Appendix A. Table of coefficients $D_{g,n}(\Lambda)$ for $g = 3$ and $n \leq 5$

We collect below the data of the normalised coefficients $24^g g! C_{g,n}(\lambda)$ for $g = 3$, $n \leq 5$, and for all $\lambda \leq 3g - 3 + n$.

In front of each coefficient we provide the value of $3g-3-(|\lambda|-\lambda_1)$ in boldface: whenever this value is negative, we get vanishing coefficients (highlighted in *green*), illustrating the vanishing predicted by lemma 3.1; whenever $\ell(\lambda) > g$, we get vanishing coefficients (highlighted in *red*), illustrating the vanishing predicted by the main conjecture.

(g,n) = (3,1)

7	$D_{3,1}(1,1,1,1,1,1,1) = 1$

(g,n) = (3,2)

7	$D_{3,2}(1,1,1,1,1,1,1,1) = 1$	**2**	$D_{3,2}(2,2,2,1,1) = -27/7$
6	$D_{3,2}(2,1,1,1,1,1,1) = -3$	**0**	$D_{3,2}(2,2,2,2) = 0$
4	$D_{3,2}(2,2,1,1,1,1) = 27/5$		

(g,n) = (3,3)

7	$D_{3,3}(1,1,1,1,1,1,1,1,1) = 1$	**2**	$D_{3,3}(3,2,2,1,1) = -81/7$
6	$D_{3,3}(2,1,1,1,1,1,1,1) = -3$	**3**	$D_{3,3}(3,3,1,1,1) = 153/35$
6	$D_{3,3}(3,1,1,1,1,1,1) = -39/5$	**0**	$D_{3,3}(2,2,2,2,1) = 0$
4	$D_{3,3}(2,2,1,1,1,1,1) = 27/5$	**1**	$D_{3,3}(3,3,2,1) = 27/7$
4	$D_{3,3}(3,2,1,1,1,1) = 594/35$	**0**	$D_{3,3}(3,2,2,2) = 0$
2	$D_{3,3}(2,2,2,1,1,1) = -27/7$	**0**	$D_{3,3}(3,3,3) = -27/35$

(g,n) = (3,4)

7	$D_{3,4}(1,1,1,1,1,1,1,1,1,1) = 1$	**1**	$D_{3,4}(3,3,2,1,1) = 27/7$
6	$D_{3,4}(2,1,1,1,1,1,1,1,1) = -3$	**2**	$D_{3,4}(4,2,2,1,1) = -54$
6	$D_{3,4}(3,1,1,1,1,1,1,1) = -39/5$	**0**	$D_{3,4}(3,2,2,2,1) = 0$
4	$D_{3,4}(2,2,1,1,1,1,1,1) = 27/5$	**3**	$D_{3,4}(4,3,1,1,1) = 324/35$
4	$D_{3,4}(3,2,1,1,1,1,1) = 594/35$	**0**	$D_{3,4}(4,2,2,2) = 0$
6	$D_{3,4}(4,1,1,1,1,1,1) = -594/35$	**2**	$D_{3,4}(4,4,1,1) = -1152/35$
2	$D_{3,4}(2,2,2,1,1,1,1) = -27/7$	**0**	$D_{3,4}(3,3,3,1) = -27/35$
2	$D_{3,4}(3,2,2,1,1,1) = -81/7$	**-1**	$D_{3,4}(3,3,2,2) = 0$
4	$D_{3,4}(4,2,1,1,1,1) = 1692/35$	**1**	$D_{3,4}(4,3,2,1) = 108/5$
3	$D_{3,4}(3,3,1,1,1,1) = 153/35$	**0**	$D_{3,4}(4,4,2) = 144/35$
0	$D_{3,4}(2,2,2,2,1,1) = 0$	**0**	$D_{3,4}(4,3,3) = -162/35$
-2	$D_{3,4}(2,2,2,2,2) = 0$		

(g,n) = (3,5)

7	$D_{3,5}(1,1,1,1,1,1,1,1,1,1,1) = 1$	**0**	$D_{3,5}(4,2,2,2,1) = 0$
6	$D_{3,5}(2,1,1,1,1,1,1,1,1,1,1) = -3$	**-2**	$D_{3,5}(3,2,2,2,2) = 0$
4	$D_{3,5}(2,2,1,1,1,1,1,1,1) = 27/5$	**0**	$D_{3,5}(3,3,3,1,1) = -27/35$
6	$D_{3,5}(3,1,1,1,1,1,1,1,1) = -39/5$	**2**	$D_{3,5}(4,4,1,1,1) = -1152/35$
2	$D_{3,5}(2,2,2,1,1,1,1,1) = -27/7$	**2**	$D_{3,5}(5,2,2,1,1) = -1458/5$
6	$D_{3,5}(4,1,1,1,1,1,1,1) = -594/35$	**3**	$D_{3,5}(5,3,1,1,1) = 2844/35$
4	$D_{3,5}(3,2,1,1,1,1,1,1) = 594/35$	**1**	$D_{3,5}(4,3,2,1,1) = 108/5$
4	$D_{3,5}(4,2,1,1,1,1,1) = 1692/35$	**-1**	$D_{3,5}(4,3,2,2) = 0$
2	$D_{3,5}(3,2,2,1,1,1,1) = -81/7$	**0**	$D_{3,5}(4,4,2,1) = 144/35$
3	$D_{3,5}(3,3,1,1,1,1,1) = 153/35$	**0**	$D_{3,5}(5,2,2,2) = 0$
0	$D_{3,5}(2,2,2,2,1,1,1) = 0$	**0**	$D_{3,5}(4,3,3,1) = -162/35$
6	$D_{3,5}(5,1,1,1,1,1,1) = -2286/35$	**1**	$D_{3,5}(5,3,2,1) = 4572/35$
-2	$D_{3,5}(2,2,2,2,2,1) = 0$	**2**	$D_{3,5}(5,4,1,1) = -5904/35$
2	$D_{3,5}(4,2,2,1,1,1) = -54$	**-2**	$D_{3,5}(3,3,3,2) = 0$
3	$D_{3,5}(4,3,1,1,1,1) = 324/35$	**1**	$D_{3,5}(5,5,1) = 432/5$
0	$D_{3,5}(3,2,2,2,1,1) = 0$	**0**	$D_{3,5}(5,4,2) = 144/5$
4	$D_{3,5}(5,2,1,1,1,1) = 8532/35$	**0**	$D_{3,5}(5,3,3) = -162/5$
1	$D_{3,5}(3,3,2,1,1,1) = 27/7$	**-1**	$D_{3,5}(4,4,3) = 0$
-1	$D_{3,5}(3,3,2,2,1) = 0$		

Appendix B. Table of all coefficients $C_g(\lambda)$ for $g = 4$

Let us apply corollary 3.5 to compute $24^g g! C_g(k,\mu)$ for $g = 4$, for all $\mu \in \mathcal{Q}_g$ and for all k . We moreover verify numerically the lower bound $k_0 = 3g - 2 - |\mu| + \mu_1 - \delta_{|\mu|,3g-3}$ for the polynomial behaviour of $C_4(k,\mu)$ in k. There are 30 partitions to consider:

$$\begin{aligned}\mathcal{Q}_4 = \{&\varnothing, (2), (3), (4), (5), (6), (7), (8), (9), (2,2), (3,2),\\ &(4,2), (3,3), (5,2), (4,3), (6,2), (5,3), (4,4), (7,2),\\ &(6,3), (5,4), (2,2,2), (3,2,2), (4,2,2), (3,3,2),\\ &(5,2,2), (4,3,2), (3,3,3), (2,2,2,2), (3,2,2,2)\}\end{aligned}$$

Start with partitions of length $\ell(\mu) \leq 1$, here $k_0 = 10$ except for the max size partition (9) where $k_0 = 9$.

$C_4(\emptyset) = 1$	$C_4(2,(2)) = 54/5$	$C_4(3,(3)) = 4842/175$	$C_4(4,(4)) = -27648/175$	$C_4(5,(5)) = -9792/35$
$C_4(2,\varnothing) = -4$	$C_4(3,(2)) = 324/7$	$C_4(4,(3)) = 14184/175$	$C_4(5,(4)) = -182592/175$	$C_4(6,(5)) = -51264/175$
$C_4(3,\varnothing) = -68/5$	$C_4(4,(2)) = 21816/175$	$C_4(5,(3)) = 105768/175$	$C_4(6,(4)) = -36288/5$	$C_4(7,(5)) = -269568/25$
$C_4(4,\varnothing) = -1144/35$	$C_4(5,(2)) = 17064/25$	$C_4(6,(3)) = 104112/25$	$C_4(7,(4)) = -11405952/175$	$C_4(8,(5)) = -10245888/175$
$C_4(5,\varnothing) = -21816/175$	$C_4(6,(2)) = 117648/25$	$C_4(7,(3)) = 227952/7$	$C_4(8,(4)) = -22470912/35$	$C_4(9,(5)) = -47778048/175$
$C_4(6,\varnothing) = -141264/175$	$C_4(7,(2)) = 266832/7$	$C_4(8,(3)) = 51469056/175$	$C_4(9,(4)) = -176332032/25$	
$C_4(7,\varnothing) = -1106064/175$	$C_4(8,(2)) = 12432384/35$	$C_4(9,(3)) = 521255808/175$		
$C_4(8,\varnothing) = -9988992/175$	$C_4(9,(2)) = 93011328/25$			
$C_4(9,\varnothing) = -102117888/175$				

$$C_4(k,\varnothing) = -\frac{(k-5)!}{2270268000}\left(3717k^9 + 345264k^8 + 10022652k^7 + 51871810k^6 - 1143710229k^5 + 8585898070k^4 - 52358293308k^3 + 196752773416k^2 - 387514181568k + 314153693568\right)$$

$$C_4(k,(2)) = \frac{(k-3)!}{756756000}\left(11151k^7 + 998761k^6 + 29709011k^5 + 269630259k^4 - 712790674k^3 + 5966416652k^2 - 20766829960k + 26179474464\right)$$

$$C_4(k,(3)) = \frac{(k-2)!}{756756000}\left(723k^6 + 199787k^5 + 16387577k^4 + 413083225k^3 - 51459908k^2 + 3514829748k - 1065697776\right)$$

$$C_4(k,(4)) = -\frac{(k-1)!}{378378000}\left(25569k^5 + 1771714k^4 + 37277747k^3 + 274984058k^2 + 184540456k + 1946520576\right)$$

$$C_4(k,(5)) = \frac{k!}{378378000}\left(12093k^4 + 930010k^3 + 14966271k^2 - 218527846k - 287488200\right)$$

$C_4(6,(6)) = 1787904/175$
$C_4(7,(6)) = 14375808/175$
$C_4(8,(6)) = 157378176/175$
$C_4(9,(6)) = 49168512/5$
$C_4(k,(6)) = \frac{(k+1)!}{15765750}\left(513k^3 + 78137k^2 + 2328884k + 15061740\right)$

$C_4(7,(7)) = -13825152/175$
$C_4(8,(7)) = -28266624/35$
$C_4(9,(7)) = -339110784/35$
$C_4(k,(7)) = -\frac{(k+2)!}{7882875}\left(2011k^2 + 66173k + 1154393\right)$

$C_4(8,(8)) = 2078208/7$
$C_4(9,(8)) = 24938496/7$

$$C_4(k,(8)) = \frac{(k+3)!}{1433250}(99k + 9868)$$
$$C_4(k,(9)) = -\frac{11\,(k+4)!}{159250}$$

$C_4(2,(2,2)) = -108/7$
$C_4(3,(2,2)) = -468/7$
$C_4(4,(2,2)) = -1800/7$
$C_4(5,(2,2)) = -57096/35$
$C_4(6,(2,2)) = -410544/35$
$C_4(7,(2,2)) = -3450672/35$

$C_4(3,(3,2)) = -180/7$
$C_4(4,(3,2)) = -1584/35$
$C_4(5,(3,2)) = -24624/35$
$C_4(6,(3,2)) = -31392/7$
$C_4(7,(3,2)) = -5870304/175$

$C_4(4,(4,2)) = 2880/7$
$C_4(5,(4,2)) = 91008/35$
$C_4(6,(4,2)) = 3836736/175$
$C_4(7,(4,2)) = 34399872/175$

$C_4(3,(3,3)) = -36/5$
$C_4(4,(3,3)) = -216/5$
$C_4(5,(3,3)) = -8712/35$
$C_4(6,(3,3)) = -51408/25$

$C_4(5,(5,2)) = -242496/175$
$C_4(6,(5,2)) = -409536/35$
$C_4(7,(5,2)) = -19225728/175$

$$C_4(k,(2,2)) = -\frac{\left(1515k^5 + 125102k^4 + 3592493k^3 + 38382106k^2 + 33387016k + 313049088\right)(k-1)!}{29106000}$$

$$C_4(k,(3,2)) = \frac{\left(450k^4 + 24362k^3 + 67239k^2 - 7592459k - 8017332\right)k!}{7276500}$$

$$C_4(k,(4,2)) = \frac{\left(371k^3 + 20182k^2 + 325180k + 2519512\right)(k+1)!}{1212750}$$

$$C_4(k,(3,3)) = -\frac{\left(69k^3 + 3077k^2 - 14k + 233736\right)(k+1)!}{882000}$$

$$C_4(k,(5,2)) = -\frac{\left(389k^2 + 29593k + 509174\right)(k+2)!}{2425500}$$

$C_4(4,(4,3)) = -1152/35$

$C_4(5,(4,3)) = -22464/175$

$C_4(6,(4,3)) = -333504/175$

$$C_4(k,(4,3)) = -\frac{\left(27k^2 + 671k + 5292\right)(k+2)!}{220500}$$

$C_4(6,(6,2)) = -10368/25$

$C_4(7,(6,2)) = -13824/7$

$$C_4(k,(6,2)) = \frac{(191k - 2532)(k+3)!}{1212750}$$

$C_4(5,(5,3)) = 119232/175$

$C_4(6,(5,3)) = 211392/35$

$$C_4(k,(5,3)) = \frac{(38k + 1673)(k+3)!}{110250}$$

$C_4(4,(4,4)) = -6912/175$

$C_4(5,(4,4)) = -55296/175$

$$C_4(k,(4,4)) = -\frac{2(7k + 188)(k+3)!}{55125}$$

$$C_4(k,(7,2)) = \frac{107(k+4)!}{242550}$$

$$C_4(k,(6,3)) = -\frac{11(k+4)!}{18375}$$

$$C_4(k,(5,4)) = \frac{2(k+4)!}{7875}$$

$C_4(2,(2,2,2)) = 9$

$C_4(3,(2,2,2)) = 36$

$C_4(4,(2,2,2)) = 216$

$C_4(5,(2,2,2)) = 7128/5$

$C_4(3,(3,2,2)) = -18$

$C_4(4,(3,2,2)) = -648/5$

$C_4(5,(3,2,2)) = -32184/35$

$C_4(4,(4,2,2)) = -1728/35$

$C_4(5,(4,2,2)) = -2880/7$

$C_4(3,(3,3,2)) = 36/5$

$C_4(4,(3,3,2)) = 1944/35$

$$C_4(k,(2,2,2)) = \frac{\left(35k^3 + 2499k^2 + 65830k + 651864\right)(k+1)!}{529200}$$
$$C_4(k,(4,2,2)) = -\frac{(7k+188)(k+3)!}{22050}$$
$$C_4(k,(3,2,2)) = -\frac{\left(35k^2 + 1701k + 22788\right)(k+2)!}{176400}$$
$$C_4(k,(3,3,2)) = \frac{(7k+180)(k+3)!}{19600}$$

$$C_4(k,(5,2,2)) = -\frac{(k+4)!}{11025} \qquad C_4(k,(4,3,2)) = \frac{(k+4)!}{2450} \qquad C_4(k,(3,3,3)) = -\frac{(k+4)!}{3920} \tag{164}$$

$$C_4(k,(2,2,2,2)) = 0 \qquad C_4(k,(3,2,2,2)) = 0 \tag{165}$$

Therefore we again verify the conjecture for $g = 4$, with $C_g((k,2,2,2,2) = 0$ for all $k \geq 2$ and $C_4(k,3,2,2,2) = 0$ for all $k \geq 3$.

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