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On generating functions of extended Jacobi polynomials

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Abstract. *In this paper we have obtained some novel generating functions of $F_n(\alpha, \beta-\alpha; x)$ – a modified form of the extended Jacobi polynomial $F_n(\alpha, \beta; x)$ – by means of Weisner's group-theoretic method with the suitable interpretation of the parameter α of the polynomial under consideration. Moreover, we have shown that the generating functions involving the extended Jacobi polynomial $F_n(\alpha, \beta; x)$ derived in [2], obtained by using the same Weisner's group-theoretic method with the suitable interpretation of (α, β), can be easily obtained from our results. Finally, a group-theoretic method of obtaining general bilateral generating relation from general unilateral generating relation is also discussed.*

Keywords: Extended Jacobi polynomial, Jacobi polynomial, Laguerre polynomial, generating function, generating relation.
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1. Introduction

Various methods viz classical, theory of Lie groups (usually known as group-theoretic method) etc. are adopted in the investigation of generating functions for the special functions. But the group-theoretic method in the study of problems on generating functions is much more important than the analytic method, because of the fact that a completely new generating function can only be discovered by the group-theoretic method, whereas a relation involving generating function can be verified and the corresponding natural extension can be made by analytic method.

The group-theoretic method in the investigation of generating functions was originally introduced by L. Weisner [9, 10, 11] while investigating hypergeometric function, Hermite function and Bessel function. Weisner's method is lucidly presented in the monograph "Obtaining generating functions" written by E. B. McBride [5].

The unified presentation for the classical orthogonal polynomials was originally introduced by I. Fujiwara [3]. This was subsequently designated by N. K. Thakare [8] as extended Jacobi polynomial, denoted by $F_n(\alpha, \beta; x)$ and defined by

$$F_n(\alpha,\beta;x) = \frac{(-1)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\left(\frac{\lambda}{b-a}\right)^n \times D^n\left[(x-a)^{n+\alpha}(b-x)^{n+\beta}\right], \qquad D \equiv \tfrac{d}{dx},$$

satisfying the following ordinary differential equation:

$$(x-a)(b-x)\tfrac{d^2y}{dx^2} + \left\{(\alpha+1)(b-x) - (\beta+1)(x-a)\right\}\tfrac{dy}{dx} + n(1+\alpha+\beta+n)y = 0\,.$$

The aims of the present article are the following:

(i) to investigate $F_n(\alpha,\beta-\alpha;x)$, a modified form of the extended Jacobi polynomial, satisfying the following ordinary differential equation:

$$(x-a)(b-x)\tfrac{d^2y}{dx^2} + \left\{(\alpha+1)(b-x) - (\beta-\alpha+1)(x-a)\right\}\tfrac{dy}{dx} + n(1+\beta+n)y = 0 \quad (1)$$

by the Weisner's group-theoretic method, with the single interpretation of the parameter α of the polynomial, for obtaining some novel generating functions,

(ii) to show that the generating functions derived in [2], while investigating $F_n(\alpha,\beta;x)$ by the Weisner's group-theoretic method with the double interpretation of the parameters (α,β), can be immediately obtained from our results by the mere replacement of β by $\beta+\alpha$,

(iii) to obtain some novel results involving Laguerre and Jacobi polynomials as special cases of our results, and finally

(iv) to discuss a group-theoretic method of obtaining general bilateral generating relation from the general unilateral generating relation.

For previous works related to extended Jacobi polynomial one can see [1, 6, 7].

2. Group-theoretic discussion and Lie algebra

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, α by $y\frac{\partial}{\partial y}$ and y by $v(x,y)$ in (1), we get the following partial differential equation:

$$\left[(x-a)(b-x)\tfrac{\partial^2}{\partial x^2} + \left\{\left(y\tfrac{\partial}{\partial y}+1\right)(b-x) - \left(\beta - y\tfrac{\partial}{\partial y}+1\right)(x-a)\right\}\tfrac{\partial}{\partial x} + n(1+\beta+n)\right]v(x,y) = 0. \quad (2)$$

Thus $v_1(x,y) = F_n(\alpha, \beta-\alpha; x)y^\alpha$ is a solution of (2), since $F_n(\alpha, \beta-\alpha; x)$ is a solution of (1).

We now define the infinitesimal operators A_i $(i = 1, 2, 3)$,

$$A_i = A_i^{(1)} \tfrac{\partial}{\partial x} + A_i^{(2)} \tfrac{\partial}{\partial y} + A_i^{(0)}, \qquad i = 1, 2, 3$$

as follows:

$$\begin{aligned} A_1 &= y \tfrac{\partial}{\partial y}\,, \\ A_2 &= (x-b)y \tfrac{\partial}{\partial x} - y^2 \tfrac{\partial}{\partial y} + \beta y\,, \\ A_3 &= (x-a)y^{-1} \tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial y}\,, \end{aligned}$$

such that

$$\begin{aligned} A_1(F_n(\alpha, \beta-\alpha; x)y^\alpha) &= \alpha F_n(\alpha, \beta-\alpha; x)y^\alpha\,, \\ A_2(F_n(\alpha, \beta-\alpha; x)y^\alpha) &= (\beta-\alpha+n)F_n(\alpha+1, \beta-\alpha-1; x)y^{\alpha+1}\,, \\ A_3(F_n(\alpha, \beta-\alpha; x)y^\alpha) &= (n+\alpha)F_n(\alpha-1, \beta-\alpha+1; x)y^{\alpha-1}. \end{aligned} \tag{3}$$

We now proceed to find the commutator relations. Using the notation:

$$[A, B]u = (AB - BA)u,$$

we have

$$[A_1, A_2] = A_2, \qquad [A_1, A_3] = -A_3, \qquad [A_2, A_3] = 2A_1 - \beta. \tag{4}$$

From the above commutator relations, we state the following theorem:

Theorem 2.1. *The set of operators $\{1, A_i : i = 1, 2, 3\}$, where 1 stands for the identity operator, generates a Lie algebra $\boldsymbol{L}$.*

It is easy to verify that the partial differential operator L, given by

$$\begin{aligned} Lv = (x-a)(b-x)\frac{\partial^2 v}{\partial x^2} &+ (b-x)y\frac{\partial^2 v}{\partial y \partial x} + (b-x)\frac{\partial v}{\partial x} \\ &- (\beta+1)(x-a)\frac{\partial v}{\partial x} + (x-a)y\frac{\partial^2 v}{\partial y \partial x} + n(1+\beta+n)v, \end{aligned}$$

can be expressed in terms of A_i $(i = 1, 2, 3)$ as follows:

$$L = -A_2A_3 - A_1^2 + (1+\beta)A_1 + n(1+\beta+n). \tag{5}$$

From (4) and (5), one can easily verify that L commutes with each $A_i (i = 1, 2, 3)$ i.e.,

$$[L, A_i] = 0, \qquad i = 1, 2, 3. \tag{6}$$

The extended form of the groups generated by $A_i (i = 1, 2, 3)$ are as follows:

$$\begin{aligned} e^{a_1 A_1} f(x,y) &= f(x, e^{a_1} y), \\ e^{a_2 A_2} f(x,y) &= (1+a_2 y)^{\beta} f\left(x + (x-b)a_2 y, \frac{y}{1+a_2 y}\right), \\ e^{a_3 A_3} f(x,y) &= f\left(x + a_3 \frac{x-a}{y}, y + a_3\right). \end{aligned} \tag{7}$$

Therefore, from above, we get

$$\begin{aligned} &e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1} f(x,y) = \left\{1 + a_2 y \left(1 + \tfrac{a_3}{y}\right)\right\}^{\beta} \\ &\times f\left(\left\{x + \tfrac{a_3}{y}(x-a)\right\}\left\{1 + a_2 y \left(1 + \tfrac{a_3}{y}\right)\right\} - b a_2 y \left(1 + \tfrac{a_3}{y}\right), \frac{y\left(1+\frac{a_3}{y}\right)}{\left\{1 + a_2 y \left(1 + \frac{a_3}{y}\right)\right\}}\right). \end{aligned} \tag{8}$$

3. Generating functions

From (2), $v(x,y) = F_n(\alpha, \beta-\alpha; x) y^{\alpha}$ is a solution of the system:

$$\begin{aligned} Lv &= 0 \\ (A_1 - \alpha) v &= 0. \end{aligned}$$

From (6), we easily get

$$SL\left(F_n(\alpha, \beta-\alpha; x) y^{\alpha}\right) = LS(F_n(\alpha, \beta-\alpha; x) y^{\alpha}) = 0,$$

where

$$S = e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1}.$$

Therefore, the transformation $S(F_n(\alpha, \beta-\alpha; x) y^{\alpha})$ is also annulled by L.

Putting $a_1 = 0$ and replacing $f(x,y)$ by $F_n(\alpha, \beta-\alpha; x) y^{\alpha}$ in (8), we get

$$\begin{aligned} e^{a_3 A_3} e^{a_2 A_2} (F_n(\alpha, \beta-\alpha; x) y^{\alpha}) &= \left\{1 + a_2 y \left(1 + \tfrac{a_3}{y}\right)\right\}^{\beta-\alpha} \left(1 + \tfrac{a_3}{y}\right)^{\alpha} y^{\alpha} \\ \times F_n &\left(\alpha, \beta-\alpha; \left\{x + \tfrac{a_3}{y}(x-a)\right\}\left\{1 + a_2 y \left(1 + \tfrac{a_3}{y}\right)\right\} - b a_2 y \left(1 + \tfrac{a_3}{y}\right)\right). \end{aligned} \tag{9}$$

On the other hand

$$\begin{aligned} &e^{a_3 A_3} e^{a_2 A_2} (F_n(\alpha, \beta-\alpha; x) y^{\alpha}) \\ &\quad = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-a_2 y)^k}{k!} \frac{(-a_3/y)^p}{p!} (-\alpha-n-k)_p (\alpha-\beta-n)_k \\ &\qquad\qquad \times F_n(\alpha+k-p, \beta-\alpha+p-k; x) y^{\alpha}. \end{aligned} \tag{10}$$

Equating (9) and (10), we get

$$\begin{aligned}
&\left\{1+a_2y\left(1+\tfrac{a_3}{y}\right)\right\}^{\beta-\alpha}\left(1+\tfrac{a_3}{y}\right)^{\alpha}\\
&\quad\times F_n\left(\alpha,\beta-\alpha;\left\{x+\tfrac{a_3}{y}(x-a)\right\}\left\{1+a_2y\left(1+\tfrac{a_3}{y}\right)\right\}-ba_2y\left(1+\tfrac{a_3}{y}\right)\right)\\
&\qquad=\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-a_2y)^k}{k!}\frac{(-a_3/y)^p}{p!}(-\alpha-n-k)_p(\alpha-\beta-n)_k\\
&\qquad\qquad\times F_n(\alpha+k-p,\beta-\alpha+p-k;x).
\end{aligned}\tag{11}$$

The above generating relation does not seem to appear in the earlier works and this, in turn, yields some particular novel generating relations for different values of a_2 and a_3.

Before discussing the particular cases of (11), it may be pointed out that the operators A_2 and A_3 being non-commutative, the relation (11) will change if we change the order of the element $e^{a_3A_3}e^{a_2A_2}$. In fact, by this change, we get the following generating relation

$$\begin{aligned}
&(1+a_2y)^{\beta-\alpha}\left(1+\tfrac{a_3}{y}(1+a_2y)\right)^{\alpha}\\
&\quad\times F_n\left(\alpha,\beta-\alpha;\{x+(x-b)a_2y\}\left\{1+\tfrac{a_3}{y}(1+a_2y)\right\}-\tfrac{aa_3}{y}(1+a_2y)\right)\\
&\qquad=\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-a_2y)^k}{k!}\frac{(-a_3/y)^p}{p!}(-n-\alpha)_p(\alpha-\beta-p-n)_k\\
&\qquad\qquad\times F_n(\alpha+k-p,\beta-\alpha-k+p;x).
\end{aligned}\tag{12}$$

The above pair of generating relations, given by (11) and (12) does not seem to appear before.

We now consider the following particular cases of the relation (11):

Case 1. Putting $a_3=0$ and then replacing $(-a_2y)$ by t in (11), we get

$$\begin{aligned}
&(1-t)^{\beta-\alpha}F_n(\alpha,\beta-\alpha;x-(x-b)t)\\
&\qquad=\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha-\beta-n)_kF_n(\alpha+k,\beta-\alpha-k;x).
\end{aligned}\tag{13}$$

Case 2. Putting $a_2=0$ and replacing $(-a_3/y)$ by t in (11), we get

$$\begin{aligned}
&(1-t)^{\alpha}F_n(\alpha,\beta-\alpha;x-(x-a)t)\\
&\qquad=\sum_{p=0}^{\infty}\frac{(-n-\alpha)_p}{p!}F_n(\alpha-p,\beta-\alpha+p;x)t^p.
\end{aligned}\tag{14}$$

Case 3. Putting $a_2 = \frac{1}{w}, a_3 = 1, y = t$ in (11), we get

$$\begin{aligned}
&\left\{1 + \tfrac{1}{wt}(1+t)\right\}^{\beta-\alpha}(1+t)^{\alpha} \\
&\quad \times F_n\left(\alpha, \beta-\alpha; \{x + t(x-a)\}\left\{1 + \tfrac{1}{wt}(1+t)\right\} - \tfrac{b}{wt}(1+t)\right) \\
&\quad = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-1/w)^k}{k!}\frac{(-1)^p}{p!}(-\alpha-n-k)_p(\alpha-\beta-n)_k \\
&\qquad \times F_n(\alpha+k-p, \beta-\alpha+p-k; x)t^{p-k}. \qquad (15)
\end{aligned}$$

The above results do not seem to appear in the earlier works.

Here we would like to remark that if one investigates $F_n(\alpha-\beta, \beta; x)$ in place of $F_n(\alpha, \beta-\alpha; x)$ by the same method of Weisner with the interpretation of β, the same results given by (13)–(15) will be obtained by virtue of the following relation

$$F_n(\alpha, \beta; a+b-x) = (-1)^n F_n(\beta, \alpha; x). \qquad (16)$$

Furthermore, it may be of interest to note that the mere replacement of β by $\beta + \alpha$ on both sides of (13)–(15) yields all the results obtained in [2] while investigating generating functions of $F_n(\alpha, \beta; x)$ by Weisner's group theoretic method with the double interpretation of the parameters (α, β) simultaneously.

Thus, while investigating $F_n(\alpha, \beta; x)$ for obtaining the results derived in [2] by Weisner's group theoretic method, we observe that the double interpretation (of the parameters (α, β)) seems to be a little bit harder as well as lengthy and may be replaced by the single interpretation of any of the parameters α or β while investigating $F_n(\alpha, \beta-\alpha; x)$ or $F_n(\alpha-\beta, \beta; x)$ by the same technique of Weisner for obtaining the same results derived in [2] in a straightforward way - making the original problem simple and easier.

Some special cases:

Special case 1. Putting $a = 0, \lambda = 1$ and $\beta = b$ in (13) and (14) and then simplifying and finally taking limit as $b \to \infty$, we get the following results on generating functions involving Laguerre polynomials:

$$e^{t_1} L_n^{(\alpha)}(x - t_1) = \sum_{k=0}^{\infty} L_n^{(\alpha+k)}(x)\frac{t_1^k}{k!},$$

$$(1-t)^{\alpha} L_n^{(\alpha)}(x(1-t)) = \sum_{p=0}^{\infty} \frac{(-n-\alpha)_p}{p!} L_n^{(\alpha-p)}(x)t^p.$$

Special case 2. Putting $-a = b = 1$ and $\lambda = 1$ in (13), (14) and (15), we get the following results involving Jacobi polynomials:

$$(1-t)^{\alpha-\beta} P_n^{(\alpha-\beta,\beta)}(x+(1-x)t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(\beta-\alpha-n)_k P_n^{(\alpha-\beta-k,\beta+k)}(x), \quad (17)$$

$$(1-t)^{\beta} P_n^{(\alpha-\beta,\beta)}(x-(1+x)t) = \sum_{p=0}^{\infty} \frac{(-n-\beta)_p}{p!} P_n^{(\alpha-\beta+p,\beta-p)}(x)t^p, \quad (18)$$

$$\left\{1+\tfrac{1}{wt}(1+t)\right\}^{\alpha-\beta}(1+t)^{\beta} P_n^{(\alpha-\beta,\beta)}\left(\{x(1+t)+t\}\left\{1+\tfrac{1}{wt}(1+t)\right\}-\tfrac{1}{wt}(1+t)\right)$$
$$= \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \tfrac{(-1/w)^k}{k!}\tfrac{(-1)^p}{p!}(-\beta-n-k)_p(\beta-\alpha-n)_k P_n^{(\alpha-\beta+p-k,\beta+k-p)}(x)t^{p-k}. \quad (19)$$

Now replacing α by $\alpha+\beta$ in (17), (18) and (19), we get the results (1.2), (1.3) and the correct version of (1.4) found in [4] while investigating Jacobi polynomials by Weisner's method with the double interpretation of (α, β).

4. Transformation of general unilateral generating relation into a general bilateral generating relation

Let us first consider a unilateral generating relation of the form:

$$G(x,w) = \sum_{\alpha=0}^{\infty} a_\alpha F_n(\alpha, \beta-\alpha; x) w^\alpha. \quad (20)$$

Now replacing w by wyv in (20) and then operating $exp(wR)$ on both sides of the derived equation, where $R = (x-b)y\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + \beta y$, and finally using (3) and (7) we obtain

$$(1+wy)^\beta G\left(x+(x-b)wy, \frac{wyv}{1+wy}\right)$$
$$= \sum_{\alpha=0}^{\infty}(-wy)^\alpha\left(\sum_{k=0}^{\alpha} a_k \frac{(k-\beta-n)_{\alpha-k}}{(\alpha-k)!}(-v)^k\right)F_n(\alpha,\beta-\alpha;x). \quad (21)$$

Now replacing wy by $(-t)$ and v by $(-v)$ on both sides of (21), we obtain

$$(1-t)^\beta G\left(x-(x-b)t, \frac{tv}{1-t}\right) = \sum_{\alpha=0}^{\infty}(t)^\alpha \sigma_\alpha(v) F_n(\alpha,\beta-\alpha;x)$$
$$\text{where} \quad \sigma_\alpha(v) = \sum_{k=0}^{\alpha} a_k \frac{(k-\beta-n)_{\alpha-k}}{(\alpha-k)!}(v)^k. \quad (22)$$

From the above discussion, we can state the following theorem.

THEOREM 4.1. *If there exists a unilateral generating relation of the form:*

$$G(x,w)=\sum_{\alpha=0}^{\infty}a_\alpha F_n(\alpha,\beta-\alpha;x)w^\alpha,$$

then

$$(1-t)^\beta G\Big(x-(x-b)t,\frac{tv}{1-t}\Big)=\sum_{\alpha=0}^{\infty}F_n(\alpha,\beta-\alpha;x)\sigma_\alpha(v)t^\alpha,$$

$$\text{where}\quad \sigma_\alpha(v)=\sum_{k=0}^{\alpha}a_k\frac{(k-\beta-n)_{\alpha-k}}{(\alpha-k)!}(v)^k.$$

REMARK 4.2. Here it is easy to observe that the above theorem is not only very important but also of general interest for its usefulness in generalizing the known results.

In fact, the importance of the above theorem lies in the fact that whenever one knows a unilateral generating relation of the form (20) for a particular value of a_α, the corresponding bilateral generating relation can at once be written down from (22). Thus one can get a large number of bilateral generating relations from (22) by attributing different values to a_α in (20).

Here we would like to mention that the above theorem, by virtue of the symmetry relation (16), yields the following analogous result.

THEOREM 4.3. *If there exists a unilateral generating relation of the form:*

$$G(x,t)=\sum_{\beta=0}^{\infty}a_\beta F_n(\alpha-\beta,\beta;x)t^\beta,$$

then

$$(1-t)^\alpha G\Big(x+(a-x)t,\frac{tv}{1-t}\Big)=\sum_{\beta=0}^{\infty}F_n(\alpha-\beta,\beta;x)\sigma_\beta(v)t^\beta,$$

$$\text{where}\quad \sigma_\beta(v)=\sum_{k=0}^{\beta}a_k\frac{(k-\alpha-n)_{\beta-k}}{(\beta-k)!}(v)^k.$$

4.0.1. Some special cases of Theorem 4.1

Case 1. Putting $a=0,\beta=b,\lambda=1$ in Theorem 4.1 and then taking limit when $b\to\infty$, we get the following theorem involving Laguerre polynomials:

THEOREM 4.4. *If there exists a unilateral generating relation of the form*

$$G(x,w)=\sum_{\alpha=0}^{\infty}\frac{a_\alpha}{\alpha!}L_n^{(\alpha)}(x)w^\alpha,$$

then

$$e^{t_1}G(x-t_1,t_1w)=\sum_{\alpha=0}^{\infty}\frac{{t_1}^\alpha}{\alpha!}L_n^{(\alpha)}(x)\sigma_\alpha(w)$$

$$\text{where}\quad \sigma_\alpha(w)=\sum_{k=0}^{\alpha}a_k\binom{\alpha}{k}w^k.$$

Case 2. Putting $-a=b=1,\lambda=1$ in Theorem 4.1, we get the following theorem involving Jacobi polynomials:

THEOREM 4.5. *If*

$$G(x,w)=\sum_{\beta=0}^{\infty}a_\beta P_n^{(\alpha-\beta,\beta)}(x)w^\beta,$$

then

$$(1-t)^\alpha G\left(x+(1-x)t,\frac{tv}{1-t}\right)=\sum_{\beta=0}^{\infty}t^\beta P_n^{(\alpha-\beta,\beta)}(x)\sigma_\beta(v)$$

$$\text{where}\quad \sigma_\beta(v)=\sum_{k=0}^{\beta}a_k\frac{(k-\alpha-n)_{\beta-k}}{(\beta-k)!}v^k.$$

The three Theorems 4.3, 4.4, 4.5 are of the same importance as the previous Theorem 4.1, as mentioned in Remark 4.2.

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