpha\, P_2) :\ \ \alpha \in C \Big\} =: \mathcal{K} \end{aligned}$$

where $P_1\, w := w_1(0)$, $P_2\, w := w_2(\pi)$ and C is the closure of the range of g'. Clearly K is a weakly sequentially compact subset of $L^1([0,\pi],\mathbb{R}^2)$, while $\mathcal{K}$ is a compact subset of $\mathcal{L}\big(C^1([a,b],\mathbb{R}^2),\mathbb{R}^2\big)$ because C is a compact subset of $\mathbb{R}$. When $A \in K$ and $L \in \mathcal{K}$ we have

$$\begin{cases} w' = A(t)\cdot w \\ L\,w = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} -u'' = h(t)\,u \\ u(0) = 0, \quad u(\pi) - \alpha\, u(\pi) = 0 \end{cases}$$

with suitable $h : [0,\pi] \to [\mu,\nu]$ and $\alpha \in C$. From $1 - \alpha \geqslant \gamma > 0$ it follows that

$$\begin{cases} -u'' = h(t)\,u \\ u(0) = 0, \quad u(\pi) - \alpha\, u(\pi) = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} -u'' = h(t)\,u \\ u(0) = 0 = u(\pi)\,. \end{cases}$$

Well-known results imply that the last BVP has only the trivial solution because m^2, $m \geqslant 1$, are the eigenvalues of

$$\begin{cases} -u'' = \lambda\, u \\ u(0) = 0 = u(\pi)\,. \end{cases}$$

Thus we can conclude that (2) has a unique solution by Theorem 2.1, hence we are done. □

3. Higher-order equations with linear boundary conditions

Standing notations of the section:

- m is a positive integer;
- $B : C^{m-1}([a,b]) \to \mathbb{R}^m$ is a bounded linear operator;

- $D_{m,B}$ is the set of all $u \in C^{m-1}([a,b])$ such that
  - $u^{(m-1)}$ is absolutely continuous and $u^{(m)} \in L^2([a,b])$,
  - $B(u) = 0$;
- $L_{m,B}$ is the unbounded linear operator in $L^2([a,b])$ defined by

$$L_{m,B}\, u := a_0(t)\, u^{(m)}$$

with domain $D_{m,B}$ and $a_0 \in C^0([a,b])$ such that $a_0(t) \neq 0$ for all t.

All arguments of the present subsection are based on the properties of the eigenvalue problem

$$L\, u = \lambda\, T\, u$$

where $L, T : D_{m,B} \to L^2([a,b])$ are arbitrary linear operators and $\lambda \in \mathbb{R}$. A $\lambda \in \mathbb{R}$ for which there is a nontrivial solution $u \in D_{m,B}$ of the above identity will be called *eigenvalue* and u its corresponding *eigenvector*.

LEMMA 3.1. *Let $L, T : D_{m,B} \to L^2([a,b])$ be linear operators. If there exists a real number $\gamma > 0$ such that*

$$(L\, u \,|\, T\, u)_{L^2} \geqslant \gamma\, \|u\|^2_{L^2} \qquad (u \in D_{m,B}),$$

then every eigenvalue λ of

$$L\, u = \lambda\, T\, u$$

is strictly positive.

Proof. Let λ be an eigenvalue and $u \in D_{m,B}$ a corresponding eigenvector. From $u \neq 0$ and $L\, u = \lambda\, T\, u$ we get

$$\lambda\, \|T\, u\|^2_{L^2} = (L\, u \,|\, T\, u)_{L^2} \geqslant \gamma\, \|u\|^2_{L^2} > 0$$

which implies that $\lambda \geqslant \gamma\, \|u\|^2_{L^2} / \|T\, u\|^2_{L^2} > 0$. □

From this we deduce the following theorem:

THEOREM 3.2. *With m, B, $D_{m,B}$, $L_{m,B}$ as above, let $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that each partial derivative f_{x_i} exists and is a uniformly bounded Carathéodory function. If there exist a linear operator $T : D_{m,B} \to L^2([a,b])$ and a real constant $\gamma > 0$ such that*

$$\Big(L_{m,B}\, u + \sum_{i=1}^{m} f_{x_i}\big(\cdot, w^i(\cdot)\big) \cdot u^{(i-1)} \,\Big|\, T\, u \Big)_{L^2} \geqslant \gamma\, \|u\|^2_{L^2}$$

whenever $u \in D_{m,B}$ and $w^1, \dots, w^m \in C^0([a,b], \mathbb{R}^m)$, then the BVP

$$\begin{cases} L_{m,B}\, u + f(t, u, u', \dots, u^{(m-1)}) = h \\ B\, u = 0 \end{cases} \tag{3}$$

has a unique solution for every $h \in L^2([a,b])$, while the BVP

$$\begin{cases} L_{m,B}\, u + f(t, u, u', \dots, u^{(m-1)}) = g(t, u, u', \dots, u^{(m-1)}) \\ B\, u = 0 \end{cases} \tag{4}$$

has at least one solution for every Carathéodory function $g : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ such that $\lim_{\|x\| \to \infty} g(t,x)/\|x\| = 0$ uniformly in t.

Proof. We start by proving uniqueness for (3). For contradiction, assume the existence of two distinct solutions u, v of (3). From

$$\begin{aligned}
f\big(t, u(t), u'(t), \dots, u^{m-1}(t)\big) &- f\big(t, v(t), v'(t), \dots, v^{(m-1)}(t)\big) \\
= f\big(t, u(t), u'(t), \dots, u^{(m-1)}(t)\big)& \\
\pm f\big(t, v(t), u'(t), &\dots, u^{(m-1)}(t)\big) \\
&\vdots \\
\pm f\big(t, v(t), v'(t), &\dots, v^{(m-2)}(t), u^{(m-1)}(t)\big) \\
&- f\big(t, v(t), v'(t), \dots, v^{(m-1)}(t)\big) \\
= \sum_{i=1}^{m} \int_0^1 f_{i,u,v}(\xi, t) &\cdot \big(u^{(i-1)}(t) - v^{(i-1)}(t)\big)\, d\xi
\end{aligned}$$

where

$$\begin{aligned}
f_{i,u,v}(\xi, t) := f_{x_i}\Big(&t, v(t), v'(t), \dots, v^{(i-1)}(t), \\
&\xi\big(u^{(i)}(t) - v^{(i)}(t)\big) + v^{(i)}(t), u^{(i+1)}(t), \dots, u^{(m-1)}(t)\Big),
\end{aligned}$$

we get

$$\begin{aligned}
(L_{m,B}\, u)(t) &- (L_{m,B}\, v)(t) \\
&+ \sum_{i=1}^{m} \int_0^1 f_{i,u,v}(\xi, t) \cdot \big(u^{(i-1)}(t) - v^{(i-1)}(t)\big)\, d\xi = 0.
\end{aligned}$$

This means that $w := u - v \neq 0$ is a solution of the linear BVP

$$\begin{cases} L_{m,B}\, w + L\, w = 0 \\ B\, w = 0 \end{cases}$$

where $L : D_{m,B} \to L^2([a,b])$ is the linear differential operator

$$L\,w := \sum_{i=1}^{m} \int_0^1 f_{i,u,v}(\xi,\cdot)\,d\xi \cdot w^{(i-1)}.$$

Consequently $\lambda = 0$ is an eigenvalue of

$$L_{m,B}\,w + L\,w = \lambda\,T\,w.$$

From the hypotheses of the theorem we have

$$\Big(L_{m,B}\,u + \sum_{i=1}^{m} f_{i,u,v}(\xi,\cdot)\cdot u^{(i-1)}\,\Big|\,T\,u\Big)_{L^2} \geqslant \gamma\,\|u\|^2_{L^2}$$

for all $u \in L^2([a,b])$. Integrating both sides on the unit interval with respect to ξ and using Fubini's theorem we deduce

$$\Big(L_{m,B}\,u + \sum_{i=1}^{m} \int_0^1 f_{i,u,v}(\xi,\cdot)\cdot u^{(i-1)}\,d\xi\,\Big|\,T\,u\Big)_{L^2} \geqslant \gamma\,\|u\|^2_{L^2}$$

for all $u \in D_{m,B}$. Therefore $L_{m,B} + L$ and T fulfill the hypotheses of the preceding lemma, so that we have contradicted it and so we can conclude that (3) has at most one solution, as desired.

To complete the proof, it suffices to show that (4) has at least one solution. To simplify notations, we set

$$\begin{aligned} F_i(t,v) &:= f_{x_i}\big(t, v(t), \ldots, v^{(m-1)}(t)\big) \\ \text{and} \qquad G(t,v) &:= g\big(t, v(t), \ldots, v^{(m-1)}(t)\big) \end{aligned}$$

so that F_i and G are maps $[a,b] \times C^{m-1}([a,b]) \to \mathbb{R}$ and

$$f(t,u,\ldots,u^{(m-1)}) = f(t,0,\ldots,0) + \sum_{i=1}^{m} \int_0^1 f_{x_i}(t,\xi\,u,\ldots,\xi\,u^{(m-1)})\,u^{(i-1)}\,d\xi,$$

hence the equation in (4) can be rewritten simply as

$$L_{m,B}\,u = -f(t,0,\ldots,0) - \sum_{i=1}^{m} \int_0^1 F_i(t,\xi\,u)\,u^{(i-1)}\,d\xi + G(t,u).$$

We claim the existence of a priori bounds in $C^{m-1}([a,b])$ of the solutions to

$$\begin{cases} L_{m,B}\,u = -f(t,0,\ldots,0) - \displaystyle\sum_{i=1}^{m} \int_0^1 F_i(t,\tau\,\xi\,u)\,u^{(i-1)}\,d\xi + \tau\,G(t,u) \\ B\,u = 0 \end{cases} \tag{5}$$

independent of $0 \leqslant \tau \leqslant 1$. For, otherwise there are $\tau_n \in [0,1]$ and $u_n \in C^{m-1}([a,b])$ such that $\tau_n \to \tau_\infty$, $\|u_n\|_{C^{m-1}} \to \infty$ and

$$\begin{cases} L_{m,B} u_n = -f(t,0,\dots,0) - \displaystyle\sum_{i=1}^{m} \int_0^1 F_i(t, \tau_n\, \xi\, u_n) u_n^{(i-1)}\, d\xi + \tau_n G(t,u_n) \\ B\, u_n = 0\,. \end{cases} \tag{6}$$

Set $v_n := u_n / \|u_n\|_{C^{m-1}}$. From

$$\begin{aligned} v_n(t) :=& \sum_{i=1}^{m-1} v_n^{(i)}(a)\, \frac{(t-a)^{(i-1)}}{i!} - \frac{f(t,0,\dots,0)}{\|u_n\|_{C^{m-1}}} \\ &- \int_a^t \frac{(t-s)^{m-1}}{(m-1)!\, a_0(s)} \Big\{ \sum_{i=1}^{m} \int_0^1 F_i(s, \tau_n\, \xi\, u_n)\, v_n^{(i-1)}\, d\xi - \frac{\tau_n\, G(s,u_n)}{\|u_n\|_{C^{m-1}}} \Big\}\, ds \end{aligned}$$

it follows that $(v_n^{(i)})_n$ is an equicontinuous sequence for $i = 1,\dots,m-1$, hence (passing to a subsequence if necessary) we assume $v_n \to v_\infty$ in $C^{m-1}([a,b])$. As every F_i is uniformly bounded, passing to a subsequence if necessary we assume that

$$- \int_0^1 F_i\big(\cdot, \tau_n\, \xi\, u_n(\cdot)\big)\, d\xi \rightharpoonup f_i \qquad (i = 1,\dots,m)$$

weakly in $L^2([a,b])$ with $f_1,\dots,f_m$ suitable. Let $(h_n)_n$ be a bounded sequence in $L^2([a,b])$ such that $h_n \rightharpoonup h_\infty$. For every v, $w \in L^2([a,b])$ we have

$$\big(h_n\, v \,\big|\, w\big)_{L^2} = \big(h_n \,\big|\, v\, w\big)_{L^2} \rightharpoonup \big(h_\infty \,\big|\, v\, w\big)_{L^2} = \big(h_\infty\, v \,\big|\, w\big)_{L^2}$$

as $\big(\cdot \,|\, v\, w\big)_{L^2}$ is a bounded linear functional on $L^2([a,b])$. Applying this remark to our case we get

$$- \int_0^1 F_i\big(\cdot, \tau_n \xi\, u_n(\cdot)\big)\, v\, d\xi \rightharpoonup f_i\, v \qquad (i = 1,\dots,m) \tag{7}$$

weakly in $L^2([a,b])$ for every $v \in L^2([a,b])$. From the hypotheses of the theorem we have

$$\Big(L_{m,B}\, u + \sum_{i=1}^{m} F_i\big(\cdot, \tau_n\, \xi\, u_n(\cdot)\big)\, u^{(i-1)} \,\Big|\, T\, u\Big)_{L^2} \geqslant \gamma\, \|u\|_{L^2}^2$$

for all $u \in D_{m,B}$ and $n \geqslant 1$. Integrating both sides on the unit interval with respect to ξ and using Fubini's theorem we deduce

$$\Big(L_{m,B}\, u + \sum_{i=1}^{m} \int_0^1 F_i\big(\cdot, \tau_n \xi\, u_n(\cdot)\big)\, u^{(i-1)}\, d\xi \,\Big|\, T\, u\Big)_{L^2} \geqslant \gamma\, \|u\|_{L^2}^2.$$

Taking limits yields

$$\Big(L_{m,B}\, u - \sum_{i=1}^{m} f_i\, u^{(i-1)} \,\Big|\, T\, u\Big)_{L^2} \geqslant \gamma\, \|u\|_{L^2}^2 \qquad (u \in D_{m,B})$$

by virtue of (7) since the involved integral corresponds to a continuous bilinear form. Applying the uniqueness result acquired in the first part of the proof we deduce that

$$\begin{cases} L_{m,B}\, u - \sum_{i=1}^{m} f_i\, u^{(i-1)} = 0 \\ B\, u = 0 \end{cases} \tag{8}$$

has only the trivial solution. If we divide (6) by $\|u_n\|_{C^{m-1}}$ and take limits we see that v_∞ solves (8). Therefore $v_\infty = 0$, a contradiction showing that the claimed a priori bounds do exist. Let $\rho > 0$ be such an a priori bound. Now define

$$X := C^{m-1}([a,b]) \times \mathbb{R}^m$$

and $M : [0,1] \times X \to X$ by

$$\begin{aligned} M\big(\tau, (u,x)\big) := \Big(& \sum_{i=1}^{m-1} u^{(i)}(a)\, \frac{(t-a)^{(i-1)}}{i!} \\ & - \int_a^t \frac{(t-s)^{m-1}}{(m-1)!\, a_0(s)} \Big\{ f(s, \tau\, u, \ldots, \tau\, u^{(m-1)}) - \tau\, g(s, u, \ldots, u^{(m-1)}) \Big\}\, ds, \\ & \qquad x - B\, u \Big) \end{aligned}$$

where $\tau \in [0,1]$, $u \in C^{m-1}([a,b])$ and $x \in \mathbb{R}^m$. The map M is completely continuous as B is a bounded linear operator and the f_{x_i}'s are bounded. Since ρ is an a priori bound for solutions to (5), for the Leray-Schauder topological degree we have

$$\deg\big(I - M(\tau,\cdot), B(0,\rho_0), 0\big) = \text{const} \qquad (0 \leqslant \tau \leqslant 1)$$

where $B(0,\rho_0)$ is the ball of X with center the origin and radius $\rho_0 > \rho$. Since $M(0,\cdot)$ is a linear operator and $\ker\big(I - M(0,\cdot)\big)$ is finite as ρ is an a priori bound for solutions to (5) when $\tau = 0$, $I - M(0,\cdot)$ is one-to-one. Then Leray-Schauder Theorem implies that

$$\deg\big(I - M(\tau,\cdot), B(0,\rho), 0\big) = \pm 1 \qquad (0 \leqslant \tau \leqslant 1)$$

and so $I - M(1,\cdot) = 0$ is solvable, hence also the given BVP. $\square$

The next corollaries and examples aim to illustrate applications of Theorem 3.2 to distinct boundary conditions by suitable interplays between the three objects $L_{m,B}$, f_{x_i}, T appearing in its statement.

Corollary 3.3. *With m, B, $D_{m,B}$, $L_{m,B}$ as above, let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ be Carathéodory functions such that each f_x exists as a uniformly bounded Carathéodory function. If*

- *for every $u \in D_{m,B}$ and every $i \in \{1,\dots,m-1\}$ there is $t_{u,i} \in [a,b]$ such that $u^{(i)}(t_{u,i}) = 0$,*
- $\|f_x\|_\infty < \sqrt{2^m}/(b-a)^m$,
- $\lim_{\|x\|\to\infty} g(t,x)/\|x\| = 0$ *uniformly in t,*

then the BVP

$$\begin{cases} u^{(m)} + f(t,u) = g(t,u,u',\dots,u^{(m-1)}) \\ B\,u = 0 \end{cases}$$

has at least one solution [the solution is unique when g depends only on t].

A typical boundary condition allowing $D_{m,B}$ to have the property mentioned in this statement, is the Nicoletti BVP

$$u(t_1) = u'(t_2) = \cdots = u^{(m-1)}(t_m) = 0$$

where $t_1,\dots,t_m$ are arbitrary points of $[a,b]$.

Note that there is no restriction neither to m nor to $b-a$.

Proof. We start by proving the following claim:

($\star$) *Let $v \in C^{n-1}([a,b])$ be such that $v^{(n-1)}$ is absolutely continuous with $v^{(n)} \in L^2([a,b])$. If for each $i = 0,\dots,n-1$ there is $t_i \in [a,b]$ such that $v^{(i)}(t_i) = 0$, then*

$$\|v\|_{L^2}^2 \leqslant \frac{(b-a)^{2\,n}}{2^n}\,\|v^{(n)}\|_{L^2}^2.$$

To prove it, fix $i \in \{0,\dots,n-1\}$. From

$$v^{(i)}(t) = \int_{t_i}^{t} v^{(i+1)}(\xi)\,d\xi$$

we get

$$|v^{(i)}(t)| \leqslant \begin{cases} \displaystyle\int_{t_i}^{t} |v^{(i+1)}(\xi)|\, d\xi \\ \displaystyle\int_{t}^{t_i} |v^{(i+1)}(\xi)|\, d\xi \end{cases} \quad [\text{according to } t_i \leqslant t \text{ or } t \leqslant t_i]$$

$$\leqslant \|v^{(i+1)}\|_{L^2} \cdot \begin{cases} (t-t_i)^{1/2} \\ (t_i - t)^{1/2} \end{cases} \quad [\text{according to } t_i \leqslant t \text{ or } t \leqslant t_i]$$

and consequently

$$\|v^{(i)}\|_{L^2}^2 \leqslant \|v^{(i+1)}\|_{L^2}^2 \cdot \max\left\{ \int_a^b (t-a)\, dt,\ \int_a^b (b-t)\, dt \right\} \leqslant \frac{(b-a)^2}{2}\, \|v^{(i+1)}\|_{L^2}^2.$$

Iterating this inequality from $i = 0$ to $i = n-1$ yields the inequality in ($\star$).

Now we define $\tilde{f} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $L_{m,B}$, $T : D_{m,B} \to L^2([a,b])$ by

$$\tilde{f}(t, x_1, \ldots, x_m) := f(t, x_1), \qquad L_{m,B}\, u := u^{(m)}, \qquad T\, u := u^{(m)}.$$

For every $u \in D_{m,B}$ and $w^1, \ldots, w^m \in C^0([a,b], \mathbb{R}^m)$ we have

$$\begin{aligned} &\Big(L_{m,B}\, u + \sum_{r=1}^{m} \tilde{f}_{x_r}\big(\cdot, w^r(\cdot)\big) \cdot u^{(r-1)} \,\Big|\, T\, u \Big)_{L^2} \\ &\quad = \Big(u^{(m)} + f_x(\cdot\big(\cdot, w^1(\cdot)\big) \cdot u \,\Big|\, u^{(m)} \Big)_{L^2} \\ &\quad \geqslant \big\|u^{(m)}\big\|_{L^2}^2 - \big\|f_x\big\|_\infty \big\|u\big\|_{L^2} \big\|u^{(m)}\big\|_{L^2} \\ &\quad \geqslant \frac{2^m}{(b-a)^{2\,m}} \big\|u\big\|_{L^2}^2 - \frac{\sqrt{2^m}}{(b-a)^m} \big\|f_x\big\|_\infty \big\|u\big\|_{L^2}^2 \qquad [\text{by } (\star)] \\ &\quad \geqslant \text{const}\, \big\|u\big\|_{L^2}^2 \end{aligned}$$

where the constant is strictly positive and independent of u and w^r. Thus an application of Theorem 3.2 yields the desired solvability of the given BVP. □

In case of two-term boundary conditions there are better estimates, as shown by the next two corollaries.

Corollary 3.4. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ be Carathéodory functions such that f_x exists as a uniformly bounded Carathéodory function. If $m = 2\,j$ with $j \geqslant 1$, then the conjugate BVP*

$$\begin{cases} (-1)^j\, u^{(m)} + f(t,u) = g(t, u, u', \ldots, u^{(m-1)}) \\ u^{(k)}(a) = 0 \quad (k = 0, \ldots, j-1) \\ u^{(k)}(b) = 0 \quad (k = 0, \ldots, j-1) \end{cases}$$

has at least one solution provided that the two conditions

- $f_x \geqslant \text{const} =: \eta > -2^j/(b-a)^m$,
- $\lim_{\|x\|\to\infty} g(t,x)/\|x\| = 0$ *uniformly in* t,

are satisfied [*the solution is unique when* g *depends only on* t].

Proof. Let $D_{m,B}$ be the set of all functions $u \in C^{m-1}([a,b])$ such that $u^{(m-1)}$ is absolutely continuous, $u^{(m)} \in L^2([a,b])$ and u satisfies the conjugate boundary condition in the statement of our BVP. Define $\tilde{f} : [a,b]\times\mathbb{R}^m \to \mathbb{R}$ and $L_{m,B}$, $T : D_{m,B} \to L^2([a,b])$ by

$$\tilde{f}(t,x_1,\ldots,x_m) := f(t,x_1), \qquad L_{m,B}\,u := (-1)^j\,u^{(m)}, \qquad T\,u := u.$$

Performing j integrations by parts we get

$$\int_0^p u^{(m)}(t)\,u(t)\,dt = (-1)^j\,\|u^{(j)}\|_{L^2}^2 \tag{9}$$

for all $u \in D_{m,B}$. For every $u \in D_{m,B}$ and $w^1,\ldots,w^m \in C^0([a,b],\mathbb{R}^m)$ we have

$$\begin{aligned}
&\Big(L_{m,B}\,u + \sum_{r=1}^m \tilde{f}_{x_r}\big(\cdot,w^r(\cdot)\big)\cdot u^{(r-1)}\,\Big|\,T\,u\Big)_{L^2}\\
&= \Big((-1)^j\,u^{(m)} + f_x(\cdot,w^1)\cdot u\,\Big|\,u\Big)_{L^2}\\
&\geqslant \big\|u^{(j)}\big\|_{L^2}^2 - \eta\,\big\|u\big\|_{L^2}^2 && \text{[by (9) and the hypotheses]}\\
&\geqslant \frac{2^j}{(b-a)^m}\,\big\|u\big\|_{L^2}^2 - \eta\,\big\|u\big\|_{L^2}^2 && \text{[by ($\star$) in the proof of Corollary 3.3]}\\
&\geqslant \text{const}\,\big\|u\big\|_{L^2}^2 && \text{[by the hypotheses]}
\end{aligned}$$

where the constant is strictly positive and independent of u and w^r. Therefore we can apply Theorem 3.2 and have the desired existence of solutions to the given BVP. $\square$

Corollary 3.5. *Let* $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ *and* $g : [a,b]\times\mathbb{R}^m \to \mathbb{R}$ *be Carathéodory functions such that* f_x *exists as a uniformly bounded Carathéodory function. If* $m = 2\,j$ *with* $j \geqslant 1$ *and* $i \in \{0,\ldots,j-1\}$, *then the BVP*

$$\begin{cases} (-1)^{j-i}\,u^{(m)} + f(t,u^{(2\,i+1)}) = g(t,u,u',\ldots,u^{(m-1)})\\ u^{(2k)}(a) = 0 = u^{(2k)}(b) \quad (k = 0,\ldots,j-1)\end{cases}$$

has at least one solution provided that the two conditions

- $f_x \geqslant \text{const} =: \eta > -2^{j-i}/(b-a)^{2\,(j-i)}$,
- $\lim_{\|x\|\to\infty} g(t,x)/\|x\| = 0$ *uniformly in* t,

are satisfied [*the solution is unique when* g *depends only on* t].

Proof. Let $D_{m,B}$ be the set of all $u \in C^{m-1}([a,b])$ such that $u^{(m-1)}$ is absolutely continuous, $u^{(m)} \in L^2([a,b])$ and u satisfies the given boundary condition. By Rolle's theorem each $u \in D_{m,B}$ has all derivatives with odd order less than or equal to $m-1$ vanishing at some interior point of $[a,b]$, while the even order derivatives vanish at the end-point of $[a,b]$ by hypotheses. Thus $(\star)$ in the proof of Corollary 3.3 applies to the members of $D_{m,B}$.

Define $B : C^{m-1}([a,b]) \to \mathbb{R}^m$ by

$$B\,u := \big(u(a), u''(a), u^{(4)}(a), \ldots, u^{(m-2)}(a), u(b), u''(b), u^{(4)}(b), \ldots, u^{(m-2)}(b)\big),$$

next $\tilde f : [a,b] \times \mathbb{R}^m \to \mathbb{R}$ by

$$\tilde f(t, x_1, \ldots, x_m) := f(t, x_{2i+1})$$

and finally $L_{m,B}$, $T : D_{m,B} \to L^2([a,b])$ by

$$L_{m,B}\,u := (-1)^{j-i}\,u^{(m)} \qquad \text{and} \qquad T\,u := u^{(2\,i)}.$$

If $u \in D_{m,B}$, $h \geqslant 1$ and $l \in \{0, \ldots, j-1\}$ with $2\,l + h \leqslant m$, then performing h integrations by parts we get

$$\int_a^b u^{(m)}(t)\,u^{(2\,l)}(t)\,dt = (-1)^h \int_a^b u^{(m-h)}(t)\,u^{(2\,l+h)}(t)\,dt. \tag{10}$$

For every $u \in D_{m,B}$ and $w^1, \ldots, w^m \in C^0([a,b], \mathbb{R}^m)$ we have

$$\begin{aligned}
&\Big(L_{m,B}\,u + \sum_{r=1}^m \tilde f_{x_r}\big(\cdot, w^r(\cdot)\big) \cdot u^{(r-1)} \Big| T\,u\Big)_{L^2} \\
&\quad = \Big((-1)^{j-i}\,u^{(m)} + f_x(\cdot, w^{2\,i+1}_{2\,i+1}) \cdot u^{(2\,i)} \;\Big|\; u^{(2\,i)}\Big)_{L^2} \\
&\quad \geqslant \big\|u^{(j+i)}\big\|^2_{L^2} - \eta\,\big\|u^{(2\,i)}\big\|^2_{L^2} \\
&\qquad \text{[applying (10) with } h = j-i \text{ and } l = i \text{ and using the assumptions]} \\
&\quad \geqslant \frac{2^{j-i}}{(b-a)^{2\,(j-i)}}\,\big\|u^{(2\,i)}\big\|^2_{L^2} - \eta\,\big\|u^{(2\,i)}\big\|^2_{L^2} \\
&\qquad \text{[by } (\star) \text{ in the proof of Corollary 3.3 with } v = u^{(2\,i)} \text{ and } n = j-i] \\
&\quad \geqslant \text{const}\,\big\|u\big\|^2_{L^2} \\
&\qquad \text{[by the hypotheses and } (\star) \text{ in the proof of Corollary 3.3]}
\end{aligned}$$

where the constant is strictly positive and independent of u and w^r. Therefore we can apply Theorem 3.2 and have the desired existence of solutions to the given BVP. $\square$

EXAMPLE 3.6. The BVP

$$\begin{cases} -u''' + (t-1)\,u'' + \alpha\,u' - \beta\,u = g(t,u,u',u'') \\ u(2) = u'(2) = u''(1) = 0 \end{cases}$$

has at least one solution when $\alpha \geqslant 1$, $\beta > 0$ and $g : [1,2] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function which is bounded on bounded sets and satisfies

$$\lim_{\|x\|\to\infty} g(t,x)/\|x\| = 0$$

uniformly in t [the solution is unique when g depends only on t].

Proof. Define $B : C^2([1,2]) \to \mathbb{R}^3$ by $B\,u := \big(u(2), u'(2), u''(1)\big)$ and let $D_{3,B}$ be the set of all $u \in C^2([1,2])$ such that u'' is absolutely continuous with $u''' \in L^2([1,2])$ and $B\,u = 0$. Define $L_{3,B}, T : D_{3,B} \to L^2([1,2])$ and $f : [1,2] \times \mathbb{R}^3 \to \mathbb{R}$ by

$$L_{3,B}\,u := -u''' - \beta\,u, \quad T\,u := u' - u, \quad f(t,x,y,z) := \alpha\,y + (t-1)\,z$$

respectively. In view of the boundary conditions, elementary computations show that

$$\begin{aligned} &\Big(L_{3,B}\,u + f_x(\cdot,w^1)\cdot u + f_y(\cdot,w^2)\cdot u' + f_z(\cdot,w^3)\cdot u'' \,\Big|\, T\,u\Big)_{L^2} \\ &= \|u''\|_{L^2}^2 + \frac{\alpha+\beta}{2}\,u^2(1) + \beta\,\|u\|_{L^2}^2 + (\alpha - \frac{1}{2})\,\|u'\|_{L^2}^2 + \int_1^2 (t-1)\,u'^2(t)\,dt \\ &\geqslant \beta\,\|u\|_{L^2}^2 \end{aligned}$$

where $u \in D_{3,B}$ and $w^i \in C^0([1,2],\mathbb{R}^3)$ are arbitrarily chosen. This means that all hypotheses of Theorem 3.2 are fulfilled, hence the existence of a solution to the given BVP follows from it. $\square$

EXAMPLE 3.7. The BVP

$$\begin{cases} -u''' + (u')^{2k+1} - \beta\,u = g(t,u,u',u'') \\ u(b) = u'(b) = u''(a) = 0 \end{cases}$$

has at least one solution when $k \geqslant 1$, $\beta > 0$ and $g : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function which is bounded on bounded sets and satisfies

$$\lim_{\|x\|\to\infty} g(t,x)/\|x\| = 0$$

uniformly in t [the solution is unique when g depends only on t].

Proof. Define $B : C^2([a,b]) \to \mathbb{R}^3$ by $B\,u := \bigl(u(b), u'(b), u''(a)\bigr)$ and let $D_{3,B}$ be the set of all $u \in C^2([a,b])$ such that u'' is absolutely continuous with $u''' \in L^2([a,b])$ and $B\,u = 0$. Define $L_{3,B}, T : D_{3,B} \to L^2([a,b])$ and $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ by

$$L_{3,B}\,u := -u''' - \beta\,u, \quad T\,u := u', \quad f(t,x,y,z) := y^{2k+1}$$

respectively. In view of the boundary conditions, elementary computations show that

$$\begin{aligned}&\Bigl(L_{3,B}\,u + f_y(\cdot, w) \cdot u' \Big| T\,u\Bigr)_{L^2} \\ &\qquad = \|u''\|^2_{L^2} + \frac{\beta\,u^2(a)}{2} + (2\,k+1) \int_a^b \bigl(w(t)\bigr)^{2k} \cdot u'^2(t)\,dt \geqslant \|u''\|^2_{L^2} \\ &\qquad \geqslant \text{const}\ \|u\|^2_{L^2} \qquad\qquad [\text{by } (\star) \text{ in the proof of Corollary 3.3}]\end{aligned}$$

where $u \in D_{3,B}$ and $w \in C^0([a,b])$ are arbitrarily chosen. This means that Theorem 3.2 is applicable, hence the existence of a solution to the given BVP follows from it. □

Example 3.8. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be Carathéodory functions such that f_x exists as a Carathéodory function and is uniformly bounded and g is bounded on bounded sets. The BVP

$$\begin{cases} u^{(4)} + f(t,u) = g(t,u,u',u'',u''') \\ u(a) = u'(b) = u''(a) = u'''(b) = 0 \end{cases}$$

has at least one solution provided that the two conditions

- $f_x(t,x) \geqslant \text{const} > \dfrac{-4}{(b-a)^4}$ for all $(t,x) \in [a,b] \times \mathbb{R}$,
- $\lim_{\|x\| \to \infty} g(t,x)/\|x\| = 0$ uniformly in t,

are satisfied [the solution is unique when g depends only on t].

Proof. Let $B : C^4([a,b]) \to \mathbb{R}^4$ be the linear operator defined by

$$B\,u := \bigl(u(a), u'(b), u''(a), u'''(b)\bigr)$$

and let $D_{4,B}$ be the set of all $u \in C^3([a,b])$ such that u''' is absolutely continuous with $u^{(4)} \in L^2([a,b])$ and $B\,u = 0$. Set

$$L_{4,B}\,u := u^{(4)} \qquad \text{and} \qquad T\,u := u.$$

The boundary conditions imply that for every $u \in D_{4,B}$ we have

$$(L_{4,B}\,u \,|\, T\,u)_{L^2} = \|u''\|^2_{L^2},$$

hence

$$(L_{4,B}\, u \,|\, T\, u)_{L^2} \geqslant \frac{4}{(b-a)^4} \|u\|_{L^2}^2$$

by $(\star)$ in the proof of Corollary 3.3 above. At this point we apply Theorem 3.2 with $\tilde{f}(t, x_1, \ldots, x_4) := f(t, x_1)$, obtaining the desired conclusion. □

4. Higher order systems with meshed linear boundary conditions

In this section we shall use freely the following notations:

- m is a positive integer;
- $B_1, \ldots, B_N : C^{2\,m-1}([a,b]) \to \mathbb{R}$ are bounded linear operators with the following two properties:
  - (i) the differential operator $v \rightsquigarrow (-1)^m\, v^{(2\,m)}$ is symmetric on the domain

    $$D_i := \Big\{ v \in C^{2\,m-1}([a,b]) : v^{(2\,m-1)} \text{ is absolutely continuous,} \\ v^{(2\,m)} \in L^2([a,b]) \text{ and } B_i\, v = 0 \Big\}$$

    for every $i \in \{1, \ldots, N\}$ and, in addition,
  - (ii) each of the scalar BVPs

    $$\begin{cases} (-1)^m\, v^{(2\,m)} = \lambda\, v \\ B_i\, v = 0 \end{cases} \qquad (i = 1, \ldots, N)$$

    has an increasing sequence $(\lambda_{i,n})_{n=1}^{\infty}$ of eigenvalues which generates a Hilbert basis $(e_{i,n})_n$ in $L^2([a,b])$ with $(-1)^m\, e_{i,n}^{(2\,m)} = \lambda_{i,n}\, e_{i,n}$ for all i and n;
- $\mathbb{R}^{N \times N}$ is the space of $N \times N$ real matrices;
- $A \leqslant B$ means that $B - A$ is positive semidefinite whenever A, $B \in \mathbb{R}^{N \times N}$ are symmetric.

In addition, eigenvalues of compact symmetric linear operators and matrices are always ordered in the increasing way counting multiplicity.

After these preliminaries we arrive to the main result of the section:

THEOREM 4.1. *Under the above notations, define* $B : C^{2m-1}([a,b],\mathbb{R}^N) \to \mathbb{R}^N$ *by*

$$B\,u := (B_1\,u_1, \dots, B_N\,u_N)$$

and set

$$\lambda_{i,0} := -\infty \qquad\qquad (i = 1,\dots,N).$$

Let $Q \in L^2_p(\mathbb{R},\mathbb{R}^{N\times N})$ *be symmetric for a.e.* t *and let* $C^{\pm} \in \mathbb{R}^{N\times N}$ *be symmetric with eigenvalues*

$$\gamma_1^{\pm} \leqslant \cdots \leqslant \gamma_N^{\pm}$$

respectively. If

$$C^- \leqslant Q(t) \leqslant C^+ \qquad\qquad (a.e.\ \ t)$$

and there exist non-negative integers $n_1,\dots,n_N$ *such that*

$$\lambda_{i,n_i} < \gamma_i^- \leqslant \gamma_i^+ < \lambda_{i,n_i+1} \qquad\qquad (i = 1,\dots,N),$$

then the BVP

$$\begin{cases} (-1)^m\,u^{(2\,m)} = Q(t)\cdot u \\ B\,u = 0 \end{cases}$$

has only the trivial solution.

This theorem generalizes [10, Theorem 1], [6, Theorem 4.1], [4, Theorem 2.1], [3, Theorem 2.10] and [13, Lemma 1] as far as ODEs are concerned. Of course, its proof is strongly inspired by Lazer's original argument.

Proof. During the proof we shall use freely the representation

$$n = n'\,N + n'' \tag{11}$$

for every integer $n \geqslant 1$, where $n'' \in \{1,\dots,N\}$ and $n' \geqslant 0$. In addition, to simplify notations we set

$$L\,u := (-1)^m\,u^{(2\,m)}$$

and look at it as an unbounded linear operator in $L^2([a,b],\mathbb{R}^N)$ with domain

$$\mathrm{dom}(L) := D_1 \times \cdots \times D_N.$$

Let $(c_i^{\pm})_i$ be an orthonormal basis of $\mathbb{R}^N$ with

$$c_i^{\pm} := (c_{i,\,1}^{\pm},\dots,c_{i,\,N}^{\pm}) \qquad\qquad (1 \leqslant i \leqslant N)$$

a normalized eigenvector of $C^{\pm}$ corresponding to $\gamma_i^{\pm}$. Using (11), we define $e_n \in L^2([a,b],\mathbb{R}^N)$ and λ_n as follows:

$$e_n := (0,\dots,0,e^0_{n'',\,n'},0,\dots,0) \qquad \text{and} \qquad \lambda_n := \lambda^0_{n'',\,n'}$$

with $e^0_{n'',n'}$ at place n'' for every $n \geqslant 1$. Obviously the e_n's are orthogonal and the vector space spanned by them is dense in $L^2([a,b],\mathbb{R}^N)$, so that $(e_n)_n$ is a Hilbert basis for $L^2([a,b],\mathbb{R}^N)$, while

$$L\,e_n = \lambda_n\,e_n \qquad\qquad (n \geqslant 1).$$

Since $(c_i^\pm)_i$ is a Hilbert basis of $\mathbb{R}^N$, for every $t \in [a,b]$ and every n we have

$$e_n(t) = \sum_{i=1}^N \big(e_n(t)\,\big|\,c_i^\pm\big)_{\mathbb{R}^N}\,c_i^\pm = \sum_{i=1}^N c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\,c_i^\pm .$$

After these preliminaries, we start with the proof. Fix any $v \in \operatorname{dom}(L)$. Since $(e_n)_n$ is a Hilbert basis of $L^2([a,b],\mathbb{R}^N)$, we have

$$v = \sum_{n=1}^\infty \hat{v}_n\,e_n \qquad \text{with } \hat{v}_n := (v\,|\,e_n)_{L^2}.$$

Then for a.e. $t \in [a,b]$ we get

$$\begin{aligned} v(t) = \sum_{n=1}^\infty \hat{v}_n\,e_n(t) = \sum_{n=1}^\infty \hat{v}_n \sum_{i=1}^N c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\,c_i^\pm \\ = \sum_{i=1}^N \sum_{n=1}^\infty \hat{v}_n\,c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\;c_i^\pm \end{aligned} \tag{12}$$

and consequently

$$\begin{aligned} (L\,v)(t) &= \sum_{i=1}^N \sum_{n=1}^\infty \widehat{(L\,v)}_n\,c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\,c_i^\pm \\ &= \sum_{i=1}^N \sum_{n=1}^\infty (L\,v\,|\,e_n)_{L^2}\,c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\,c_i^\pm \\ &= \sum_{i=1}^N \sum_{n=1}^\infty (v\,|\,L\,e_n)_{L^2}\,c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\,c_i^\pm \\ &= \sum_{i=1}^N \sum_{n=1}^\infty \lambda_n\,\hat{v}_n\,c^\pm_{i,\,n''}\,e_{n'',\,n'}(t)\,c_i^\pm . \end{aligned} \tag{13}$$

Inspired by these formulas, we define three subspaces of $\operatorname{dom}(L)$. First we set

$$e_0 := 0 \qquad \text{and} \qquad \hat{v}_0 = 0$$

to cope with the possibility to have $n_i = 0$ for some i. The mentioned subspaces are:

$$X := \Big\{ v \in \mathrm{dom}(L) : \;\; v = \sum_{i=1}^{N} \sum_{n=n_i+1}^{\infty} \alpha_n \, c^{+}_{i,\,n''} \, e_{n'',\,n'}(\cdot)\, c^{+}_{i} \,, \qquad \alpha_n \in \mathbb{R}, \sum_{n=1}^{\infty} \alpha_n^2 < \infty \Big\},$$

$$Y := \Big\{ v \in \mathrm{dom}(L) : \;\; v = \sum_{i=1}^{N} \sum_{n=1}^{n_i} \beta_n \, c^{+}_{i,\,n''} \, e_{n'',\,n'}(\cdot)\, c^{+}_{i} \,, \;\; \beta_n \in \mathbb{R} \Big\},$$

$$Z := \Big\{ v \in \mathrm{dom}(\tilde{L}_N) : \;\; v = \sum_{i=1}^{N} \sum_{n=1}^{n_i} \gamma_n \, c^{-}_{i,\,n''} \, e_{n'',\,n'}(\cdot)\, c^{-}_{i} \,, \;\; \gamma_n \in \mathbb{R} \Big\}.$$

By Parseval's identity, every $v \in L^2([a,b], \mathbb{R}^N)$ satisfies $\sum_n \hat{v}_n^2 < \infty$. By this and (12) we have $\mathrm{dom}(L) = X \oplus Y$. Clearly Y and Z are isomorphic.

For every $v = \sum_{i} \sum_{n \geqslant n_i+1} \alpha_{v,\,n} \; c^{+}_{i,\,n''} \, e_{n'',\,n'}(\cdot)\, c^{+}_{i} \; \in \; X \setminus \{0\}$, from (12) and (13) we get:

$$\begin{aligned}
\big(L\, v - C^{+} \cdot v \,\big|\, v\big)_{L^2} &= \int_a^b \Big(\sum_{i=1}^{N} \sum_{n=n_i+1}^{\infty} \alpha_{v,\,n} \, c^{+}_{i,\,n''} \, e_{n'',\,n'}(t) \, \big\{ \lambda_n \, c^{+}_{i} - C^{+} \cdot c^{+}_{i} \big\} \\
&\qquad\qquad \Big| \sum_{h=1}^{N} \sum_{m=n_h+1}^{\infty} \alpha_{v,m} \, c^{+}_{h,\,m''} \, e_{m'',\,m'}(t) \, c^{+}_{h} \Big)_{\mathbb{R}^N} \, dt \\
&= \int_a^b \sum_{i=1}^{N} \sum_{m,\,n=n_i+1}^{\infty} \alpha_{v,n} \, \alpha_{v,\,m} \, c^{+}_{i,\,n''} \, c^{+}_{i,\,m''} \, \big\{ \lambda_n - \gamma^{+}_{i} \big\} \, e_{n'',\,n'}(t) \, e_{m'',\,m'}(t) \, dt \\
&\qquad \text{[by } C^{+} \cdot c^{+}_{i} = \gamma^{+}_{i} \, c^{+}_{i} \text{ and the orthonormality of the } c^{+}_{i}\text{'s in } \mathbb{R}^N] \\
&= \sum_{i=1}^{N} \sum_{m,\,n=n_i+1}^{\infty} \alpha_{v,\,n} \, \alpha_{v,\,m} \, c^{+}_{i,\,n''} \, c^{+}_{i,\,m''} \, \big\{ \lambda_n - \gamma^{+}_{i} \big\} \int_a^b e_{n'',\,n'}(t) \, e_{m'',\,m'}(t) \, dt \\
&\qquad \text{[since the two series converges in } L^2([a,b], \mathbb{R}^N) \text{ and the integral is a} \\
&\qquad\qquad \text{continuous linear functional on } L^2([a,b], \mathbb{R}^N)] \\
&= \sum_{i=1}^{N} \sum_{n=n_i+1}^{\infty} \alpha^2_{v,\,n} \, (c^{+}_{i,\,n''})^2 \, \big\{ \lambda_n - \gamma^{+}_{i} \big\} \\
&\qquad \text{[by the orthonormality of the } e_{i,\,n}\text{'s in } L^2([a,b])] \\
&> 0 \,,
\end{aligned}$$

the last inequality being due to two facts: first, $v \neq 0$ implies $\alpha_{v,\,n}\, c^{+}_{i,\,n''} \neq 0$ for some $n \geqslant n_i+1$ in view of the definition of X; second, $\lambda_n - \gamma_i^+ \geqslant \lambda_{n_i+1} - \gamma_i^+ > 0$ for all $n \geqslant n_i + 1$.

A similar computation shows that

$$\big(L\,v - C^- \cdot v \,\big|\, v\big)_{L^2} \leqslant \Big(\max_i \,\{\lambda_{n_i} - \gamma_i^-\}\Big) \sum_{i=1}^{N} \sum_{n=1}^{n_i} \alpha_n^2\, (c^-_{i,\,n''})^2 < 0 \qquad (v \in Z \setminus \{0\})$$

since the definition of Z guarantees that if $v \neq 0$, then $\alpha_n\, c^-_{i,\,n''} \neq 0$ for some $n \leqslant n_i$.

As

$$\big(-C^- \cdot v(t) \,\big|\, v(t)\big)_{\mathbb{R}^N} \geqslant \big(-Q(t) \cdot v(t) \,\big|\, v(t)\big)_{\mathbb{R}^N} \geqslant \big(-C^+ \cdot v(t) \,\big|\, v(t)\big)_{\mathbb{R}^N}$$

we conclude that

$$\begin{aligned} &\big(L\,v - Q(\cdot) \cdot v \,\big|\, v\big)_{L^2} > 0 \ \text{ when } \ v \in X \setminus \{0\} \\ \text{and} \qquad &\big(L\,v - Q(\cdot) \cdot v \,\big|\, v\big)_{L^2} < 0 \ \text{ when } \ v \in Z \setminus \{0\}. \end{aligned}$$

In particular, $X \cap Z = \{0\}$ and consequently $\operatorname{dom}(L) = X \oplus Z$ algebraically by virtue [10, Lemma 2]. Then [10, Lemma 1] implies that the bilinear form

$$(u, v) \rightsquigarrow \big(L\,u - Q(\cdot) \cdot u \,\big|\, v\big)_{L^2}$$

is non-degenerate on $\operatorname{dom}(L)$, i.e.

$$\big(L\,u - Q(\cdot) \cdot u \,\big|\, v\big)_{L^2} = 0 \ \text{ for all } \ v \in \operatorname{dom}(L) \qquad \Rightarrow \qquad u = 0.$$

Now, if u is a solution of $L\,u = Q(\cdot) \cdot u$, then $\big(L\,u - Q(\cdot) \cdot u \,\big|\, v\big)_{L^2} = 0$ for all $v \in \operatorname{dom}(L)$, hence $u = 0$ by the above. $\square$

The existence part of the proof of Theorem 3.2 can be easily adapted to derive from Theorem 4.1 the following corollary:

Corollary 4.2. *Assume that $A : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{N \times N}$ and $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory maps with*

$$\lim_{\|x\| \to \infty} \frac{g(t, x)}{\|x\|} = 0$$

uniformly in t. Assume further the notations of the preceding theorem.

If there exists $\rho > 0$, non-negative integers $n_1, \ldots, n_M$ and symmetric matrices $C^\pm \in \mathbb{R}^{N \times N}$ with eigenvalues $\gamma_1^\pm \leqslant \cdots \leqslant \gamma_N^\pm$ such that:

- $\lambda_{i,\,n_i} < \gamma_i^- \leqslant \gamma_i^+ < \lambda_{i,\,n_i+1}$ *for every* i*;*
- *for each* $x \in \mathbb{R}^N$ *satisfying* $\|x\| \geqslant \rho$ *we have:*
  - $*$ $A(t,x)$ *is symmetric for a.e.* t*,*
  - $*$ $C^- \leqslant A(t,x) \leqslant C^+$ *for a.e.* t*;*

then the BVP

$$\begin{cases} (-1)^m\, u^{(2\,m)} = A(t,u)\cdot u + g(t,u) \\ B\,u = 0 \end{cases}$$

has a solution.

There is another corollary:

Corollary 4.3. *Assume that* $f,\, g : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ *are Carathéodory maps such that* f_x *exists as a bounded Carathéodory map and*

$$\lim_{\|x\|\to\infty} \frac{g(t,x)}{\|x\|} = 0$$

uniformly in t*. Assume further the notations of the preceding theorem.*

If there exist $\rho > 0$*, non-negative integers* $n_1,\ldots,n_N$ *and symmetric matrices* $C^\pm \in \mathbb{R}^{N\times N}$ *with eigenvalues* $\gamma_1^\pm \leqslant \cdots \leqslant \gamma_N^\pm$ *such that:*

- $\lambda_{i,\,n_i} < \gamma_i^- \leqslant \gamma_i^+ < \lambda_{i,\,n_{i+1}}$ *for every* i*;*
- *for each* $x \in \mathbb{R}^N$ *satisfying* $\|x\| \geqslant \rho$ *we have:*
  - $*$ $f_x(t,x)$ *is symmetric for a.e.* t*,*
  - $*$ $C^- \leqslant f_x(t,x) \leqslant C^+$ *for a.e.* t*;*

then the BVP

$$\begin{cases} (-1)^m\, u^{(2\,m)} = f(t,u) + g(t,u) \\ B\,u = 0 \end{cases}$$

has a solution.

Proof. Set

$$A(t,x) := \int_0^1 f_x(t,\xi\,x)\,d\xi \qquad \text{and} \qquad G(t,x) := f(t,0) + g(t,x)$$

so that

$$f(t,x) + g(t,x) = A(t,x)\cdot x + G(t,x).$$

Let

$$\gamma := \|f_x(\cdot)\|_\infty$$

and let ε be a number in $]0,1[$ to be fixed later. For any x with $\|x\| > \rho/\varepsilon$ and any $y \in \mathbb{R}^N$ we have

$$\begin{aligned}
\big(A(t,x)\cdot y \,\big|\, y\big)_{\mathbb{R}^N} &= \int_0^1 \big(f_x(t,\xi\, x)\cdot y \,\big|\, y\big)_{\mathbb{R}^N}\, d\xi \\
&= \int_0^\varepsilon \big(f_x(t,\xi\, x)\cdot y \,\big|\, y\big)_{\mathbb{R}^N}\, d\xi + \int_\varepsilon^1 \big(f_x(t,\xi\, x)\cdot y \,\big|\, y\big)_{\mathbb{R}^N}\, d\xi \\
&\geqslant \int_0^\varepsilon \big(-\gamma\, y \,\big|\, y\big)_{\mathbb{R}^N}\, d\xi + \int_\varepsilon^1 \big(C^- \cdot y \,\big|\, y\big)_{\mathbb{R}^N}\, d\xi \\
&\qquad [\text{as } \|\xi x\| > \rho \text{ when } \xi \geqslant \varepsilon] \\
&= \big(\{-\varepsilon\,\gamma\, I + (1-\varepsilon)\, C^-\}\cdot y \,\big|\, y\big)_{\mathbb{R}^N}
\end{aligned}$$

which means

$$-\varepsilon\,\gamma\, I + (1-\varepsilon)\, C^- \leqslant A(t,x).$$

In a similar manner we get

$$A(t,x) \leqslant \varepsilon\,\gamma\, I + (1-\varepsilon)\, C^+$$

when $\|x\| > \rho/\varepsilon$. Calling

$$\gamma^\pm_{\varepsilon,1} \leqslant \cdots \leqslant \gamma^\pm_{\varepsilon,N}$$

the eigenvalues of the symmetric matrix $\pm\varepsilon\, I + (1-\varepsilon)\, C^\pm$ ordered in the increasing way counting multiplicity, we have

$$\gamma^\pm_{\varepsilon,i} = \pm\varepsilon\,\gamma + (1-\varepsilon)\,\gamma^\pm_i \qquad (i = 1,\dots,N).$$

Therefore we can select $\varepsilon > 0$ so small that

$$\lambda_{n_i} < \gamma^-_{\varepsilon,i} < \gamma^-_i \leqslant \gamma^+_i < \gamma^+_{\varepsilon,i} < \lambda_{n_{i+1}}$$

for every i. With this ε and with ρ/ε in place of ρ, the system

$$(-1)^m\, u^{(2\,m)} = A(t,u)\cdot u + G(t,u) = f(t,u) + g(t,u)$$

satisfies assumptions of the preceding corollary with each matrix $C^\pm$ replaced by $\pm\varepsilon\, I + (1-\varepsilon)\, C^\pm$. Then the solvability of our BVP follows from the preceding corollary. $\square$

The following examples involve simultaneously a two-point and a periodic boundary condition.

Example 4.4. Let $f : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable and $g : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ continuous such that

$$\lim_{\|x\|\to\infty} \frac{g(t,x)}{\|x\|} = 0$$

uniformly in t. If

- $f_2(\cdot, x)$ and $g_2(\cdot, x)$ are 2π-periodic,
- for each $i \in \{1, 2\}$ there are $\mu_i, \nu_i \in \mathbb{R}$ and $n_i \in \mathbb{N}$ such that
$$n_i^2 < \mu_i \leqslant \frac{\partial}{\partial x_i} f_i(t, x) \leqslant \nu_i < (n_i + 1)^2$$
for all t and x,
- $f_1(t, 0, x_2)$ and $f_2(t, x_1, 0)$ are uniformly bounded,

then the BVP
$$\begin{cases} -u'' = f(t, u) + g(t, u) \\ u_1(0) = 0 = u_1(2\pi) \\ u_2(0) = u_2(2\pi) \text{ and } u_2'(0) = u_2'(2\pi) \end{cases}$$
has a solution.

Proof. Defining
$$A(t, x) := \begin{pmatrix} \displaystyle\int_0^1 \frac{\partial}{\partial x_1} f_1(t, \xi x_1, x_2)\, d\xi & 0 \\ 0 & \displaystyle\int_0^1 \frac{\partial}{\partial x_2} f_2(t, x_1, \xi x_2)\, d\xi \end{pmatrix}$$
$$G(t, x) := \big(f_1(t, 0, x_2), f_2(t, x_1, 0)\big) + g(t, x),$$
the given system can be rewritten equivalently as
$$-u'' = A(t, u) \cdot u + G(t, u).$$
Clearly
$$\lim_{\|x\| \to \infty} \frac{G(t, x)}{\|x\|} = 0.$$
As is well-known, the integers n^2 are the eigenvalues of the scalar symmetric BVPs
$$\begin{cases} -v'' = \lambda v \\ v(0) = 0 = v(2\pi) \end{cases} \qquad \text{and} \qquad \begin{cases} -v'' = \lambda v \\ v \ \ 2\pi\text{-periodic} \end{cases}$$
while the eigenvalues of diagonal matrices are the diagonal entries and a symmetric matrix is positive semidefinite if and only if all of its eigenvalues are nonnegative. Thus with
$$C^- := \operatorname{diag}(\mu_1, \mu_2) \qquad \text{and} \qquad C^+ := \operatorname{diag}(\nu_1, \nu_2)$$
we are in position to apply Corollary 3.3 to Theorem 4.1 and get the solvability of our BVP. ☐

Example 4.5. Suppose that $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable, $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and that $f_1(\cdot, x)$, $g_1(\cdot, x)$ are 2π-periodic in t with

$$\lim_{\|x\|\to\infty} \frac{g(t,x)}{\|x\|} = 0$$

uniformly in t. If there exist μ_1, μ_2, ν_1, ν_2, ρ, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ such that

- $n^4 < \mu_1 \leqslant \dfrac{\partial}{\partial x_1} f_1(t,x) \leqslant \nu_1 < (n+1)^4$ whenever $\|x\| \geqslant \rho$,
- $-\infty < \mu_2 \leqslant \dfrac{\partial}{\partial x_2} f_2(t,x) \leqslant \nu_2 < 0$ whenever $\|x\| \geqslant \rho$,
- $\dfrac{\partial f_1}{\partial x_2} = \dfrac{\partial f_2}{\partial x_1}$,
- $\left|\dfrac{\partial}{\partial x_1} f_2(t,x)\right| < \alpha < \min\left\{\mu_1 - n^4,\, -\mu_2,\, (n+1)^4 - \nu_1,\, -\nu_2\right\}$ whenever $\|x\| \geqslant \rho$,

then the BVP

$$\begin{cases} u^{(4)} = f(t,u) + g(t,u) \\ u_1^{(k)}(0) = u_1^{(k)}(2\pi) \ \text{ for } \ k = 0, \dots, 3 \\ u_2''(0) = u_2'''(0) = u_2(2\pi) = u_2'(2\pi) = 0 \end{cases}$$

has a solution.

Proof. Set

$$f_{ij} := \frac{\partial f_i}{\partial x_j} \qquad (i,\, j = 1,\, 2)$$

and define the following matrices

$$C^- := \operatorname{diag}(\mu_1 - \alpha,\, \mu_2 - \alpha),\ C^+ := \operatorname{diag}(\nu_1 + \alpha,\, \nu_2 + \alpha),\ f_x := \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}.$$

[9, Corollary 7.2.3] says that a matrix $(\alpha_{ij})_{ij}$ is positive definite when $\alpha_{ii} > \sum_{i\neq j} |\alpha_{ij}|$ for all i. This criterion shows that

$$f_x - C^- \qquad \text{and} \qquad C^+ - f_x$$

are positive definite when $\|x\| \geqslant \rho$, so that

$$C^- \leqslant f_x(t,x) \leqslant C^+ \qquad (\|x\| \geqslant \rho).$$

It is well-known that the integers m^4 are the eigenvalues of the scalar symmetric BVP

$$\begin{cases} v^{(4)} = \lambda\, v \\ v \;\; 2\,\pi\text{-periodic}\,, \end{cases}$$

while the eigenvalues of the scalar symmetric BVP

$$\begin{cases} v^{(4)} = \lambda\, v \\ v''(0) = v'''(0) = v(2\,\pi) = v'(2\,\pi) = 0 \end{cases}$$

are positive (as is easily seen by integrating $v^{(4)}\, v = \lambda\, v^2$ and using the boundary conditions).

As the eigenvalues of diagonal matrices are the diagonal entries, we have

$$\gamma_1^- = \mu_2 - \alpha, \quad \gamma_1^+ = \nu_2 + \alpha, \quad \gamma_2^- = \mu_1 - \alpha, \quad \gamma_2^+ = \nu_1 + \alpha$$

hence we are in condition to apply Corollory 3.4 to Theorem 4.1 since $\gamma_1^- \leqslant \gamma_1^+ \leqslant 0$ and $n^4 < \gamma_2^- \leqslant \gamma_2^+ < (n+1)^4$. □

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