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Topology of the space of measure-preserving transformations of the circle

Houssam Boukhecham and Hamza Ounesli

Abstract. *In this paper we prove that the space of circle expanding maps of degree 2 preserving Lebesgue measure is an arc-connected space homeomorphic to an infinite-dimensional Lie group whose fundamental group is $\mathbb{Z}$. The techniques involved in the proof are rather unexpected and lead to a formulation of a conjecture generalizing this result to higher dimensional infra-nilmanifolds.*

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1. Introduction and statement of results

One of the classical problems in topology, dynamics, and geometry is studying properties of the group of *diffeomorphisms* of a closed manifold M, preserving a given smooth volume form ω. Questions about the topology of this space, dynamics-rigidity phenomenons, and algebraic properties can be addressed. There has been extensive work in this direction, as in [3, 6]. In particular, in [4] J.Moser has shown that these groups are locally arc-connected. In this paper, we generalize Moser's result on arc-connectedness to a space of non-invertible volume preserving maps in dimension 1. More precisely, we consider our manifold to be the circle, and we study the space of C^1 orientation preserving uniformly expanding maps of degree 2, preserving the natural volume form on the circle i.e Lebesgue measure. We denote this space by Λ_{Leb}. Our results suggest that the facts known for volume preserving *diffeomorphism* groups can be extended to spaces of non-invertible volume preserving maps. The only topological information we know about Λ_{Leb} is that it is of first category in the space $C^1(\mathbb{S}^1, \mathbb{S}^1)$ of all C^1 maps of the circle, this was shown in [2].

Our result shows that Λ_{Leb} is indeed arc-connected, with fundamental group $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$. Moreover, we show that this space is homeomorphic to a natural infinite dimensional Lie group.

Remark 1.1. We always denote by $D_+(\mathbb{S}^1)$ the group of circle diffeomorphisms which preserves the orientation and $D_+(I, J)$ for the space of orientation preserving interval diffeomorphisms and $D_{+,exp}(I,J)$ for the expanding ones (i.e $f' \geq \gamma > 1$). $\mathbb{T}^2$ denotes the torus $S^1 \times S^1$.

THEOREM 1.2. *The space Λ_{Leb} endowed with the C^1-topology is homeomorphic to $\mathbb{T}^2 \setminus diag(\mathbb{T}^2) \times D_+(\mathbb{S}^1, 0$ is fixed), in particular, Λ_{Leb} is arc-connected, and $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$.*

This theorem, as mentioned before, is an extension of Moser result on local arc-connectedness of the group of volume preserving diffeomorphisms. However, our result extends it only in dimension one. Intuitively the result says that for any two Lebesgue preserving uniformly expanding circle maps f, g there exists a deformation between each other $\gamma(t) : [0,1] \to \Lambda_{Leb}$ which preserves Lebesgue along the deformation. The fact that the fundamental group is isomorphic to $\mathbb{Z}$ signifies that any deformation is generated by a fixed deformation in Λ_{Leb}. On the other hand, we show that the space Λ_{Leb} is huge in a sense albeit being meagre in $C^1(\mathbb{S}^1, \mathbb{S}^1)$, as we have partially proven in [5]. We conjecture that our result can be extended to arbitrary dimensions.

CONJECTURE 1.3. *Let (M, g) be a closed Riemannian manifold and ω its volume form. The space $\Lambda^r_\omega(M)$ of C^1 expanding r-folds of M, preserving the volume form, is locally arc-connected.*

2. Proof of Theorem 1.2

2.1. Uniformly expanding circle maps

Denote by $E^1(\mathbb{S}^1)$ the space of uniformly expanding maps of the circle, and by Λ_{Leb} the sub-space of maps f of degree 2 and preserving the Lebesgue measure λ (i.e $f_*\lambda = \lambda$) and the orientation. We endow this space with the C^1-topology. The circle is seen as the natural quotient space $[0,1]/(0 \sim 1)$. Circle maps of degree 2 which are orientation preserving, up to conjugacy with a rotation, can be regarded as interval maps with two full branches.

We recall that uniformly expanding circle maps of degree 2 have two main characteristics: a unique fixed point $p \in S^1$ and two branch-arcs determined by two distinct points $x_1 \neq x_2 \in S^1$.

2.2. The transfer operator

Let $f \in E^1(\mathbb{S}^1)$. We define the transfer operator P associated to f, and acting on $L^1_\lambda(\mathbb{S}^1)$ as: if $h \in L^1_\lambda(\mathbb{S}^1)$ then:

$$Ph = \frac{d\big(f_*\mu_h\big)}{d\lambda}\,, \tag{1}$$

where $\mu_h = h \cdot d\lambda$. The transfer operator provides the density of the push-forward of a given absolutely continuous measure with respect to Lebesgue. The transfer operator for maps of degree 2 has an explicit formula:

$$Ph(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{f'(y)}. \tag{2}$$

The main property of this operator is the following Folklore proposition:

PROPOSITION 2.1. *The set of absolutely continuous invariant measures of f corresponds to the non-negative fixed points of the operator P.*

2.3. Proof of Theorem 1.2

The proof of the theorem will be based on the following proposition, which we consider to be of independent interest:

PROPOSITION 2.2. *Let $a \in (0,1)$ and $f_1 : [0,a] \to [0,1]$ be an expanding C^1-diffeomorphism, then there exists a unique extension of f_1 to a Lebesgue-preserving full branch expanding transformation of the unit interval.*

Proof. Consider the differential equation

$$f_2'(x) = \frac{f_1'\Big(f_1^{-1}(f_2(x))\Big)}{f_1'\Big(f_1^{-1}(f_2(x))\Big) - 1}, \; x \in [a,1]. \tag{3}$$

Since f_1 is C^1, by Peano's existence theorem the Cauchy problem with the initial condition $f_2(a) = 0$ admits a maximal solution f_2 defined on the interval $[a,1]$. Let's show that f_2 maps diffeomorphically onto $[0,1]$. Notice that $f_2'(x) > 1$ for all $x \in [a,1]$, therefore it only remains to show that $f_2(1) = 1$. Assume that $f_2(1) < 1$ and consider $I = [0,b]$ where $b = f_2(1)$. We notice that for every $y \in I$ we get:

$$\frac{1}{f_1'\big(f_1^{-1}(y)\big)} + \frac{1}{f_2'\big(f_2^{-1}(y)\big)} = 1. \tag{4}$$

This implies in particular:

$$\begin{aligned} f_\star\lambda([0,b]) &= \lambda(f_1^{-1}([0,b])) + \lambda(f_2^{-1}([0,b])) \\ &\qquad - \int_{[0,b]} \frac{1}{f_1'\big(f_1^{-1}(y)\big)} + \frac{1}{f_2'\big(f_2^{-1}(y)\big)} d\lambda = \lambda([0,b]) \end{aligned}$$

On the other hand, we know that $f_\star\lambda([b,1]) = \lambda(f_1^{-1}([b,1])) < \lambda([b,1])$ which implies that $\lambda(f_\star([0,1])) < \lambda([0,1])$, resulting in a contradiction. The case

$b > 1$ results in the same contradiction, hence $b = 1$, this implies in particular that (4) is satisfied for every $x \in [0, 1]$ and hence the Lebesgue measure is preserved. Since $b = 1$, we also get that (4) is satisfied on all the interval and hence f preserves λ.

Uniqueness cannot be deduced directly from the equation (3), because Peano's existence theorem provides only existence, we will deduce it using the fact that the solution preserves λ. Let $f, g : [0, 1] \to [0, 1]$ be two full branch interval maps which preserve Lebesgue measure, assume they have the same first branches (i,e $f_1 = g_1$) on an interval $[0, a]$, then for every $y \in [0, 1]$ we have

$$\lambda([0, y]) = \lambda(f^{-1}([0, y])) = \lambda(g^{-1}([0, y])),$$

which implies by assumption that

$$\lambda([a, f_2^{-1}(y)]) = \lambda([a, g_2^{-1}(y)]),$$

this implies that $f_2^{-1}(y) = g_2^{-1}(y)$, thus $f = g$. □

LEMMA 2.3. *The extension of an expanding diffeomorphism* $f_1 : [0, a] \to [0, 1]$ *to a full branch interval map preserving Lebesgue is a* C^1 *circle map, if and only if the following holds:*

$$f_1'(0) = \frac{f_1'(a)}{f_1'(a) - 1}. \tag{5}$$

Proof. This is because for a full branch map to lift to a circle map, the derivatives at the end points must coincide, as well as the left and right derivatives at the point a, and so by equation (4), we need (5) to hold. □

We will use the previous results to show that Λ_{Leb} is arc connected.

COROLLARY 2.4. Λ_{Leb} *is arc-connected.*

Proof. Let f be the doubling map of the circle, and $g \in \Lambda_{Leb}$. Up to composing g with a rotation, we can assume that g and f have the same fixed point 0. Denote by x_g the point in $\mathbb{S}^1$ such that $\int_0^{x_g} g'(t)\, dt = 1$, we will construct a homotopy between g and $\tilde{g}$ in Λ_{Leb}, such that $x_{\tilde{g}} = \frac{1}{2}$. Without loss of generality, let us assume that $x_g > \frac{1}{2}$. For $x_g > \epsilon > \frac{1}{2}$, translate horizontally the graph of $g|_{(\epsilon, x_g)}$ to $(\frac{1}{2} - x_g + \epsilon, \frac{1}{2})$ by a linear homotopy $T(t, \cdot)$. Now let z close enough to 0, more precisely, choose $z < \frac{1}{2} - x_g + \epsilon$. Construct a homotopy $H(t, x)$ as follows: for every t define $H(t, \cdot)|_{[0,z]} = g$ and $H(t, \cdot)|_{[\epsilon - t, x_g - t]} = T(t, \cdot)$, and for every t extend it in a C^1 and expanding way to the whole interval $[0, x_g - t]$, as represented on Figure 1. This yields a homotopy between g and $\tilde{g}$ in Λ_{Leb}, because condition (5) is satisfied for every t, also $\tilde{g}$ satisfies $x_{\tilde{g}} = \frac{1}{2}$.

The second step is to construct an appropriate homotopy between $\tilde{g}$ and f. This is straight forward by considering a continuous family of expanding C^1

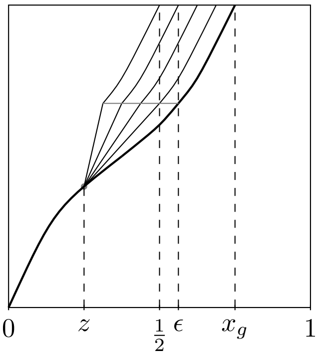

Figure 1: A representation of the homotopies H and T.

maps $(h_c : [0, \frac{1}{2}] \to [0, 1])_{c\in[2,g'(0)]\text{or}[g'(0),2]}$ with $h'_c(0) = c$ and $h'_c(\frac{1}{2}) = \frac{c}{c-1}$. Notice in this case that $\tilde{g}|_{[0,\frac{1}{2}]}$ is homotopic to $h_{g'(0)}$ by simply taking $H(t, x) = t\tilde{g}|_{[0,\frac{1}{2}]}(x)+(1-t)h_{g'(0)}(x)$ and same for $f|_{[0,\frac{1}{2}]}$ and h_2 by $G(t, x) = tf|_{[0,\frac{1}{2}]}(x)+ (1-t)h_2(x)$, this homotopies satisfy (5), and so they extend to a homotopy in Λ_{Leb} between $\tilde{g}$ and f by concatenating the extension of the homotopy H with the extension of the family $(h_c)_c$ and the extension of G in Λ_{Leb}, this finishes the proof of arc-connectedness. □

Proposition 2.5. *The space Λ_{Leb} is homeomorphic to the infinite dimensional Lie group $\mathbb{T}^2 \setminus diag(\mathbb{T}^2) \times D(\mathbb{S}^1, 0)$.*

Proof. Let Γ be the space

$$\Gamma = \bigcup_{0\leq x-y<1} \left\{ f \in \mathrm{D}^1_{+,exp}([x, y], [0, 1]) \text{ such that } f'(x) = \frac{f'(y)}{f'(y) - 1} \right\}.$$

Proposition 2.2 results naturally in a map $\mathcal{F}$

$$\mathcal{F} : \Gamma \to \Lambda_{Leb},$$

defined by sending an element $f \in \Gamma$ to a Lebesgue preserving circle map, by extension after translating $[x, y]$ to $[0, x - y]$, and translating the solution back.

Proposition 2.6. *The map $\mathcal{F}$ is a homeomorphism (in the C^1-topology).*

Proof. By proposition 2.2 and Lemma 2.3, the map is well defined and for every $f \in \Gamma$, there exists a unique extension of f to a circle expanding map preserving Lebesgue measure. Continuity follows from the fact that the unique solutions to a continuous family of Cauchy problems $(ODE_t)_{t\in I}$, with a continuous family of initial conditions form a continuous family $(f_t)_{t\in I}$ in the C^1-topology and this shows that $\mathcal{F}$ is a continuous injection.

The image of the operator $\mathcal{F}$ covers all Lebesgue preserving circle maps f, whose fixed point p_f is inside the branch interval $[x, y]$ of the specific element, hence it is surjective, the inverse is clearly continuous and hence is a homeomorphism. □

To finish the proof, notice that Γ is homeomorphic to

$$\mathbb{T}^2 \setminus diag(\mathbb{T}^2) \times \left\{ f \in D_+ \left([0, \tfrac{1}{2}], [0, 1]\right) \text{ such that } f(0) = \frac{f(\frac{1}{2})}{f(\frac{1}{2}) - 1} \right\}$$

and that

$$\left\{ f \in D_+ \left([0, \tfrac{1}{2}], [0, 1]\right) \text{ such that } f'(0) = \frac{f'(\frac{1}{2})}{f'(\frac{1}{2}) - 1} \right\}$$
$$\simeq D_+([0, 1], [0, 1] \text{ such that } f'(0) = f'(1)) \simeq D_+(\mathbb{S}^1, 0 \text{ is fixed}).$$

Now remark that $\mathbb{T}^2 \setminus diag(\mathbb{T}^2)$ inherits the Lie group structure of $\mathbb{C} \setminus \{0\}$ and $D_+(\mathbb{S}^1, 0 \text{ is fixed})$ is an infinite dimensional Lie group. □

COROLLARY 2.7. $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$.

Proof. First, notice that $\pi_1(\mathbb{T}^2 \setminus diag(\mathbb{T}^2)) = \pi_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}$, on the other hand, by results of [1], we know that the injection of $SO(2)$ in $D_+(\mathbb{S}^1)$ induces a splitting of the fundamental group $\pi_1(D_+(\mathbb{S}^1)) = \pi_1(SO(2)) \oplus \pi_1(D_+([0, 1], \partial[0, 1]))$, and since we know that $\pi_1(SO(2)) = \mathbb{Z}$, and that $D_+([0, 1], \partial[0, 1])$ is contractible, we deduce that $\pi_1(D_+(\mathbb{S}^1)) = \mathbb{Z}$ and that $D_+(\mathbb{S}^1, 0 \text{ is fixed})$ is simply connected. So we have $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$. □

REMARK 2.8. Arc-connectedness can be deduced again by the fact that our space is homeomorphic to an infinite dimensional Lie group. However, we consider our prove of arc-connectedness to be of independent interest since we believe the idea can be generalized to higher dimensions as we conjectured in the statement of results.

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Authors' addresses:

Houssam Boukhecham
Université de Paris 12, LAMA. Paris, France.
E-mail: `houssam-eddine.boukhecham@u-pec.fr`

Hamza Ounesli
Scuola Internazionale Superiorie di Studi Avanzati
Via Bonomea 265
34136 Trieste, Italy
and
Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11
34151 Trieste, Italy
E-mail: `hounesli@sissa.it`

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