Rend. Istit. Mat. Univ. Trieste
Vol. 55 (2023), Art. No. 3, 27 pages
DOI: 10.13137/2464-8728/35484

Integrability aspects of the dynamical forest model

Evan Sulaiman, Azad Amen and Waleed Aziz

Abstract. *In this paper, we study the integrability problem of a mathematical models of forests with two age classes of the form,* $\dot{x} = \rho y - (y-1)^2 x - sx$, $\dot{y} = x - hy$, *where* $\rho, h, s \in \mathbb{R}$. *We proved that the system has a unique Darboux polynomial if and only if* $\rho = 0$. *The model has only two or three exponential factors if* $h \neq 0$ *or* $h = 0$, *respectively. It is also, showed that the system admits a Darboux first integral if and only if* $\rho = h = 0$ *and has no analytic first integral in any neighborhood of fixed point except when* $\rho = h = 0$.

Keywords: Forest system, Darboux polynomial, polynomial first integral, Darboux first integrals, analytic first integral.
MS Classification 2020: 34A05, 34A34, 37J35.

1. Introduction

One of the most interesting problems in environmental science and mathematical ecology is modeling the dynamics of forest age structure. The forest age structure dynamics, means the change of space and time of tree numbers in different age classes, which affect by internal and external factors [11]. The works in [3, 5, 9, 10], are devoted to model such dynamics in the simplest case of just two age classes, young and old tress, of the form

$$\dot{x} = \rho y - (y-1)^2 x - sx, \quad \dot{y} = x - hy, \tag{1}$$

in which the densities of young and old trees at time t are denoted by $x(t)$ and $y(t)$, respectively. Note that the parameters ρ, s and h are real numbers. The parameter ρ is fertility, s and h are ageing and death rates. Note that the system (1) has been studied in the papers [1, 2, 4, 6, 11, 21] but none of these papers are devoted to investigate the integrability or non-integrability problem. The local stability and dynamics near singularities have been studies in [20]. In particular, they used first Lyapunov coefficient and averaging theory to study the bifurcation phenomena and Hopf bifurcation occurs at singular points. In [15], authors demonstrated that the Brusselator system have no Darboux polynomial and polynomial first integral. The local and global integrability of Chua circuit system are studied in [12]. They prove that under

some conditions on parameters, the Chua system has no local analytic first integrals at the origin as well as the system eventually admits no global analytic first integrals the problem of finding Darboux polynomials and Darboux first integrals, are also considered in [13]. In [16], Llibre and Valls, showed that Muthuswamy-Chua system admits no Darboux polynomial, polynomial first integral and Darboux first integral. The existence of local analytic first integrals of Liénard system has been studied in [14, 17].
The aim of this paper, is to characterize the existence and nonexistence of polynomial and Darboux first integrals of system (1). We also study the existence of local analytic first integrals of system (1). Note that all calculations were performed by the computer algebra system Maple.

2. Preliminary Results

Consider the system of differential equations

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{2}$$

where P and Q are polynomials of degree at most d. The associated vector field of system (2) is denoted by

$$\mathcal{X} = P\,\frac{\partial}{\partial x} + Q\,\frac{\partial}{\partial y}.$$

Definition 2.1. *Let M be an open subset of $\mathbb{R}^2$. A non-constant analytic function $F : M \to \mathbb{R}$ is a first integral of a vector field $\mathcal{X}$ on M if it is constant on all solutions of system* (2) *which contained in M. That is, F is a first integral of $\mathcal{X}$ on M if and only if*

$$\mathcal{X}(F) = P\,\frac{\partial F}{\partial x} + Q\,\frac{\partial F}{\partial y} = 0.$$

Note that F is a polynomial first integral when it is a polynomial.

Definition 2.2. *We say that $g(x,y) = 0$, is an invariant algebraic curve of the system* (2) *if there exists a polynomial $K \in \mathbb{C}[x,y]$ such that*

$$\mathcal{X}(g) = P\,\frac{\partial g}{\partial x} + Q\,\frac{\partial g}{\partial y} = K\,g,$$

where K is a cofactor of the system (2) *of degree at most $d-1$. Note that, $g(x,y)$ is also known as a Darboux polynomial.*

Definition 2.3. *Let $f, g \in \mathbb{C}[x,y]$ be coprime, a non-constant function $E = \exp(f/g)$ is said to be an exponential factor of the system* (2) *if it satisfies*

$$\mathcal{X}(E) = P\,\frac{\partial E}{\partial x} + Q\,\frac{\partial E}{\partial y} = E\,L.$$

The polynomial L is a cofactor of the exponential factor with degree at most $d-1$.

DEFINITION 2.4. *A function $R : M \to \mathbb{R}$ is an integrating factor of $\mathcal{X}$, if it satisfies*

$$\mathcal{X}(R) = -R\,\mathrm{div}(\mathcal{X}),$$

where $\mathrm{div}(\mathcal{X}) = \mathrm{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is the divergent of $\mathcal{X}$. The first integral F, which is related to the integrating factor R, is $F(x, y) = -\int R(x, y)\,P(x, y)dy + T(x)$ satisfying $\frac{\partial F}{\partial x} = -RQ$. Then

$$\dot{x} = P\,R = -\frac{\partial F}{\partial y}, \quad \dot{y} = Q\,R = \frac{\partial F}{\partial x}.$$

DEFINITION 2.5. *A polynomial $f(x, y)$ is called a weight homogeneous polynomial if there exist $r = (r_1, r_2) \in \mathbb{N}^2$ and $m \in \mathbb{N}$, such that for all $\alpha > 0$, $f(\alpha^{r_1}x, \alpha^{r_2}y) = \alpha^m f(x, y)$, where $\mathbb{N}$ the set of all positive integers. The variable $r = (r_1, r_2)$ refers to the weight exponent of f and m denotes the weight degree of f with the weight exponent r.*

DEFINITION 2.6. *Let F be a first integral. Then F is said to be analytic first integral, if F is an analytic function. If M is a neighborhood of a singular point (x_0, y_0), then F is called a local analytic first integral of $\mathcal{X}$ at (x_0, y_0). If $M = \mathbb{R}^2$, then F is called a global analytic first integral of $\mathcal{X}$.*

REMARK 2.7. Let w be a finite generated vector subspace of $\mathbb{C}[x, y]$. The extactic algebraic curve of $\mathcal{X}$, denoted by $\varepsilon_w(\mathcal{X})$, is a polynomial defined by

$$\varepsilon_w(\mathcal{X}) = \det \begin{pmatrix} u_1 & u_2 & \cdots & u_l \\ \mathcal{X}(u_1) & \mathcal{X}(u_2) & \cdots & \mathcal{X}(u_l) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(u_1) & \mathcal{X}^{l-1}(u_2) & \cdots & \mathcal{X}^{l-1}(u_l) \end{pmatrix} = 0,$$

where $\{u_1, u_2, \ldots, u_l\}$ is a basis of w, $l = \dim(w)$ is the dimension of w and $\mathcal{X}^i(u_i) = \mathcal{X}^{i-1}(\mathcal{X}(u_i))$.

PROPOSITION 2.8 ([7]). *Let w be a finitely generated vector subspace of $\mathbb{C}[x, y]$, with $\dim(w) > 1$, and $\mathcal{X}$ be a polynomial vector field $\mathbb{C}^2$. Then every Darboux polynomial $g = 0$ for the vector field $\mathcal{X}$, with $g \in w$, is a factor of $\varepsilon_w(\mathcal{X})$.*

THEOREM 2.9 ([8]). *Assume that a polynomial vector field $\mathcal{X}$ of degree d in $\mathbb{C}^2$ admits p irreducible Darboux polynomial $g_i = 0$, with cofactor K_i for $i = 1, \ldots, p$ and q exponential factors $E_j = \exp(f_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$. Then the following statements hold.*

a. *There exist certain complex numbers λ_i and μ_j, not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

if and only if the function

$$F = g_1^{\lambda_1} g_2^{\lambda_2} \dots g_p^{\lambda_p} E_1^{\mu_1} E_2^{\mu_2} \dots E_q^{\mu_q},$$

is the Darboux first integral for $\mathcal{X}$.

b. *The function F is an integrating factor of $\mathcal{X}$ provided that the condition*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\mathrm{div}(\mathcal{X}),$$

is satisfied.

PROPOSITION 2.10 ([18, 19]). *The following statements hold.*

a. *If $E = \exp(\frac{f}{g})$ is an exponential factor for system (2), and g is not a constant polynomial, then $g = 0$ is an invariant algebraic curve.*

b. *Eventually, $E = \exp(f)$ can be an exponential factor, derived from the multiplicity of the infinite invariant straight line.*

THEOREM 2.11 ([17]). *Assume that the eigenvalues $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ at some singular point (x_0, y_0) of $\mathcal{X}$ do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0, \quad for \;\; k_1, k_2 \in \mathbb{Z}^+ \; with \;\; k_1 + k_2 > 0.$$

Then system (2) has no local analytic first integrals in a neighborhood of the singular point (x_0, y_0).

THEOREM 2.12 ([17]). *Assume that the eigenvalues λ_1 and λ_2 at some singular point (x_0, y_0) of $\mathcal{X}$ satisfy that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then system (2) has no local analytic first integrals if the singular point (x_0, y_0) is isolated.*

3. Darboux first integrals

In this section, we prove that system (1) has a unique Darboux polynomial when the parameter $\rho = 0$. It is also proved that system (1) has only two exponential factors if $h \neq 0$ and has only three exponential factors when $h = 0$. Finally, it is proved that system (1) has a Darboux first integral if and only if $\rho = h = 0$.

LEMMA 3.1. *If $g = g(x, y)$ is a Darboux polynomial of system* (1) *with cofactor $K \neq 0$, then $K = K(y) = b_0 + b_1 y + b_2 y^2$ for some $b_0, b_1, b_2 \in \mathbb{C}$.*

Proof. Assume that $g = g(x, y)$ is a Darboux polynomial of system (1) with non-zero cofactor $K = K(x, y) = \sum_{i=0}^{2} K_i(y) x^i$, for each i, $K_i(y)$ is a polynomial in the variable y of degree at most $2 - i$. Then g satisfies the partial differential equation

$$(\rho y - (y-1)^2 x - sx)\,\frac{\partial g}{\partial x} + (x - hy)\,\frac{\partial g}{\partial y} = Kg. \tag{3}$$

Without loss of generality, we can write $g(x, y) = \sum_{i=0}^{n} g_i(y) x^i$, where $g_i(y)$ is a polynomial in the variable y for each i and $n \in \mathbb{N} \cup \{0\}$ is the degree of g. In equation (3), the terms x^{n+2} satisfy

$$g_n(y) K_2(y) = 0. \text{ This implies that, } K_2(y) = 0.$$

Next, computing the terms x^{n+1} in (3), we obtain

$$\frac{dg_n(y)}{dy} = g_n(y)\, K_1(y).$$

The solution of this equation is $g_n(y) = C_1\, e^{\int k_1(y) dy}$, where C_1 is an arbitrary constant. Since g_n is a polynomial in y then it must be $K_1(y) = 0$. Eventually, $K(x, y) = K_0(y) = b_0 + b_1 y + b_2 y^2$ with $b_0, b_1, b_2 \in \mathbb{C}$. □

LEMMA 3.2. *Assume $g = g(x, y)$ is a Darboux polynomial of system* (1)*, then it is cofactor K is $K(y) = b_0 + b_1 y + b_2 y^2$, where $b_0 = -m(1+s) - lh, b_1 = 2m, b_2 = -m$ and $m \in \mathbb{N} \cup \{0\}$.*

Proof. We first use the weight-change of variables

$$x = \alpha^{-2} x_1, \quad y = \alpha^{-1} y_1, \quad t = \alpha^2 r.$$

Then system (1) becomes

$$\dot{x_1} = \alpha^3\, \rho\, y_1 - x_1\, y_1^2 + 2\alpha\, x_1\, y_1 - \alpha^2\, (1+s)\, x_1, \quad \dot{y_1} = \alpha\, x_1 - h\, \alpha^2\, y_1, \tag{4}$$

where $\alpha > 0$ and the primes denote the derivatives of variables with respect to r. We set $G(x_1, y_1) = \alpha^l\, g(\alpha^{-2} x_1, \alpha^{-1} y_1)$, and Lemma 3.1 implies

$$K = \alpha^2 K(\alpha^{-2} x_1, \alpha^{-1} y_1) = b_0 \alpha^2 + \alpha b_1 y_1 + b_2 y_1^2,$$

where l is the highest weight degree in the weight homogeneous components of g in x_1 and y_1. Note that $G = 0$ is a Darboux polynomial of system (1) with cofactor K. Indeed

$$\frac{dG}{dr} = \alpha^{l+2}\, \frac{dg}{dt} = (b_0 \alpha^2 + \alpha b_1 y_1 + b_2 y^2) G = K\, G.$$

Assume that $G = \sum_{i=0}^{n} \alpha^i\, G_i(x_1, y_1)$, G_i is a weight homogeneous polynomial in x_1 and y_1 with weight degree $n-i$, for $i = 0, \dots, n$. In particular

$$G_i(x_1, y_1) = \alpha^n g_{n-i}(x, y), \text{ for } i = 0, \dots, n. \tag{5}$$

The polynomial G must satisfies

$$\begin{aligned}(\alpha^3 \rho y_1 - x_1 y_1^2 + 2\alpha x_1 y_1 - (1+s)\alpha^2 x_1) \sum_{i=0}^{n} \alpha^i \frac{\partial G_i}{\partial x_1} \\ + (\alpha x_1 - h\alpha^2 y_1) \sum_{i=0}^{n} \alpha^i \frac{\partial G_i}{\partial y_1} = \sum_{i=0}^{n} \alpha^i (b_0 \alpha^2 + \alpha b_1 y_1 + b_2 y_1^2) G_i.\end{aligned} \tag{6}$$

The coefficients of α^0 in equation (6) is

$$-x_1\, y_1^2\, \frac{\partial G_0}{\partial x_1} = b_2\, y_1^2\, G_0.$$

Since $G_0 \neq 0$, otherwise g would be a constant, then the solution of the above partial differential equation is $G_0 = G_0(y)\, x_1^{-b_2}$. Since G_0 is a weight homogeneous polynomial with weight degree n, then $b_2 = -m$ and $m \in \mathbb{N} \cup \{0\}$. This implies that $G_0 = C_0\, x_1^m\, y_1^l$, where C_0 is non-zero constant. Note that $n = 2m + l$.
Calculating the coefficients of α^1 in (6), which satisfy

$$-x_1 y_1^2\, \frac{\partial G_1}{\partial x_1} + 2x_1 y_1\, \frac{\partial G_0}{\partial x_1} + x_1\, \frac{\partial G_0}{\partial y_1} = b_1 y_1\, G_0 - m y_1^2\, G_1.$$

Solving it, yields $G_1 = C_0\, (l\, x_1 + 2(m - \frac{b_1}{2}) y_1^2\, \ln(x_1)) x_1^m\, y_1^{l-3} + F_1(y_1) x_1^m$, where $F_1(y_1)$ is an arbitrary polynomial in the variable y_1. Since G_1 is a weight homogeneous polynomial with weight degree $n-1$, then $b_1 = 2m$ and we obtain

$$G_1 = l\, C_0\, x_1^{m+1}\, y_1^{l-3} + C_1\, x_1^m\, y_1^{l-1}, \quad C_1 \in \mathbb{C}. \tag{7}$$

The coefficients of α^2 in (6), are

$$\begin{aligned}- x_1 y_1^2\, \frac{\partial G_2}{\partial x_1} + 2x_1 y_1\, \frac{\partial G_1}{\partial x_1} - (1+s) x_1\, \frac{\partial G_0}{\partial x_1} + x_1\, \frac{\partial G_1}{\partial y_1} - h y_1\, \frac{\partial G_0}{\partial y_1} \\ = b_0 G_0 + 2m y_1\, G_1 - m y_1^2\, G_2,\end{aligned}$$

whose solution is

$$\begin{aligned}G_2 = \frac{1}{2y_1^3} \big(2((C_1 + 2C_0) l - C_1) x_1^{m+1}\, y_1^{l-1} + C_0\, l\, (l-3)\, x_1^{m+2}\, y_1^{l-3} \\ - 2\big(C_0 \ln(x_1)(lh + (1+s)m + b_0)\, y_1^{l+1} - F_2(y_1)\, y_1^3\big)\, x_1^m\big),\end{aligned}$$

where $F_2(y_1)$ is an arbitrary polynomial in the variable y_1. Since G_2 is a weight homogeneous polynomial of degree $n-2 = 2m+l-2$, then it must be $b_0 = -lh - (1+s)m$ with $m \in \mathbb{N} \cup \{0\}$. □

THEOREM 3.3. *The system* (1) *has a unique Darboux polynomial if and only if $\rho = 0$. Moreover, a Darboux polynomial is $g = x$ with non-zero cofactor $K = -(1+s) + 2y - y^2$.*

Proof. Let g be the Darboux polynomial of system (1) and by Lemma 3.2 it is cofactor is $K = -lh - m(1+s) + 2my - my^2$ and G_0, G_1 and G_2 are calculated can be as before. The terms of the coefficient α^3 in equation (6), satisfy

$$\begin{aligned} -x_1y_1^2\frac{\partial G_3}{\partial x_1} + 2x_1y_1\frac{\partial G_2}{\partial x_1} - (1+s)x_1\frac{\partial G_1}{\partial x_1} + \rho y_1\frac{\partial G_0}{\partial x_1} + x_1\frac{\partial G_2}{\partial y_1} - hy_1\frac{\partial G_1}{\partial y_1} \\ = (-lh - m(s+1))G_1 + 2my_1G_2 - my_1^2G_3. \end{aligned}$$

Solving this differential equation, gives

$$\begin{aligned} G_3 = & \frac{1}{6y_1^6}\big(18\,(((h - \tfrac{s}{3} + 1)\,C_0 + \tfrac{2C_1}{3} + \tfrac{C_2}{3})\,l - \tfrac{2C_1}{3} - \tfrac{2C_2}{3})\,x_1^{m+1}\,y_1^{l+1} \\ & + 3\,((4C_0 + C_1)\,l^2 + (-5\,C_1 - 14\,C_0)\,l + 4\,C_1)\,x_1^{m+2}\,y_1^{l-1} + 6\,F_3(y_1)\,y_1^6\,x_1^m \\ & + l\,C_0\,(l-3)\,(l-6)\,y_1^{l-3}\,x_1^{m+3} - 6\,\rho\,C_0\,m\,x_1^{m-1}\,y_1^{l+5} \\ & + 6\,C_1\,h\,\ln(x_1)\,y_1^{l+3}\,x_1^m\big), \end{aligned} \tag{8}$$

where $C_2 \in \mathbb{C}$ and $F_3(y_1)$ is an arbitrary polynomial in the variable y_1. Since G_3 is a weight homogeneous polynomial of degree $n-3 = 2m+l-3$, then must be $C_1\,h = 0$. We distinguish the following two cases.

Case 1: If $C_1 \neq 0$, $h = 0$. From equation (8) we obtain

$$\begin{aligned} G_3 = & (((-s+3)C_0 + C_2 + 2C_1)\,l - 2\,C_2 - 2\,C_1)\,x_1^{m+1}\,y_1^{l-5} - \rho\,m\,C_0\,y_1^{l-1}\,x_1^{m-1} \\ & + 2\,((C_0 + \tfrac{C_2}{4})\,l^2 + (\tfrac{-7\,C_0}{2} - \tfrac{5\,C_1}{4})\,l + C_1)\,x_1^{m+2}\,y_1^{l-7} + C_3\,x_1^m\,y_1^{l-3} \\ & + \tfrac{1}{6}l\,C_0(l-3)(l-6)\,x_1^{m+3}\,y_1^{l-9}, \quad C_3 \in \mathbb{C}. \end{aligned}$$

The coefficients of α^4 in equation (6) are

$$\begin{aligned} -x_1y_1^2\frac{\partial G_4}{\partial x_1} + 2x_1y_1\frac{\partial G_3}{\partial x_1} - (1+s)x_1\frac{\partial G_2}{\partial x_1} + \rho y_1\frac{\partial G_1}{\partial x_1} + x_1\frac{\partial G_3}{\partial y_1} \\ = -m(s+1)G_2 + 2my_1G_3 - my_1^2G_4. \end{aligned}$$

Solving it, we obtain

$$
\begin{aligned}
G_4 =& \frac{1}{24y_1^5}\big(24((C_0+\tfrac{C_1}{6})l^3+(-10C_0-2C_1)\,l^2+(\tfrac{67C_0}{3}+\tfrac{13C_1}{2})\,l-\tfrac{14\,C_1}{3})\\
& x_1^{m+3}\,y_1^{l-5}+48(((-2s+2)C_0+(\tfrac{-s}{2}+\tfrac{3}{2})C_1+C_2+\tfrac{C_3}{2})l+(\tfrac{s}{2}-\tfrac{3}{2})C_1-\\
& 2C_2-\tfrac{3\,C_3}{2})y_1^{l-1}x_1^{m+1}+12(((-2s+10)C_0+C_2+4C_1)l^2+((8s-40)C_0\\
& -7C_2-22C_1)l+10C_2+18C_1)\,y_1^{l-3}\,x_1^{m+2}-48(C_0+\tfrac{C_1}{2})\,\rho\,m\,y_1^{l+3}\,x_1^{m-1}\\
& +24\,F_4(y_1)\,x_1^m\,y_1^5+24\,C_0\,\ln(x_1)\,\rho\,(l+m)\,y_1^{l+1}\,x_1^m\\
& +l\,C_0\,(l-3)(l-6)(l-9)\,y_1^{l-7}\,x_1^{m+4}\big),
\end{aligned}
\tag{9}
$$

where $F_4(y_1)$ is an arbitrary polynomial in the variable y_1. Since G_4 must be a weight homogeneous polynomial of degree $n-4=2m+l-4,\ l+m\neq 0$ and $C_0\neq 0$, then must be $\rho=0$. From equation (9), we get that $h=0$ and $\rho=0$. Simple calculation shows

$$
\varepsilon_w(\mathcal{X})=\det\begin{pmatrix}1 & x & y\\ 0 & -(y-1)^2\,x-s\,x & x\\ 0 & ((y-1)^2+s)^2\,x-2\,x^2\,(y-1) & -(y-1)^2\,x-s\,x\end{pmatrix}=0.
$$

Then $\varepsilon_w(\mathcal{X})=2\,x^3\,(y-1)$. So by Proposition 2.8, then $g=x$ is a unique Darboux polynomial with cofactor $-s-(y-1)^2$. The polynomial $(y-1)$ is not Darboux polynomial of system (1).

Case 2: If $h\neq 0$, $C_1=0$. The equation (7), gives

$$
G_1=l\,C_0\,x_1^{m+1}\,y_1^{l-3}. \tag{10}
$$

Let $g=\sum_{i=0}^{n}g_i(x,y)$, where g_i is a homogeneous polynomial in the variable x and y. Without loss of generality we can assume that $g_n\neq 0$ and $n\geq 1$. Then g satisfies the partial differential equation

$$
(\rho y-(y-1)^2x-sx)\,\frac{\partial g}{\partial x}+(x-hy)\,\frac{\partial g}{\partial y}=(b_0+b_1y+b_2y^2)\,g. \tag{11}
$$

The terms of degree $n+2$ in equation (11), satisfy

$$
-x\,y^2\,\frac{\partial g_n}{\partial x}=b_2\,y^2\,g_n,
$$

and its solution is $g_n=x^{-b_2}\,f_n(y)$, where $f_n(y)$ is an arbitrary polynomial in variable y. Since g_n is a polynomial of degree n, then must be $b_2=-m$, where m is a non-negative integer. Therefore

$$
g_n=C_n\,x^m\,y^{n-m},\quad C_n\in\mathbb{C}\setminus\{0\}. \tag{12}
$$

The coefficients of degree $n+1$ in equation (11) satisfy

$$-xy^2\,\frac{\partial g_{n-1}}{\partial x}+2xy\,\frac{\partial g_n}{\partial x}=b_1 y g_n - m y^2 g_{n-1}.$$

Solving the differential equation above, we obtain

$$g_{n-1}=2\,x^m\,\left(C_n\,\ln(x)\,(m-\frac{b_1}{2})\,y^{n-m-1}+\frac{f_{n-1}(y)}{2}\right),$$

where $f_{n-1}(y)$ is an arbitrary polynomial in the variables y. Since g_{n-1} is a polynomial of degree $n-1$, we must have that $b_1=2m$. Then

$$g_{n-1}=C_{n-1}\,x^m\,y^{n-m-1},\quad C_{n-1}\in\mathbb{C}. \tag{13}$$

Computing the terms of degree n in equation (11), gives

$$\begin{aligned}-\,xy^2\,\frac{\partial g_{n-2}}{\partial x}+2xy\,\frac{\partial g_{n-1}}{\partial x}+(\rho y-(s+1)x)\,\frac{\partial g_n}{\partial x}+(x-hy)\,\frac{\partial g_n}{\partial y}&\\ =b_0\,g_n+2my\,g_{n-1}-my^2\,g_{n-2},&\end{aligned}$$

whose solution is

$$\begin{aligned}g_{n-2}=-C_n\,x^{m-1}\,((-n+m)\,x^2+\rho\,m\,y^2)\,y^{n-m-3}+C_{n-2}\,x^m y^{n-m-2},&\\ C_{n-2}\in\mathbb{C}.&\end{aligned}$$

Now, computing the degree $n-1$ in equation (11), we see

$$\begin{aligned}-\,xy^2\,\frac{\partial g_{n-3}}{\partial x}+2xy\,\frac{\partial g_{n-2}}{\partial x}+(\rho y-(s+1)x)\,\frac{\partial g_{n-1}}{\partial x}+(x-hy)\,\frac{\partial g_{n-1}}{\partial y}&\\ =((h-s-1)m-n\,h)\,g_{n-1}+2my\,g_{n-2}-my^2\,g_{n-3},&\end{aligned}$$

and solving it, yields

$$\begin{aligned}g_{n-3}=&\frac{1}{y^3}(-x^{m+1}((-n+m+1)C_{n-1}+2C_n(-n+m))y^{n-m-1}-x^{m-1}\\ &\rho m(C_{n-1}+2C_n)\,y^{1+n-m}+(hC_{n-1}y^{n-m}\ln(x)+f_{n-3}(y)y^3)\,x^m),\end{aligned}$$

where $f_{n-3}(y)$ is an arbitrary polynomial in variable y. Since g_{n-3} is a polynomial of degree $n-3$, then must be $h\,C_{n-1}=0$. By hypothesis $h\neq 0$ then must be $C_{n-1}=0$. Hence

$$g_{n-1}=0. \tag{14}$$

From (5), (10), (14) and since $C_0\neq 0$, then must be $l=0$ and we obtain

$$G_1=0,\quad G_2=C_{n-2}\,x_1^m\,y_1^{-2}. \tag{15}$$

Since G_2 is a weight homogeneous of degree $n-2$, then $G_0 = C_0\, x^m$, $G_2 = 0$ and $G_3 = -\rho\, m\, C_0\, x^{m-1}\, y^{-1}$. Since G_3 is a weight homogeneous of weight degree $n-3$, then $m = 0$ or $\rho = 0$. If $m = 0$, then $K = 0$ which is contradiction. Hence, must be $\rho = 0$, $G_3 = 0$ and $l = 0$. To prove all $G_i = 0$ for $i = 1, \ldots, n$, we use mathematical induction. For $i = 1$, directly we get from equation (15). Assume it is true for $i = n-1$, which mean that $G_i = 0$ for $i = 1, \ldots, n-1$. Now computing the term α^n in equation (6), we have

$$-x_1 y_1^2\, \frac{\partial G_n}{\partial x_1} = -m y_1^2\, G_n.$$

Solving it, we obtain $G_n = x_1^m\, f_0(y_1)$, where $f_0(y_1)$ is an arbitrary polynomial in variable y_1. Since G_n is a weight homogeneous polynomial of degree zero, then $f_0(y_1) = 0$. Therefore, $G_i = 0$ for $i = 1, \ldots, n$. We get that $G = G_0 + G_1 + \cdots + G_n = C_0\, x_1^m$. Hence, x_1^m is a Darboux polynomial with cofactor $m(-(1+s) + 2y_1 - y_1^2)$. Then the result follows. $\square$

We note that if $h = 0$ and $\rho = 0$ then system (1) is integrable, see Theorem 3.5, so in the following result we consider h and ρ are not zero simultaneously.

Theorem 3.4. *The following statements hold.*

i. *For $h \neq 0$, the exponential factors of system* (1) *are* e^y *and* e^{y^2} *with respective cofactors* $x - hy$ *and* $2xy - 2hy^2$.

ii. *For $h=0$, the exponential factors of system* (1) *are* $\mathrm{e}^y, \mathrm{e}^{y^2}$ *and* $\mathrm{e}^{4xy - \frac{8}{3}y^3 + y^4}$ *with respective cofactors* x, $2xy$ *and* $4x^2 + 4\rho y^2 - 4(s+1)xy$.

Proof. By Theorem 3.3 and Proposition 2.10 the exponential factor of system (1) can be expressed as $E = \exp(\frac{f}{x^n})$ for some non-negative integer n, note that f and x^n are coprime. Then E satisfies

$$(\rho y - (y-1)^2 x - sx)\, \frac{\partial E}{\partial x} + (x - hy)\, \frac{\partial E}{\partial y} = L\, E,$$

and this implies

$$(\rho y - (y-1)^2 x - sx)\, \frac{\partial f}{\partial x} + (x - hy)\, \frac{\partial f}{\partial y} + n\, f\, ((y-1)^2 + s) = L\, x^n. \quad (16)$$

First, for $n \geq 1$. In this case, by denoting the restriction of f to $x = 0$ by $\hat{f}$ in equation (16), we can derive $\hat{f} \neq 0$, otherwise, f becomes divisible by x, which is impossible. The function $\hat{f}$ satisfies

$$-hy\, \frac{d\hat{f}(y)}{dy} + n\, \hat{f}(y)\, \big((y-1)^2 + s\big) = 0. \quad (17)$$

Solving (17), we obtain $\hat{f}(y) = C_1\, y^{\frac{n\,(s+1)}{h}} \exp(\frac{n\,y\,(y-4)}{2\,h})$. Since $\hat{f}(y)$ is a polynomial, if $h \neq 0$ then must be $n = 0$, we get the result. On the other hand, if $h = 0$ then we get $n\hat{f}(y)\,((y-1)^2 + s) = 0$. Then must be $n = 0$.
Second, for $n = 0$, directly $E = \mathrm{e}^f$ where $f \in \mathbb{C}[x, y]$ is a polynomial of degree $m \in \mathbb{N}$. Then E satisfies

$$(\rho y - (y-1)^2 x - sx)\,\frac{\partial E}{\partial x} + (x - hy)\,\frac{\partial E}{\partial y} = E\,L, \tag{18}$$

and since $E \neq 0$

$$(\rho y - (y-1)^2 x - sx)\,\frac{\partial f}{\partial x} + (x - hy)\,\frac{\partial f}{\partial y} = L, \tag{19}$$

where $f = f(x, y) \in \mathbb{C}[x, y]$, with a cofactor $L = L(x, y)$ of degree at most two. That we can write $L = b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2$ for some $b_i \in \mathbb{C}$, $i = 0, \ldots, 5$. Assume $f = \sum_{i=0}^{m} f_i(x, y)$, where each f_i is a homogeneous polynomial of degree i. Suppose that $f_m \neq 0$ for $m \geq 5$.
The terms of degree $m + 2$ in equation (19) is

$$-xy^2\,\frac{\partial f_m}{\partial x} = 0, \text{ and this implies that } f_m = f_m(y).$$

Since f_m is a homogeneous polynomial of degree m, then $f_m = C_m y^m$, $C_m \in \mathbb{C} \setminus \{0\}$. The terms of degree $m + 1$ in equation (19) satisfy

$$-xy^2\,\frac{\partial f_{m-1}}{\partial x} + 2xy\,\frac{\partial f_m}{\partial x} = 0.$$

Solving it, we obtain $f_{m-1} = C_{m-1}\, y^{m-1}$, $C_{m-1} \in \mathbb{C}$. Now compute the terms of degree m in equation (19), which are

$$-xy^2\,\frac{\partial f_{m-2}}{\partial x} + 2xy\,\frac{\partial f_{m-1}}{\partial x} + (-x - sx + \rho y)\,\frac{\partial f_m}{\partial x} + (x - hy)\,\frac{\partial f_m}{\partial y} = 0.$$

The solution of differential equation above is

$$f_{m-2} = -hmC_m \ln(x)\, y^{m-2} + mC_m y^{m-3} x + C_{m-2}\, y^{m-2}, \quad C_{m-2} \in \mathbb{C}.$$

Since f_{m-2} is a homogeneous polynomial, then it must be $hC_m m = 0$. By hypothesis $m\,C_m \neq 0$. Then considering two different cases.

i. If $h \neq 0$, then we get contradiction.Then f must be a polynomial of the degree four satisfying equation (19). Suppose that $f(x, y) = c_0 + c_1 x + c_2 y + c_3 xy + c_4 x^2 + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3 + c_{10} x^4 + c_{11} y^4 + c_{12} x^3 y + c_{13} xy^3 + c_{14} x^2 y^2$ for some $c_i \in \mathbb{C}$, and for $i = 0, \ldots, 14$. From equation (19) we have

$$(\rho y - (y-1)^2 x - sx)\,\frac{\partial f}{\partial x} + (x - hy)\,\frac{\partial f}{\partial y} = b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2,$$

after some calculations, we see $f = y + y^2$. Then e^{y+y^2} is the exponential factor with cofactor $x - hy + 2xy - 2hy^2$. In particular, e^y and e^{y^2} are only exponential factors of system (1) with respective cofactors $x - hy$ and $2xy - 2hy^2$.

ii. If $h = 0$, then $f_{m-2} = mC_m xy^{m-3} + C_{m-2}y^{m-2}$. The equation, which encompasses terms of degree $m-1$ in (19) are

$$-xy^2\,\frac{\partial f_{m-3}}{\partial x} + 2xy\,\frac{\partial f_{m-2}}{\partial x} + (\rho y - x(1+s))\,\frac{\partial f_{m-1}}{\partial x} + x\,\frac{\partial f_{m-1}}{\partial y} = 0,$$

and its solution is

$$f_{m-3} = (2mC_m + mC_{m-1} - C_{m-1})\,xy^{m-4} + C_{m-3}\,y^{m-3}, \quad C_{m-3} \in \mathbb{C}.$$

The terms of degree $m-2$ in equation (19) satisfy

$$-xy^2\,\frac{\partial f_{m-4}}{\partial x} + 2xy\,\frac{\partial f_{m-3}}{\partial x} + (\rho y - (1+s)x)\,\frac{\partial f_{m-2}}{\partial x} + x\,\frac{\partial f_{m-2}}{\partial y} = 0,$$

and whose solution is

$$\begin{aligned} f_{m-4} = {} & (\,((-s+3)\,C_m + C_{m-2} + 2\,C_{m-1})m - 2\,C_{m-2} - 2\,C_{m-1})\,x\,y^{m-5} \\ & + \tfrac{1}{2}mx^2C_m(m-3)y^{m-6} + \rho m\ln(x)C_m y^{m-4} + C_{m-4}y^{m-4}, \\ & \qquad C_{m-4} \in \mathbb{C}. \end{aligned}$$

Since f_{m-4} is a homogeneous polynomial of degree $m-4$, then must be $\rho m C_m = 0$. Since m, C_m and ρ are non-zero, then we get a contradiction. Then f must be a polynomial of the degree four. Proceeding as in the proof of case i, we obtain that e^y, e^{y^2} and $e^{4xy - \frac{8}{3}y^3 + y^4}$ are exponential factors with respective cofactors $x, 2xy$ and $4x^2 + 4\rho y^2 - 4(s+1)xy$.

□

THEOREM 3.5. *System* (1) *has a polynomial first integrals if and only if* $\rho = h = 0$. *In particular the polynomial first integral is* $H(x,y) = -\frac{1}{3}y^3 - sy + y^2 - x - y$.

Proof. The change of variables

$$y_1 = y - 1, \quad x_1 = x,$$

transform system (1) to

$$\dot{x_1} = \rho y_1 + \rho - x_1 y_1^2 - s x_1, \quad \dot{y_1} = x_1 - h y_1 - h. \tag{20}$$

Since the change is linear, clearly it is equivalent to look for polynomial first integral $H(x,y)$ of system (1) that to look for polynomial first integrals

$\bar{H}(x_1, y_1) = H(x, y)$ of system (20). We can write $\bar{H} = \sum_{i=1}^{n} \bar{H}_i(x_1) y_1^i$, where $\bar{H}_i$ is a polynomials in the variables x_1 for each i for $i = 1, \dots, n$. Since $\bar{H}_n \neq 0$ for $n > 0$. The polynomial $\bar{H}$ satisfies

$$(\rho y_1 + \rho - x_1 y_1^2 - s x_1) \frac{\partial \bar{H}}{\partial x_1} + (x_1 - h y_1 - h) \frac{\partial \bar{H}}{\partial y_1} = 0. \tag{21}$$

The coefficients of y_1^{n+2} in equation (21), satisfies

$$-x_1 \frac{d\bar{H}_n}{dx_1} = 0.$$

This implies that

$$\bar{H}_n(x_1) = B_n, \quad B_n \in \mathbb{C} \setminus \{0\}.$$

Again, the coefficients of y_1^{n+1} in equation (21), satisfy

$$-x_1 \frac{d\bar{H}_{n-1}}{dx_1} + \rho \frac{d\bar{H}_n}{dx_1} = 0.$$

The solution of the equation above is

$$\bar{H}_{n-1} = B_{n-1}, \quad B_{n-1} \in \mathbb{C}.$$

Next the coefficient of y_1^n in equation (21), gives

$$-x_1 \frac{d\bar{H}_{n-2}}{dx_1} + \rho \frac{d\bar{H}_{n-1}}{dx_1} - nh\, \bar{H}_n = 0.$$

Solving it, we obtain

$$\bar{H}_{n-2} = -nh \ln(x_1) B_n + B_{n-2}, \quad B_{n-2} \in \mathbb{C}.$$

Since $\bar{H}_{n-2}$ is a polynomial, then $nhB_n = 0$. By hypothesis $n > 0$ and $B_n \neq 0$, then must $h = 0$. We obtain $\bar{H}_{n-2} = B_{n-2}$. Computing the coefficients of y_1^{n-1} in equation (21), we obtain

$$-x_1 \frac{d\bar{H}_{n-3}}{dx_1} + \rho \frac{d\bar{H}_{n-2}}{dx_1} + (\rho - s x_1) \frac{d\bar{H}_{n-1}}{dx_1} + n x_1\, \bar{H}_n = 0.$$

Solving differential equation above, yields

$$\bar{H}_{n-3} = n B_n x_1 + B_{n-3}, \quad B_{n-3} \in \mathbb{C}.$$

Also the coefficients of y_1^{n-2} in equation (21), satisfies

$$-x_1 \frac{d\bar{H}_{n-4}}{dx_1} + \rho \frac{d\bar{H}_{n-3}}{dx_1} + (\rho - s x_1) \frac{d\bar{H}_{n-2}}{dx_1} + x_1 (n-1) \bar{H}_{n-1} = 0,$$

which has a solution

$$\bar{H}_{n-4} = \rho n B_n \ln(x_1) + (n-1)x_1 B_{n-1} + B_{n-4}, \quad B_{n-4} \in \mathbb{C}.$$

Since $\bar{H}_{n-4}$ is a polynomial, we get $\rho n B_n = 0$. Then it is obvious that must be $\rho = 0$. Therefore, the polynomial first integral of system (1) is $H(x,y) = -\frac{1}{3}y^3 - sy + y^2 - x - y$ when $h = \rho = 0$. □

THEOREM 3.6. *The following statements hold.*

1. *The system* (1) *has no Darboux first integrals if* $h\rho \neq 0$.
2. *If* $\rho = 0$ *and* $h \neq 0$, *then the system* (1) *has no Darboux first integrals.*
3. *If* $\rho = 0$ *and* $h = 0$, *then the system has a Darboux first integrals. More precisely the Darboux first integral is* $\frac{y^3}{3} - y^2 + (1+s)y + x$.
4. *If* $h = 0$ *and* $\rho \neq 0$, *then the system* (1) *has no Darboux first integrals.*

Proof. 1. Suppose that system (1) has a Darboux first integral. Applying Theorem 3.3 system (1) does not admits any Darboux polynomial with non-zero cofactor. Also Theorem 3.4 implies that system (1) has only two exponential factors e^y and e^{y^2} with cofactors $x - hy$ and $2xy - 2hy^2$, respectively. Applying Theorem 2.9, if there exists $\mu_1, \mu_2 \in \mathbb{C}$, not all zero such that

$$\mu_1 (x - hy) + \mu_2 (2xy - 2hy^2) = 0.$$

It is clear that the equation above has only trivial solution which is a contradiction. Hence, the system (1) has no a Darboux first integral.

2. Suppose that system (1) has a Darboux first integrals. Again Theorem 3.3 implies that system (1) has a unique Darboux polynomial $g = x$ with cofactor $K = -(y-1)^2 - s$ and Theorem 3.4 showed that system (1) has only two exponential factors e^y and e^{y^2} with cofactors $x - hy$ and $2xy - 2hy^2$, respectively. By Theorem 2.9, if there exists $\lambda_1, \mu_1, \mu_2 \in \mathbb{C}$, not all zero such that

$$\lambda_1 (-(y-1)^2 - s) + \mu_1 (x - hy) + \mu_2 (2xy - 2hy^2) = 0.$$

The equation above has no non-zero solution which gives a contradiction. Hence, the system has no Darboux first integral.

3. In this case x is a Darboux polynomial with cofactor $K_1 = -(y-1)^2 - s$ and e^y, e^{y^2} and $e^{4xy - \frac{8}{3}y^3 + y^4}$ are exponential factors with respective cofactors $L_1 = x$, $L_2 = 2xy$ and $L_3 = 4x^2 - 4(s+1)xy$. In this case, $\operatorname{div}(\mathcal{X}) = -(y-1)^2 - s$. If there exists $\lambda_1, \mu_1, \mu_2, \mu_3 \in \mathbb{C}$, not all zero and

such that the relation, $\lambda_1(-(y-1)^2-s)+\mu_1(x)+\mu_2(2xy)+\mu_3(4x^2-4(s+1)xy) = -\mathrm{div}(\mathcal{X})$ satisfied, then there is an integrating factor. Direct calculation shows that $\mu_1 = \mu_2 = \mu_3 = 0$ and $\lambda_1 = -1$. Then $R = x^{-1}$ is an integrating factor. So by using Definition 2.4 we obtain that $\frac{y^3}{3} - y^2 + (1+s)\,y + x$ is a first integral.

4. The proof is similar to case 1.

□

4. Analytic first integrals

This section is devoted to investigate the analytic first integral of system (1). Note that, system (1) is a special case of Liénard polynomial differential system. The change of coordinates

$$X = y \quad \text{and} \quad Y = x - hy,$$

transform system (1) to

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= (\rho - h(s+1))X - hX^3 + 2hX^2 - ((h+s+1) + X^2 - 2X)Y. \end{aligned} \tag{22}$$

The system (22) has unique singular point at origin if $h = 0$ and $\rho \neq 0$, while $h \neq 0$ it has three singular points $(0,0)$ and $\left(\frac{h\pm\sqrt{-h^2s+h\rho}}{h}, 0\right)$. Furthermore, the system (22) has infinite singular point if $h = \rho = 0$. Here $g(X) = (\rho - h(s+1))X - hX^3 + 2hX^2$ and $f(X) = (h+s+1) + X^2 - 2X$. The eigenvalues of the system (22) at the origin are

$$\lambda_1, \lambda_2 = \frac{-(h+s+1) \pm \sqrt{(h+s+1)^2 + 4\rho - 4(s+1)h}}{2}.$$

The investigation and calculations at the singular point $\left(\frac{h-\sqrt{-h^2s+h\rho}}{h}, 0\right)$ are similar to the singular point $\left(\frac{h+\sqrt{-h^2s+h\rho}}{h}, 0\right)$, so we consider just one of them. We move the singular point $\left(\frac{h+\sqrt{-h^2s+h\rho}}{h}, 0\right)$ into the origin. We use the linear change of coordinates

$$X_1 = X - \frac{h+\sqrt{-h^2s+h\rho}}{h} \quad \text{and} \quad Y_1 = Y,$$

which gives

$$\dot{X}_1 = Y_1, \quad \dot{Y}_1 = 2(sh-\rho)X_1 - hX_1^2 + (-3X_1^2 - 2X_1)\sqrt{-h^2 s + h\rho} \\ - hX_1^3 - \left(\frac{h^2+\rho}{h} + \frac{2X_1}{h}\sqrt{-h^2 s + h\rho} + X_1^2\right) Y_1. \quad (23)$$

Since $h \neq 0$, $-h^2 s + h\,\rho > 0$, $g(X_1) = 2(sh-\rho)X_1 - X_1^2 h + (-3X_1^2 - 2X_1)\sqrt{-h^2 s + h\rho} - hX_1^3$ and $f(X_1) = \frac{h^2+\rho}{h} + \frac{2X_1}{h}\sqrt{-h^2 s + h\rho} + X_1^2$. Then eigenvalues of system (23), at the origin are

$$\lambda_1, \lambda_2 = \frac{-(h^2+\rho) \pm \sqrt{(h^2+\rho)^2 + 8h^2(hs - \rho - \sqrt{-h^2 s + h\rho})}}{2h}.$$

THEOREM 4.1. *The system* (22) *has no local analytic first integral in a neighborhood of the origin if* $h+s+1 \neq 0$ *and one of the following conditions hold.*

- **i.** $\rho = h(s+1)$,
- **ii.** $-\rho + h(s+1) \neq 0$ *and* $\frac{(h+s+1)^2}{h(s+1)-\rho} \notin \mathbb{Q}^-$,
- **iii.** $-\rho + h(s+1) \neq 0$ *and* $\frac{h(s+1)-\rho}{(h+s+1)^2} = -\alpha \in \mathbb{Q}^-$, $\alpha \neq \frac{pq}{(p-q)^2}$ *for some* $p, q \in \mathbb{Z}^+$ *and* $p \neq q$,
- **iv.** $\rho = -\frac{(h-(s+1))^2}{4}$,
- **v.** $-(h-(s+1))^2 - 4\rho \neq 0$.

Proof. **i.** Then the eigenvalues of the system (22) where $h+s+1 \neq 0$ and $\rho = h(s+1)$ are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -(h+s+1).$$

Since $(0,0)$ is an isolated singular point of system (22), Theorem 2.12 guarantees that system (22) has no local analytic first integrals in a neighborhood of the origin.

ii. Now $f(0) = h+s+1$, $g(0)' = -\rho + h(s+1)$ and $\frac{f(0)^2}{g(0)'} = \frac{(h+s+1)^2}{h(s+1)-\rho} \notin \mathbb{Q}^-$. It is obvious

$$\lambda_1 + \lambda_2 = -(h+s+1) \quad \text{and} \quad \lambda_1\lambda_2 = -\rho + h\,(s+1).$$

It is sufficient to show $k_1\lambda_1 + k_2\lambda_2 \neq 0$ for $k_1, k_2 \in \mathbb{Z}^+$. Suppose that $k_1\lambda_1 + k_2\lambda_2 = 0$. Then $\lambda_1 = -\alpha\lambda_2$ for some $\alpha \in \mathbb{Q}^+$. We obtain

$$\lambda_2(1-\alpha) = -(h+s+1) \quad \text{and} \quad -\alpha\lambda_2^2 = -\rho + h(s+1).$$

We have, $\frac{(h+s+1)^2}{h(s+1)-\rho} = -\frac{(1-\alpha)^2}{\alpha} \in \mathbb{Q}^-$. Since $(h+s+1) \neq 0$ then $\alpha \neq 1$ and $\alpha \neq 0$ because $h(s+1) - \rho \neq 0$. Note that $\frac{(h+s+1)^2}{h(s+1)-\rho} \notin \mathbb{Q}^-$, then $k_1\lambda_1 + k_2\lambda_2 \neq 0$. Hence Theorem 2.11, guarantee that system (22) has no local analytic first integrals in a neighborhood of the origin.

iii. We write $g(0)' = -\alpha(h+s+1)^2$ with $\alpha \in \mathbb{Q}^+ \setminus \{0\}$. The rescaling $(X_1, Y_1, T_1) = ((h+s+1)X, Y, (h+s+1)t)$, system (22) becomes

$$X_1' = Y_1, \quad Y_1' = \frac{1}{(h+s+1)^2}(\rho - h(s+1))X_1 - Y_1 + O(X_1, Y_1),$$

where $O(X_1, Y_1)$ means terms of higher order and without loss of generality we write (X, Y, t) instead of (X_1, Y_1, T_1), then system above becomes of the form

$$X' = Y, \quad Y' = \alpha X - Y + O(X, Y).$$

We proceed the proof as the proof of Lemma 12 in [17].

iv. The eigenvalues of system (22) with $\rho = -\frac{(h-(s+1))^2}{4}$ and $h+s+1 \neq 0$ are repeated eigenvalues $\lambda_1 = \lambda_2 = -(\frac{h+s+1}{2})$. So does not satisfy resonance condition in Theorem 2.11. Therefore, the system (22) has no local analytic first integrals in a neighborhood of origin.

v. We know that a necessary condition in order that system (22) has analytic first integral is that the linear part of system (22) with $h+s+1 \neq 0$ and $-(h-(s+1))^2 - 4\rho \neq 0$, admits a polynomial first integral.

$$Y\frac{\partial H_1}{\partial X} + ((\rho - h(s+1))X - (h+s+1)Y)\frac{\partial H_1}{\partial Y} = 0.$$

Solving it, we obtain

$$H_1 = \frac{1}{\sqrt{-(h-(s+1))^2 - 4\rho}}\left((h+s+1)\arctan\left(\frac{(h+s+1)X+2Y}{\sqrt{-(h-(s+1))^2 - 4\rho}X}\right)\right.$$
$$\left. - \frac{\sqrt{-(h-(s+1))^2 - 4\rho}\,\ln\left((hs+h-\rho)X^2 + (h+s+1)XY + Y^2\right)}{2}\right),$$

then the linear part of system (22) has no polynomial first integrals in a neighborhood of the origin, hence the result follows directly via Hartman-Grobman Theorem.

□

THEOREM 4.2. *System (22) with $h+s+1 = 0$ and $\rho + (s+1)^2 \neq 0$ has no analytic first integrals in a neighborhood of the origin.*

Proof. Suppose that $H = H(X, Y)$ is a local analytic first integral at the origin of system (22) where $h + s + 1 = 0$ and $\rho + (s + 1)^2 \neq 0$. We write $H = \sum_{k\geq 0} H_k(X, Y)$, where each H_k is a homogeneous polynomial of degree k for $k = 1, 2, \dots$. We use induction to show that

$$H_k = 0 \quad \text{for all} \quad k \geq 1. \tag{24}$$

If $H = H_0 =$ constant, then system (22) has no local analytic first integral at the origin. Since H is a first integral of system (22), it must satisfy

$$Y\,\frac{\partial H}{\partial X} + \big(\rho X + (s{+}1)(X^3{-}2X^2{+}(s{+}1)X) - (X^2{-}2X)Y\big)\,\frac{\partial H}{\partial Y} = 0. \tag{25}$$

The equation, which encompasses terms of degree one in the variables X and Y in (25) are

$$Y\,\frac{\partial H_1}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial H_1}{\partial Y} = 0.$$

Thus H_1 is either zero or it is polynomial first integral of linear part of system (22). The calculations shows that $H_1 = 0$.
The terms of degree two in the variables X and Y of (25), satisfy

$$Y\,\frac{\partial H_2}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial H_2}{\partial Y} = 0.$$

Then again either H_2 is zero or it is polynomial first integral. The solution of the partial differential equation above is $H_2 = c_2\,F_2$, where $c_2 \in \mathbb{C}$ and

$$F_2 = (-(\rho + (s+1)^2)X^2 + Y^2).$$

The terms of degree 3 in the variables X and Y in (25), satisfy

$$Y\,\frac{\partial H_3}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial H_3}{\partial Y} + c_2\,(2XY - 2(s+1)X^2)\,\frac{\partial F_2}{\partial Y} = 0.$$

Computing the homogeneous polynomial H_3, we obtain $H_3 = \frac{4}{3}\,c_2\,G_3$, where

$$G_3 = (s+1)X^3 - \frac{1}{(\rho + (s+1)^2)}Y^3.$$

Calculating the terms of degree 4 in the variables X and Y in (25), we obtain

$$\begin{aligned} Y\frac{\partial H_4}{\partial X} &+ (\rho + (s+1)^2)X\frac{\partial H_4}{\partial Y} \\ &+ (2XY - 2(s+1)X^2)\frac{\partial H_3}{\partial Y} + ((s+1)X^3 - X^2Y)\frac{\partial H_2}{\partial Y} = 0. \end{aligned}$$

Computing the homogeneous polynomials H_4, implies that $c_2 = 0$, then $H_2 = H_3 = 0$ and $H_4 = c_4\,F_2^2,\; c_4 \in \mathbb{C}$. The terms of degree 5 in the variables X and Y in (25), satisfy

$$Y\,\frac{\partial H_5}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial H_5}{\partial Y} + (2XY - 2(s+1)X^2)(2\,c_4\,F_2)\frac{\partial F_2}{\partial Y} = 0. \quad (26)$$

Computing the homogeneous polynomials H_5 in (26), which gives $H_5 = \frac{8}{3}\,c_4\,F_2\,G_3$. Computing the terms of degree 6 in variables X and Y in (25), gives

$$\begin{aligned} Y\frac{\partial H_6}{\partial X} &+ (\rho + (s+1)^2)X\frac{\partial H_6}{\partial Y} \\ &+ (2XY - 2(s+1)X^2)\frac{\partial H_5}{\partial Y} + ((s+1)X^3 - X^2Y)\frac{\partial H_4}{\partial Y} = 0. \end{aligned}$$

Computing the homogeneous polynomials H_6, implies that $c_4 = 0$, then $H_4 = H_5 = 0$ and $H_6 = c_6\,F_2^3,\;\; c_6 \in \mathbb{C}$. The terms of degree 7 in the variables X and Y in (25), satisfy

$$Y\,\frac{\partial H_7}{\partial X} + (\rho + (s+1)^2)\,X\,\frac{\partial H_7}{\partial Y} + (2XY - 2(s+1)\,X^2)\,(3\,c_6\,F_2^2)\,\frac{\partial F_2}{\partial Y} = 0. \quad (27)$$

Computing the homogeneous polynomials H_7 in (27), which is

$$H_7 = c_6\,F_2^2\,G_3. \quad (28)$$

The terms of degree 8 in the variables X and Y in (25), which satisfy

$$\begin{aligned} Y\frac{\partial H_8}{\partial X} + (\rho + (s+1)^2)X\frac{\partial H_8}{\partial Y} &+ (2XY - 2\,(s+1)X^2)\frac{\partial H_7}{\partial Y} \\ &+ ((s+1)X^3 - X^2Y)\frac{\partial H_6}{\partial Y} = 0. \end{aligned}$$

Computing the homogeneous polynomials H_8, we obtain $c_6 = 0$, then $H_6 = H_7 = 0$ and $H_8 = c_8\,F_2^4,\; c_8 \in \mathbb{C}$. We now prove by induction for $n \geq 3$.

$$\begin{aligned} &H_{2n} = c_{2n}\,F_2^n, \\ &H_{2n+1} = F_2^{n-1}\,g_3 \;\text{ and }\; H_i = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots, 2n-1, \end{aligned} \quad (29)$$

where $c_{2n} \in \mathbb{C}$ and $g_3 = g_3(X, Y)$ is a homogeneous polynomial of degree 3. From equation (28) is true for $n = 3$. Next, we assume that (29) is true for $n = 4, \ldots, N-1$ and we will prove it for $n = N$. By induction assumption the terms of degree $2N-2$ in the variables X and Y in (25), we obtain $H_{2N-2} =$

$c_{2N-2}\,F_2^{N-1}$, $c_{2N-2} \in \mathbb{C}$. Calculating the terms of degree $2N-1$ in the variables X and Y in (25), which are

$$Y\,\frac{\partial H_{2N-1}}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial H_{2N-1}}{\partial Y} + \big(2XY - 2(s+1)X^2\big)\big((N-1)\,c_{2N-2}\,F_2^{N-2}\big)\,\frac{\partial F_2}{\partial Y} = 0. \tag{30}$$

We consider two cases.

Case 1: H_{2N-1} is not divisible by F_2. Repeating the argument for passing from (27) and (28). We obtain $H_{2N-1} = F_2\,\tilde{G}_{2N-1}$, where $\tilde{G}_{2N-1} = \tilde{G}_{2N-1}(X,Y)$ is a homogeneous polynomial of degree $2N-3$ which is contradiction.

Case 2: H_{2N-1} is divisible by F_2. In this case we can write $H_{2N-1} = F_2^l\ g_3$ with $1 \le l \le N-3$ and g_3 is a homogeneous polynomial of degree $2N-1-2l$, otherwise the results would be obtained. Again, eventually we get a contradiction. In that case H_{2N-1} satisfies

$$Y\,\frac{\partial H_{2N-1}}{\partial X} + (\rho + (s+1)^2)\,X\,\frac{\partial H_{2N-1}}{\partial Y} + (2XY - 2(s+1)X^2)\left((N-1)\,c_{2N-2}\,F_2^{N-2-l}\,\frac{\partial F_2}{\partial Y}\right) = 0.$$

As $l \le N-3$, then the same argument used in Case 1, imply a contradiction. From the claim $H_{2N-1} = F_2^{N-2} g_3$, where $g_3 = g_3(X,Y)$ is a homogeneous polynomial of degree 3. Then equation (30) becomes

$$\begin{aligned}(N-2)F_2^{N-3}g_3&\left(Y\,\frac{\partial F_2}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial F_2}{\partial Y}\right)\\ &+\left(Y\frac{\partial g_3}{\partial X} + (\rho + (s+1)^2)\,X\frac{\partial g_3}{\partial Y}\right)F_2^{N-2}\\ &+(2\,X\,Y - 2(s+1)\,X^2)\left((N-1)\,c_{2N-2}\,F_2^{N-2}\,\frac{\partial F_2}{\partial Y}\right) = 0.\end{aligned} \tag{31}$$

Since F_2 is a first integral of linear part of system (22), then we can rewrite equation (31), as

$$Y\,\frac{\partial g_3}{\partial X} + (\rho + (s+1)^2)X\,\frac{\partial g_3}{\partial Y} + (2XY - 2(s+1)X^2)\left((N-1)\,c_{2N-2}\,\frac{\partial F_2}{\partial Y}\right) = 0. \tag{32}$$

The terms of degree $2N$ in the variables X and Y in (25) satisfy

$$\begin{aligned} &Y\frac{\partial H_{2N}}{\partial X}+(\rho+(s+1)^2)X\frac{\partial H_{2N}}{\partial Y}\\ &+(2XY-2(s+1)X^2)\left((N-2)g_3\,F_2^{N-3}\frac{\partial F_2}{\partial Y}\right)+(2XY-2(s+1)X^2)F_2^{N-2}\frac{\partial g_3}{\partial Y}\\ &\quad+(N-1)c_{2N-2}((s+1)X^3-YX^2)F_2^{N-2}\frac{\partial F_2}{\partial Y}=0. \end{aligned}$$

We now proceed similarly as calculating H_{2N-1}, we obtain $H_{2N}=F_2^{N-3}\,U_6$, where $U_6=U_6(X,Y)$ is a homogeneous polynomial of degree 6. Then U_6 satisfies

$$\begin{aligned} &Y\frac{\partial U_6}{\partial X}+(\rho+(s+1)^2)X\frac{\partial U_6}{\partial Y}+(2XY-2(s+1)X^2)\left((N-2)g_3\frac{\partial F_2}{\partial Y}\right)\\ &\qquad+(2XY-2(s+1)\,X^2)\,F_2\,\frac{\partial g_3}{\partial Y}\\ &\qquad\qquad+(N-1)\,c_{2N-2}\,((s+1)X^3-YX^2)\,F_2\,\frac{\partial F_2}{\partial Y}=0. \qquad (33) \end{aligned}$$

Computing the homogeneous polynomials g_3 and U_6 from (32) and (33), therefore $c_{2N-2}=0$ and $g_3=0$. This implies that $H_{2N-2}=H_{2N-1}=0$ and $H_{2N}=c_{2N}\,F_2^N$, $c_{2N}\in\mathbb{C}$. The terms of degree $2N+1$ in equation (25), gives

$$\begin{aligned} &Y\frac{\partial H_{2N+1}}{\partial X}+(\rho+(s+1)^2)X\frac{\partial H_{2N+1}}{\partial Y}\\ &\qquad\qquad+(2XY-2(s+1)X^2)\left(N\,c_{2N}\,F_2^{N-1}\,\frac{\partial F_2}{\partial Y}\right)=0. \end{aligned}$$

Then the same arguments used for calculating H_{2N-1} imply that H_{2N+1} is of the form $H_{2N+1}=F_2^{N-1}\,T_3$ where $T_3=T_3(X,Y)$ is a homogeneous polynomial of degree 3. Hence (29) holds true when $n=N$. $\square$

THEOREM 4.3. *System (22) has no local analytic first integral in a neighborhood of the singular point* $\left(\frac{h+\sqrt{-h^2s+h\rho}}{h},0\right)$ *if one of the following conditions hold.*

a. $\rho=hs$, $h\neq 0$ *and* $h+s\neq 0$,

b. $\rho-hs>0$ *and* $h>0$.

Proof. **a.** The eigenvalues of system (23), where $\rho=hs$ and $(h+s)\neq 0$, are $\lambda_1=0$ and $\lambda_2=-(h+s)$. Since $(0,0)$ is isolated singular point of system (23), by Theorem 2.12, we obtain the system (23) has no local analytic first integral in a neighborhood of the origin.

b. Computing the eigenvalues of system (22) at the singular point $\left(\frac{h+\sqrt{-h^2s+h\rho}}{h},0\right)$, which are

$$\lambda_1,\lambda_2=\frac{-(h^2+\rho)\pm\sqrt{h^4+8\,h^3s-8\,\sqrt{-h^2\,s+h\,\rho}\,h^2-6\,\rho\,h^2+\rho^2}}{2h}.$$

Suppose that there exist positive integers k_1,k_2 such that $k_1\lambda_1+k_2\lambda_2=0$. Computing $\lambda_1\lambda_2=2\big((\rho-h\,s)+\sqrt{h(\rho-hs)}\big)$. Note that by Theorem 2.11, if such integers do not exist we are done. Then $\lambda_1=-\alpha\lambda_2$ with α is a positive rational, and hence in particular $\lambda_1\lambda_2=-\alpha\lambda_2^2<0$. But $\lambda_1\lambda_2=2\big((\rho-hs)+\sqrt{h(\rho-hs)}\big)>0$ if $(\rho-h\,s)h>0$. The theorem's proof is now complete.

□

THEOREM 4.4. *If $h\neq 0$, $-h^2s+h\,\rho>0$ and $h^2+\rho\neq 0$ satisfies one of the following conditions*

i. $hs-\rho-\sqrt{-h^2s+h\rho}=0.$

ii. $hs-\rho-\sqrt{-h^2s+h\rho}\neq 0$ *and* $\frac{\frac{(h^2+\rho)^2}{h^2}}{2(hs-\rho-\sqrt{-h^2s+h\rho})}\notin\mathbb{Q}^-$.

iii. $hs-\rho-\sqrt{-h^2s+h\rho}\neq 0$, $\frac{2(hs-\rho-\sqrt{-h^2s+h\rho})}{\frac{(h^2+\rho)^2}{h^2}}=-\alpha\in\mathbb{Q}^-$ *and* $\alpha\neq\frac{p\,q}{(p-q)^2}$ *for some* $p,q\in\mathbb{Z}^+$.

iv. $(h^2+\rho)^2+8h^2(hs-\rho-\sqrt{-h^2s+h\rho})=0.$

v. $(h^2+\rho)^2+8h^2(hs-\rho-\sqrt{-h^2s+h\rho})\neq 0.$

Then the system (23) *has no analytic first integrals in a neighborhood of the origin.*

Proof. **i.** If the conditions are satisfied in this case, then the eigenvalues of the system (23), are $\lambda_1=0$ and $\lambda_2=-\frac{(h^2+\rho)}{h}$. Since $(0,0)$ is an isolated singular point of system (23), Theorem 2.12 guarantee that the system (23) has no local analytic first integrals in a neighborhood of the origin.

ii. Now $f(0)=\frac{h^2+\rho}{h}$, $g(0)'=2(hs-\rho-\sqrt{-h^2s+h\rho})$ and $\frac{f(0)^2}{g(0)'}=\frac{\frac{(h^2+\rho)^2}{h^2}}{2(hs-\rho-\sqrt{-h^2\,s+h\,\rho})}\notin\mathbb{Q}^-$. It is obvious

$$\lambda_1+\lambda_2=-(\frac{h^2+\rho}{h})\quad\text{and}\quad\lambda_1\lambda_2=2(hs-\rho-\sqrt{-h^2s+h\rho}).$$

It is sufficient to show $k_1\lambda_1 + k_2\lambda_2 \neq 0$ for $k_1, k_2 \in \mathbb{Z}^+$. Suppose that $k_1\lambda_1 + k_2\lambda_2 = 0$, then $\lambda_1 = -\alpha\lambda_2$ for some $\alpha \in \mathbb{Q}^+$. We obtain

$$\lambda_2(1-\alpha) = -\left(\tfrac{h^2+\rho}{h}\right) \quad \text{and} \quad -\alpha\lambda_2^2 = 2(hs - \rho - \sqrt{-h^2 s + h\rho}).$$

We have, $\frac{\frac{(h^2+\rho)^2}{h^2}}{2(hs-\rho-\sqrt{-h^2s+h\rho})} = -\frac{(1-\alpha)^2}{\alpha} \in \mathbb{Q}^-$. Since $\frac{h^2+\rho}{h} \neq 0$, then $\alpha \neq 1$ and $\alpha \neq 0$ because $2(h\,s - \rho - \sqrt{-h^2\,s + h\,\rho}) \neq 0$. Note that $\dfrac{\frac{(h^2+\rho)^2}{h^2}}{2(hs - \rho - \sqrt{-h^2 s + h\rho})} \notin \mathbb{Q}^-$, then $k_1\lambda_1 + k_2\lambda_2 \neq 0$. Hence by Theorem 2.11, the system (23) has no local analytic first integrals in a neighborhood of the origin.

iii. We write $g(0)' = -\alpha\,(\frac{h^2+\rho}{h})^2$ with $\alpha \in \mathbb{Q}^+ \setminus \{0\}$. With the rescaling $(X, Y, T) = \left(\left(\frac{h^2+\rho}{h}\right) X_1, Y_1, \left(\frac{h^2+\rho}{h}\right) t\right)$, system (23) becomes of the form

$$X' = Y, \quad Y' = -\frac{2(h\,s - \rho - \sqrt{-h^2\,s + h\rho})}{(\frac{h^2+\rho}{h})^2} X - Y + O(X, Y).$$

where $O(X, Y)$ denotes terms of higher order and without loss of generality we write (X_1, Y_1, t) instead of (X, Y, T_1), the system above becomes of the form

$$X_1' = Y_1, \quad Y_1' = \alpha X_1 - Y_1 + O(X_1, Y_1).$$

We proceed the proof as the proof of Lemma 12 in [17].

iv. The eigenvalues of system (23) with

$$(h^2+\rho)^2 + 8\,h^2\big(h\,s - \rho - \sqrt{-h^2 s + h\rho}\,\big) = 0 \text{ and } h^2 + \rho \neq 0$$

are repeated eigenvalues $\lambda_1 = \lambda_2 = -\frac{(h^2+\rho)}{2h}$. Then the resonance condition does not hold and by Theorem 2.11, the system (23) has no local analytic first integral in a neighborhood of origin.

v. We know that a necessary condition in order that system (23) has analytic first integral is that the linear part of system (23) with $h^2 + \rho \neq 0$ and $(h^2+\rho)^2 + 8h^2(hs - \rho - \sqrt{-h^2 s + h\rho}) \neq 0$, admits a polynomial first integral.

$$Y_1\frac{\partial H_1}{\partial X_1} + \left(-2X_1\sqrt{-h^2s+h\rho} + 2hsX_1 - 2\rho X_1 - \frac{h^2+\rho}{h}Y_1\right)\frac{\partial H_1}{\partial Y_1} = 0.$$

Solving it, we obtain

$$H_1 = \frac{1}{\sqrt{-((h^2+\rho)^2+8h^2(-\rho+hs-\sqrt{-h^2s+h\rho}))}}\Big((h^2+\rho)$$
$$\arctan\left(\frac{(h^2+\rho)X_1+2hY_1}{\sqrt{-((h^2+\rho)^2+8h^2(-\rho+hs-\sqrt{-h^2s+h\rho}))}X_1}\right)-$$
$$\frac{1}{2}\sqrt{-((h^2+\rho)^2+8h^2(-\rho+hs-\sqrt{-h^2s+h\rho}))}\ln\big(-2h^2sX_1^2$$
$$+2\sqrt{-h^2s+h\rho}hX_1^2+h^2X_1Y_1+2h\rho X_1^2+hY_1^2+\rho X_1Y_1\big)\Big).$$

Since the linear part of system (23) has no polynomial first integrals in a neighborhood of the origin, hence the result follows directly.

□

THEOREM 4.5. *The system* (23) *has no local analytic first integrals in a neighborhood of the origin if* $h^2+\rho=0$ *and* $h(h+s)-\sqrt{-h^2(h+s)})\neq 0$.

Proof. Suppose that $H=H(X_1,Y_1)$ is a local analytic first integral at the origin of system (23) where $h^2+\rho=0$ and $h(h+s)-\sqrt{-h^2(h+s)})\neq 0$. We write $H=\sum_{i\geq 0} H_i(X_1,Y_1)$, where each H_i is a homogeneous polynomial of degree i for $i=1,2,\ldots$. We use induction to show that

$$H_i = 0 \quad \text{for all} \quad i \geq 1. \tag{34}$$

If equation (34) implies that $H=H_0=$ constant, then system (23) has no local analytic first integral at the origin. Since H is a first integral of system (23) with $h^2+\rho=0$ and $h(h+s)-\sqrt{-h^2(h+s)})\neq 0$, it must satisfy

$$Y_1\frac{\partial H}{\partial X_1}+\bigg(-hX_1^3+2h^2X_1+2hsX_1-hX_1^2-3X_1^2\sqrt{-h^2(h+s)}$$
$$-2X_1\sqrt{-h^2(h+s)}-\left(X_1^2Y_1+\frac{2X_1Y_1}{h}\sqrt{-h^2(h+s)}\right)\bigg)\frac{\partial H}{\partial Y_1}=0. \tag{35}$$

The terms of degree one in the variables X_1 and Y_1 in (35) satisfy

$$Y_1\frac{\partial H_1}{\partial X_1}+2\big(h^2+h\,s-\sqrt{-h^2\,(h+s)}\,\big)\,X_1\,\frac{\partial H_1}{\partial Y_1}=0.$$

Thus H_1 is either zero or it is polynomial first integral of linear part of system (23). As before, we obtain $H_1=0$. Computing the terms of degree two in the variables X_1 and Y_1 of (35), which satisfy

$$Y_1\frac{\partial H_2}{\partial X_1}+2\big(h^2+h\,s-\sqrt{-h^2\,(h+s)}\,\big)\,X_1\,\frac{\partial H_2}{\partial Y_1}=0.$$

Then either H_2 is zero or it is polynomial first integral of linear system. Computing the homogeneous polynomial H_2, we obtain $H_2 = C_2\, T_2, \quad C_2 \in \mathbb{C}$, where

$$T_2 = Y_1^2 + 2\big(\sqrt{-h^2\,(h+s)} - h\,(h+s)\,\big)\, X_1^2.$$

The terms of degree 3 in the variables X_1 and Y_1 in (35), satisfy

$$\begin{aligned} &Y_1 \frac{\partial H_3}{\partial X_1} + 2\left(h^2 + h\,s - \sqrt{-h^2\,(h+s)}\right) X_1\, \frac{\partial H_3}{\partial Y_1} \\ &\quad + C_2\left(-hX_1^2 - 3X_1^2\sqrt{-h^2(h+s)} - \frac{2X_1\,Y_1}{h}\sqrt{-h^2\,(h+s)}\right)\,\frac{\partial T_2}{\partial Y_1} = 0. \end{aligned}$$

Computing the homogeneous polynomial H_3, we obtain $H_3 = \frac{2}{3}\, C_2\, G_3$, where

$$G_3 = \left(3\sqrt{-h^2\,(h+s)} + h\right)\; X_1^3 - \left(\frac{h - \sqrt{-h^2\,(h+s)}}{h^2\,(h+s+1)}\right) Y_1^3.$$

Calculating the terms of degree 4 in the variables X_1 and Y_1 in (35), which are

$$\begin{aligned} &Y_1\, \frac{\partial H_4}{\partial X_1} + 2\left(h^2 + hs - \sqrt{-h^2(h+s)}\right) X_1\, \frac{\partial H_4}{\partial Y_1} \\ &\quad + \left(-hX_1^2 - 3X_1^2\sqrt{-h^2(h+s)} - \frac{2X_1Y_1}{h}\sqrt{-h^2(h+s)}\right)\,\frac{\partial H_3}{\partial Y_1} \\ &\qquad\qquad + (-hX_1^3 - X_1^2Y_1)\,\frac{\partial H_2}{\partial Y_1} = 0. \end{aligned}$$

Computing the homogeneous polynomials H_4, we obtain $C_2 = 0$ and $H_2 = H_3 = 0,\; H_4 = C_4\, T_2^2,\; C_4 \in \mathbb{C}$. By the same argument in Theorem 4.2 we can show that $C_4 = 0$, $H_4 = H_5 = 0$ and $H_6 = C_6\, T_2^3$. The terms of degree 7 in the variables X_1 and Y_1 in (35), satisfy

$$\begin{aligned} &Y_1\, \frac{\partial H_7}{\partial X_1} + 2\left(h^2 + hs - \sqrt{-h^2(h+s)}\right) X_1\, \frac{\partial H_7}{\partial Y_1} \\ &\quad + \left(-hX_1^2 - 3X_1^2\sqrt{-h^2(h+s)} - \frac{2X_1Y_1}{h}\sqrt{-h^2(h+s)}\right)\,\frac{\partial H_6}{\partial Y_1} = 0. \end{aligned}$$

Computing the homogeneous polynomials H_7, which is

$$H_7 = C_6\; T_2^2\; G_3. \tag{36}$$

Now we will prove by induction for $n \geq 3$.

$$\begin{aligned} &H_{2n} = C_{2N}\, T_2^n, \\ &H_{2n+1} = T_2^{n-1}\, g_3 \;\text{ and }\; H_i = 0 \;\text{ for }\; i = 1, 2, 3, \ldots, 2n-1, \end{aligned} \tag{37}$$

where $C_{2n} \in \mathbb{C}$ and $g_3 = g_3(X, Y)$ is a homogeneous polynomial of degree 3. From equation (36) it is true for $n = 3$. Next we assume that (37) is true for $n = 4, \ldots, N-1$ and we will prove it for $n = N$. By induction assumption the terms of degree $2N-2$, in (37), $H_{2N-2} = C_{2N-2}\, T_2^{N-1}$, $C_{2N-2} \in \mathbb{C}$. By the same argument in Theorem 4.2 we can show that $H_i = 0$ for all $i \geq 1$. $\square$

References

[1] M.Y. Antonovsky, E.A. Aponina, and Y.A. Kuznetsov, *Spatial-temporal structure of mixed-age forest boundary: The simplest mathematical model*, IIASA Working Paper WP-89-054, 1989.

[2] M.Y. Antonovsky, E.A. Aponina, and Y.A. Kuznetsov, *On the stability analysis of the standing forest boundary*, IIASA Working Paper WP-91-010, 1991.

[3] M.Y. Antonovsky, W.C. Clark, and Y.A. Kuznetsov, *The influence of pests on forest age structure dynamics: The simplest mathematical models*, IIASA Working Paper WP-87-070, 1987.

[4] M.Y. Antonovsky, R.A. Fleming, Y.A. Kuznetsov, and W.C. Clark, *Forest-pest interaction dynamics: The simplest mathematical models*, Theor. Population Biol. **37** (1990), no. 2, 343–367.

[5] M.Y. Antonvsky and M. Korzukhin, *Mathematical modelling of economic and ecological economic processes*, Integrated Global Monitoring of Environmental pollution, 1983, pp. 353–358.

[6] G. Cantin and N. Verdière, *Mathematical modeling of complex forest ecosystems: impacts of deforestation*, hal-02496187, 2020.

[7] C. Christopher, J. Llibre, and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. Math. **229** (2007), no. 1, 63–117.

[8] F. Dumortier, J. Llibre, and J.C. Artés, *Qualitative theory of planar differential systems*, vol. 2, Springer, 2006.

[9] R.A. Fleming, M.Y. Antonovsky, and Y.A. Kuznetsov, *The response of the balsam fir forest to a spruce budworm invasion: A simple dynamical model*, IIASA Working Paper WP-87-071, 1987.

[10] M.D. Korzukhin, *Age structure dynamics of high edification ability tree population*, Problems of ecological monitoring and ecosystem modelling, vol. 3, 1980, pp. 162–178.

[11] Y.A. Kuznetsov, M.Y. Antonovsky, V.N. Biktashev, and E.A. Aponina, *A cross-diffusion model of forest boundary dynamics*, J. Math. Biol. **32** (1994), 219–232.

[12] J. Llibre and C. Valls, *Analytic integrability of a chua system*, J. Math. Phys. **49** (2008), no. 10, 102701.

[13] J. Llibre and C. Valls, *Integrability of a sis model*, J. Math. Anal. Appl. **344** (2008), no. 1, 574–578.

[14] J. Llibre and C. Valls, *On the local analytic integrability at the singular point of a class of liénard analytic differential systems*, Proc. Amer. Math. Soc. **138**

(2010), no. 1, 253–261.

[15] J. Llibre and C. Valls, *Liouvillian and analytic first integrals for the brusselator system*, J. Nonlinear Math. Phys. **19** (2012), no. 2, 247–254.

[16] J. Llibre and C. Valls, *On the integrability of a muthuswamy–chua system*, J. Nonlinear Math. Phys. **19** (2012), no. 4, 477–488.

[17] J. Llibre and C. Valls, *On the analytic integrability of the liénard analytic differential systems*, Discrete Contin. Dyn. Syst. Ser. B **21** (2016), no. 2, 557–573.

[18] J. Llibre and X. Zhang, *Darboux theory of integrability for polynomial vector fields in rn taking into account the multiplicity at infinity*, Bull. Sci. Math. **133** (2009), no. 7, 765–778.

[19] J. Llibre and X. Zhang, *Darboux theory of integrability in cn taking into account the multiplicity*, J. Differential Equations **246** (2009), no. 2, 541–551.

[20] K.B. Othman and A.I. Amen, *Periodic solutions of the forest pest system via hopf bifurcation and averaging theory*, Iraqi J. Sci. (2022), 5496–5509.

[21] T.V. Ta, L.T.H. Nguyen, and A. Yagi, *A sustainability condition for stochastic forest model*, arXiv:1509.00527, 2015.

Authors' addresses:

Evan Sulaiman
Mathematics Department
College of Science
Salahaddin University–Erbil
Kurdistan Region, Iraq
E-mail: evan.sulaiman@su.edu.krd

Azad Amen
Department of Mathematics
College of Basic Education, Salahaddin University–Erbil
Basic Education College, Raparin University–Ranya, Iraq
Faculty of Science, Soran University Soran, Iraq
College of Science, Duhok University, Iraq
E-mail: azad.amen@su.edu.krd

Waleed Aziz
Mathematics Department
College of Science
Salahaddin University–Erbil
Kurdistan Region, Iraq
E-mail: waleed.aziz@su.edu.krd

Received November 21, 2022
Revised June 5, 2023
Accepted July 7, 2023