Rend. Istit. Mat. Univ. Trieste
Vol. 55 (2023), Art. No. 2, 24 pages
DOI: 10.13137/2464-8728/34711

On meromorphic solutions of a certain type of nonlinear differential-difference equation

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Abstract. *The main objective of the paper is to give the specific forms of the meromorphic solutions of the equation* $f^n(z)f(z+c)+P_d(z,f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$*, where* $c \in \mathbb{C}\setminus\{0\}$*,* $P_d(z,f)$ *is a differential-difference polynomial in* f *of degree* $d \leq n-1$ *with small functions of* f *as its coefficients,* $p_1, p_2(\not\equiv 0)$ *are rational functions and* α_1*,* α_2 *are non-constant polynomials.*

Keywords: Nevanlinna's value distribution theory, nonlinear differential equation, differential-difference polynomial.
MS Classification 2020: 30D35, 33E30, 30D30.

1. Introduction

Let $\mathscr{M}(\mathbb{C})$ be the set of all non-constant meromorphic functions in $\mathbb{C}$, whereas $\mathscr{E}(\mathbb{C})$ denotes the set of all non-constant entire functions. On the other hand we denote by $\mathscr{M}_T(\mathbb{C})$ and $\mathscr{E}_T(\mathbb{C})$ the set of all transcendental meromorphic and entire functions respectively. Let $f \in \mathscr{M}(\mathbb{C})$ and $a \in \mathscr{M}(\mathbb{C}) \cup \mathbb{C}$ such that $f \not\equiv a$. We denote by $n(t,a;f) = n(t,0;f-a)$ the number of roots of the equation $f(z)-a(z) = 0$ in $|z| \leq t$, multiple roots being counted multiplely and by $\overline{n}(t,a;f)$ the number of distinct roots of $f(z)-a(z) = 0$ in $|z| \leq t$. Correspondingly we define

$$N(r,a;f) = \int_0^r \frac{n(t,a;f)-n(0,a;f)}{t}dt + n(0,a;f)\log r,$$

$$\overline{N}(r,a;f) = \int_0^r \frac{\overline{n}(t,a;f)-\overline{n}(0,a;f)}{t}dt + \overline{n}(0,a;f)\log r.$$

Also we use the standard notations of Nevanlinna's value distribution theory such as $N(r,f)$, $m(r,f)$, $T(r,f)$, ... (see, e.g., [3, 11]). By $S(r,f)$ we denote any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside of an exceptional set of finite linear measure. A meromorphic function

a is said to be a small function of f if $T(r,a)=S(r,f)$. We denote by $\mathscr{S}(f)$ the set of all small functions of f. Also we use $\rho(f)$ and $\rho_2(f)$ to denote the order and hyper-order of a meromorphic function f respectively.

By a differential polynomial $P_d(z,f)$ in f of degree d, we mean it is a polynomial in f and its derivatives with a total degree d and small functions of f as the coefficients. Note that $P_d(z,f)$ is said to be an algebraic differential polynomial if the coefficients are polynomials. By a differential-difference polynomial $P_d(z,f)$ in f of degree d, we mean it is a polynomial in f, $f(z+c)$ and their derivatives with a total degree d and small functions of f as the coefficients.

It is difficult to prove the existence of solutions of a given differential equation and it is also interesting to find out the solutions if the solutions exist.

A special type of nonlinear differential equation $f^n+P_d(z,f)=h$, where $h\in\mathscr{M}(\mathbb{C})\cup\mathbb{C}$ and $P_d(z,f)$ is a differential polynomial in f of degree d, has become a matter of increasing interest among the researchers.

It is easy to verify that $f_1(z)=\sin z$ is a solution of the differential equation $4f^3(z)+3f''(z)=-\sin 3z$. In [4], it was proved that $f_2(z)=-\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$ is also a solution of this equation. In 2004, Yang and Li [10] proved that this equation admits exactly three entire solutions namely $f_1(z)$, $f_2(z)$ and $f_3(z)=\frac{\sqrt{3}}{2}\cos z-\frac{1}{2}\sin z$. Since $-\sin 3z$ is a linear combination of e^{i3z} and e^{-i3z}, so it is interesting to find out all entire solutions of the following general equation

$$f^n(z)+P_d(z,f)=p_1e^{\lambda z}+p_2e^{-\lambda z}, \tag{1.1}$$

where $p_1,p_2,\lambda\in\mathbb{C}\setminus\{0\}$ and $d\leq n-1$.

In this direction, Yang and Li [10] obtained the following result.

Theorem 1.1 ([10]). *Let $P_d(z,f)$ be a differential polynomial such that $d\leq n-3$, where $n\geq 3$, $b\in\mathscr{S}(f)$ and $\lambda,p_1,p_2\in\mathbb{C}\setminus\{0\}$. Then there does not exist $f\in\mathscr{E}_T(\mathbb{C})$ such that $f^n(z)+P_d(z,f)=b(z)\left(p_1e^{\lambda z}+p_2e^{-\lambda z}\right)$.*

In 2006, Li and Yang [7] further generalized Theorem A and obtained the following result.

Theorem 1.2 ([7]). *Let $P_d(z,f)$ be an algebraic differential polynomial such that $d\leq n-3$, where $n\geq 4$. Let p_1, p_2 be non-zero polynomials, $\alpha_1,\alpha_2\in\mathbb{C}\setminus\{0\}$ such that $\frac{\alpha_1}{\alpha_2}\notin\mathbb{Q}$. Then there does not exist $f\in\mathscr{E}_T(\mathbb{C})$ such that $f^n(z)+P_d(z,f)=p_1(z)e^{\alpha_1 z}+p_2(z)e^{\alpha_2 z}$.*

In 2011, Li [6] derived the possible forms of the solutions of the equation (1.1) when $d\leq n-2$ and obtained the following result.

Theorem 1.3 ([6]). *Let $P_d(z,f)$ be a differential polynomial such that $d\leq n-2$, where $n\geq 2$ and $p_1,p_2,\alpha_1,\alpha_2\in\mathbb{C}\setminus\{0\}$ with $\alpha_1\neq\alpha_2$. If $f\in\mathscr{M}_T(\mathbb{C})$ is a solution of the equation $f^n(z)+P_d(z,f)=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}$ satisfying $N(r,f)=S(r,f)$, then one of the following holds*

(i) $f(z)=c_0(z)+c_1e^{\frac{\alpha_1}{n}z}$, *where* $c_0\in\mathscr{S}(f)$ *and* $c_1^n=p_1$;

(ii) $f(z)=c_0(z)+c_2e^{\frac{\alpha_2}{n}z}$, *where* $c_0\in\mathscr{S}(f)$ *and* $c_2^n=p_2$;

(iii) $f(z)=c_1e^{\frac{\alpha_1}{n}z}+c_2e^{\frac{\alpha_2}{n}z}$, *where* $\alpha_1+\alpha_2=0$ *and* $c_i^n=p_i$, $i=1,2$.

In 2013, Liao, Yang and Zhang [8] further extended and improved the above results by giving the following result.

THEOREM 1.4 ([8]). *Let $P_d(z,f)$ be a differential polynomial with rational functions as its coefficients. Let $p_1,p_2(\not\equiv 0)$ be rational functions, α_1, α_2 be polynomials and $n\geq 3$. If $d\leq n-2$, then the differential equation $f^n+P_d(z,f)=p_1e^{\alpha_1}+p_2e^{\alpha_2}$ admits a solution $f\in\mathscr{M}(\mathbb{C})$ with finitely many poles and $\frac{\alpha_1'}{\alpha_2'}\in\mathbb{Q}$. Furthermore only one of the following four cases holds:*

(1) *$f=qe^p$ and $\frac{\alpha_1'}{\alpha_2'}=1$, where $q(\not\equiv 0)$ is a rational function and p is a polynomial with $np'=\alpha_1'=\alpha_2'$;*

(2) *$f=qe^p$ and either $\frac{\alpha_1'}{\alpha_2'}=\frac{k}{n}$ or $\frac{\alpha_1'}{\alpha_2'}=\frac{n}{k}$, where $q(\not\equiv 0)$ is a rational function, $k\in\mathbb{N}$ with $1\leq k\leq d$ and p is a polynomial with $np'=\alpha_1'$ or $np'=\alpha_2'$;*

(3) *f satisfies one of the differential equations* (1) $f'=\frac{1}{n}\left(\frac{p_2'}{p_2}+\alpha_2'\right)f+\psi$ *and* $\frac{\alpha_1'}{\alpha_2'}=\frac{n-1}{n}$ *and* (2) $f'=\frac{1}{n}\left(\frac{p_1'}{p_1}+\alpha_1'\right)f+\psi$ *and* $\frac{\alpha_1'}{\alpha_2'}=\frac{n}{n-1}$, *where ψ is a rational function;*

(4) *$f=\gamma_1e^{\beta_1}+\gamma_2e^{-\beta_1}$ and $\frac{\alpha_1'}{\alpha_2'}=-1$, where γ_1, $\gamma_2(\not\equiv 0)$ are rational functions and β_1 is a polynomial with $n\beta_1'=\alpha_1'$ or $n\beta_1'=\alpha_2'$.*

Now it is interesting to find out all the meromorphic solutions of the following nonlinear differential-difference equation:

$$f^n(z)f(z+c)+P_d(z,f)=p_1(z)e^{\alpha_1(z)}+p_2(z)e^{\alpha_2(z)}, \tag{1.2}$$

where $c\in\mathbb{C}\setminus\{0\}$, $P_d(z,f)$ is a differential-difference polynomial with small functions of f as its coefficients, $p_1,p_2(\not\equiv 0)$ are rational functions and α_1, α_2 are non-constant polynomials.

The objective of the paper is threefold. Our first objective is to find out the possible solution of the nonlinear differential-difference equation given by (1.2), when the right side of the equation (1.2) contains only one term. Now we state one of our main results.

THEOREM 1.5. *Let $c\in\mathbb{C}\setminus\{0\}$ and $P_d(z,f)$ be a differential-difference polynomial with small functions of f as its coefficients and $n\geq d+2$. Let $p(\not\equiv 0)$*

be a rational function and α be a non-constant polynomial. If $f \in \mathscr{M}(\mathbb{C})$ is a solution of the equation

$$f^n(z)f(z+c) + P_d(z,f) = p(z)e^{\alpha(z)} \tag{1.3}$$

satisfying $\rho_2(f) < 1$ and $N(r,f) = O(\log r)$, then $P_d(z,f) \equiv 0$ and $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c) = p(z)$ and $np'(z) + p'(z+c) = \alpha'(z)$.

Let us take $f^2(z)f(z+c) + P_d(z,f) = p(z)e^{\alpha(z)}$, where

$$P_d(z,f) = -\frac{3}{2}f'(z) - 1,$$

$p(z) = -\frac{1}{2}$, $\alpha(z) = 3z$ and $c \in \mathbb{C} \setminus \{0\}$ such that $e^c = -\frac{1}{2}$. Here $n = 2$ and $d = 1$. Clearly $f(z) = e^z + 1$ is a solution of the given equation and so the given equation admits a solution which is not of the form $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial.

Our second objective is to find out the possible forms of meromorphic solutions of the differential-difference equation (1.2), when $p_1, p_2(\not\equiv 0)$ are rational functions and α_1, α_2 are non-constant polynomials. In this regard, we obtain the following result.

THEOREM 1.6. *Let $c \in \mathbb{C} \setminus \{0\}$ and $P_d(z,f)$ be a differential-difference polynomial with small functions of f as its coefficients and $n \geq d+3$. Suppose $p_1, p_2(\not\equiv 0)$ are rational functions and α_1, α_2 are non-constant polynomials. If $f \in \mathscr{M}(\mathbb{C})$ is a solution of the equation (1.2) satisfying $\rho_2(f) < 1$ and $N(r,f) = O(\log r)$, then one of the following cases holds:*

(1) $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c) = b_1p_1(z) + p_2(z)$, where $b_1 \in \mathbb{C}$ and $np'(z) + p'(z+c) = \alpha_1'(z) = \alpha_2'(z)$.

(2) $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that either $q^n(z)q(z+c) = p_1(z)$ and $np'(z)+p'(z+c) = \alpha_1'(z)$ or $q^n(z)q(z+c) = p_2(z)$ and $np'(z) + p'(z+c) = \alpha_2'(z)$.

Let us take $f^2(z)f(z+c) + P_d(z,f) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$, where

$$P_d(z,f) = -if(z+c),\ p_1(z) = p_2(z) = i,\ \alpha_1(z) = 3z,\ \alpha_2(z) = -3z$$

and $c \in \mathbb{C}\setminus\{0\}$ such that $e^c = i$. Here $n = 2$ and $d = 1$. Clearly $f(z) = e^z + e^{-z}$ is a solution of the given equation and so the given equation admits a solution which is not of the form $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial.

REMARK 1.7. In Theorem 1.6 we study the existence of meromorphic solutions of equation (1.2) having finitely many poles. Now our next purpose is to study the existence of meromorphic solutions of equation (1.2) satisfying $N(r,f) = S(r,f)$.

For further study, it is quite natural to ask the following question.

Question 1. How to find the solutions of the equation (1.2) under the condition $n \geq d+1$?

DEFINITION 1.8. *Let $f,\ g \in \mathscr{M}(\mathbb{C})$ and $a \in \mathscr{S}(f) \cap \mathscr{S}(g)$. Denote by $N_E(r,a)$ the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities. If $N(r,a;f)+N(r,a;g)-2N_E(r,a) = S(r,f)$, then we say f and g share a CM$_*$.*

Our third objective for writing this paper is to find out the possible answer to the above question. In the paper we have been able to solve **Question 1** at the cost of considering the fact that $f(z)$ and $f(z+c)$ share 0 CM$_*$ and obtain the following result.

THEOREM 1.9. *Let $c \in \mathbb{C}\setminus\{0\}$ and $P_d(z,f)$ be a differential-difference polynomial with small functions of f as its coefficients and $n \geq d+1$. Suppose $p_1, p_2 (\not\equiv 0)$ are rational functions and α_1, α_2 are non-constant polynomials. If $f \in \mathscr{M}(\mathbb{C})$ is a solution of the equation (1.2)*

such that $\rho_2(f) < 1$, $N(r,f) = S(r,f)$ and $f(z)$, $f(z+c)$ share 0 CM$_*$, then one of the following cases holds:

1. *$f = qe^{\frac{\alpha_2}{n+1}}$, $q \in \mathscr{S}(f)\setminus\{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_2(z+c)-\alpha_2(z)}{n+1}} = c_0p_2(z)$, where $e^{\alpha_1-\alpha_2} \in \mathscr{S}(f)$, $c_0 \in \mathbb{C}\setminus\{0\}$;*
2. *$f = qe^{\frac{\alpha_1}{n+1}}$, $q \in \mathscr{S}(f)\setminus\{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}} = p_1(z)+\varphi(z)p_2(z)$, where $\varphi = e^{\alpha_2-\alpha_1} \in \mathscr{S}(f)$;*
3. *$f = qe^{\frac{\alpha_1}{n+1}}$, $q \in \mathscr{S}(f)\setminus\{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}} = p_1(z)$ and $e^{\frac{k\alpha_1-(n+1)\alpha_2}{n+1}} \in \mathscr{S}(f)$, where $k \in \{0,1,2,\ldots,d\}$;*
4. *$f = u_1e^{\frac{\alpha_1}{n+1}} - v_1$, where $u_1, v_1 \in \mathscr{S}(f)\setminus\{0\}$ such that $u_1^n(z)u_1(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}} = p_1(z)$, $u_1^{n+1}(z)v_1(z+c) = p_1(z)v_1(z)$ and $e^{n\alpha_1-(n+1)\alpha_2} \in \mathscr{S}(f)$;*
5. *$f = \delta_1e^{\gamma}+\delta_2e^{-\gamma}$, where $e^{\alpha_1+\alpha_2} \in \mathscr{S}(f)$, $\delta_1, \delta_2 \in \mathscr{S}(f)\setminus\{0\}$ and γ is a non-constant polynomial such that either $e^{(n+1)\gamma+\alpha_1} \in \mathscr{S}(f)$ or $e^{(n+1)\gamma+\alpha_2} \in \mathscr{S}(f)$.*

From Theorems 1.5, 1.6 and 1.9, we have the following corollary.

COROLLARY 1.10. *Equations (1.2) and (1.3) do not have any solution* $f \in \mathscr{M}(\mathbb{C})$ *satisfying* $N(r,f) = O(\log r)$ $(S(r,f))$, $\rho(f) = +\infty$ *and* $\rho_2(f) < 1$.

The following example shows that conclusion (4) in Theorem 1.9 cannot be removed.

EXAMPLE 1.11. Let us take $f^2(z)f(z+c)+P_d(z,f) = p_1(z)e^{\alpha_1(z)}+p_2(z)e^{\alpha_2(z)}$, where $P_d(z,f) = -\frac{1}{3}f'(z)+\frac{1}{27}$, $p_1(z) = p_2(z) = 1$, $\alpha_1(z) = 3z$, $\alpha_2(z) = 2z$ and $c \in \mathbb{C} \setminus \{0\}$ such that $e^c = 1$. Here $n = 2$ and $d = 1$. Clearly $f = u_1 e^{\frac{\alpha_1}{n+1}} - v_1$, where $u_1 = 1$ and $v_1 = \frac{1}{3}$ is a solution of the given equation. Note that $u_1^n(z)u_1(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}} = p_1(z)$, $u_1^{n+1}(z)v_1(z+c) = p_1(z)v_1(z)$ and $f(z)$, $f(z+c)$ share 0 CM_*.

The following example shows that conclusion (5) in Theorem 1.9 cannot be removed.

EXAMPLE 1.12. Let us take $f(z)f(z+c)+P_d(z,f) = p_1(z)e^{\alpha_1(z)}+p_2(z)e^{\alpha_2(z)}$, where $P_d(z,f) \equiv 2$, $p_1(z) = p_2(z) = -1$, $\alpha_1(z) = 2z$, $\alpha_2(z) = -2z$ and $c \in \mathbb{C}\setminus\{0\}$ such that $e^c = -1$. Here $n = 1$ and $d = 0$. Clearly $f = \delta_1 e^{\gamma} + \delta_2 e^{-\gamma}$ is a solution of the given equation, where $\delta_1 = \delta_2 = 1$ and $\gamma(z) = z$. Note that $f(z)$ and $f(z+c)$ share 0 CM_* and $e^{(n+1)\gamma(z)+\alpha_2(z)} \in \mathscr{S}(f)$.

2. Auxiliary lemmas

LEMMA 2.1 ([5]). *Let* $f \in \mathscr{M}_T(\mathbb{C})$ *be a solution of finite order* ρ *of the equation* $H(z,f)P(z,f) = Q(z,f)$, *where* $H(z,f), P(z,f)$ *and* $Q(z,f)$ *are difference polynomials such that the total degree of* $H(z,f)$ *in* f *and its shifts is* n *and that the total degree of* $Q(z,f)$ *is at most* n. *If* $H(z,f)$ *just contains one term of maximal total degree, then* $m(r,P(z,f)) = O\left(r^{\rho-1+\varepsilon}\right) + S(r,f)$ *holds possibly outside of an exceptional set of finite logarithmic measure, where* $\varepsilon > 0$.

REMARK 2.2. Particularly, if $H(z,f) = f^n(z)$, then a similar conclusion holds when $P(z,f)$ and $Q(z,f)$ are differential-difference polynomials in f.

LEMMA 2.3 ([1]). *Let* $f \in \mathscr{M}_T(\mathbb{C})$ *and* $f^n(z)P(z,f) = Q(z,f)$, *where* $P(z,f)$ *and* $Q(z,f)$ *are polynomials in* f *and its derivatives with meromorphic coefficients, say* $\{a_\lambda(z)\,|\lambda \in I\}$ *such that* $m(r,a_\lambda) = S(r,f)$ *for all* $\lambda \in I$. *If the total degree of* $Q(z,f)$ *as a polynomial in* f *and its derivatives is less than or equal to* n, *then* $m(r,P(z,f)) = S(r,f)$.

LEMMA 2.4 ([3]). *Let* $f \in \mathscr{M}(\mathbb{C})$ *and* $a_i \in \mathscr{S}(f)$, $i = 1,2$. *Then* $T(r,f) \le \overline{N}(r,f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f)$.

LEMMA 2.5 ([2]). *Let* $c \in \mathbb{C} \setminus \{0\}$, $\varepsilon > 0$ *and* $f \in \mathscr{M}(\mathbb{C})$ *such that* $\rho_2(f) < 1$. *Then*

$$m\Big(r,\frac{f(z+c)}{f(z)}\Big) + m\Big(r,\frac{f(z)}{f(z+c)}\Big) = o\Big(\frac{T(r,f)}{r^{1-\rho_2(f)-\varepsilon}}\Big)$$

outside of an exceptional set of finite logarithmic measure.

LEMMA 2.6 ([9]). *Let $f \in \mathscr{M}_T(\mathbb{C})$ such that $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Then $T(r, f(z+c)) = T(r,f) + S(r,f)$ and $N(r, f(z+c)) = N(r,f) + S(r,f)$.*

LEMMA 2.7. *Let $c \in \mathbb{C} \setminus \{0\}$ and $P_d(z,f)$ be a differential-difference polynomial with small functions of f as its coefficients and $d \leq n-1$. Suppose $p_1, p_2 (\not\equiv 0)$ are rational functions and α_1, α_2 are non-constant polynomials. If $f \in \mathscr{M}(\mathbb{C})$ is a solution of (1.2) satisfying $\rho_2(f) < 1$ and $N(r,f) = S(r,f)$, then $f \in \mathscr{M}_T(\mathbb{C})$ and $\rho(f) < +\infty$.*

Proof. Let $f \in \mathscr{M}(\mathbb{C})$ be a solution of the equation (1.2). We claim that $f \in \mathscr{M}_T(\mathbb{C})$. If not, suppose f is a rational function. In this case $p_1 e^{\alpha_1} + p_2 e^{\alpha_2}$ must be a rational function, say $R_1 (\not\equiv 0)$ and so $-p_1 e^{\alpha_1} = p_2 e^{\alpha_2} - R_1$. Consequently $p_2 e^{\alpha_2} - R_1$ has finitely many zeros and so by Lemma 2.4 we get

$$\begin{aligned} T(r, p_2 e^{\alpha_2}) &\leq \overline{N}(r, p_2 e^{\alpha_2}) + \overline{N}(r, 0; p_2 e^{\alpha_2}) + \overline{N}(r, R_1; p_2 e^{\alpha_2}) + S(r, p_2 e^{\alpha_2}) \\ &= S(r, p_2 e^{\alpha_2}), \end{aligned}$$

which is impossible. Hence $f \in \mathscr{M}_T(\mathbb{C})$. Note that

$$P_d(z,f) = \sum_{\mu} b_\mu(z) G_\mu(z,f),$$

where $b_\mu \in \mathscr{S}(f)$ and

$$\begin{aligned} &G_\mu(z,f) \\ &= (f(z))^{p_0^\mu} (f'(z))^{p_1^\mu} \ldots (f^{(k)}(z))^{p_k^\mu} (f(z+c))^{q_0^\mu} (f'(z+c))^{q_1^\mu} \ldots (f^{(k)}(z+c))^{q_k^\mu}, \end{aligned}$$

$p_0^\mu, p_1^\mu, \ldots, p_k^\mu, q_0^\mu, q_1^\mu, \ldots, q_k^\mu \in \mathbb{N} \cup \{0\}$ such that $\sum\limits_{j=0}^{k} p_j^\mu + \sum\limits_{j=0}^{k} q_j^\mu = \mu \leq d$. Therefore we have

$$P_d(z,f) = \sum_{\mu} b_\mu(z) \frac{G_\mu(z,f)}{f^\mu(z)} f^\mu(z). \tag{2.1}$$

Now applying the lemma on the logarithmic derivative and Lemma 2.5 we obtain

$$\begin{aligned} &m\Big(r, b_\mu(z) \frac{G_\mu(z,f)}{f^\mu(z)}\Big) \\ &= m\Big(r, b_\mu(z) \left(\frac{f'(z)}{f(z)}\right)^{p_1^\mu} \ldots \left(\frac{f^{(k)}(z)}{f(z)}\right)^{p_k^\mu} \left(\frac{f(z+c)}{f(z)}\right)^{q_0^\mu} \ldots \left(\frac{f^{(k)}(z+c)}{f(z)}\right)^{q_k^\mu}\Big) \\ &= S(r,f). \end{aligned}$$

Therefore (2.1) takes the form $P_d(z,f) = c_d(z)f^d(z) + c_{d-1}(z)f^{d-1}(z) + \ldots + c_0(z)$, where $c_d \not\equiv 0$ and $m(r,c_i) = S(r,f)$ for $i = 0,1,2,\ldots,d$. Now by mathematical induction we can prove that $m(r,P_d(z,f)) \leq d\, m(r,f) + S(r,f)$. Note that $f \in \mathscr{M}_T(\mathbb{C})$ and $N(r,f) = S(r,f)$. Then by Lemma 2.6 we get $N(r,f(z+c)) = S(r,f)$. We know that $N(r,f^{(j)}) \leq (1+j)N(r,f)$. Therefore $N(r,G_\mu(z,f)) \leq \big(\sum_{j=0}^{k}(1+j)p_j^\mu\big)N(r,f) + \big(\sum_{j=0}^{k}(1+j)q_j^\mu\big)N(r,f(z+c)) = S(r,f)$. Since $N(r,b_\mu) = S(r,f)$ one can easily deduce that $N(r,P_d(z,f)) = S(r,f)$. Consequently

$$\begin{aligned} T(r,P_d(z,f)) &= m(r,P_d(z,f)) + S(r,f) \\ &\leq d\, m(r,f) + S(r,f) = d\, T(r,f) + S(r,f). \end{aligned} \tag{2.2}$$

On the other hand from Lemma 2.5, we have

$$\begin{aligned} T(r,f^n(z)f(z+c)) &= m(r,f^n(z)f(z+c)) + S(r,f) \\ &\leq m(r,f^{n+1}(z)) + m\big(r,\frac{f(z+c)}{f(z)}\big) + S(r,f) \\ &= (n+1)\,T(r,f) + S(r,f). \end{aligned}$$

Again from Lemma 2.5, we see that

$$\begin{aligned} (n+1)\,T(r,f) &= m(r,f^{n+1}) + S(r,f) \\ &\leq m\big(r,f^n(z)f(z+c)\big) + m\big(r,\frac{f(z)}{f(z+c)}\big) + S(r,f) \\ &\leq T(r,f^n(z)f(z+c)) + S(r,f). \end{aligned}$$

Therefore $T(r,f^n(z)f(z+c)) = (n+1)\,T(r,f) + S(r,f)$ and so from (2.2), we get

$$\begin{aligned} T\big(r,p_1e^{\alpha_1} + p_2e^{\alpha_2}\big) &\leq T\big(r,f^n(z)f(z+c)\big) + T\big(r,P_d(z,f)\big) \\ &\leq (n+d+1)\,T(r,f) + S(r,f), \\ T\big(r,p_1e^{\alpha_1} + p_2e^{\alpha_2}\big) &\geq T\big(r,f^n(z)f(z+c)\big) - T\big(r,P_d(z,f)\big) \\ &\geq (n-d+1)\,T(r,f) + S(r,f). \end{aligned}$$

Consequently $(n-d+1)\,T(r,f) + S(r,f) \leq T\big(r,p_1e^{\alpha_1} + p_2e^{\alpha_2}\big) \leq (n+d+1)\,T(r,f) + S(r,f)$, which implies that $\rho(f) < +\infty$. This completes the proof. □

From Lemma 2.7, we immediately have the following lemma.

LEMMA 2.8. *Let $c \in \mathbb{C}\setminus\{0\}$ and $P_d(z,f)$ be a differential-difference polynomial with small functions of f as its coefficients and $d \leq n-1$. Suppose $p(\not\equiv 0)$ is a rational function and α is a non-constant polynomial. If $f \in \mathscr{M}(\mathbb{C})$ is a solution of (1.3) satisfying $\rho_2(f) < 1$ and $N(r,f) = S(r,f)$, then $f \in \mathscr{M}_T(\mathbb{C})$ and $\rho(f) < +\infty$.*

LEMMA 2.9 ([6]). *Let* $f \in \mathscr{M}_T(\mathbb{C})$ *and* $q_1, q_2, q_3, a \in \mathscr{S}(f)$ *such that* $q_1q_3a \not\equiv 0$. *If* $q_1f^2 + q_2ff' + q_3(f')^2 = a$, *then* $q_3\left(q_2^2 - 4q_1q_3\right)\frac{a'}{a} + q_2\left(q_2^2 - 4q_1q_3\right) - q_3\left(q_2^2 - 4q_1q_3\right)' + \left(q_2^2 - 4q_1q_3\right)q_3' \equiv 0$.

LEMMA 2.10 ([3]). *Let* $f \in \mathscr{M}(\mathbb{C})$. *Suppose* $g(z) = f^n(z) + P_{n-1}(z, f)$, *where* $P_{n-1}(z, f)$ *is a differential polynomial with small functions of* f *as its coefficients and* $N(r, f) + N\big(r, \frac{1}{g}\big) = S(r, f)$. *Then* $g = (f + \gamma)^n$, *where* $\gamma \in \mathscr{S}(f)$.

LEMMA 2.11. *Let* $f \in \mathscr{M}(\mathbb{C})$. *Suppose* $g(z) = f^{n+1}(z) + P_{n-1}(z, f)$, *where* $P_{n-1}(z, f)$ *is a differential polynomial with small functions of* f *as its coefficients and* $N(r, f) + N\big(r, \frac{1}{g}\big) = S(r, f)$. *Then* $g = f^{n+1}$ *and* $P_{n-1}(z, f) \equiv 0$.

Proof. From Lemma 2.10 we get $g = (f + \gamma)^{n+1}$, where $\gamma \in \mathscr{S}(f)$. If possible, suppose $\gamma \not\equiv 0$. Then we have $(f(z) + \gamma(z))^{n+1} = f^{n+1}(z) + P_{n-1}(z, f)$ and so $(n+1)\gamma(z)f^n(z) + Q_{n-1}(z, f) = P_{n-1}(z, f)$, where $Q_{n-1}(z, f)$ is a differential polynomial with small functions of f as its coefficients. Therefore $f^{n-1}(z).(n+1)\gamma(z)f(z) = P_{n-1}(z, f) - Q_{n-1}(z, f)$ and so by Lemma 2.3 we conclude that $m(r, f) = S(r, f)$. Since $N(r, f) = S(r, f)$, it follows that $f \in \mathscr{S}(r, f)$, which is impossible. Hence $\gamma \equiv 0$. Consequently $g = f^{n+1}$ and $P_{n-1}(z, f) \equiv 0$. □

3. Proofs of the theorems

Proof of Theorem 1.5. Let $f \in \mathscr{M}(\mathbb{C})$ be a solution of the equation (1.3). Then by Lemma 2.8 we conclude that $f \in \mathscr{M}_T(\mathbb{C})$ and $\rho(f) < +\infty$. Now differentiating (1.3) once we get

$$f^{n-1}(z)\left(nf'(z)f(z+c) + f(z)f'(z+c)\right) + P_d'(f(z)) = (p(z)\alpha'(z) + p'(z))e^{\alpha(z)}, \quad (3.1)$$

where $P_d(f(z)) = P_d(z, f)$.

We claim that $p\alpha' + p' \not\equiv 0$. If not, suppose $p\alpha' + p' \equiv 0$. On integration we get $e^{\alpha} = \frac{a_0}{p}$, where $a_0 \in \mathbb{C} \setminus \{0\}$, which is impossible. Now eliminating e^{α} from (1.3) and (3.1) we get

$$\begin{aligned} f^{n-1}(z)\big(p(z)\left(nf'(z)f(z+c) + f(z)f'(z+c)\right) \\ - \left(p(z)\alpha'(z) + p'(z)\right)f(z)f(z+c)\big) \\ = \left(p(z)\alpha'(z) + p'(z)\right)P_d(f(z)) - p(z)P_d'(f(z)). \quad (3.2) \end{aligned}$$

Suppose $p(z)\left(nf'(z)f(z+c) + f(z)f'(z+c)\right) - \left(p(z)\alpha'(z) + p'(z)\right)f(z)f(z+c) \not\equiv 0$. Then by Lemma 2.1 we get

$$\begin{aligned} m\big(r, p(z)\left(nf'(z)f(z+c) + f(z)f'(z+c)\right) \\ - \left(p(z)\alpha'(z) + p'(z)\right)f(z)f(z+c)\big) = O\left(r^{\rho-1+\varepsilon}\right) + S(r, f) \quad (3.3) \end{aligned}$$

and

$$\begin{aligned}m\big(r, p(z)\left(nf(z)f'(z)f(z+c)+f^2(z)f'(z+c)\right) \\ -\left(p(z)\alpha'(z)+p'(z)\right)f^2(z)f(z+c)\big) = O\left(r^{\rho-1+\varepsilon}\right)+S(r,f). \quad (3.4)\end{aligned}$$

Since $N(r,f)=O(\log r)$, from (3.3) and (3.4) we have

$$\begin{aligned}T(r,f) \leq & T\big(r, p(z)\left(nf(z)f'(z)f(z+c)+f^2(z)f'(z+c)\right) \\ & -\left(p(z)\alpha'(z)+p'(z)\right)f^2(z)f(z+c)\big) \\ & + T\big(r, p(z)\left(nf'(z)f(z+c)+f(z)f'(z+c)\right) \\ & -\left(p(z)\alpha'(z)+p'(z)\right)f(z)f(z+c)\big) = O\left(r^{\rho-1+\varepsilon}\right)+S(r,f),\end{aligned}$$

which is impossible. Therefore

$$p(z)\left(nf'(z)f(z+c)+f(z)f'(z+c)\right)-\left(p(z)\alpha'(z)+p'(z)\right)f(z)f(z+c)\equiv 0$$

and so on integration we get

$$f^n(z)f(z+c)=a_1p(z)e^{\alpha(z)}, \quad (3.5)$$

where $a_1\in\mathbb{C}\setminus\{0\}$. Now from (1.3) we have $\big(1-\frac{1}{a_1}\big)f^n(z)f(z+c)=-P_d(z,f)$. If $a_1\neq 1$, then by Lemma 2.1 we get $m(r,f(z+c))=O\big(r^{\rho-1+\varepsilon}\big)+S(r,f)$. Since $N(r,(z+c))=O(\log r)$, we have $T(r,f(z+c))=O\big(r^{\rho-1+\varepsilon}\big)+S(r,f)$, which is impossible. Hence $a_1=1$ and so $P_d(z,f)\equiv 0$. Also from (3.5) we deduce that $N(r,0;f)=O(\log r)$ and so we let $f=qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c)=p(z)$ and $np'(z)+p'(z+c)=\alpha'(z)$. This completes the proof. □

Proof of Theorem 1.6. Let $f\in\mathscr{M}(\mathbb{C})$ be a solution of the equation (1.2). Then by Lemma 2.7 we conclude that $f\in\mathscr{M}_T(\mathbb{C})$ and $\rho(f)<+\infty$. Differentiating (1.2) once we get

$$\begin{aligned}f^{n-1}(z)\left(nf'(z)f(z+c)+f(z)f'(z+c)\right)+P_d'(f(z)) \\ =(p_1\alpha_1'+p_1')e^{\alpha_1}+(p_2\alpha_2'+p_2')e^{\alpha_2}\,. \quad (3.6)\end{aligned}$$

Now eliminating e^{α_2} from (1.2) and (3.6) we get

$$\begin{aligned}f^{n-1}(z)\big(p_2(z)\left(nf'(z)f(z+c)+f(z)f'(z+c)\right) \\ -\left(p_2(z)\alpha_2'(z)+p_2'(z)\right)f(z)f(z+c)\big) \\ +p_2(z)P_d'(f(z))-\left(p_2(z)\alpha_2'(z)+p_2'(z)\right)P_d(f(z))=A(z)e^{\alpha_1(z)}\,, \quad (3.7)\end{aligned}$$

where

$$A(z)=p_2(z)(p_1(z)\alpha_1'(z)+p_1'(z))-p_1(z)(p_2(z)\alpha_2'(z)+p_2'(z)).$$

First we suppose $A \equiv 0$. Then $\alpha_1' - \alpha_2' \equiv \frac{p_2'}{p_2} - \frac{p_1'}{p_1}$ and so $\alpha_1' \equiv \alpha_2'$. Now from (3.7) we get

$$\begin{aligned} f^{n-1}(z)\big(p_2(z)\big(nf'(z)f(z+c)+f(z)f'(z+c)\big)-\big(p_2(z)\alpha_2'(z)+p_2'(z)\big)f(z)f(z+c)\big) \\ = \left(p_2(z)\alpha_2'(z)+p_2'(z)\right)P_d(f(z)) - p_2(z)P_d'(f(z)). \quad (3.8) \end{aligned}$$

Then proceeding in the same way as done in the proof of Theorem 1.5, one can easily conclude that $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c) = p_2(z)$ and $np'(z)+p'(z+c) = \alpha_1'(z) = \alpha_2'(z)$.

Next we suppose $A \not\equiv 0$. Now differentiating (3.7) once we get

$$\begin{aligned} f^{n-2}(z)\,\big((n-1)np_2(z)(f'(z))^2f(z+c) + 2np_2(z)f(z)f'(z)f'(z+c) \\ -np_2(z)\alpha_2'(z)f(z)f'(z)f(z+c) + np_2(z)f(z)f''(z)f(z+c) \\ +p_2(z)f^2(z)f''(z+c) - (p_2(z)\alpha_2'(z)+p_2'(z))'f^2(z)f(z+c) \\ -p_2(z)\alpha_2'(z)f^2(z)f'(z+c)\big) + Q_d'(f(z)) \\ = (A'(z)+A(z)\alpha_1'(z))e^{\alpha_1(z)}, \quad (3.9) \end{aligned}$$

where

$$Q_d(f(z)) = p_2(z)P_d'(f(z)) - \left(p_2(z)\alpha_2'(z)+p_2'(z)\right)P_d(f(z)). \quad (3.10)$$

Eliminating e^{α_1} from (3.7) and (3.9) we get

$$f^{n-2}(z)\varphi(z) = A(z)Q_d'(f(z)) - (A'(z)+A(z)\alpha_1'(z))Q_d(f(z)), \quad (3.11)$$

where

$$\begin{aligned} \varphi(z) =\, & h_1(z)(f'(z))^2f(z+c) + h_2(z)f(z)f'(z)f'(z+c) \\ & + h_3(z)f(z)f'(z)f(z+c) + h_4(z)f(z)f''(z)f(z+c) \\ & + h_5(z)f^2(z)f''(z+c) + h_6(z)f^2(z)f(z+c) \\ & + h_7(z)f^2(z)f'(z+c), \quad (3.12) \end{aligned}$$

and

$$\begin{cases} h_1(z) = n(n-1)p_2(z)A(z) \\ h_2(z) = -2np_2(z)A(z) \\ h_3(z) = np_2(z)\left((A(z)\alpha_1'(z)+A'(z)) + \alpha_2'(z)A(z)\right) \\ h_4(z) = -np_2(z)A(z) \\ h_5(z) = -p_2(z)A(z) \\ h_6(z) = \big(p_2(z)\alpha_2'(z)+p_2'(z)\big)'A(z) \\ \qquad\quad - (A(z)\alpha_1'(z)+A(z))\big(p_2(z)\alpha_2'(z)+p_2'(z)\big) \\ h_7(z) = p_2(z)\left(A(z)(\alpha_1'(z)-\alpha_2'(z))+A'(z)\right). \end{cases}$$

If $\varphi \not\equiv 0$, then by Lemma 2.1 we get

$$\begin{cases} m(r,\varphi) = O\left(r^{\rho-1+\varepsilon}\right) + S(r,f) \\ m(r,f\varphi) = O\left(r^{\rho-1+\varepsilon}\right) + S(r,f). \end{cases} \tag{3.13}$$

Since $N(r,f) = O(\log r)$, from (3.13) we get $T(r,f) \leq T(r,\varphi) + T(r,f\varphi) = O\left(r^{\rho-1+\varepsilon}\right) + S(r,f)$, which is impossible. Hence $\varphi \equiv 0$ and so from (3.11) we have

$$AQ_d' \equiv (A' + A\alpha_1')Q_d. \tag{3.14}$$

Suppose $Q_d \equiv 0$. Then from (3.10) we have

$$p_2 P_d' \equiv (p_2\alpha_2' + p_2')\, P_d. \tag{3.15}$$

If $P_d \equiv 0$, then from (1.2) we get

$$\begin{aligned} f^n(z)f(z+c) &= p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)} \\ &= e^{\alpha_2(z)}\big(p_1(z)e^{\alpha_1(z)-\alpha_2(z)} + p_2(z)\big). \end{aligned} \tag{3.16}$$

We claim that $\alpha_1 - \alpha_2 \in \mathbb{C}$. If not, suppose $\alpha_1 - \alpha_2 \notin \mathbb{C}$. Since $N(r,f) = O(\log r)$, from (3.16) we get $\overline{N}\big(r,0;p_1e^{\alpha_1-\alpha_2}+p_2\big) \leq \frac{1}{n}N\big(r,0;p_1e^{\alpha_1-\alpha_2}+p_2\big) + O(\log r)$. Now by Lemma 2.4 we get

$$\begin{aligned} T\big(r,e^{\alpha_1-\alpha_2}\big) &= T\big(r,p_1e^{\alpha_1-\alpha_2}\big) + S\big(r,e^{\alpha_1-\alpha_2}\big) \\ &\leq \overline{N}\big(r,0;p_1e^{\alpha_1-\alpha_2}\big) + \overline{N}\big(r,\infty;p_1e^{\alpha_1-\alpha_2}\big) \\ &\quad + \overline{N}\big(r,-p_2;p_1e^{\alpha_1-\alpha_2}\big) + S\big(r,e^{\alpha_1-\alpha_2}\big) \\ &\leq \frac{1}{n}N\big(r,0;p_1e^{\alpha_1-\alpha_2}+p_2\big) + S\big(r,e^{\alpha_1-\alpha_2}\big) \\ &\leq \frac{1}{n}T\big(r,e^{\alpha_1-\alpha_2}\big) + S\big(r,e^{\alpha_1-\alpha_2}\big), \end{aligned}$$

which is impossible. Hence $\alpha_1 - \alpha_2 \in \mathbb{C}$ and so we let $e^{\alpha_1} = b_1e^{\alpha_2}$, where $b_1 \in \mathbb{C}\setminus\{0\}$. Therefore from (3.16), we have $f^n(z)f(z+c) = \big(b_1p_1(z) + p_2(z)\big)e^{\alpha_2(z)}$. This shows that f has finitely many zeros. In this case one can easily conclude that $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c) = b_1p_1(z) + p_2(z)$ and $np'(z) + p'(z+c) = \alpha_1'(z) = \alpha_2'(z)$.

If $P_d \not\equiv 0$, then from (3.15) we have $\frac{P_d'}{P_d} \equiv \alpha_2' + \frac{p_2'}{p_2}$. On integration, we get $P_d = b_2p_2e^{\alpha_2}$, where $b_2 \in \mathbb{C}\setminus\{0\}$ and so from (1.2) we get

$$f^n(z)f(z+c) + \big(1 - \frac{1}{b_2}\big)P_d(f(z)) = p_1(z)e^{\alpha_1(z)}.$$

Now by Theorem 1.5, we conclude that $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c) = p_1(z)$ and $np'(z) + p'(z+c) = \alpha_1'(z)$.

Suppose $Q_d \not\equiv 0$. Then from (3.14) we have $\frac{Q_d'}{Q_d} \equiv \frac{A'}{A} + \alpha_1'$. On integration we get $Q_d = b_3 A e^{\alpha_1}$, where $b_3 \in \mathbb{C} \setminus \{0\}$ and so from (3.7) we have

$$f^{n-1}(z)\big(p_2(z)\big(nf'(z)f(z+c) + f(z)f'(z+c)\big) - \big(p_2(z)\alpha_2'(z) + p_2'(z)\big)f(z)f(z+c)\big) = \big(1 - \frac{1}{b_3}\big)Q_d. \quad (3.17)$$

Let

$$\varphi_1(z) = p_2(z)\left(nf'(z)f(z+c) + f(z)f'(z+c)\right) - \left(p_2(z)\alpha_2'(z) + p_2'(z)\right)f(z)f(z+c).$$

If $b_3 = 1$, then from (3.17) we get

$$p_2(z)\big(nf'(z)f(z+c) + f(z)f'(z+c)\big) - \big(p_2(z)\alpha_2'(z) + p_2'(z)\big)f(z)f(z+c) \equiv 0$$

and so on integration we have $f^n(z)f(z+c) = b_4 p_2(z) e^{\alpha_2(z)}$, where $b_4 \in \mathbb{C}\setminus\{0\}$. Now by Theorem 1.5, we conclude that $f = qe^p$, where $q(\not\equiv 0)$ is a rational function and p is a non-constant polynomial such that $q^n(z)q(z+c) = p_2(z)$ and $np'(z) + p'(z+c) = \alpha_2'(z)$.

If $b_3 \neq 1$, then from Lemma 2.1 and (3.17) we get $m(r,\varphi_1) = O\big(r^{\rho-1+\varepsilon}\big) + S(r,f)$ and $m(r,\varphi_1 f) = O\big(r^{\rho-1+\varepsilon}\big) + S(r,f)$. Since $N(r,\infty;f) = O(\log r)$ we get $T(r,\varphi_1) = O\big(r^{\rho-1+\varepsilon}\big) + S(r,f)$ and $T(r,\varphi_1 f) = O\big(r^{\rho-1+\varepsilon}\big) + S(r,f)$. Note that

$$T(r,f) \leq T(r,\varphi_1 f) + T\big(r,\frac{1}{\varphi_1}\big) + O\big(r^{\rho-1+\varepsilon}\big) + S(r,f) = O\big(r^{\rho-1+\varepsilon}\big) + S(r,f),$$

which is impossible. This completes the proof. $\square$

Let $k \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. We use the notation $N_{(k+1}(r,a;f)$ to denote the counting function of a-points of f with multiplicity greater than k. Also $\overline{N}_{(k+1}(r,a;f)$ is the reduced counting function.

Proof of Theorem 1.9. Let $f \in \mathscr{M}(\mathbb{C})$ be a solution of (1.2). Now using Lemma 2.7 we conclude that $f \in \mathscr{M}_T(\mathbb{C})$ and $\rho(f) < +\infty$. We have $N(r,f) = S(r,f)$ and so from Lemma 2.6 we get $N(r, f(z+c)) = S(r,f)$. Since $f(z)$ and $f(z+c)$ share 0 CM$_*$, we have $N\big(r, \frac{f(z+c)}{f(z)}\big) = S(r,f)$. Also by Lemma 2.5 we get $m\big(r,\frac{f(z+c)}{f(z)}\big)=S(r,f)$. Consequently $\frac{f(z+c)}{f(z)}\in\mathscr{S}(f)$ and so $f(z+c)=\phi(z)f(z)$, where $\phi\in\mathscr{S}(f)$. Therefore (1.2) reduces to

$$f^{n+1}+Q_d=p_3e^{\alpha_1}+p_4e^{\alpha_2}, \tag{3.18}$$

where $Q_d(f)$ $(d\leq n-1)$ is a differential polynomial with small functions of f as its coefficients,

$$p_3(z)=\frac{p_1(z)}{\phi(z)}=\frac{p_1(z)f(z)}{f(z+c)}\in\mathscr{S}(f) \tag{3.19}$$
$$\text{and}\;\; p_4(z)=\frac{p_2(z)}{\phi(z)}=\frac{p_2(z)f(z)}{f(z+c)}\in\mathscr{S}(f).$$

Now differentiating both sides of (3.18) once we get

$$(n+1)f^nf'+Q_d'=(p_3\alpha_1'+p_3')e^{\alpha_1}+(p_4\alpha_2'+p_4')e^{\alpha_2}. \tag{3.20}$$

Eliminating e^{α_2} from (3.18) and (3.20) we get

$$f^n\big((n+1)p_4f'-(p_4\alpha_2'+p_4')\,f\big)+p_4Q_d'-(p_4\alpha_2'+p_4')\,Q_d=A_1e^{\alpha_1}, \tag{3.21}$$

where $A_1=p_4(p_3\alpha_1'+p_3')-p_3(p_4\alpha_2'+p_4')$. Again eliminating e^{α_1} from (3.18) and (3.20) we get

$$f^n\big((n+1)p_3f'-(p_3\alpha_1'+p_3')\,f\big)+p_3Q_d'-(p_3\alpha_1'+p_3')\,Q_d=-A_1e^{\alpha_2}. \tag{3.22}$$

First we suppose $A_1\equiv 0$. Then we have $\alpha_1'-\alpha_2'=\frac{p_4'}{p_4}-\frac{p_3'}{p_3}$ and so $e^{\alpha_1-\alpha_2}\in\mathscr{S}(f)$. Now from (3.21) we get

$$f^n\big((n+1)p_4f'-(p_4\alpha_2'+p_4')\,f\big)=(p_4\alpha_2'+p_4')\,Q_d-p_4Q_d'. \tag{3.23}$$

If $(n+1)p_4f'-(p_4\alpha_2'+p_4')f\not\equiv 0$, then from Lemma 2.3 we get

$$\begin{cases} m\big(r,(n+1)p_4f'-(p_4\alpha_2'+p_4')f\big)=S(r,f)\\ m\big(r,(n+1)p_4ff'-(p_4\alpha_2'+p_4')f^2\big)=S(r,f).\end{cases} \tag{3.24}$$

Since $N(r,f)=S(r,f)$, from (3.24) we get $f\in\mathscr{S}(r,f)$, which is impossible. Therefore $(n+1)p_4f'-(p_4\alpha_2'+p_4')f\equiv 0$ and so $f^{n+1}=c_1p_4e^{\alpha_2}$, where $c_1\in\mathbb{C}\setminus\{0\}$. Therefore we let $f=qe^{\frac{\alpha_2}{n+1}}$, where $q\in\mathscr{S}(f)\setminus\{0\}$ such that $q^{n+1}(z)f(z+c)=c_0p_2(z)f(z)$, i.e., $q^n(z)q(z+c)e^{\frac{\alpha_2(z+c)-\alpha_2(z)}{n+1}}=c_0p_2(z)$, where $c_0\in\mathbb{C}\setminus\{0\}$.

Next we suppose $A_1\not\equiv 0$. Now differentiating (3.21) once we get

$$\begin{aligned} f^{n-1}\big(&-(p_4\alpha_2'+p_4')'\,f^2-(n+1)p_4\alpha_2'ff'\\ &+n(n+1)p_4(f')^2+(n+1)p_4ff''\big)+R_d'=(A_1'+A_1\alpha_1')e^{\alpha_1}, \end{aligned} \tag{3.25}$$

where

$$R_d = p_4Q'_d - \big(p_4\alpha'_2 + p'_4\big)Q_d. \tag{3.26}$$

Eliminating e^{α_1} from (3.21) and (3.25) we get

$$f^{n-1}\big(h_{21}f^2 + h_{22}ff' + h_{23}(f')^2 + h_{24}ff''\big) = R^*_d, \tag{3.27}$$

where

$$\begin{cases} R^*_d &=& (A'_1 + A_1\alpha'_1)R_d - A_1R'_d \\ h_{21} &=& (p_4\alpha'_2 + p'_4)(A'_1 + A_1\alpha'_1) - A_1\,(p_4\alpha'_2 + p'_4)' \\ h_{22} &=& -(n+1)(\alpha'_1 + \alpha'_2)p_4A_1 - (n+1)p_4A'_1 \\ h_{23} &=& n(n+1)p_4A_1 \not\equiv 0 \\ h_{24} &=& (n+1)p_4A_1 \not\equiv 0. \end{cases} \tag{3.28}$$

Clearly $h_{2j} \in \mathscr{S}(f)$ for $j = 1, 2, 3, 4$.

Suppose $h_{21} \equiv 0$. Then we have $\frac{\left(p_4\alpha'_2+p'_4\right)'}{p_4\alpha'_2+p'_4} - \frac{A'_1}{A_1} \equiv \alpha'_1$. On integration we get $p_4\alpha'_2+p'_4 = c_2A_1e^{\alpha_1}$, $c_2 \in \mathbb{C}\setminus\{0\}$ and so $A_1e^{\alpha_1} \in \mathscr{S}(f)$. Then from (3.21), we have

$$f^n\big((n+1)p_4f' - (p_4\alpha'_2 + p'_4)\,f\big) = (p_4\alpha'_2 + p'_4)\,Q_d - p_4Q'_d + A_1e^{\alpha_1}. \tag{3.29}$$

In this case also, we conclude that $f = qe^{\frac{\alpha_2}{n+1}}$, where $q \in \mathscr{S}f) \setminus \{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_2(z+c)-\alpha_2(z)}{n+1}} = c_0p_2(z)$, where $c_0 \in \mathbb{C}\setminus\{0\}$.

Suppose $h_{21} \not\equiv 0$. Let

$$h_{21}f^2 + h_{22}ff' + h_{23}(f')^2 + h_{24}ff'' = a. \tag{3.30}$$

Now we consider the following two cases.

Case 1. Suppose $a \equiv 0$. Then from (3.30), we have

$$-h_{21}f^2 \equiv h_{22}ff' + h_{23}(f')^2 + h_{24}ff''. \tag{3.31}$$

Let z_1 be a zero of f of multiplicity l_1 such that $h_{2i}(z_1) \neq 0, \infty$ for $i = 1, 2, 3, 4$. Clearly z_1 is a zero of multiplicity $2l_1$ of the left hand side of (3.31) and a zero of multiplicity $2l_1 - 2$ of the right hand side of (3.31). Therefore we arrive at a contradiction from (3.31). Now from (3.31) we deduce that $N(r, 0; f) = S(r, f)$. Since $a \equiv 0$, from (3.27) and (3.28) we get

$$R^*_d \equiv 0, \;\text{ i.e., }\; (A'_1 + A_1\alpha'_1)R_d \equiv A_1R'_d. \tag{3.32}$$

First we suppose $R_d \equiv 0$. Then from (3.26) we have

$$\big(p_4\alpha'_2 + p'_4\big)Q_d \equiv p_4Q'_d. \tag{3.33}$$

Suppose $Q_d \equiv 0$. Then from (3.18) and (3.21) we have respectively

$$f^{n+1} = p_3 e^{\alpha_1} + p_4 e^{\alpha_2} \text{ and } f^n\big((n+1)p_4 f' - \big(p_4\alpha_2' + p_4'\big)f\big) = A_1 e^{\alpha_1}. \quad (3.34)$$

Clearly (3.34) gives $(n+1)p_4\frac{f'}{f} - (p_4\alpha_2' + p_4') = A_1\frac{e^{\alpha_1}}{f^{n+1}}$ and so $m\big(r, \frac{e^{\alpha_1}}{f^{n+1}}\big) = S(r,f)$. Since $N(r,0;f) = S(r,f)$ we have $\frac{e^{\alpha_1}}{f^{n+1}} \in \mathscr{S}(f)$ and so $\frac{f^{n+1}}{e^{\alpha_1}} \in \mathscr{S}(f)$. Again from (3.34) we have $\frac{f^{n+1}}{e^{\alpha_1}} = p_3 + p_4 e^{\alpha_2-\alpha_1}$ and so $e^{\alpha_2-\alpha_1} \in \mathscr{S}(f)$. Let $e^{\alpha_2} = \phi_1 e^{\alpha_1}$, where $\phi_1 \in \mathscr{S}(f)$. Then from (3.34) we get $f^{n+1} = \phi_2 e^{\alpha_1}$, where $\phi_2 = p_3 + \phi_1 p_4 \in \mathscr{S}(f)$. In this case also we get $f = qe^{\frac{\alpha_1}{n+1}}$, where $q \in \mathscr{S}(f)\setminus\{0\}$ such that $q^{n+1}(z)f(z+c) = (p_1(z) + \varphi(z)p_2(z))f(z)$, i.e., $q^n(z)q(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}} = p_1(z) + \varphi(z)p_2(z)$, where $\varphi = e^{\alpha_2-\alpha_1}$.

Suppose $Q_d \not\equiv 0$. Then (3.33) gives $\frac{Q_d'}{Q_d} \equiv \alpha_2' + \frac{p_4'}{p_4}$. On integration we get $Q_d = c_3 p_4 e^{\alpha_2}$, where $c_3 \in \mathbb{C}\setminus\{0\}$ and so from (3.18) we get $f^{n+1} + \big(1 - \frac{1}{c_3}\big)Q_d = p_3 e^{\alpha_1}$. If $c_3 \neq 1$, then by Lemma 2.11 we have $f^{n+1} = p_3 e^{\alpha_1}$ and $Q_d \equiv 0$. Therefore we get a contradiction since $Q_d \not\equiv 0$. Hence $c_3 = 1$ and so $f^{n+1} = p_3 e^{\alpha_1}$ and $Q_d = p_4 e^{\alpha_2} \not\equiv 0$. In this case also, we have $f = qe^{\frac{\alpha_1}{n+1}}$, where $q \in \mathscr{S}(f)\setminus\{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}} = p_1(z)$. Now substituting $f = qe^{\frac{\alpha_1}{n+1}}$ into $Q_d(f(z)) = p_4(z)e^{\alpha_2(z)}$ we get

$$\sum_{k=0}^{d} a_{2k}(z)e^{\frac{k\alpha_1(z)}{n+1}} = p_4(z)e^{\alpha_2(z)}, \quad (3.35)$$

where $a_{2k} \in \mathscr{S}(f)$ $(k = 0,1,\ldots,d)$. Since $T(r,f) = T\big(r, e^{\frac{\alpha_1}{n+1}}\big) + S(r,f)$, it follows that $a_{2k} \in \mathscr{S}\big(e^{\frac{\alpha_1}{n+1}}\big)$ $(k = 0,1,\ldots,d)$ and so $a_{2k} \in \mathscr{S}\big(e^{\frac{k\alpha_1}{n+1}}\big)$ $(k = 0,1,\ldots,d)$, where $k \in \{1,2,\ldots,d\}$. Since $p_4 \not\equiv 0$, from (3.35) we conclude that there exists at least one value of $k \in \{0,1,\ldots,d\}$ such that $a_{2k} \not\equiv 0$. We now claim that there exists exactly one value of $k \in \{0,1,\ldots,d\}$ such that $a_{2k} \not\equiv 0$. If $d = 0$, then our claim is true. Next we suppose that $d \geq 1$. If possible suppose that there exist at least two values of $k \in \{0,1,\ldots,d\}$ such that $a_{2k} \not\equiv 0$. For the sake of simplicity we may assume that $a_{20} \not\equiv 0$ and $a_{2d} \not\equiv 0$. Clearly

$$T\Big(r, \sum_{k=1}^{d} a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big) = d\,T\big(r, e^{\frac{\alpha_1}{n+1}}\big) + S\big(r, e^{\frac{\alpha_1}{n+1}}\big). \quad (3.36)$$

Also from (3.35) we have

$$N\Big(r, -a_{20}; \sum_{k=1}^{d} a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big) = N(r,0;p_4) \leq S\big(r, e^{\frac{\alpha_1}{n+1}}\big). \quad (3.37)$$

Now from Lemma 2.4, (3.36) and (3.37) we get

$$\begin{aligned} d\,T\big(r,e^{\frac{\alpha_1}{n+1}}\big) &\leq \overline{N}\Big(r,0;\sum_{k=1}^{d}a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big)+\overline{N}\Big(r,\sum_{k=1}^{d}a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big)\\ &\quad+\overline{N}\Big(r,-a_{20};\sum_{k=1}^{d}a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big)+S\big(r,e^{\frac{\alpha_1}{n+1}}\big)\\ &\leq \overline{N}\Big(r,0;\sum_{k=0}^{d-1}a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big)+S\big(r,e^{\frac{\alpha_1}{n+1}}\big)\\ &\leq T\Big(r,\sum_{k=0}^{d-1}a_{2k}e^{\frac{k\alpha_1}{n+1}}\Big)+S\big(r,e^{\frac{\alpha_1}{n+1}}\big)\\ &=(d-1)\,T\big(r,e^{\frac{\alpha_1}{n+1}}\big)+S\big(r,e^{\frac{\alpha_1}{n+1}}\big), \end{aligned}$$

which is impossible. Therefore there exists exactly one value of $k\in\{0,1,\ldots,d\}$ such that $a_{2k}\not\equiv 0$ and so from (3.35), we conclude that there must exist exactly one value of $k\in\{0,1,2,\ldots,d\}$ such that $e^{\frac{k\alpha_1-(n+1)\alpha_2}{n+1}}\in\mathscr{S}(f)$.

Next we suppose $R_d\not\equiv 0$. Then (3.32) gives $\frac{R_d'}{R_d}\equiv\frac{A_1'}{A_1}+\alpha_1'$ and so $R_d=c_4A_1e^{\alpha_1}$, where $c_4\in\mathbb{C}\setminus\{0\}$. Also from (3.21) we get $f^n\big((n+1)p_4f'-\big(p_4\alpha_2'+p_4'\big)f\big)\equiv\big(\frac{1}{c_4}-1\big)R_d$.

Let $\phi_3=(n+1)p_4f'-\big(p_4\alpha_2'+p_4'\big)f$. If $c_4\neq 1$, then by Lemma 2.3 we have $m(r,\phi_3)=S(r,f)$ and $m(r,\phi_3f)=S(r,f)$. Since $N(r,f)=S(r,f)$, it follows that $\phi_3\in\mathscr{S}(f)$ and $\phi_3f\in\mathscr{S}(f)$. Note that $T(r,f)\leq T(r,\phi_3f)+T\big(r,\frac{1}{\phi_3}\big)+S(r,f)=S(r,f)$, which is impossible. Hence $c_4=1$ and so $\phi_3\equiv 0$. Then we have $(n+1)\frac{f'}{f}=\frac{p_4'}{p_4}+\alpha_2$ and so $f^{n+1}=c_5p_4e^{\alpha_2}$, where $c_5\in\mathbb{C}\setminus\{0\}$. If $c_5\neq 1$, then from (3.18) we have $\big(1-\frac{1}{c_5}\big)f^{n+1}+Q_d=p_3e^{\alpha_1}$. Now by Lemma 2.11 we conclude that $Q_d\equiv 0$ and so $R_d\equiv 0$, which contradicts the fact that $R_d\not\equiv 0$. Hence $c_5=1$ and so $f^{n+1}=p_4e^{\alpha_2}$. Also from (3.18) we have $Q_d=p_3e^{\alpha_1}$. In this case, we have $f=qe^{\frac{\alpha_2}{n+1}}$, where $q\in\mathscr{S}(f)\setminus\{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_2(z+c)-\alpha_2(z)}{n+1}}=p_2(z)$. Also there must exist exactly one $k\in\{0,1,2,\ldots,d\}$ such that $e^{\frac{k\alpha_2-(n+1)\alpha_1}{n+1}}\in\mathscr{S}(f)$.

Case 2. Suppose $a\not\equiv 0$. Then Lemma 2.3 gives $a\in\mathscr{S}(f)$. Also from (3.30) we have

$$\frac{1}{f^2}=\frac{h_{21}}{a}+\frac{h_{22}}{a}\frac{f'}{f}+\frac{h_{23}}{a}\left(\frac{f'}{f}\right)^2+\frac{h_{24}}{a}\frac{f''}{f}. \tag{3.38}$$

Therefore from (3.38) we deduce that $m\big(r,\frac{1}{f^2}\big)=S(r,f)$, i.e., $m\big(r,\frac{1}{f}\big)=S(r,f)$. Consequently $T(r,f)=N(r,0;f)+S(r,f)$. This shows that f has infinitely many zeros. Let z_2 be a multiple zero of f such that $h_{2i}(z_2) \neq 0, \infty$ for $i = 1, 2, \ldots, 4$. Then from (3.30) we conclude that z_2 is also a zero of a. Therefore $N_{(2}(r, 0; f) \leq T(r, a) = S(r, f)$, i.e., $N_{(2}(r, 0; f) = S(r, f)$. Consequently f has infinitely many simple zeros. Differentiating (3.30) once we have

$$a' = h'_{21}f^2 + (2h_{21} + h'_{22})ff' + (h_{22} + h'_{23})(f')^2 + (h_{22} + h'_{24})ff'' \\ + (2h_{23} + h_{24})f'f'' + h_{24}ff'''. \quad (3.39)$$

Now from (3.30) and (3.39) we get

$$(ah'_{21} - a'h_{21})f^2 + (2ah_{21} + ah'_{22} - a'h_{22})ff' \\ + (ah_{22} + ah'_{23} - a'h_{23})(f')^2 + (ah_{22} + ah'_{24} - a'h_{24})ff'' \\ + a(2h_{23} + h_{24})f'f'' + ah_{24}ff''' \equiv 0. \quad (3.40)$$

Let z_3 be a simple zero of f which is not a zero or pole of the coefficients in (3.40). Now from (3.40) we see that z_3 is also a zero of $(2ah_{23} + ah_{24})f'' - (a'h_{23} - ah_{22} - ah'_{23})\, f'$. Let

$$\alpha = \frac{(2ah_{23} + ah_{24})f'' - (a'h_{23} - ah_{22} - ah'_{23})\, f'}{f}. \quad (3.41)$$

Since $N(r, f) + N_{(2}(r, 0; f) = S(r, f)$, from (3.41) we see that $N(r, \alpha) = S(r, f)$. Since $m(r, \alpha) = S(r, f)$, we get $\alpha \in \mathscr{S}(f)$. Therefore from (3.41) we have

$$f'' = \frac{a'h_{23} - ah_{22} - ah'_{23}}{2ah_{23} + ah_{24}} f' + \frac{\alpha}{2ah_{23} + ah_{24}} f. \quad (3.42)$$

Now from (3.30) and (3.42) we get

$$a = q_1 f^2 + q_2 ff' + q_3 (f')^2, \quad (3.43)$$

where

$$q_1 = h_{21} - \frac{\beta}{2ah_{23} + ah_{24}},\ q_2 = h_{22} + \frac{a'h_{23} - ah_{22} - ah'_{23}}{2ah_{23} + ah_{24}} h_{24} \text{ and } q_3 = h_{23}$$

are small functions of f. Also from (3.28) we see that

$$\frac{q_2}{q_3} = -\frac{2}{2n+1}(\alpha'_1 + \alpha'_2) - \frac{3}{2n+1}\frac{A'_1}{A_1} + \frac{1}{2n+1}\frac{a'}{a} - \frac{1}{2n+1}\frac{p'_4}{p_4}. \quad (3.44)$$

Then by Lemma 2.7 we get

$$q_3(q_2^2 - 4q_1q_3)\frac{a'}{a} + q_2(q_2^2 - 4q_1q_3) - q_3(q_2^2 - 4q_1q_3)' + (q_2^2 - 4q_1q_3)q'_3 \equiv 0. \quad (3.45)$$

Let $\delta = q_2^2 - 4q_1q_3$. Clearly $\delta \in \mathscr{S}(f)$. Now we consider the following two sub-cases.

Sub-case 2.1. Suppose $\delta = q_2^2 - 4q_1q_3 \equiv 0$. Then (3.43) gives $q_3\big(f' + \frac{q_2}{2q_3}f\big)^2 = a$. This shows that $f' + \frac{q_2}{2q_3}f \in \mathscr{S}(f)$. Let $b = f' + \frac{q_2}{2q_3}f$. Since $a \not\equiv 0$, it follows that $b \not\equiv 0$. Now substituting $f' = b - \frac{q_2}{2q_3}f$ into (3.21) and (3.22) we get respectively

$$f^{n+1}\big(p_4\alpha_2' + p_4' + (n+1)p_4\frac{q_2}{2q_3}\big) - (n+1)p_4bf^n + R_{1d} = A_1e^{\alpha_1} \tag{3.46}$$

$$\text{and } f^{n+1}\big(p_3\alpha_1' + p_3' + (n+1)p_3\frac{q_2}{2q_3}\big) - (n+1)p_3bf^n + R_{2d} = -A_1e^{\alpha_2}, \tag{3.47}$$

where $R_{1d} = p_4Q_d' - (p_4\alpha_2' + p_4')Q_d$ and $R_{2d} = p_3Q_d' - (p_3\alpha_1' + p_3')Q_d$. Let

$$\gamma_1 = p_4\alpha_2' + p_4' + (n+1)p_4\frac{q_2}{2q_3} \ \text{ and } \ \gamma_2 = p_3\alpha_1' + p_3' + (n+1)p_3\frac{q_2}{2q_3}.$$

First we suppose $\gamma_1 \equiv 0$. Then using (3.44) we get

$$(2n+1)\Big(\frac{p_4'}{p_4} + \alpha_2'\Big) = (n+1)\Big(\alpha_1' + \alpha_2' + \frac{3}{2}\frac{A_1'}{A_1} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{p_4'}{p_4}\Big).$$

On integration we get

$$(p_4e^{\alpha_2})^{2n+1} = c_6\frac{A_1^{\frac{3(n+1)}{2}}p_4^{\frac{n+1}{2}}}{a^{\frac{n+1}{2}}}e^{(n+1)(\alpha_1+\alpha_2)},$$

where $c_6 \in \mathbb{C}\setminus\{0\}$ and so $e^{n\alpha_2-(n+1)\alpha_1} \in \mathscr{S}(f)$.

Next we suppose $\gamma_2 \equiv 0$. Then using (3.44) we get

$$(2n+1)\Big(\frac{p_3'}{p_3} + \alpha_1'\Big) = (n+1)\Big(\alpha_1' + \alpha_2' + \frac{3}{2}\frac{A_1'}{A_1} - \frac{1}{2}\frac{a'}{a} + \frac{1}{2}\frac{p_4'}{p_4}\Big).$$

On integration we get

$$(p_3e^{\alpha_1})^{2n+1} = c_7\frac{A_1^{\frac{3(n+1)}{2}}p_4^{\frac{n+1}{2}}}{a^{\frac{n+1}{2}}}e^{(n+1)(\alpha_1+\alpha_2)},$$

where $c_7 \in \mathbb{C}\setminus\{0\}$ and so $e^{n\alpha_1-(n+1)\alpha_2} \in \mathscr{S}(f)$. Next we discuss the following four sub-cases.

Sub-case 2.1.1. Suppose $\gamma_1 \equiv 0$ and $\gamma_2 \equiv 0$.

Then $e^{n\alpha_2-(n+1)\alpha_1}, e^{n\alpha_1-(n+1)\alpha_2} \in \mathscr{S}(f)$. Clearly $e^{\alpha_1+\alpha_2} \in \mathscr{S}(f)$ and so $e^{\alpha_2} = \phi_4e^{-\alpha_1}$, where $\phi_4 \in \mathscr{S}(f)$. Now from (3.46) and (3.47) we have respectively

$$-(n+1)p_4bf^n + R_{1d} = A_1e^{\alpha_1} \ \text{ and } \\ -(n+1)p_3bf^n + R_{2d} = -A_1\phi_4e^{-\alpha_1}. \tag{3.48}$$

Eliminating e^{α_1} and $e^{-\alpha_1}$, from (3.48) we get

$$\begin{aligned} & f^{2n-1}\left((n+1)^2b^2p_3p_4f\right)+R_{3d}=-A_1^2\phi_4,\\ & \quad\text{where } R_{3d}=-(n+1)p_4bR_{2d}f^n-(n+1)p_3bR_{1d}f^n+R_{1d}R_{2d} \end{aligned} \tag{3.49}$$

is a differential polynomial in f of degree $\leq 2n-1$ with small functions as its coefficients. Then from Lemma 2.3 and (3.49) we obtain $m(r,f)=S(r,f)$ and so $f\in\mathscr{S}(f)$, which is impossible.

Sub-case 2.1.2. Suppose $\gamma_1\not\equiv 0$ and $\gamma_2\equiv 0$. Then we have $e^{n\alpha_1-(n+1)\alpha_2}\in\mathscr{S}(f)$ and so

$$e^{\alpha_2}=\phi_5 e^{\frac{n}{n+1}\alpha_1},\ \text{ where } \phi_5\in\mathscr{S}(f). \tag{3.50}$$

Now from Lemma 2.10 and (3.46) we conclude that there exists $v_1\in\mathscr{S}(f)$ such that

$$(f+v_1)^{n+1}=\frac{A_1}{\gamma_1}e^{\alpha_1},\ \text{ i.e., }\ f=u_1e^{\frac{\alpha_1}{n+1}}-v_1, \tag{3.51}$$

where $u_1\in\mathscr{S}(f)\setminus\{0\}$. Since f has infinitely many zeros, it follows that $v_1\not\equiv 0$. Now from (3.18), (3.50) and (3.51) we have $\left(u_1e^{\frac{\alpha_1}{n+1}}-v_1\right)^{n+1}+Q_d=p_3e^{\alpha_1}+c_5p_4e^{\frac{n\alpha_1}{n+1}}$. Using Lemma 2.11 we obtain $u_1^{n+1}=p_3$ and so from (3.19) we get $u_1^{n+1}(z)f(z+c)=p_1(z)f(z)$, i.e.,

$$\begin{aligned} & u_1(z)e^{\frac{\alpha_1(z)}{n+1}}\left(u_1^n(z)u_1(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}}-p_1(z)\right)\\ & \qquad = u_1^{n+1}(z)v_1(z+c)-p_1(z)v_1(z). \end{aligned} \tag{3.52}$$

Note that $p_1,u_1,v_1,e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}}\in\mathscr{S}\left(e^{\frac{\alpha_1(z)}{n+1}}\right)$. Therefore from (3.52), one can easily conclude that $u_1^n(z)u_1(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}}=p_1(z)$ and $u_1^{n+1}(z)v_1(z+c)=p_1(z)v_1(z)$.

Sub-case 2.1.3. Suppose $\gamma_1\equiv 0$ and $\gamma_2\not\equiv 0$. Since $\gamma_1\equiv 0$, we have $e^{n\alpha_2-(n+1)\alpha_1}\in\mathscr{S}(f)$ and so $e^{\alpha_1}=\phi_6e^{\frac{n}{n+1}\alpha_2}$, where $\phi_6\in\mathscr{S}(f)$. Now proceeding in the same way as in Sub-case 2.1.2, one can easily conclude that $f=u_2e^{\frac{\alpha_2}{n+1}}-v_2$, where $u_2,v_2\in\mathscr{S}(f)\setminus\{0\}$ such that $u_2^{n+1}=p_4$. Also from (3.19) we get $u_2^{n+1}(z)f(z+c)=p_2(z)f(z)$. In this case we can conclude that $u_2^n(z)u_2(z+c)e^{\frac{\alpha_2(z+c)-\alpha_2(z)}{n+1}}=p_2(z)$ and $u_2^{n+1}(z)v_2(z+c)=p_2(z)v_2(z)$.

Sub-case 2.1.4. Suppose $\gamma_1\not\equiv 0$ and $\gamma_2\not\equiv 0$. Now from Lemma 2.10, (3.46) and (3.47) we conclude that there exist $v_3,v_4\in\mathscr{S}(f)$ such that $(f+v_3)^{n+1}=\frac{A_1}{\gamma_1}e^{\alpha_1}$ and $(f+v_4)^{n+1}=-\frac{A_1}{\gamma_2}e^{\alpha_2}$. From these we have respectively

$$f=u_3e^{\frac{\alpha_1}{n+1}}-v_3\quad\text{and}\quad f=u_4e^{\frac{\alpha_2}{n+1}}-v_4, \tag{3.53}$$

where $u_3^{n+1}=\frac{A_1}{\gamma_1}$, $u_4^{n+1}=-\frac{A_1}{\gamma_2}$. Since f has infinitely many zeros we have $v_3\not\equiv 0$ and $v_4\not\equiv 0$.

First we suppose $e^{\alpha_1-\alpha_2}\in\mathscr{S}(f)$. Therefore $e^{\alpha_2}=\phi_7 e^{\alpha_1}$, where $\phi_7\in\mathscr{S}(f)$. Now from (3.18) we have $f^{n+1}+Q_d=p_5e^{\alpha_1}$, where $p_5=p_3+\phi_7p_4$. If $p_5\equiv 0$, then we have $f^{n+1}=-Q_d$ and so by Lemma 2.3 we get $m(r,f)=S(r,f)$. Therefore $f\in\mathscr{S}(f)$, which is impossible. Hence $p_5\not\equiv 0$. Now by Lemma 2.11 we conclude that $f^{n+1}=p_5e^{\alpha_1}$ and $Q_d\equiv 0$. In this case we have $f=qe^{\frac{\alpha_1}{n+1}}$, $q\in\mathscr{S}(f)\setminus\{0\}$ such that $q^n(z)q(z+c)e^{\frac{\alpha_1(z+c)-\alpha_1(z)}{n+1}}=p_1(z)+\varphi(z)p_2(z)$, where $\varphi=e^{\alpha_2-\alpha_1}\in\mathscr{S}(f)$.

Next we suppose $e^{\alpha_1-\alpha_2}\not\in\mathscr{S}(f)$. Note that $T(r,f)\leq T\big(r,e^{\frac{\alpha_1}{n+1}}\big)+S(r,f)$. Also we have $T\big(r,e^{\frac{\alpha_1}{n+1}}\big)\leq T\big(r,u_3e^{\frac{\alpha_1}{n+1}}-v_3\big)+S(r,f)=T(r,f)+S(r,f)$. Therefore $T(r,f)=T\big(r,u_3e^{\frac{\alpha_1}{n+1}}\big)+S(r,f)$. Similarly we have $T(r,f)=T\big(r,u_4e^{\frac{\alpha_2}{n+1}}\big)+S(r,f)$. Consequently $S(r,f)=S\big(r,u_3e^{\frac{\alpha_1}{n+1}}\big)=S\big(r,u_4e^{\frac{\alpha_2}{n+1}}\big)$. Clearly $u_3,u_4,v_3,v_4\in\mathscr{S}\big(e^{\frac{\alpha_1}{n+1}}\big)\cap\mathscr{S}\big(e^{\frac{\alpha_2}{n+1}}\big)$. On the other hand from (3.53) we have

$$u_3e^{\frac{\alpha_1}{n+1}}-u_4e^{\frac{\alpha_2}{n+1}}=v_3-v_4. \tag{3.54}$$

We claim that $v_3\equiv v_4$. If not, suppose $v_3\not\equiv v_4$. Then by Lemma 2.4 we get

$$\begin{aligned}T(r,f)&=T\big(r,u_3e^{\frac{\alpha_1}{n+1}}\big)+S(r,f)\\&\leq\overline{N}\big(r,0;u_3e^{\frac{\alpha_1}{n+1}}\big)+\overline{N}\big(r,v_3-v_4;u_3e^{\frac{\alpha_1}{n+1}}\big)+S\big(r,u_3e^{\frac{\alpha_1}{n+1}}\big)+S(r,f)\\&=S(r,f),\end{aligned}$$

which is absurd. Hence $v_3\equiv v_4$ and so from (3.54) we get $e^{\alpha_1-\alpha_2}\equiv\big(\frac{u_4}{u_3}\big)^{n+1}$. Hence $e^{\alpha_1-\alpha_2}\in\mathscr{S}(f)$, which is impossible.

Sub-case 2.2. Suppose $\delta=q_2^2-4q_1q_3\not\equiv 0$. Then (3.45) gives $\frac{q_2}{q_3}\equiv\frac{\delta'}{\delta}-\frac{q_3'}{q_3}-\frac{a'}{a}$. Therefore from (3.28) and (3.44) we get $2(\alpha_1'+\alpha_2')\equiv(2n-2)\frac{A_1'}{A_1}+(2n+2)\frac{a'}{a}+2n\frac{p_4'}{p_4}-(2n+1)\frac{\delta'}{\delta}$. On integration we get $e^{2(\alpha_1+\alpha_2)}=c_8\frac{A_1^{2n-2}a^{2n+2}p_4^{2n}}{\delta^{2n+1}}$, where $c_8\in\mathbb{C}$. This shows that $e^{\alpha_1+\alpha_2}\in\mathscr{S}(f)$ and so $e^{\alpha_2}=\phi_8e^{-\alpha_1}$, where $\phi_8\in\mathscr{S}(f)$. Now from (3.21) and (3.22) we have respectively

$$f^n\left((n+1)p_4f'-(p_4\alpha_2'+p_4')f\right)+R_{1d}=A_1e^{\alpha_1} \tag{3.55}$$

$$\text{and}\quad f^n\left((n+1)p_3f'-(p_3\alpha_1'+p_3')f\right)+R_{2d}=-\phi_8A_1e^{-\alpha_1}. \tag{3.56}$$

Eliminating e^{α_1} and $e^{-\alpha_1}$, from (3.55) and (3.56) we get

$$\begin{aligned}f^{2n}\left((n+1)p_4f'-(p_4\alpha_2'+p_4')f\right)\left((n+1)p_3f'-(p_3\alpha_1'+p_3')f\right)&\\+\mathcal{Q}_d^{**}=-\phi_8A_1^2,&\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}\mathcal{Q}_d^{**}=f^n\left((n+1)p_4f'-(p_4\alpha_2'+p_4')f\right)R_{2d}&\\+f^n\left((n+1)p_1f'-(p_1\alpha_1'+p_1')f\right)R_{1d}+R_{1d}R_{2d}&\end{aligned}$$

is a differential polynomial of degree $\leq 2n$ with small functions of f as its coefficients. Now by Lemma 2.3 we get

$$((p_3\alpha_1'+p_3')f-(n+1)p_3f')\,((p_4\alpha_2'+p_4')f-(n+1)p_4f')=b_{11},$$

where $b_{11}\in\mathscr{S}(f)$. If $b_{11}\equiv 0$, then we have either $(p_3\alpha_1'+p_3')f-(n+1)p_3f'\equiv 0$ or $(p_4\alpha_2'+p_4')f-(n+1)p_4f'\equiv 0$. Thus in either case one can easily conclude that $N(r,0;f)=S(r,f)$, which is impossible. Hence $b_{11}\not\equiv 0$. Therefore we can assume that

$$\begin{aligned} &(p_4\alpha_2'+p_4')f-(n+1)p_4f'=b_1e^{\gamma} \\ \text{and}\;\; &(p_3\alpha_1'+p_3')f-(n+1)p_3f'=b_2e^{-\gamma}, \end{aligned} \tag{3.58}$$

where $b_1,b_2\in\mathscr{S}(f)$ such that $b_1b_2=b_{11}$ and γ is an entire function. Since f is of finite order, it follows that γ is a polynomial.

First we suppose $\gamma\in\mathbb{C}$. Then (3.58) gives $f'=\frac{1}{n+1}\big(\alpha_2'+\frac{p_4'}{p_4}\big)f-\frac{b_1e^{\gamma}}{(n+1)p_4}$ and $f'=\frac{1}{n+1}\big(\alpha_1'+\frac{p_3'}{p_3}\big)f-\frac{b_2e^{-\gamma}}{(n+1)p_3}$. These imply that

$$\Big(\alpha_1'-\alpha_2'+\frac{p_3'}{p_3}-\frac{p_4'}{p_4}\Big)f=\frac{b_2e^{-\gamma}}{p_3}-\frac{b_1e^{\gamma}}{p_4}. \tag{3.59}$$

If $\alpha_1'-\alpha_2'+\frac{p_3'}{p_3}-\frac{p_4'}{p_4}\equiv 0$, then on integration we get $e^{\alpha_1-\alpha_2}=c_9\frac{p_4}{p_3}$, where $c_9\in\mathbb{C}\setminus\{0\}$ and so $e^{\alpha_1-\alpha_2}\in\mathscr{S}(f)$. Since $e^{\alpha_2}=\phi_8e^{-\alpha_1}$, we have $e^{\alpha_2}\in\mathscr{S}(f)$. Certainly $e^{\alpha_1}\in\mathscr{S}(f)$. Then from Lemma 2.3 and (3.18) we deduce that $m(r,f)=S(r,f)$ and so $f\in\mathscr{S}(f)$, which is absurd. Also if $\alpha_1'-\alpha_2'+\frac{p_3'}{p_3}-\frac{p_4'}{p_4}\not\equiv 0$, then from (3.59) we get $f\in\mathscr{S}(f)$, which is absurd.

Next we suppose $\gamma\notin\mathbb{C}$. Then (3.58) gives $(p_3p_4(\alpha_2'-\alpha_1')+p_3p_4'-p_3'p_4)\,f=p_3b_1e^{\gamma}-p_4b_2e^{-\gamma}$. With the similar argument, we can prove that $p_3p_4(\alpha_2'-\alpha_1')+p_3p_4'-p_3'p_4\not\equiv 0$. Clearly we have $f(z)=\delta_1(z)e^{\gamma(z)}+\delta_2(z)e^{-\gamma(z)}$, where $\delta_1=\frac{p_3b_1}{p_3p_4'-p_3'p_4-p_3p_4(\alpha_1'-\alpha_2')}$ and $\delta_2=\frac{-p_4b_2}{p_3p_4'-p_3'p_4-p_3p_4(\alpha_1'-\alpha_2')}$. Then (3.58) can be rewritten as

$$A_2f-(n+1)p_4f'=b_1e^{\gamma},\;\; A_2=p_4\alpha_2'+p_4'. \tag{3.60}$$

Differentiating (3.60) we get $A_2'f+(A_2-(n+1)p_4')\,f'-(n+1)p_4f''=(b_1'+b_1\gamma')e^{\gamma}$. Then from (3.42) we have

$$\begin{aligned}&\left(A_2'-\frac{(n+1)p_4\alpha}{2ah_{23}+ah_{24}}\right)f\\ &\quad+\left(A_2-(n+1)p_4'-(n+1)\frac{a'h_{23}-ah_{22}-ah_{23}'}{2ah_{23}+ah_{24}}p_4\right)f'=(b_1'+b_1\gamma')e^{\gamma}\end{aligned}$$

and so from (3.28) we get

$$\left(A_2' - \frac{1}{2n+1}\frac{\alpha}{aA_1}\right)f + \left(A_2 - (n+1)p_4' - \frac{1}{2n+1}(\alpha_1' + \alpha_2')p_4 \right.$$
$$\left. - \frac{n(n+1)}{2n+1}\frac{a'}{a}p_4 + \frac{n(n+1)}{2n+1}p_4' - \frac{n^2-1}{2n+1}\frac{A_1'}{A_1}p_4\right)f' = (b_1' + b_1\gamma')e^{\gamma}. \quad (3.61)$$

Dividing (3.61) by (3.60) we get $\zeta_1 f + \zeta_2 f' \equiv 0$, where

$$\zeta_1 = A_2' - \frac{1}{2n+1}\frac{\alpha}{aA_1} - A_2\left(\frac{b_1'}{b_1} + \gamma'\right)$$

and $$\zeta_2 = A_2 - (n+1)p_4' - \frac{1}{2n+1}(\alpha_1' + \alpha_2')p_4 - \frac{n(n+1)}{2n+1}\frac{a'}{a}p_4 + \frac{n(n+1)}{2n+1}p_4'$$
$$- \frac{n^2-1}{2n+1}\frac{A_1'}{A_1}p_4 + (n+1)\left(\frac{b_1'}{b_1} + \gamma'\right)p_4.$$

It is clear that either $\zeta_1 \not\equiv 0$ and $\zeta_2 \not\equiv 0$ or $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 0$. If $\zeta_1 \not\equiv 0$ and $\zeta_2 \not\equiv 0$, then we have $N(r,0;f) = S(r,f)$, which is impossible. Hence $\zeta_1 \equiv 0$ and $\zeta_2 \equiv 0$. Now $\zeta_2 \equiv 0$ yields,

$$\alpha_2' - \frac{n^2}{2n+1}\frac{p_4'}{p_4} - \frac{1}{2n+1}(\alpha_1' + \alpha_2') - \frac{n(n+1)}{2n+1}\frac{a'}{a} - \frac{n^2-1}{2n+1}\frac{A_1'}{A_1}$$
$$+ (n+1)\frac{b_1'}{b_1} + (n+1)\gamma' \equiv 0,$$

which implies that

$$e^{(2n+1)((n+1)\gamma+\alpha_2)} = c_{10}\frac{p_4^{n^2}e^{\alpha_1+\alpha_2}a^{n(n+1)}A_1^{n^2-1}}{b_1^{n+1}}, \ c_{10} \in \mathbb{C} \setminus \{0\}.$$

So $e^{(n+1)\gamma+\alpha_2} \in \mathscr{S}(f)$. Finally $f = \delta_1 e^{\gamma} + \delta_2 e^{-\gamma}$, $e^{\alpha_1+\alpha_2} \in \mathscr{S}(f)$, where $\delta_1, \delta_2 \in \mathscr{S}(f) \setminus \{0\}$ and γ is a non-constant polynomial such that either $e^{(n+1)\gamma+\alpha_2} \in \mathscr{S}(f)$ or $e^{(n+1)\gamma+\alpha_1} \in \mathscr{S}(f)$. □

Acknowledgements

The authors are grateful to the referee for his/her valuable comments and suggestions towards the improvement of the paper.

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Received September 29, 2022
Revised March 9, 2023
Accepted March 19, 2023