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Prym varieties and Prym map

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Abstract. *These are introductory notes to the theory of Prym varieties. Subsequently, we focus on the description and geometry of the fibres of the Prym map for étale double coverings over genus 6 curves.*

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Contents

1. Introduction

Given a finite morphism between smooth curves one can associate to it a polarized abelian variety (not necessarily principally polarized), called Prym variety. This construction induces a map from the moduli space of coverings to the moduli space of polarized abelian varieties, know as Prym map, depending on the genus of the base curve, the degree of the map and its ramification pattern. The classical Prym varieties revisited by Mumford in [25] are principally polarized obtained from double coverings (étale or ramified in two points). Since then they have been studied not only as a way of understanding abelian varieties and their moduli space, but also as interesting objects on their own, see for instance the recent work in [1, 9, 20, 21, 30, 31]. Prym maps in low genera often display very rich geometry and interesting structure. These notes summarize the early developments in the theory of Prym varieties which continue to inspire recent work in algebraic geometry. As an introduction to the subject, we chose to focus on the structure of the fibres of the Prym $\mathcal{P}_6$ for étale double coverings over a genus 6 curve, which is generically finite of degree 27. The computation of the degree of $\mathcal{P}_6$ is the ideal occasion to encounter classical algebraic objects (cubic surfaces and threefolds, plane quintics, Fano surface of lines, etc.), geometric constructions (tetragonal and trigonal constructions, conic bundles), as well as moduli spaces (of coverings, abelian varieties, intermediate Jacobians). We tried to put together the main ingredients for a good understanding of the geometric structure of the fibres of $\mathcal{P}_6$.

These notes cover the material presented in the course "Prym Varieties" of the Trieste Algebraic Summer School (TAGSS) 2021 given by the second author. The series of lectures included exercise sessions run by the first author. Some of the exercises can be found here. The main references are Beauville, Donagi and Donagi-Smith papers [4, 5, 14, 15].

2. Basics on abelian varieties

Through this notes we work over $\mathbb{C}$.

DEFINITION 2.1. *A complex torus A is a quotient V/Λ, with $V \simeq \mathbb{C}^g$ a $\mathbb{C}$-vector space and $\Lambda \simeq \mathbb{Z}^{2g}$ a full rank lattice inside V. A polarization on A is an ample line bundle*[1] *L on A. An abelian variety is a complex torus admitting a polarization, so (A, L) is polarized abelian variety.*

REMARK 2.2. In particular, with the addition operation inherited from V, an abelian variety is an abelian group.

[1] In fact, the polarization depends only on the first Chern class $c_1(L)$

By definition of ampleness, given a line bundle L on A we have that the map

$$\begin{aligned}\varphi_{L^{\otimes k}} : A &\hookrightarrow \mathbb{P}H^0(A, L^{\otimes k})^* \\ x &\mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)],\end{aligned}$$

defined by the sections of $L^{\otimes k}$ is an embedding for some $k > 1$. In fact, in the case of polarized abelian varieties it suffices to take $k = 3$. Then an abelian variety is also a projective variety.

Different incarnations of a polarization on A. The following data are equivalent:

- A first Chern class $c_1(L) \in H^2(A, \mathbb{Z})$ of an ample line bundle L on A.
- A non degenerated alternating form $E : V \times V \to \mathbb{R}$ such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(iv, iw) = E(v, w)$.
- A positive definite Hermitian form $H : V \times V \to \mathbb{C}$ with $H(\Lambda, \Lambda) \subset \mathbb{Z}$.
- An isogeny $\phi_L : A \to \widehat{A} := \mathrm{Pic}^0(A)$ that satisfies 'positivity' properties.
- An effective Weil divisor $\Theta \subset A$ such that the subgroup $\{x \in A \mid t_x^*\Theta \sim \Theta\}$ is finite.

Let E be an alternating form representing a polarization on $A = V/\Lambda$. There exists a basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots \mu_g$ of Λ with respect to which E is given by the matrix $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$, where D is the diagonal matrix with positive integer entries $d_1, \dots, d_g$ satisfying $d_i | d_{i+1}$ for $i = 1, \dots, g-1$.

Definition 2.3. *The vector $(d_1, \dots, d_g)$ is called the type of the polarization of L and when it is of the form $(1, \dots, 1)$ the polarization is principal and the variety is called ppav.*

Let C be a smooth curve and let $H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ be the group of closed paths in C (which does not depend on the starting point) modulo homology. This group can be seen as a full rank lattice inside of $H^0(C, \omega_C)^*$, via the injective map

$$\gamma \mapsto \left\{\omega \mapsto \int_\gamma \omega\right\}$$

assigning to a path γ the functional which integrates the holomorphic differentials along γ.

Definition 2.4. *The Jacobian of a smooth algebraic curve C (or a compact Riemann surface) is the complex torus*

$$JC = H^0(C, \omega_C)^* / H_1(C, \mathbb{Z}).$$

The intersection product on $H_1(C,\mathbb{Z})$ induces an alternating form E on $V := H^0(C,\omega_C)^*$. More precisely, if we choose a basis over $\mathbb{Z}$, $\gamma_1,...,\gamma_{2g}$ of $H_1(C,\mathbb{Z})$ as in the Figure 1, the intersection product has a matrix of the form $\begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}$. As $H_1(C,\mathbb{Z})$ is a full rank lattice in V, the $\{\gamma_i\}$ also form a basis of V as an $\mathbb{R}$-vector space. One verifies then that, with respect to this basis, the intersection matrix gives an alternating form E on V defining a principal polarization Θ.

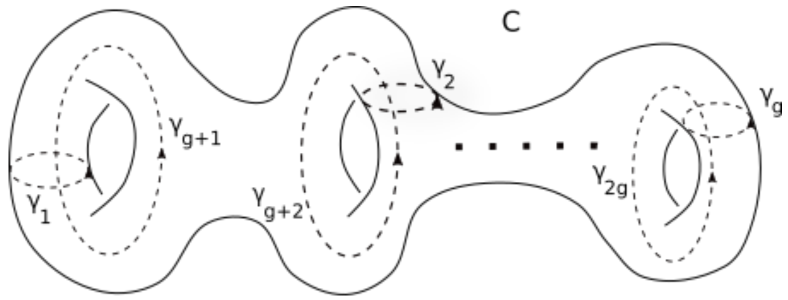


Figure 1: Curve of genus g

A one-dimensional abelian variety also is an algebraic curve of genus one (with a distinguished point), that is, an elliptic curve. The Jacobian of a genus one curve is then isomorphic to the curve itself.

2.1. Abel-Jacobi map

Let $\mathrm{Pic}^0(C)$ be the group of line bundles of degree 0 on C, it is the quotient of the group of divisors $Div^0(C)$ of degree 0 modulo principal divisors. Define the *Abel-Jacobi* map

$$\mathrm{Div}^0(C) \to \mathrm{Pic}^0(C), \quad D = \sum (p_\nu - q_\nu) \mapsto \left\{ \omega \mapsto \sum \int_{q_\nu}^{p_\nu} \omega \right\} \mathrm{mod}\ H_1(C,\mathbb{Z}).$$

THEOREM 2.5. *The Abel-Jacobi map induces an isomorphism* $\mathrm{Pic}^0(C) \simeq JC$.

A variation of this Abel-Jacobi map is given by

$$\alpha_{D_n} : C^{(n)} \to JC, \qquad \sum n_\nu p_\nu \mapsto \left\{ \omega \mapsto \sum \int_c^{p_\nu} \omega \right\} \mathrm{mod}\ H_1(C,\mathbb{Z}),$$

where $D_n = nc$ for a fixed point $c \in C$ and $C^{(n)}$ denotes the cartesian product C^n of the curve modulo the symmetric group S_n, so its elements can be seen as effective divisors of degree n on C. For $n = 1$, we denote the map by α_c. Let $\beta : C^{(n)} \to \mathrm{Pic}^n(C)$ be the map $D \mapsto \mathcal{O}_C(D)$, so for a line bundle L of degree n the fibre $\beta^{-1}(L)$ consists of all divisors in the linear system $|L|$. We have the following commutative diagram

$$\begin{array}{ccc} C^{(n)} & \xrightarrow{\ \beta\ } & \mathrm{Pic}^n(C) \\ & \searrow^{\alpha_{D_n}} & \downarrow{\scriptstyle \alpha_{\mathcal{O}(D_n)}} \\ & & JC \end{array}$$

Proposition 2.6. *The projectivized differential of the Abel-Jacobi map* $\alpha_c : C \to JC$ *is the canonical map* $\varphi_{\omega_C} : C \to \mathbb{P}^{g-1}$.

Corollary 2.7. *For any* $g \geq 1$ *and* $c \in C$ *the Abel-Jacobi map* $\alpha_c : C \to JC$ *is an embedding.*

Remark 2.8. Note that for any $c, c' \in C$ we have $\alpha_c = t^*_{c'-c}\alpha_{c'}$, where $t_D : JC \to JC$ is the translation map $t_D(D') = D' + D$, so we sometimes omit a base point c of the Abel-Jacobi map.

Algebraic geometers typically gather their objects of study in families to investigate a *general* property or single out interesting elements. Ideally, the set of all the objects forms itself an algebraic variety where one can apply known tools. This leads to the notion of moduli space, which is the variety parametrizing the objects. Fortunately, there exists a nice parameter space for all principally polarized abelian varieties (ppav) of fixed dimension g (up to isomorphism classes). Let $\mathfrak{h}_g$ be the Siegel upper half plane

$$\mathfrak{h}_g := \{\tau \in M_{g\times g}(\mathbb{C}) \ \mid \ \tau^t = \tau,\ \mathrm{Im}\,\tau > 0\}$$

(where $\mathrm{Im}\,\tau > 0$ means that the imaginary part is a positive definite 2-form) and

$$Sp_{2g}(\mathbb{Z}) = \left\{ M \in GL_{2g}(\mathbb{Z}) \ : \ M \begin{pmatrix} 0 & \mathbf{1_g} \\ -\mathbf{1_g} & 0 \end{pmatrix} {}^t M = \begin{pmatrix} 0 & \mathbf{1_g} \\ -\mathbf{1_g} & 0 \end{pmatrix} \right\}$$

the symplectic group, which acts on $\mathfrak{h}_g$ by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2g}(\mathbb{Z}), \quad M \cdot \tau = (a + b\tau)(c + d\tau)^{-1}.$$

Thus, every point in the quotient $\mathfrak{h}_g / Sp_{2g}(\mathbb{Z})$ represents an isomorphism class of a principally polarized abelian variety of dimension g: for each $\tau \in \mathfrak{h}_g$ set $A_\tau = \mathbb{C}^g / \tau\mathbb{Z}^g \oplus \mathbb{Z}^g$, then

$$A_\tau \simeq A_{\tau'} \text{ as ppav } \ \Leftrightarrow\ \exists\ M \in Sp_{2g}(\mathbb{Z}) \quad \text{s.t.} \quad \tau' = M \cdot \tau.$$

In the sequel, we denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g. Observe that the dimension of this space is the same as the dimension of the space of symmetric matrices of size g, thus $\dim \mathcal{A}_g = \frac{g(g+1)}{2}$.

Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus $g > 1$, it is an irreducible algebraic variety of dimension $3g-3$. By associating to each smooth curve $[C] \in \mathcal{M}_g$ its Jacobian we get the *Torelli map:*

$$\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g, \qquad [C] \mapsto (JC, \Theta).$$

THEOREM 2.9. *The Torelli map* $\mathfrak{t}$ *is injective.*

Comparing the dimensions of both spaces, one deduces that a general principally polarized abelian variety of dimension 2 and 3 is the Jacobian of some curve.

2.2. The theta divisor

Let $W_n := \beta(C^{(n)}) \subset \mathrm{Pic}^n(C)$ for $n \geq 1$; it consists of the line bundles of degree n with non-empty linear system. According to Riemann-Roch Theorem $W_n = \mathrm{Pic}^n(C)$ for $n \geq g$. For a general divisor D of degree $1 \leq n \leq g$, $h^0(C, \mathcal{O}_C(D)) = 1$, that is, in this range β is birational onto W_n. Since β is proper W_n is an irreducible closed subvariety of $\mathrm{Pic}^n(C)$ of dimension n, so in particular W_{g-1} is a divisor in $\mathrm{Pic}^{g-1}(C)$. For a fixed point $c \in C$ we set $\widetilde{W}_n = \alpha_{\mathcal{O}_C(nc)}(Pic^n(C)) \subset JC$. Recall that the fundamental class $[Y]$ of an n-dimensional subvariety Y of a variety X, $\dim X = g$ is the element in $H^{2g-2n}(X, \mathbb{Z})$, Poincaré dual to the homology class of Y in $H_{2n}(X, \mathbb{Z})$.

THEOREM 2.10 (Poincaré's Formula). $[\widetilde{W}_n] = \frac{1}{(g-n)!} \bigwedge^{g-n} [\Theta]$ *for any* $1 \leq n \leq g$.

COROLLARY 2.11. *There is a line bundle* $\eta \in \mathrm{Pic}^{g-1}(C)$ *such that* $W_{g-1} = \alpha_\eta^* \Theta$.

Proof. By Poincaré Formula $[\widetilde{W}_{g-1}] = [\Theta]$ so $c_1(\mathcal{O}_{JC}(\widetilde{W}_{g-1}) = c_1(\mathcal{O}_{JC}(\Theta))$. There exists $x \in JC \simeq \mathrm{Pic}^0(C)$ such that $\widetilde{W}_{g-1} = t_x^* \Theta$. Hence

$$W_{g-1} = \alpha^*_{\mathcal{O}_C((g-1)c)} \widetilde{W}_{g-1} = \alpha^*_\eta \Theta$$

with $\eta = \mathcal{O}_C((g-1)c) \otimes x^{-1}$. □

We recall that a *theta characteristic* on C is a line bundle κ such that $\kappa^{\otimes 2} \simeq \omega_C$. A divisor D is called *symmetric* if $(-1)^* D \sim D$.

THEOREM 2.12. *Riemann's Theorem] For any symmetric theta divisor* Θ *there is a theta characteristic* κ *on* C *such that*

$$W_{g-1} = \alpha^*_\kappa \Theta\,.$$

The divisor W_{g-1} is called the canonical theta divisor.

Given a theta characteristic κ, the map $\alpha_\kappa : \mathrm{Pic}^{g-1} \to JC$ induces a bijection between the set of theta characteristics on C and the subgroup $JC[2] = \{a \in JC[2] \mid 2a = 0\}$

THEOREM 2.13 (Riemann's Singularity Theorem). *For every* $L \in \mathrm{Pic}^{(g-1)}(C)$

$$mult_L W_{g-1} = h^0(C, L).$$

3. Prym varieties

Consider a finite covering $\pi : \widetilde{C} \to C$ of degree d between two smooth projective curves and let g and $\tilde{g}$ denote the genera of C and $\widetilde{C}$ respectively. By the Hurwitz formula these genera are related by

$$\tilde{g} = d(g-1) + \frac{\deg R}{2} + 1\,, \tag{1}$$

where R denotes the ramification divisor of f, that is the set of points in $\widetilde{C}$ (counted with multiplicities) where the map is not locally a homeomorphism. The map π induces a map between the Jacobians of the curves, the *norm map*. As a group, the Jacobian JC is generated by the points of the curve $\alpha(C)$, and in fact, by Theorem 2.5, JC parametrizes classes of linear equivalence of divisors of degree zero. With this in mind, one can simply define the norm map as the push forward of divisors from $\widetilde{C}$ to C:

$$\mathrm{Nm}_\pi : J\widetilde{C} \to JC, \qquad \left[\sum_i n_i p_i\right] \mapsto \left[\sum_i n_i \pi(p_i)\right],$$

where the sum is finite, $\sum n_i = 0$ with $n_i \in \mathbb{Z}$ and the bracket denotes the class of linear equivalence. The kernel of Nm_π is not necessarily connected but since Nm_π is a group homomorphism the connected component containing the zero is naturally a subgroup of $J\widetilde{C}$. This subgroup is the *Prym variety of f* denoted by

$$P(\pi) := (\mathrm{Ker}\,\mathrm{Nm}_\pi)^0 \subset J\widetilde{C}. \tag{2}$$

Moreover, the restriction Ξ of the principal polarization Θ on $J\widetilde{C}$ to $P(\pi)$, defines a polarization so $(P(\pi), \Xi)$ is an abelian subvariety of the Jacobian $J\widetilde{C}$ of dimension

$$\dim P(\pi) = \dim J\widetilde{C} - \dim JC = \tilde{g} - g.$$

The Prym variety can be regarded as the complementary abelian subvariety to the image of $\pi^* : JC \to J\widetilde{C}$ inside $J\widetilde{C}$, see [8, p. 125].

THEOREM 3.1. *Let $\pi : \widetilde{C} \to C$ be of degree $d \geq 2$ with $g \geq 1$. Then Ξ defines a principal polarization if and only if one of the following cases occur:*

(a) π is étale of degree 2, in this case $\Theta_{|P} \equiv 2\Xi$, with Ξ principal;

(b) π is a double covering ramified in exactly 2 points, so $\Theta_{|P} \equiv 2\Xi$;

(c) $g(\widetilde{C}) = 2, g = 1$ (any degree);

(d) $g = 2$, $d = 3$, π is non-cyclic.

Proof. Uses that $(\pi^*)^*\tilde{\Theta} \equiv n\Theta$ and that P and π^*JC are complementary subvarieties of a ppav. The cases (a),(b),(c) can be found in [8, Thm 12.3.3], where the case (d) is omitted by mistake. The case (d) is considered in [19]. □

From now on, we assume that the covering $\pi : \widetilde{C} \to C$ is étale of degree 2. Then the dimension of the corresponding Prym variety is $\dim P(\pi) = 2g - 1 - g = g - 1$. If ι denotes the involution on $\widetilde{C}$ exchanging the sheets of the covering f, it induces an automorphism ι^* on $J\widetilde{C}$. We can also describe the Prym variety of π as

$$P = \mathrm{Im}(1 - \iota^*) \subset J\widetilde{C}.$$

So P is the ι^*-anti-invariant part of $J\widetilde{C}$ orthogonal to π^*JC. Further, the addition map defines an isogeny

$$\pi^*JC \times P \to J\widetilde{C}.$$

Let

$$\mathcal{R}_g := \{[C, \eta] \ \mid \ [C] \in \mathcal{M}_g,\ \eta \in \mathrm{Pic}^0(C) \setminus \{\mathcal{O}_C\},\ \eta^{\otimes 2} \simeq \mathcal{O}_C\}$$

be the moduli space parametrizing all étale double coverings over curves of genus g up to isomorphism. Given a pair $[C, \eta] \in \mathcal{R}_g$ the isomorphism $\eta^{\otimes 2} \simeq \mathcal{O}_C$ endows $\mathcal{O}_C \oplus \eta$ with a ring structure (actually with a structure of $\mathcal{O}_C$-algebra). Thus, the corresponding double covering is given by taking the spectrum $\widetilde{C} := \mathrm{Spec}(\mathcal{O}_C \oplus \eta)$ and the map π is just the natural projection $\mathrm{Spec}(\mathcal{O}_C \oplus \eta) \to C = \mathrm{Spec}\,\mathcal{O}_C$, induced by the inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C \oplus \eta$. There are finitely many "square roots" of $\mathcal{O}_C$, that is, line bundles η with $\eta^{\otimes 2} \simeq \mathcal{O}_C$. In other words, the forgetful map

$$\mathcal{R}_g \to \mathcal{M}_g, \qquad [C, \eta] \mapsto \eta$$

is finite of degree $2^{2g} - 1$ and hence $\dim \mathcal{R}_g = \dim M_g = 3g - 3$. The *Prym map* is then defined as

$$\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1} \qquad [C, \eta] \mapsto (P(\pi), \Xi).$$

By comparing the dimensions on both sides, one sees that for $g \leq 6$, we have $\dim \mathcal{R}_g \geq \dim \mathcal{A}_{g-1} = \frac{g(g-1)}{2}$ so it makes sense to ask if for low values of g the Prym map is dominant, i.e. if we can realize a (general) principally polarized abelian varieties of dimension ≤ 6 as the Prym variety of some covering.

In order to investigate when the map $\mathcal{P}_g$ is generically finite one has to check if the differential map is injective at a generic point of $\mathcal{R}_g$ or equivalently, when the codifferential map $d^*\mathcal{P}_g$ is surjective. On one side, the tangent space at $0 \in P(\pi)$ to the Prym variety can be identified with

$$T_0P \simeq H^0(C, \omega_C \otimes \eta)^*,$$

which is the (-1)-eigenspace for the action of ι on $H^0(\widetilde{C}, \omega_{\widetilde{C}})^*$. Further, we have the identification $T^*_{[P,\Xi]}\mathcal{A}_{g-1} \simeq \operatorname{Sym}^2(T_0P)^*$ of the cotangent space to $[P, \Xi]$. On the other hand, notice that the forgetful map $[C, \eta] \mapsto [C]$ is finite over the moduli space $\mathcal{M}_g$ of smooth curves of genus g. Therefore the cotangent space to a generic point $[C, \eta] \in \mathcal{R}_g$ can be identified to the cotangent space to $\mathcal{M}_g$ at $[C]$, that is,

$$T^*_{[C,\eta]}\mathcal{R}_g \simeq T^*_{[C]}\mathcal{M}_g \simeq H^0(C, \omega_C^2).$$

Via these identifications one obtains that the codifferential of $\mathcal{P}_g$ at a generic point $[C, \eta]$ is given by the multiplication of sections

$$d^*\mathcal{P}_g : \operatorname{Sym}^2(T_0P)^* \to H^0(C, \omega_C^2 \otimes \mathcal{O}),$$

which is surjective for $g \geq 6$ at a generic point $[(C,\ \eta)]$. More precisely, the following theorem summarizes the situation for the classical Prym map:

THEOREM 3.2. *(a) The Prym map is dominant if $g \leq 6$.*
(b) The Prym map is generically injective if $g \geq 7$.
(c) The Prym map is never injective.

Proof. Let $\mathcal{B}_{g-1}$ denote the image of $\mathcal{P}_g$. Wirtinger showed [34] that the closure $\overline{\mathcal{B}}_{g-1}$ is an irreducible subvariety in $\mathcal{A}_{g-1}$ of dimension $3g-3$, so $\overline{\mathcal{B}}_{g-1} = \mathcal{A}_{g-1}$ for $g \leq 6$, which implies part (a). Moreover, he also proved that the Jacobian locus in $\mathcal{A}_{g-1}$ (i.e. the image of the Torelli map $\mathfrak{t}$) is contained in $\overline{\mathcal{B}}_{g-1}$. In this sense, Pryms are a generalization of Jacobians. Part (b) was first proved by R. Friedman and R. Smith [16] and for $g \geq 8$, by V. Kanev [18] by using degeneration methods. More geometric proofs were given by G. Welters [33] and later by O. Debarre [12], in the spirit of the proof of Torelli's theorem.

The fact that the Prym map is non-injective was first observed by Beauville [5]. Donagi's tetragonal construction [13] provides examples for the non-injectivity in any genus. □

OPEN QUESTION. What is exactly the non-injectivity locus of the Prym map $\mathcal{P}_g$?

4. Allowable covers

We will focus now on the rich geometry of the fibres of the Prym map $\mathcal{P}_6 : \mathcal{R}_6 \to \mathcal{A}_5$. The following theorem is proved in [15].

THEOREM 4.1. *The degree of* $\mathcal{P}_6 : \mathcal{R}_6 \to \mathcal{A}_5$ *is 27.*

This number encodes the geometry of the fibres, which have the structure of the 27 lines in a smooth cubic surfaces. Although the degree is the number of étale double coverings mapping to a general element in $\mathcal{A}_5$, which in this case is a Prym variety, the count is done over loci with positive dimensional fibers, involving a very precise description of the blow ups. In order to compute the degree we shall

1. extend the $\mathcal{P}_g$ to a proper map (Theorem 4.7),
2. study the Prym varieties on the boundary,
3. compute the local degree along the different loci.

Parts (2) and (3) will be treated in Sections 6,7 and 8.

Beauville introduced the notion of *generalized Prym varieties* in [4] to denote Prym varieties associated with double coverings of stable curves, that is, lying on the boundary of $\mathcal{R}_g$. Since the objects in the compactification of $\mathcal{R}_g$ are known as *admissible covers*, we use the terminology in [15] and denote those covers in the boundary that give rise to a generalized Prym variety, *allowable covers.*

Let $\overline{\mathcal{M}}_g$ be the compactification of $\mathcal{M}_g$ by stable curves of genus g. Recall that a (complete) curve C is stable if it is connected, the only singularities are ordinary double points and $|Aut(C)| < \infty$. In particular, $\rho_a(C) = g \neq 1$ (arithmetic genus) and every non singular rational component meet other components in at least 3 points. Let $C \in \overline{\mathcal{M}}_g$, $\widetilde{C} \in \overline{\mathcal{M}}_{2g-1}$ and $\pi : \widetilde{C} \to C$ be a (possibly branched) double covering with an involution $\iota : \widetilde{C} \to \widetilde{C}$. In order to analyse the "good" coverings for the Prym map, we make the following assumption:

(*) The fixed points of $\widetilde{C}$ under the involution ι are exactly the singular points and at a singular point the two branches are not exchanged under ι.

The reason for this assumption is that in this case the quotient $C := \widetilde{C}/\iota$ has only ordinary double points as singularities. We have the following commutative diagram

$$\begin{array}{ccc} \widetilde{N} & \xrightarrow{\tilde{f}} & \widetilde{C} \\ {\scriptstyle \pi'}\downarrow {\scriptstyle 2:1} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & C \end{array}$$

where f and $\tilde{f}$ are the normalization maps. Thus, π' is ramified at the points $x_i, y_i \in \widetilde{N}$ lying over a singular point $z_i \in \widetilde{C}$. One can also show that $\pi^*\omega_C \simeq \omega_{\widetilde{C}}$; as a consequence,

$$\rho_a(\widetilde{C}) = 2\rho_a(C) - 1.$$

Let $\widetilde{K}$ (resp. K) be the ring of rational functions on $\widetilde{C}$ (resp. C). The group of Cartier divisors on $\widetilde{C}$ can be described as

$$\operatorname{Div}\widetilde{C} = \bigoplus_{x\in\widetilde{C}_{sm}} \mathbb{Z}x \oplus \bigoplus_{s\in\widetilde{C}_{sing}} \widetilde{K}_s^*/\mathcal{O}_s^*.$$

Let $s_1, s_2 \in \widetilde{N}$ be the points over a singular point $s \in \widetilde{C}$. By choosing parameters t_1, t_2 around s_1 and s_2, we have the following isomorphism

$$\widetilde{K}_s^*/\mathcal{O}_s^* \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}, \qquad a \mapsto (\tfrac{u}{v}, m, n)$$

where $a = ut_1^m$ and $a = vt_2^n$ are the local descriptions of a around s_1, resp. s_2. Assuming that $\iota^*t_1 = -t_1$ and $\iota^*t_2 = -t_2$, the action of ι on $\widetilde{K}_s^*/\mathcal{O}_s^*$ is

$$\iota^*(z, m, n)_s = ((-1)^{m+n}z, m, n)_s$$

which yields the commutative diagram

$$\begin{array}{ccccccc} \widetilde{K}^* & \longrightarrow & \operatorname{Div}(\widetilde{C}) & \longrightarrow & \operatorname{Pic}(\widetilde{C}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle N_{\widetilde{K}/K}} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \\ K^* & \longrightarrow & \operatorname{Div}(C) & \longrightarrow & \operatorname{Pic}(C) & \longrightarrow & 0 \end{array}$$

where $\pi_*(\sum_i x_i) = \sum_i \pi(x_i)$ for $x_i \in \widetilde{C}_{sm}$ and for singularities we have $\pi_*((z, m, n)_s) = ((-1)^{m+n}z^2, m, n)_{\pi(s)}$. The norm map $\operatorname{Pic}(\widetilde{C}) \to \operatorname{Pic}(C)$ restricts to a norm map between the generalized Jacobians $\mathrm{Nm} : J\widetilde{C} \to JC$. We want to consider coverings such that the kernel of this map is an abelian variety.

Example 4.2. Let X be a smooth genus g curve with two marked points p, q and $X_1 = X_2 = X$ be two copies with marked points $p_i, q_i \in X_i$. Define the Wirtinger cover $\pi : \widetilde{C} \to C$ (see Figure 2) by

$$\widetilde{C} := X_1 \cup X_2/p_1 \sim q_2, p_2 \sim q_1, \qquad C = X/p \sim q.$$

Let $\nu : X \to C$ denote the normalization map and $s \in C$ be the node. To specify a line bundle L on C one has to specify $\tilde{L} := \nu^*L$ and a descent data, i.e. when a section of ν^*L is the pullback of a section of L. In this case it suffices to give the identification of the fibers $\varphi_s : \tilde{L}_p \xrightarrow{\sim} \tilde{L}_q$ over p and q, so $\varphi_s \in \mathbb{C}^*$. More generally, we have a short exact sequence

$$0 \to (\mathbb{C}^*)^b \to JC \xrightarrow{\nu^*} JN \to 0$$

where b is the first Betti number of the dual graph of C. As will see, this is an example of an allowable cover.

Lemma 4.3. *If L is a line bundle on $\widetilde{C}$ such that $\mathrm{Nm}_\pi L = \mathcal{O}_C$ then $L = M \otimes \iota^* L^{-1}$ for some line bundle M on $\widetilde{C}$ which can be chosen of multidegree $\underline{\deg} M = (0, \dots, 0)$ or $(1, 0, \dots, 0)$.*

We define Prym variety of the covering $\pi : \widetilde{C} \to C$ as the connected algebraic subgroup

$$P := \{M \otimes \iota^* L^{-1} \;\mid\; \underline{\deg} M = (0, \dots, 0)\}.$$

Proposition 4.4. *The variety P is an abelian variety of dimension $\rho_a(C) - 1$.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{T}_2 & \to & P \times \mathbb{Z}/2 & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{T} & \to & J\widetilde{C} & \to & J\widetilde{N} & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \\
0 & \to & T & \to & JC & \to & JN & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Notice that $\mathrm{Nm} \circ \pi^*$ is the multiplication by 2 and since $\pi^* : T \to \widetilde{T}$ is an isomorphism the left vertical arrow Nm is surjective and its kernel $\widetilde{T}_2$ corresponds to the points of order 2 in $\widetilde{T}$. The Kernel R is a complete subvariety of $J\widetilde{N}$, so P is also complete. □

Choose a line bundle $L \in \mathrm{Pic}(C)$ with multidegree satisfying $2\underline{\deg} L = \underline{\deg} \omega_{\widetilde{C}}$ and define

$$\Theta_L := \{M \in J\widetilde{C} \;\mid\; H^0(\widetilde{C}, L \otimes M) \neq 0\}.$$

It turns out that, as in the smooth case, $\Theta_{L|P} \equiv 2\Xi$, with $\Xi \in \mathrm{NS}(P)$ a principal polarization. Thus (P, Ξ) is a ppav associated to $(\widetilde{C}, \iota)$.

Definition 4.5. *A covering* $(\widetilde{C}, \iota)$ *with* $\widetilde{C} \in \overline{\mathcal{M}}_{2g-1}$ *and* ι *an involution, is allowable if its associated Prym variety* P *is an abelian variety and* $\rho_a(\widetilde{C}/\iota) = g$.

This definition is equivalent to any of these properties:

(a) The only fixed points of ι are the nodes where the two branches are not exchanged and the number of nodes exchanged under ι equals the number of irreducible components exchanged under ι.

(b) The components of $\widetilde{C}$ can be grouped as $\widetilde{C} = A \cup A' \cup \widetilde{B}$ where ι interchanges A and A' and fixes $\widetilde{B}$, each A is tree-like and either

- $\widetilde{B} = \emptyset$, A connected and $|A \cap A'| = 2$, or
- $A \cap A' = \emptyset$, $|\widetilde{B} \cap A_i| = 1$ for each connected component A_i of A the fixed points of ι in $\widetilde{B}$ are precisely the nodes and the two branches there are never exchanged (so that $\widetilde{B}/\iota$) also has nodes at the corresponding points.

Remark 4.6. The condition (*) is equivalent to (a) and (b) if there is no exchanged components under ι.

Let us denote

$$\overline{\mathcal{R}}_g := \{[\pi : \widetilde{C} \to C] \ | \ [\widetilde{C}] \in \overline{\mathcal{M}}_{2g-1}, \ [C] \in \overline{\mathcal{M}}_g, \ \pi \text{ is an allowable cover}\},$$

which is an open subspace in the compactification by admissible coverings of the moduli space $\mathcal{R}_g$.

Theorem 4.7. *The Prym map* $\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$*, extends to a proper map*

$$\overline{\mathcal{P}}_g : \overline{\mathcal{R}}_g \to \mathcal{A}_{g-1}$$

For the proof of this theorem we refer [15, Theorem 1.1]. Now the aim is to compute the local degree of $\overline{\mathcal{P}}_g$ along the relevant divisors (those which are not contracted under the Prym map).

5. Computation of the local degree

Let $f : Y \to X$ be a proper dominant map between two varieties, with $\dim X = \dim Y = n$, so f is generically finite. Set $d = \deg f$. Let $W \subset Y$ be an irreducible closed subvariety of codimension k, thus $f^{-1}(W)$ consists of finitely many irreducible components Z_i of codimension l_i in X. The local degree of d_i of f along Z_i is the degree of the map obtained from f by localizing X at Z_i; thus $d = \sum_i d_i$. Let $Z \subset X$ be one of these components, $\widetilde{X} = Bl_Z X$ (resp.

$\widetilde{Y} = Bl_Z Y$) the blow up fo X (resp. Y) along Z. Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{Z} \subset \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \supset \widetilde{W} \\ \downarrow & & \downarrow \\ Z \subset X & \xrightarrow{f} & Y \supset W \end{array}$$

The map $\tilde{f}$ induces a map between the exceptional divisors $f_* : \widetilde{Z} \to \widetilde{W}$, described as follows. Recall that $\widetilde{Z} = \mathbb{P}(\mathcal{N}_{Z \backslash X})$ and $\widetilde{W} = \mathbb{P}(\mathcal{N}_{W \backslash Y})$ are the projectivized normal bundles. Let $z \in Z$ and $w = f(z) \in W$. The differential $df_z : T_z X \to T_w W$ at z maps $T_z Z$ to $T_w W$. therefore, this induces a map

$$f_{*,z} : \mathcal{N}_{Z \backslash X} \to \mathcal{N}_{W \backslash Y}.$$

This lemma follows from the universal property of blow ups.

Lemma 5.1.

(a) The map $\tilde{f}$ is regular at a generic $z \in \widetilde{Z}$ if and only if $f_{*,z}$ is not identically zero at a generic $z \in Z$.

(b) The map $\tilde{f}$ is regular for all $\tilde{z}$ in the fiber over $z \in \widetilde{Z}$ if and only if $f_{*,z}$ is injective on the normal space $\mathcal{N}_{Z \backslash X, z}$ to Z at z. In this case $\tilde{f}_{|\text{fiber over } z}$ is the projectivization of the linear map $f_{*,z}$.

Lemma 5.2. *Assume $f_{*,z}$ is injective on $\mathcal{N}_{Z \backslash X, z}$ at each $z \in Z$. Then the local degree of f along Z equals the degree of the map $f_* : \widetilde{Z} \to \widetilde{W}$ on the exceptional divisors.*

Remark 5.3 (Warning). The lemma requires the injectivity **for all** $z \in Z$. Otherwise $\tilde{f}$ is not regular on a neighbourhood of $\widetilde{Z}$ and could involve a blow up of some small dimensional subvariety onto $\widetilde{W}$, implying that $\widetilde{Z}$ is only one of several components of the graph $\tilde{f}$ over Z. In this case the degree of f_* is possibly smaller than the degree of $\tilde{f}$ restricted to f^{-1}(neighbourhood of z), that is, smaller than the local degree of f on Z.

Consider the locus $\mathcal{J}_5$ of Jacobians of smooth curves of genus 5, which is of codimension 3 in $\mathcal{A}_5$. Given a generic curve $X \in \mathcal{M}_5$, Mumford [25] provided a list of smooth double covers such that their Prym variety $P(\widetilde{C}/C) \simeq JX$. Later Donagi and Smith [15] extended this list to the allowable covers with the same Jacobian. From this list only four cases are relevant for the computation of the degree, since the rest of the cases involve covers mapping to smaller loci in $\mathcal{J}_5$, that is whose image is of dimension $< 12 = \dim \mathcal{J}_5$, hence these loci do not contain the generic Jacobian. These are the four relevant loci in $\overline{\mathcal{R}}_g$:

(a) $[C] \in \mathcal{M}_6$ is a smooth plane quintic and $\pi : \widetilde{C} \to C$ is an even double cover, that is, with the property $h^0(\widetilde{C}, \pi^*\mathcal{O}_C(1)) = 0 \bmod 2$ [2]. We will denote this locus by $\mathcal{R}_Q$[3].

(b) Double covers over trigonal curves, denoted $\mathcal{R}_{\mathcal{T}}$.

(c) Wirtinger covers, later denoted by $\mathcal{R}_S$.

(d) Elliptic tails, $\mathcal{R}_E$, given by : $\pi : \widetilde{C} \to C$ where

$$\widetilde{C} := X_1 \cup \tilde{E} \cup X_2 / p_1 \sim 0, p_2 \sim a, \quad C := X \cup E/p \sim 0,$$

with $[X] \in \mathcal{M}_5$, $X_1 \simeq X_2$ copies of X and $\widetilde{E}$ an étale double cover of an elliptic curve E. Here $p_i \in X_i$, $i = 1, 2$ map to $p \in X$ and intersection points $0, a \in \widetilde{E}$ map to $0 \in E$.

The loci (a) and (b) are of dimension 12, whereas (c) and (d) are of dimension 14. In the next section we shall apply Lemma 5.2 to the computation of the local degree along these loci mapping onto the Jacobi locus. We will prove that their contributions to the degree are $1, 10, 16$ and 0 respectively.

6. Plane quintics

Recall that a theta characteristic on a curve C of genus g is a line bundle $\kappa \in \mathrm{Pic}^{g-1}(C)$ such that $\kappa^{\otimes 2} \simeq \Omega_C$; κ is even or odd according to the parity of $h^0(C, \kappa)$. Let $[C] \in \mathcal{M}_6$ be a smooth plane quintic. There is a natural odd theta characteristic κ given by the pullback of the hyperplane class ℓ under the embedding $C \hookrightarrow \mathbb{P}^2$. Define

$$\mathcal{R}'_Q = \{[C, \eta] \in \mathcal{R}_6 \ \mid \ C \text{ is a plane quintic}\}.$$

We distinguish two types of coverings in $\mathcal{R}'_Q$, if $h^0(\eta \otimes \kappa) = 0 \bmod 2$ (respectively $= 1 \bmod 2$) we say that the cover $[C, \eta]$ is even (respectively odd). This gives a decomposition of $\mathcal{R}'_Q$ into two irreducible components

$$\mathcal{R}'_Q = \mathcal{R}_Q \sqcup \mathcal{R}_{\mathcal{C}}.$$

We will show that the locus of even coverings, $\mathcal{R}_Q$, maps onto $\mathcal{J}_5$ and odd coverings, $\mathcal{R}_{\mathcal{C}}$, maps to the locus of intermediate Jacobians of cubic threefolds.

Assume now, that $JX = \mathcal{P}_6([C, \eta]) \in \mathcal{J}_5$, with $[X] \in \mathcal{M}_5$ generic (non hyperelliptic, nor trigonal), so $(JX, \Theta) \in \mathcal{A}_5$. According to the Riemann Singularity Theorem

$$\Theta_{\mathrm{Sing}} = \{L \in \mathrm{Pic}^4(X) \ \mid \ h^0(X, L) \geq 2\}.$$

[2] $\mathcal{O}_C(1)$ denotes the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ to C.

[3] In order to facilitate further reading, we keep the notation as in [15].

On the other hand, the genericity of X implies that the image of the canonical embedding $X \hookrightarrow \mathbb{P}^4$ is given by the intersections of three smooth quadrics

$$X = Q_0 \cap Q_1 \cap Q_2.$$

Any g^1_4 on X is cut out by a 1-parameter family of 2-planes sweeping out a quadric (of rank 3 or 4) in $\mathbb{P}^4$ containing X. Set

$$\Pi = \langle Q_0, Q_1, Q_2 \rangle = \{\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 \;\mid\; \lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{P}^2\}$$

the net parametrizing all the quadrics containing X. The discriminant locus

$$\{\lambda \in \Pi \;\mid\; Q_\lambda \text{ is a singular quadric}\}$$

is a plane quintic $C \subset \mathbb{P}^2$ defined by the vanishing of 5×5 linear determinant. For a given $\lambda \in C$, the quadric Q_λ possesses two 1-parameter families of planes cutting a g^1_4. Let $\widetilde{C}$ be the curve parametrizing the g^1_4's. By construction, this defines an étale double cover $\widetilde{C} \to C$. Actually, $\widetilde{C} \simeq \Theta_{\mathrm{Sing}}$.

In conclusion, one can recover uniquely a double covering $[C, \eta] \in \mathcal{R}_Q$ from the Jacobian of a generic genus 5 curve.

Remark 6.1. For X generic, $\widetilde{C}$ is smooth and π is étale. This fails when X possesses a vanishing thetanull, i.e., an even theta characteristic κ with $h^0(C, \kappa) \geq 2$. In this case $\widetilde{C}$ is singular. Masiewicki showed [24] that the corresponding cover is allowable, extending the result to all the curves in $\mathcal{M}_5$.

In order to show that the local degree of $\mathcal{P}_6$ on $\mathcal{R}_Q$ equals 1, we have to show that $\mathcal{P}_6$ is not ramified on $\mathcal{R}_Q$. This is equivalent to showing that the codifferential map

$$d\mathcal{P}_6^* : \mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C^{\otimes 2})$$

is injective on the generic element of $\mathcal{R}_Q$. Using the identification $(T_0 P)^* \simeq H^0(C, \omega_C \otimes \eta)$ one can show that the projectivized of the Abel-Prym map $\widetilde{C} \to P$ is the composition [8, Prop. 12.5.3]

$$\widetilde{C} \xrightarrow{\pi} C \to \mathbb{P}(H^0(C, \omega_C \otimes \eta)) \simeq \mathbb{P}^5.$$

Therefore, the injectivity of the map follows from:

Proposition 6.2. *The Prym-canonical image $\Psi(C) \subset \mathbb{P}^5$ for a generic $[C, \eta] \in \mathcal{R}_Q$ is contained in no quadrics.*

Proof. Beauville proved [5, Prop. 7.10] that for a non-hyperelliptic curve $X \in \mathcal{M}_5$, the corresponding plane quintic C is contained in no quadrics. Since $\mathcal{R}_Q$ is irreducible [15, II.3.3], this suffices to to prove the proposition. □

7. Trigonal curves

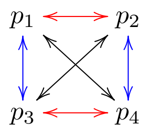
Let us recall Recillas construction [32] that shows that the Jacobian of a tetragonal curve is the Prym variety of a covering of a trigonal curve.

Let (X, g^1_4) be a tetragonal curve of genus $g-1$. Consider

$$\widetilde{C} := \{p_1 + p_2 \in X^{(2)} \ \mid \ \exists\, p_3, p_4 \in X,\ p_1 + p_2 + p_3 + p_4 \in g^1_4\}\,.$$

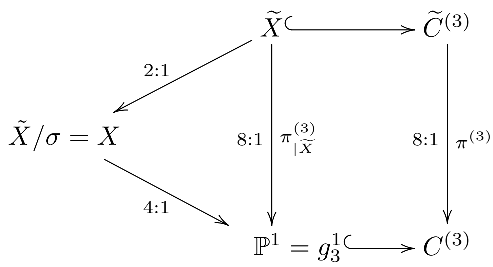
Note that there exists a natural involution $\sigma : \widetilde{C} \to \widetilde{C}$, $\sigma(p_1 + p_2) = p_3 + p_4$, so we can define $C = \widetilde{C}/\sigma$. For the construction, we assume that X is general tetragonal, i.e. a map $f : X \to \mathbb{P}^1$ induced by the g^1_4 has in any fibre at least three points. The assumption implies that σ is a fixed point free involution. A technical lemma shows that $\widetilde{C}$ is smooth [8, Lemma 12.7.1].

Note that C is trigonal. This is because 4 points can be divided to pairs in 3 different ways, as in the following Diagram.



This gives a g^1_3 and a map $h : C \to \mathbb{P}^1$. The map h is ramified in exactly the same locus as f, so one can apply Hurwitz formula to get $g(C) = g-1+1 = g$. Then $g(\widetilde{C}) = 2g-1$ and the aim will be to show that $P(\widetilde{C}/C) = JX$.

Now, we will show an inverse construction that will be drawn in Diagram (3). Let $\pi : \widetilde{C} \to C$ be a double covering of a trigonal curve C of genus g. Let $\pi^{(3)} : \widetilde{C}^{(3)} \to C^{(3)}$ be an induced $8:1$ covering. Note that one can embed $\mathbb{P}^1 = g^1_3 \ni p_1 + p_2 + p_3 \hookrightarrow [p_1 + p_2 + p_3] \in C^{(3)}$ and restrict $\pi^{(3)}$ to the preimage of $\mathbb{P}^1$, called $\tilde{X}$. An involution σ acts on $\widetilde{C}$, hence on $\widetilde{C}^{(3)}$ and $\tilde{X}$ is σ-invariant. Hence, $\pi^{(3)}|_{\tilde{X}}$ factorises via $X = \tilde{X}/\sigma$ which is a tetragonal curve of genus $g-1$.



(3)

Trigonal construction allows us to define a map:

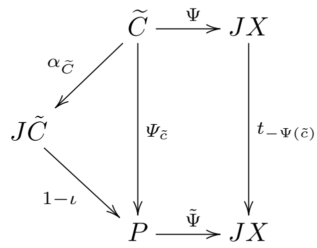
$$\begin{aligned} \tau\colon \mathcal{J}^1_{4,g-1} &\longrightarrow \overline{\mathcal{R}}_g \\ (X, g^1_4) &\longmapsto [\widetilde{C} \to C] \end{aligned}$$

that gives us an allowable covering. We have the following proposition.

PROPOSITION 7.1. *Recall that* $\widetilde{C} \subseteq X^{(2)}$ *and let* $\alpha = \alpha_{g^1_4} : X \longrightarrow JX$ *be the Abel map chosen such that* $\alpha(x) = 4x - g^1_4$. *We get:*

1. $P_g(\tau(X, g^1_4)) = JX$.
2. *The map* $\Psi : \widetilde{C} \longrightarrow JX = P_g(\widetilde{C} \to C)$ *given by* $(a, b) \mapsto \alpha(a) + \alpha(b)$ *is the Abel-Prym map of the covering* $\widetilde{C} \to C$.

Proof. Fix $\tilde{c} \in \widetilde{C}$. We will use the universal property of Prym varieties [8, Thm 12.5.1] to get the bottom row of the diagram:



(4)

and to show the $\tilde{\Psi}$ is an isomorphism.

Firstly, in order to get the diagram, we need to show that $\Psi \circ \iota = -\Psi$. This is satisfied since

$$\Psi \circ \iota(a, b) = \Psi(c, d) = \alpha(c) + \alpha(d) = -\alpha(a) - \alpha(b),$$

for $a + b + c + d \in g^1_4$.

Now, by Matsusaka's criterion [8, Rmk 12.2.5] it is enough to show that

$$\psi(\widetilde{C}) = \frac{2}{(g-2)!} \bigwedge^{(g-2)} \Theta_{JX} \in H^{2g-4}(JX, \mathbb{Z})$$

(note that $g(X) = g - 1$, both JX and P are of the same dimension and the polarisation on P is twice the principal one).

To prove it we will use a degeneration method. Let X_t degenerate to $X_0 \cup \mathbb{P}^1$ with X_0 being trigonal curve and g^1_4 degenerates to g^1_3 on X_0 and the intersection point $X_0 \cap \mathbb{P}^1 = p_0$. Let $p_0 + p_1 + p_2 \in g^1_3$ and consider $C = X_0/p_1 \sim p_2$ with the Wirtinger cover $\widetilde{C}$ where $q_1 = p_2, q_2 = p_1$ as in Example 4.2. Note that the class $[\Psi(\widetilde{C})]$ does not change in the degeneration. We compute

$$[\Psi(\widetilde{C})] = [\alpha(X_0) + \alpha(X_0)] = 2[\alpha(X_0)] = \frac{2}{(g-2)!} \bigwedge^{(g-2)} \Theta_{JX_0}. \qquad \square$$

Denote by

$$\mathcal{R}_{T,g} = \{[\widetilde{C} \to C] \in \mathcal{R}_g : \ C \text{ trigonal } \}.$$

Then $\overline{\mathcal{R}}_{T,g} \subset \overline{\mathcal{R}}_g$ and $\mathrm{Im}(\tau) = \overline{\mathcal{R}}_{T,g}$.

Remark 7.2. By Brill-Noether theory, every curve of genus 5 has got a g^1_4.

We have the following diagram:

$$\begin{array}{ccccccc}
\widetilde{\mathcal{R}}_T & \hookrightarrow & Bl_{\overline{\mathcal{R}}_T}\overline{\mathcal{R}}_6 = \widetilde{\mathcal{R}}_6 & \xrightarrow{\widetilde{\mathcal{P}}} & \widetilde{\mathcal{A}}_5 = Bl_{\mathcal{J}_5}\mathcal{A}_5 & \hookleftarrow & \widetilde{\mathcal{J}}_5 \\
\downarrow{\scriptstyle \mathbb{P}^1-bundle} & & \downarrow{\scriptstyle bl} & & \downarrow{\scriptstyle bl} & & \downarrow{\scriptstyle \mathbb{P}^2-bundle} \\
\overline{\mathcal{R}}_T & \hookrightarrow & \overline{\mathcal{R}}_6 & \xrightarrow{\overline{\mathcal{P}}_6} & \mathcal{A}_5 & \hookleftarrow & \mathcal{J}_5
\end{array} \qquad (5)$$

where $\widetilde{\mathcal{R}}_T$ and $\tilde{J}_5$ are exceptional divisors of the blow ups.

Recall that for X of genus $g-1$

$$\Phi : X \longrightarrow \mathbb{P}H^0(X, \omega_X)^* \simeq \mathbb{P}^{g-2}$$

is the canonical map and for $(C, \eta) \in \mathcal{R}_g$ of genus g

$$\Psi : C \longrightarrow \mathbb{P}H^0(C, \omega_C \otimes \eta)^* \simeq \mathbb{P}^{g-2}$$

is called the Prym-canonical map. Consider again the map

$$d\mathcal{P}^*_g : \mathrm{Sym}^2 H^0(C, \omega_C \otimes \eta) \longrightarrow H^0(C, \omega^2_C).$$

Lemma 7.3. *Let $\widetilde{C} \to C$ be a double covering of a trigonal curve C of genus g, and X its corresponding tetragonal curve of genus $g-1$. Then*

(i) The image in $\mathbb{P}^{g-2}$ of the 4 points $a, b, c, d \in X$ under the canonical embedding of each divisor $D \in g^1_4$ are coplanar.

(ii) On each of these planes the 3 points of intersection of opposite lines, that is, $\overline{ab} \cap \overline{cd}$, $\overline{ac} \cap \overline{bd}$, $\overline{ad} \cap \overline{bc}$ are on $\Psi(C)$ and as D varies in g^1_4, they trace $\Psi(C)$ once, giving the g^1_3 on C.

Lemma 7.4. *The intersection of $\Phi(X) \cap \Psi(C)$ in $\mathbb{P}^{g-2}$ consists of $2g+4$ points, corresponding to the ramification points of the g^1_4 and g^1_3.*

Proposition 7.5. *Let $[C, \eta] \in \widetilde{\mathcal{R}}_T$, then $ker(d\mathcal{P}^*_6)$ is the one-dimensional subspace corresponding to the unique quadric in $\mathbb{P}^4$ containing $\Phi(X)$ and the family of planes cutting the given g^1_4 on X.*

Proof. Recall that $\mathrm{Ker}(d\mathcal{P}_6^*) = \{\text{quadrics in } \mathbb{P}^4 \text{ containing } \Psi(C)\}$. A g^1_4 is given by cutting out X with 1-dimensional family of plane of a quadric $Q \subset \mathbb{P}^4$. By Lemma 7.3, Q contains $\Psi(C)$ since $\Psi(C)$ is contained in the union of these planes. Moreover, every quadric in $\mathrm{Ker}(d\mathcal{P}_6^*)$ also contains $\Phi(X)$. Suppose that $\Psi(C)$ is contained in another quadric Q', so $\Psi(C) \subset Q \cap Q'$ and let Q'' be the quadric so that

$$\Phi(X) = Q \cap Q' \cap Q''.$$

In the smooth case $\Psi(C)$ has degree $2g - 2 = 10$. Hence $\Psi(C) \cap \Phi(X) = \Psi(C) \cap Q''$ has degree 20, but this contradicts Lemma 7.4, since the trigonal map has $2g + 4 = 16$ ramification points. □

THEOREM 7.6. *The local degree of* $\overline{\mathcal{P}}_6$ *at* $\overline{\mathcal{R}}_T$ *equals 10.*

Proof. Identifying $\mathcal{J}_5$ to $\mathcal{M}_5$ via the Torelli map, we denote by $\mathcal{N}(\mathcal{M}_5 \setminus \mathcal{A}_5)$ the normal subbundle to $\mathcal{J}_5$ in $\mathcal{A}_5$, and $\mathcal{N}(\widetilde{\mathcal{R}}_T \setminus \widetilde{\mathcal{R}}_6)$ denotes the normal subbundle to the exceptional divisor $\widetilde{\mathcal{R}}_T$ in the blowup $\widetilde{\mathcal{R}}_6$. Consider the codifferential map on the conormal subbundles

$$\mathcal{N}^*(\mathcal{M}_5 \setminus \mathcal{A}_5) \to \mathcal{N}^*(\widetilde{\mathcal{R}}_T \setminus \widetilde{\mathcal{R}}_6).$$

Notice that the source bundle is of rank 3 and the target one is of rank 2. By Proposition 7.5, the kernel of this map is at most of rank one, therefore the map is surjective. According to Lemma 5.2 the local degree equals the degree of the

$$\widetilde{P}_e : \widetilde{\mathcal{R}}_T \to \widetilde{\mathcal{M}}_5,$$

where $\widetilde{P}_e$ denotes the projectivization of the conormal map on the exceptional divisors. Let $[X] \in \mathcal{M}_5$ be a generic curve and $\mathbb{P}^2$ the fiber over $[X]$ in $\widetilde{\mathcal{M}}_5$ and $\mathcal{R} = \widetilde{P}_e^{-1}([X])$ in $\widetilde{\mathcal{R}}_T$. We shall describe the map

$$\widetilde{P}_e : \mathcal{R} \to \mathbb{P}^2 = \mathbb{P}(\mathcal{N}_X(\mathcal{M}_5 \setminus \mathcal{A}_5)).$$

The target plane is dual to the plane Π containing the discriminant plane quintic F parametrizing the singular quadrics containing the canonical embedding of X. Thus, a point of $\mathbb{P}^2$ corresponds to a line in Π, that is a pencil of quadrics. And viceversa, a line in $\mathbb{P}^2$ corresponds to a point in Π, that is a quadric Q containing $\Phi(X)$. The quadric Q is singular if and only if, $p \in F \subset \Pi$. We have that $\mathcal{R}$ is a $\mathbb{P}^1$-bundle over $[\widetilde{F} \to F] \in \mathcal{R}_T$. Moreover, $\widetilde{F} = \mathrm{Sing}\,\Theta_X$ parametrizes the fiber of the Prym map over JX. Indeed, an element $L \in \mathrm{Sing}\,\Theta_X$ corresponds to a g_1^4 on X and the data (X, g_4^1) produces, via the trigonal construction, a double covering $[\widetilde{C} \to C] \in \mathcal{R}_T$ over a trigonal curve whose Prym variety is isomorphic to JX. The map

$$\widetilde{P}_e : \mathbb{P}^1 \to \mathbb{P}^2$$

on the fibers is injective and its image is a line in $\mathbb{P}^2$, that is a point in Π corresponding to a singular quadric in $\ker dP^*_{6\,|[\widetilde{C}\to C]}$.

Now, let $p \in \mathbb{P}^2$ be a generic point, then

$$\begin{aligned}\deg \widetilde{P}_e &= |\{\widetilde{P}_e\}| \\ &= |\{[\widetilde{C} \to C] \in \widetilde{F} \ \mid \ p \in \widetilde{P}_e(\mathbb{P}^1_C)\}| \\ &= |\{[\widetilde{C} \to C] \in \widetilde{F} \ \mid \ \pi(C) \in \ell(p)\}|\,,\end{aligned}$$

where $\ell(p)$ is the dual line to p in Π and $\pi : \widetilde{F} \to F$ is the double covering. Hence,

$$\deg \widetilde{P}_e = \deg(\widetilde{F} \to \Pi) = \deg(\pi) \cdot \deg F = 2 \cdot 5 = 10\,. \qquad \square$$

8. Boundary components

In this section we will describe the Prym map on the boundary $\overline{\mathcal{R}}_6 \backslash \mathcal{R}_6$ mapping onto the Jacobian locus $\mathcal{J}_5$. Denote by:

$$\mathcal{R}_S = \{[\widetilde{C} \to C] \in \overline{\mathcal{R}}_6 : [C] \in \overline{\mathcal{M}}_g \ \text{ is irreducible with one node and its degenerations}\}$$

$$\mathcal{R}_E = \{[\widetilde{C} \to C] \in \overline{\mathcal{R}}_6 : [C] \in \overline{\mathcal{M}}_g \ \text{ has an irreducible component of genus } g-1 \text{ and an elliptic tail and its degenerations}\}$$

The general element of $\mathcal{R}_S$ is a Wirtinger cover and in fact $\mathcal{R}_S$ is a boundary component over $\mathcal{M}_5$ (see Figure 2). Let $\mathcal{R}_{E,S}$ denote the intersection of $\mathcal{R}_S$ and $\mathcal{R}_E$.

Lemma 8.1. *The only irreducible components of* $\overline{\mathcal{R}}_6 \setminus \mathcal{R}_6$ *whose image contains* $\mathcal{J}_5$ *are* $\mathcal{R}_S$, $\mathcal{R}_E$ *and* $\overline{\mathcal{R}}_T \setminus \mathcal{R}_T$.

A fine analysis is required to study the Prym map near the singular curves (for instance one needs to distinguish the dualising sheaf ω_C from the Kähler differentials Ω_C). Donagi and Smith avoid the difficulties arising along the locus $\mathcal{R}_E$ and $\mathcal{R}_{E,S}$ by constructing a new compactification $\mathcal{M}'_6$ of $\mathcal{M}_6$ and a space $\mathcal{R}'$ over $\mathcal{M}'_6$, such that $\overline{\mathcal{P}}_6$ factorises through $\mathcal{R}'$:

$$\begin{array}{ccc} \overline{\mathcal{R}}_6 & \xrightarrow{\ \overline{\mathcal{P}}_6\ } & \mathcal{A}_5 \\ & {\scriptstyle\beta}\searrow \quad \nearrow & \\ & \mathcal{R}' & \end{array}$$

such that the map β blow downs the component $\mathcal{R}_E$. This shows that $\mathcal{R}_E$ has no contribution to the degree of $\overline{\mathcal{P}}_6$. The details of these constructions can be found in [15, IV.§2,§4]

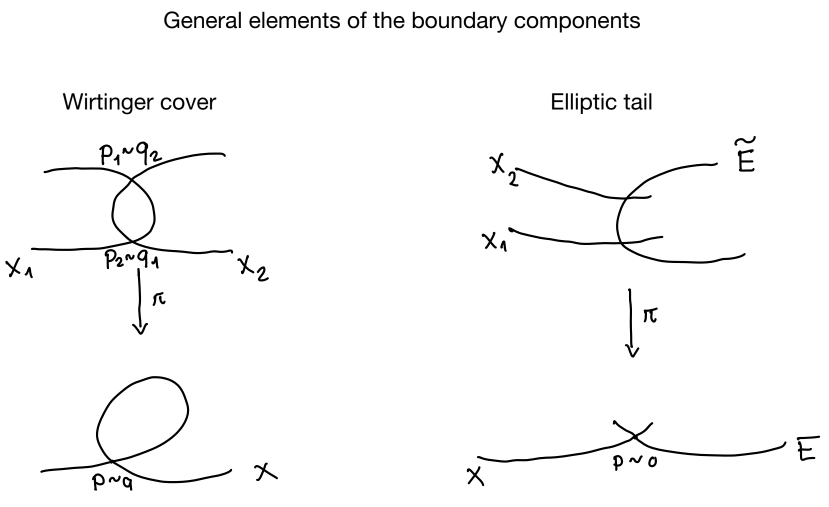


Figure 2: Stable covers

In this section we will use the genericity of $X \in \mathcal{M}_5$ (1) to ignore families in $\overline{\mathcal{R}}_6$ of dimension smaller than 12 ($= \dim \mathcal{M}_5$) and (2) to assume that $[X] \in \mathcal{M}_5$ is smooth (in particular has no automorphisms).

Note that $\overline{\mathcal{P}}_6|_{\mathcal{R}_S} : \mathcal{R}_S \longrightarrow \mathcal{J}_5$ is proper and surjective. The fibre of a general JX is naturally isomorphic to $S^2(X)$, since all you need is to choose $p, q \in X$ where you glue.

LEMMA 8.2. *For an element* $[C, \eta] \in \mathcal{R}_S \setminus \mathcal{R}_{E,S}$ *over a generic* $[X] \in \mathcal{M}_{g-1}$ *we have*

$$\operatorname{Ker} dP^*_{g-1} = \{\textit{Quadrics } Q \textit{ containing } \Phi(X) \textit{ and the chord } \overline{\Phi(p)\Phi(q)}\}.$$

For $[C, \eta] \in \mathcal{R}_S$ *with* $p = q$

$$\operatorname{Ker} dP^*_{g-1} = \{\textit{Quadrics } Q \textit{ containing } \Phi(X) \textit{ and its tangent line at the normalisation of the cusp}\}.$$

PROPOSITION 8.3. *For* $C = X/p \sim q$ *we have that* $\ker dP^*$ *is two dimensional.*

Proof. Recall that for the canonical model $X \subset \mathbb{P}^4$ we have $X = Q_1 \cap Q_2 \cap Q_3$, for some quadrics Q_1, Q_2, Q_3. The secant $\overline{pq}$ (or the tangent line if $p = q$) imposes 1 linear condition on the quadrics. The proposition follows from Lemma 8.2. □

THEOREM 8.4. *The local degree of the Prym map* P_6 *at the boundary of* $\overline{\mathcal{R}}_6$ *equals 16.*

Proof. The degree can be computed after blowing up $\mathcal{M}_5 \hookrightarrow \mathcal{A}_5$ and restricting the map to the exceptional divisor. On the fiber over a fixed generic $[X] \in \mathcal{M}_5$ the map becomes

$$f : S^2X \to \mathbb{P}^2 = \mathbb{P}(\mathcal{N}_X(\mathcal{M}_5 \setminus \mathcal{A}_5)),$$

sending the $(p, q) \in S^2X$ to the pencil of quadrics trough the line $\overline{\Phi(p)\Phi(q)}$. Therefore the degree is computed by the number of chords of $\Phi(X)$ contained in the intersection of two quadrics in general position. The theorem follows from Lemma 8.5 and 8.6. □

LEMMA 8.5. *The intersection of two quadrics in general position in $\mathbb{P}^4$ contains 16 lines.*

This is the number of lines on a del Pezzo surface of degree 4 obtained as the blow up of 5 points in general position on $\mathbb{P}^2$.

LEMMA 8.6. *The canonical curve $\Phi(X) \subset \mathbb{P}^4$ meets each of the 16 lines twice.*

Proof. Recall that $\Phi(X)$ is a complete intersection of $Q_0 \cap Q_1 \cap Q_2 \in \mathbb{P}^4$ of three quadrics. Let ℓ be a line in $Q_1 \cap Q_2$. The result follows from

$$|(\Phi(X) \cap \ell)_{|Q_1 \cap Q_2}| = |(Q_0 \cap \ell)_{|\mathbb{P}^4}| = 2. \qquad \square$$

9. Cubic threefolds and their intermediate Jacobians

In this section we study the fiber of the Prym map on the locus of intermediate Jacobians. Although the preimage of an intermediate Jacobian is 2-dimensional, after blow up the Prym map displays the structure of the finite fiber. The tetragonal construction provides a beautiful geometric way of recovering the fiber starting from one element in the preimage identifying it with the structure of the 27 lines on a smooth cubic surface. The original references for the theory of cubic threefolds are [11, 27, 28].

Let $X \subset \mathbb{P}^4$ be a smooth cubic hypersurface. Since a generic hyperplane section intersects X in 27 lines, there is a 2-dimensional family of lines lying in X parmetrized by the Fano surface $F(X)$. The intermediate Jacobian

$$JX = H^{1,2}(X, \mathbb{C})/H^3(X, \mathbb{Z})$$

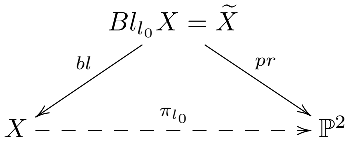
of X is isomorphic as a ppav to the Albanese variety $\mathrm{Alb}(F(X))$. The theta divisor Θ in JX is the image of the map

$$F(X) \times F(X) \to JX, \quad (\ell, \ell') \mapsto [\ell] - [\ell'],$$

which collapses the diagonal to $0 \in JX$ giving the only singularity in Θ (a triple point). One can identify the projectivized tangent space $\mathbb{P}(T_0(JX))$ with the ambient $\mathbb{P}^4$.

Let $\mathcal{C}$ be the 10-dimensional moduli space parametrizing the smooth cubic threefolds. The construction of the intermediate Jacobian yields a map $\mathcal{C} \to \mathcal{A}_5$. Since one can recover X from its tangent cone to Θ at 0, this map is an embedding (Torelli Theorem for cubic threefolds [11], see [6] for a proof involving Prym varieties).

Let (X, l_0) be a pair consisting of a cubic threefold X and a generic line $l_0 \subset X \subset \mathbb{P}^4$. Let π_{l_0} be a projection from l_0 to $\mathbb{P}^2$ and bl the blowing up map of X along ℓ_0. We have the following diagram



where $(\widetilde{X}, pr)$ can be seen as a conic bundle over $\mathbb{P}^2$. It is because a point $p \in \mathbb{P}^2$ is the image of a plane that contain l_0, so its intersection with X (that is of degree 3) is the union of l_0 and a conic, either smooth, or degenerated to two lines.

We define the discriminant locus and denote it as

$$C := \{p \in \mathbb{P}^2 : pr^{-1}(p) \text{ contains 2 lines}\}.$$

Note that C is a plane quintic, smooth for generic l_0, since a generic hyperplane section of X contains 5 pairs of lines coplanar with l. We also have a natural line bundle $L = \mathcal{O}_{\mathbb{P}^2}(1)|_C$ of degree 5.

Let

$$\widetilde{C} = \{l \in F(X) : l \cap l_0 \neq \emptyset\}$$

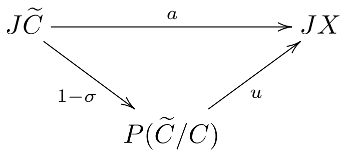
be the curve of lines intersecting l_0. The plane generated by l_0, l intersects X in the third line l' and hence there is a natural $2:1$ map $\pi = pr_{|\widetilde{C}} : \widetilde{C} \to C$ that sends a line l to $l \cap l' \in C$. One checks that $\widetilde{C}$ is also smooth and the covering π is unramified. For any line l_0 one obtains an allowable cover.

Proposition 9.1. *The Prym variety of the covering $\widetilde{C} \to C$ is isomorphic to $P(\widetilde{C}/C) \simeq JX$. Moreover the fibre $\mathcal{P}_6^{-1}(JX) \simeq F(X)$ is 2-dimensional.*

Proof. We give a sketch of the proof (a complete proof is available in [5]). Since $\widetilde{C}$ parametrizes a family of lines on X, there exists a map (an Abel-Prym map)

$$\psi : \widetilde{C} \to JX, \quad l \mapsto [l] - [l_0]$$

defined up to translation. This induces a homomorphism $a : J\widetilde{C} \to JX$. The family of fibres of *pr* is parametrized by $\mathbb{P}^2$, hence its corresponding Abel-Jacobi map is constant. Therefore, a is zero on $\pi^* JC$ and it gets factorized through a map u:



were σ is the involution exchanging the lines on the fiber of π. One shows that u is an isomorphism by means of cohomology properties and that u pulls back the principal polarization of JX to the principal polarization of the Prym variety [5, §2.6].

Since $\widetilde{C}$ is defined via a line $l_0 \subset X$ one can get that $\mathcal{P}_6^{-1}(JX) \simeq F(X)$. □

For a generic $p \in C$, $pr^{-1}(p)$ is a conic in X meeting l_0 in two points. For $p \in C \subset \mathbb{P}^2$, we denote

$$pr^{-1}(p) = l_1(p) \cup l_2(p)$$

with $l_1(p) \cup l_2(p)$ coplanar to l_0.

Proposition 9.2. *The map $C \to X \subset \mathbb{P}^4$ sending*

$$p \mapsto l_1(p) \cap l_2(p)$$

is the Prym-canonical map of (C, η)

Proof. The Abel-Prym map $\psi : \widetilde{C} \to P(\widetilde{C}, C) \simeq JX \simeq \mathrm{Alb}(F(X))$ is just the restriction to $\widetilde{C}$ of the map

$$F(X) \to JX, \quad l \mapsto [l] - [l_0].$$

The Prym-canonical image of $l_1(p) \in \widetilde{C}$ is the projectivized of the derivative of ψ at $l_1(p)$ and corresponds to a point of $l_1(p) \subset \mathbb{P}^4$ and similarly for $l_2(p)$. Hence this point should be the intersection of both lines. □

Proposition 9.3. *The 2-torsion point $\eta \in JC$ defining the covering $\pi : \widetilde{C} \to C$ satisfies $h^0(C, \eta \otimes L) = 1$, so $(C, \eta) \in \mathcal{R}_C$.*

Proof. The Prym-canonical map Ψ of (C, η) is given by the line bundle

$$\omega_C \otimes \eta \simeq \mathcal{O}_{\mathbb{P}^2}(2)_{|C} \otimes \eta$$

and after projecting from l, C is mapped to $\mathbb{P}^2$ by $\mathcal{O}_{\mathbb{P}^2}(1)_{|C} = L$. Hence, the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)_{|C} \otimes \eta$ has a unique effective divisor given by the 5-points intersection of the image of l with $\Psi(C)$. □

Let $\mathcal{R}_X := \mathcal{P}_6^{-1}(JX) \cap \mathcal{R}_{\mathcal{C}}$. We will see that the codifferential is of maximal rank so $\mathcal{R}_X$ is isolated in $\mathcal{P}_6^{-1}(JX)$, that is, it is a connected and irreducible component. Therefore $\mathcal{R}_X = \mathcal{P}_6^{-1}(JX)$.

As we have seen the choice of a line in the Fano variety $F(X)$ produces an étale double covering whose Prym variety is isomorphic to the intermediate Jacobian JX. Thus $F(X)$ parametrizes a subvariety $\mathcal{R}'_X \subset \mathcal{R}_X$. It can be shown that the closure of the union of $\cup_X \mathcal{R}'_X$ for all the smooth cubic threefolds X equals the locus $\mathcal{R}_{\mathcal{C}}$ of pairs (C, η) with η odd. Let $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}_5$ be the closure of the locus of intermediate Jacobians of cubic threefolds. We have the following blow up diagram:

$$\begin{array}{ccccccc} (C,\eta,L) \in \widetilde{\mathcal{R}}_{\mathcal{C}} & \hookrightarrow & \widetilde{\mathcal{R}}_6 & \xrightarrow{\widetilde{\mathcal{P}}_6} & \tilde{\mathcal{A}}_5 & \hookleftarrow & \widetilde{\mathcal{C}} \ni (X,H) \\ {\scriptstyle \mathbb{P}^2-bundle}\ \downarrow{\scriptstyle \pi_1} & & \downarrow & & \downarrow & & {\scriptstyle \pi_2}\downarrow\ {\scriptstyle \mathbb{P}^4-bundle} \\ (C,\eta) \in \mathcal{R}_{\mathcal{C}} & \hookrightarrow & \mathcal{R}_6 & \xrightarrow{\mathcal{P}_6} & \mathcal{A}_5 & \hookleftarrow & \mathcal{C} \ni X \end{array} \tag{6}$$

where $\widetilde{\mathcal{R}}_{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ are the exceptional divisors, $H = \mathcal{O}_{\mathbb{P}^4}(1)|_X$ is a hyperplane section and $L \in F(X)$. We will see that $\widetilde{\mathcal{P}}_e^{-1}(X, H) = \{l \in F(X) : l \in X \cap H\}$. We can find the following dimensions of spaces and general fibres of maps that appears in Diagram (6)

$$\begin{array}{ccccccc} 14 & \hookrightarrow & 15 & \xrightarrow{\widetilde{\mathcal{P}}_6} & 15 & \hookleftarrow & 14 \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 4} \\ 12 & \hookrightarrow & 15 & \xrightarrow{\mathcal{P}_6} & 15 & \hookleftarrow & 10 \end{array}$$

The cotangent space

$$T^*_{JX}\mathcal{A}_5 = \operatorname{Sym}^2 T^*_0(JX)$$

consists of all quadrics in $\mathbb{P}^4 = \mathbb{P}(T^*_0(JX))$. The quadrics corresponding to the conormal space $\mathcal{N}^*_{JX}(\mathcal{A}_{\mathcal{C}} \setminus \mathcal{A}_5)$ are those X_p polar to points $p \in \mathbb{P}^4$ with respect to X (see [17]). Thus

$$\pi_2^{-1}(JX) \simeq \mathbb{P}(\mathcal{N}) \simeq (\mathbb{P}^4)^*.$$

Since $\mathcal{R}'_{\mathcal{C}}$ is an unramified cover over the moduli space of plane quintics, we can identify the fiber of π_1 over $(C, \eta) \in \mathcal{R}'_{\mathcal{C}}$ with the dual of the ambient $\mathbb{P}^2$ of C. In terms of given pair (X, l) with $l \in F(X)$ this $\mathbb{P}^2_l$ is the space of planes through l in $\mathbb{P}^4$ and $(\mathbb{P}^2_l)^*$ is the subspace of $(\mathbb{P}^4)^*$ dual to l.

LEMMA 9.4. *Let $\widetilde{\mathcal{R}}_X = \pi^{-1}(\mathcal{R}'_X)$ and $\mathcal{P}_X$ be the restricted map. Then*

$$\mathcal{P}_X : \widetilde{\mathcal{R}}_X = \cup_{l \in F(X)} (\mathbb{P}^2_l)^* \to (\mathbb{P}^4)^*$$

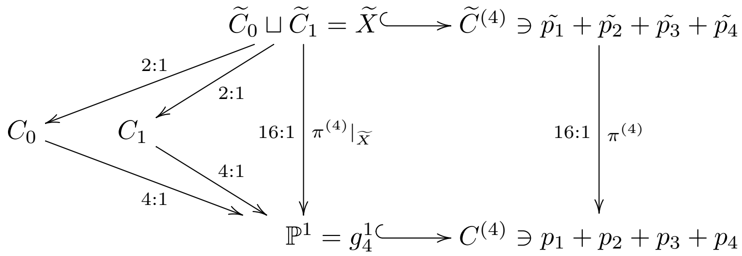
is the natural injection on each $(\mathbb{P}^2_l)^*$.

This lemma shows that $\widetilde{\mathcal{P}}_e$ is of maximal rank and it is of degree 27, since a generic hyperplane section of X contains 27 lines. Hence, an element in $(\mathbb{P}^4)^*$ has 27 planes in its preimage.

10. Tetragonal construction

As we have seen, the fiber of $\widetilde{\mathcal{P}}_6$ over an intermediate Jacobian corresponds to the 27 lines on a smooth cubic surface, so it carries also a structure of the incidence correspondence of the lines. The tetragonal construction on elements $(C, \eta) \in \mathcal{R}_{\mathcal{C}}$ on the fiber reflects this correspondence.

Let C denote a tetragonal curve of genus g (with $f : C \to \mathbb{P}^1$ given by a g^1_4) and let $\pi : \widetilde{C} \to C$ be an étale double covering. As usual, we have the following construction.



(7)

Note that for $D, D' \in Pic(\widetilde{X})$ we have $D \sim D'$ if and only if they push down to the same divisor on $C^{(4)}$ and they share an even number of points in each orbit. This shows in particular that $\widetilde{X}$ has two connected components $\widetilde{C}_0$ and $\widetilde{C}_1$.

We have so called triality $(\widetilde{C}, C, f)$, $(\widetilde{C}_0, C_0, f_0)$, $(\widetilde{C}_1, C_1, f_1)$ because the construction does not depend from which curve we have started. This phenomenon can be explained by the monodromy representation $\pi(\mathbb{P}^1 \setminus \{\text{branch points}\}) \longrightarrow WD_4$, where $W(D_4)$ is the Weyl group of D_4. Note that S_3 acts on $W(D_4)$ as the group of outer automorphisms. The outer automorphism of order 3 is responsible for the appearance of the three tetragonally related double covers.

Theorem 10.1. *The tetragonal construction commutes with the Prym map, that is,*

$$\mathcal{P}_g(\widetilde{C}, C) \simeq \mathcal{P}_g(\widetilde{C}_0, C_0) \simeq \mathcal{P}_g(\widetilde{C}_1, C_1)$$

are isomorphic as ppav (for any genus $g \geq 5$).

Proof. One can use Masiewicki's criterion to prove the isomorphisms. Instead, we sketch here the degeneration argument given in [14]. Consider $\mathcal{R}^{Tet}_g \subset$

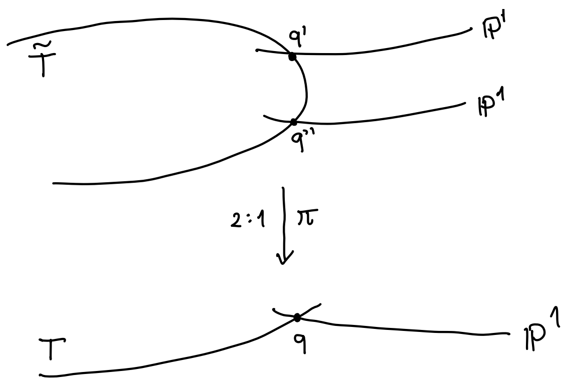


Figure 3: Stable cover

$\mathcal{R}_g$ the space parametrizing pairs $(\widetilde{C}, C)$ of étale double coverings with C a tetragonal curve of genus g. This is an irreducible space and the construction varies continuously with $(\widetilde{C}, C)$, so we can make the computation for a single pair. Consider the allowable covering

$$\widetilde{C} := \mathbb{P}^1 \cup_{q'} \widetilde{T} \cup_{q''} \mathbb{P}^1, \quad C := T \cup_q \mathbb{P}^1$$

with $\widetilde{T} \to T$ an étale double cover over a trigonal curve T as in Figure 3.

The tetragonal construction applied to the cover produces other two Wirtinger covers $\widetilde{C}_i, C_i,\ i = 0, 1,$ such that the normalization of C_i is the tetragonal curve N associated to $(\widetilde{T}, T)$ via the trigonal construction. In this sense, the trigonal is a degeneration of the tetragonal construction. We have then isomorphisms of ppav

$$JN \simeq \mathcal{P}_g(\widetilde{T}, T) \simeq \mathcal{P}_g(\widetilde{C}, C)$$

such that image of the Abel-Prym map $\alpha_i : C_i \to \mathcal{P}_g(\widetilde{C}, C)$ consists of the image of the Abel-Jacobi map $\varphi : N \to JN$ and its involution. Thus the fundamental class is twice that of $\varphi(N)$. □

Curves of genus 6 are tetragonal and the generic one possess 5 g^1_4's. Let $\mathcal{M}_6^{Tet}$ denote the moduli space parametrizing pairs of genus-6 curves with a g^1_4. So the forgetful map $\mathcal{M}_6^{Tet} \to \mathcal{M}_6$ is generically finite of degree 5. By base change we get the following diagram

$$\begin{array}{ccc} \mathcal{R}_6^{Tet} & \longrightarrow & \mathcal{R}_6 \\ \downarrow & & \downarrow \\ \mathcal{M}_6^{Tet} & \longrightarrow & \mathcal{M}_6 \end{array}$$

The tetragonal construction induces a $(2,2)$ correspondence on $\mathcal{R}_6^{Tet}$ whose image in $\mathcal{R}_6$ is a $(10,10)$ correspondence $\mathrm{Tet} \subset \mathcal{R}_6 \times \mathcal{R}_6$.

THEOREM 10.2. *The correspondence Tet on the fiber $\mathcal{P}_6^{-1}(A)$ for a generic $A \in \mathcal{A}_5$ is isomorphic the the incidence correspondence of the lines on a smooth cubic surface. Moreover, the Galois group of the Galois closure of $\mathcal{R}_6 \to \mathcal{A}_5$ is the Weyl group $W(E_6)$, the symmetry group of the incidence of the 27 lines on the cubic surface.*

Proof. The generically finite map $\mathcal{R}_6^{Tet} \to \mathcal{R}_6$ has 1-dimesional fibers over the locus of double coverings $\widetilde{C} \to C$ with C trigonal or a plane quintic. After blowing up and normalizing one gets generically finite fibers over the corresponding exceptional loci. One checks that the tetragonal correspondence lifts to a generically finite $(10,10)$ correspondence

$$\widetilde{\mathrm{Tet}} \subset \widetilde{\mathcal{R}}_6 \times \widetilde{\mathcal{R}}_6.$$

It suffices to identify the structure over a point over which $\widetilde{\mathcal{P}}_6$ and $\widetilde{\mathrm{Tet}}$ are étale. For instance over a generic $(X,H) \in \widetilde{\mathcal{C}}$, where the group $W(E_6)$ acts on the line of the cubic surface $X \cap H$. So the monodromy is contained in $W(E_6)$. □

For instance, for an element $(C,\eta,l) \in \widetilde{\mathcal{R}}_{\mathcal{C}}$ (that is a plane quintic C with an odd 2-torsion point η and l a line in $\mathbb{P}^2$), the 5 g_4^1's correspond to the projections of the plane quintic C from one of the 5 points of the intersection $C \cap l$. We have the identification of $\widetilde{\mathcal{P}}_6(X,H)$ with the set of lines of the cubic surface $X \cap H$, which for generic X and H there are 27 lines. For each l of these lines the conic bundle construction (blow up of the projection from l) gives a double cover $\pi : \widetilde{C} \to C$, with $L = \pi(H) \subset \mathbb{P}^2$. In order to corroborate Theorem 10.2, we need to check that for two given lines $l, l' \in F(X)$ they intersect each other if and only if the corresponding objects $(C,\eta,l), (C'\eta', l')$ are tetragonally related, that is, the pair belongs to $\widetilde{\mathrm{Tet}}$. If $l \cap l' \neq \emptyset$, let $A \subset \mathbb{P}^4$ be the plane containing l, l' and l'' the line such that $A \cap X = l \cup l' \cup l''$. The conic bundle construction gives then 3 plane quintics C, C', C'' with their respective double covers $\widetilde{C}, \widetilde{C}', \widetilde{C}''$. Note that the l, l' map to a point $p \in C$ and this point determines a 4:1 map $f : C \to \mathbb{P}^1$ by projecting from it. Similarly, for C', C'' we obtain tetragonal maps f', f''. These 3 maps can be realised simultaneously via the pencil of hyperplanes $S_\lambda \subset \mathbb{P}^4$ containing A. For a generic λ, $S_\lambda \cap X =: Y_\lambda$ is a smooth cubic surface. A line $m \in Y_\lambda$, with $m \notin A$, $m \cap l' \neq \emptyset$ also meets 4 of the 8 lines in $Y_\lambda \setminus A$ meeting l. This gives the injection

$$\widetilde{C}' \hookrightarrow \widetilde{C}^{(4)}, \quad m \mapsto \{m' : m' \cap l \neq \emptyset,\ m' \cap m \neq \emptyset\}.$$

This shows that the three covers are tetragonally related, hence

$$(\widetilde{C}, C, f), (\widetilde{C}', C', f') \in \widetilde{\mathrm{Tet}}.$$

Since both, the line incidence and the tetragonal correspondence are of bidegree $(10, 10)$ and we have the inclusion, they must be equal.

11. Exercises

The course has been supplemented with the exercise sessions. The idea was to compute some examples in low genera and show that Prym theory combines constructions from curve theory and theory abelian varieties. We would like to show some ideas of what was covered.

EXERCISE 11.1. Show that if C is a genus 2 curve and $f : C \to E$ is an $n : 1$ covering of an elliptic curve, then there is another $n : 1$ covering $g : C \to E'$.

Proof. By dimension count, one gets that the Prym variety $P(f)$ is of dimension 1, hence an elliptic curve, say E'. Since we have an inclusion $j : E' \to JC$, we can dualize it to get $\widehat{j} : JC \to E'$ and restrict to an image of an Abel Jacobi map to get a map $g = \widehat{j} \circ \alpha_C : C \to E'$. Here, we have used the fact that both JC and E' are principally polarized, hence isomorphic to their duals. It is also worth noting that a change of the base point of an Abel Jacobi map results in a map that differs by a translation on E', so the map g is (up to translation on E') unique. Since f is $n : 1$, we have that E' has restricted polarization being n times the principal one, so g is of order n. ☐

REMARK 11.2. The locus of Jacobians of curves mentioned in Exercise 11.1 coincides with the locus of abelian surfaces that are polarized isogenous to a product of elliptic curves of exponent n and is called the Humbert surface of degree n^2.

From the proof of Exercise 11.1, we get an immediate corollary.

COROLLARY 11.3. *An elliptic curve E can be embedded in a Jacobian JC with exponent n if and only if there exists an $n : 1$ covering $C \to E$ (that does not factorize via $C \to E' \to E$ with $E' \to E$ an isogeny).*

Before showing next exercise, we need to recall result from [7] that deals with curves on surfaces.

LEMMA 11.4 ([7, Prop 4.3]). *Let C be a smooth curve and (JC, Θ) its Jacobian. Let (A, H) be a polarised abelian surface and suppose $f_C : C \to A$ is a morphism and $f : JC \to A$ is the canonical homomorphism defined by the universal property of Jacobians. Then the following are equivalent:*

- $\hat{f}^*(\Theta) \equiv \hat{H}$;
- $(f_C)_*[C] = H$ *in* $H^2(A, \mathbb{Z})$.

If C is of genus 3, we can use the fact that if JC contains an abelian subvariety, then it contains an elliptic curve and therefore C is a covering of an elliptic curve. We will recall Barth's result [3] in the following exercise.

EXERCISE 11.5. Show that a smooth genus 3 curve C can be embedded in a $(1, 2)$ polarized abelian surface if and only if it is a double covering of an elliptic curve. In such a case, the curve C is hyperelliptic if and only if C is an étale double covering of a genus 2 curve.

Proof. Note that by [2], a general section of a polarization of type $(1, d)$ is a smooth curve and by Riemann-Roch, it is of genus $d+1$. Hence, if $f_C : C \to A$ is an embedding of a genus 3 curve, then $f_C(C)$ has to generate A as a group and hence $\mathcal{O}(f_C(C))$ is a $(1, 2)$ polarization. Now, by Universal Property of Jacobians, we can extend f_C to a map $f : JC \to A$ which will be surjective and hence $\mathrm{Ker}(f) = E$ is an elliptic curve. Since E is complementary to $\hat{A}$ in JC, its exponent equals 2 and so there exists a double covering $C \to E$.

On the other hand, if $g : C \to E$ is a double covering then $\mathrm{Nm}_g : JC \to E$ has kernel $\mathrm{Ker}(\mathrm{Nm}_g) = A$ that is a $(1, 2)$ polarized abelian surface. If we take the dual map to the inclusion, we get a map $JC \to \hat{A}$ and by composing with an Abel-Jacobi map, using Lemma 11.4 we get that the image is of arithmetic genus 3 and hence it is a desired embedding of C.

As for the second part, it is well known that an étale double covering of a genus 2 curve is bielliptic, i.e. hyperelliptic and a double cover of an elliptic curve (see [25]). On the other hand, if C is hyperelliptic, we can use the hyperelliptic involution ι to show that for any degree 0 divisor D, we have that $D + \iota^* D$ is a principal divisor. In particular ι extends to -1 on the Jacobian JC. Now, if τ is the involution defining the covering $f : C \to E$, then $\iota\tau$ defines another double covering $\pi : C \to C'$ and one can compute

$$E = \mathrm{Im}(\mathrm{Nm}(f)) = \mathrm{Im}(1 + \tau) = \mathrm{Im}(1 - (-\tau)) = P(C/C')$$

and in particular C' is of genus 2 and π an étale double covering. □

REMARK 11.6. A trick of composing an involution with the hyperelliptic involution used in the proof of Exercise 11.5 can be generalised to any genus. If C is hyperelliptic and $f' : C \to C'$ and $f'' : C \to C''$ are double coverings given by involutions τ and $\iota\tau$ respectively then the Prym varieties equals $P(f') = (f'')^*(JC'')$ and $P(f'') = (f')^*(JC')$.

One may suppose that if a Jacobian contains an abelian subvariety, then there is a covering of curves involved. The aim of the last exercise is to show that it may not be the case.

EXERCISE 11.7. Show that there exists a Jacobian of a curve that contains abelian subvarieties but does not come from the Prym construction (i.e. the curve is not a covering of a positive genus curve).

Proof. Here, we will show a heuristic argument. Let C be a smooth genus 4 curve embedded in a $(1,3)$ polarised abelian surface A. Then we can construct the exact sequence $0 \to K \to JC \to A \to 0$ and its dual sequence $0 \to \hat{A} \to JC \to \hat{K} \to 0$. Since K and $\hat{A}$ are complementary to each other in JC and therefore of the same type $(1,3)$ we get that C is also embedded in $\hat{K}$. The moduli of abelian surfaces is three dimensional and on a fixed surface there is a two dimensional family of genus 4 curves (since $h^0(A,\mathcal{L}) = 3$), hence there is a five dimensional family of such curves (locally). Note that any curve can be embedded in only finitely many surfaces, since for a fixed abelian variety (in this case a Jacobian) there is only finitely many abelian subvarieties of a fixed exponent. Because of that, we can assume that all $A, \hat{A}, K, \hat{K}$ do not contain elliptic curves. In such a case, C is not a covering of an elliptic curve. Now, the only possible covering is a triple étale covering of a genus 2 curve but there are only finitely many such curves on a fixed A.

An explicit example when we additionally assume that C is hyperelliptic and get that $K = A$ and a precise proof that C is not a covering of a positive genus curve can be found in [10]. □

To show that the inverse to the Torelli map is mysterious even in dimensions 2 and 3, we have finished the exercises with two open questions:

Exercise 11.8. Can you find an explicit example (i.e. a hyperelliptic equation) of a smooth genus 2 curve that is a 2021 : 1 covering of an elliptic curve? Can you find an explicit example (i.e. a bivariate quartic equation) of a smooth genus 3 curve that is a 2021 : 1 covering of an elliptic curve?

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References

[1] V. Alexeev, R. Donagi, G. Farkas, E. Izadi, and A. Ortega, *The uniformization of the moduli space of principally polarized abelian 6-folds*, J. Reine Angew. Math. **761** (2020), 163–217.

[2] A. Andreotti and A. L. Mayer, *On period relations for abelian integrals on algebraic curves*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **21** (1967), 189–238.

[3] W. Barth, *Abelian surfaces with* $(1,2)$*-polarization*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math. **10** (1987), 41–84.

[4] A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), 149–196.

[5] A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) **4** (1977), 309–391.

[6] A. Beauville, *Les singularités du diviseur θ de la jacobienne intermédiaire de l'hypersurface cubique dans $\mathbb{P}^4$*, Algebraic threefolds (Varenna, 1981), Lecture Notes in Math., vol. 947, Springer, Berlin-New York, 1982, pp. 190–208.

[7] C. Birkenhake. and H. Lange, *Moduli spaces of abelian surfaces with isogeny*, Geometry and analysis (Bombay, 1992), Tata Inst. Fund. Res., 1995, pp. 225–243.

[8] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. 302, Springer, 2004.

[9] P. Borówka and A. Ortega, *Klein coverings of genus 2 curves*, Trans. Amer. Math. Soc. **373** (2020), 1885–1907.

[10] P. Borówka and G. K. Sankaran, *Hyperelliptic genus 4 curves on abelian surfaces*, Proc. Amer. Math. Soc. **145** (2017), 5023–5034.

[11] C.H. Clemens and P.A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95** (1972), 281–356.

[12] O. Debarre, *Sur le problème de Torelli pour les variétés de Prym*, Amer. J. Math. **111** (1989), 111–134.

[13] R. Donagi, *The tetragonal construction*, Bull. Amer. Math.Soc. **4** (1981), 181–185.

[14] R. Donagi, *The fibers of the Prym map*, Curves, Jacobians, and abelian varieties (Amherst), Contemp. Math. **136** (1992), 55–125.

[15] R. Donagi and R. Smith, *The structure of the Prym map*, Acta Math. **146** (1981), 25–102.

[16] R. Friedman and R. Smith, *The generic Torelli theorem for the Prym map*, Invent. Math. **67** (1982), 473–490.

[17] P.A. Griffiths, *Periods of integrals on algebraic manifolds II. Local study of the period mapping*, Amer. J. Math. **90** (1968), 805–865.

[18] V.I. Kanev, *A global Torelli theorem for Prym varieties at a general point*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 244–268.

[19] H. Lange and A. Ortega, *Prym varieties of triple coverings*, Int. Math. Res. Not. **22** (2011), 5045–5075.

[20] H. Lange and A. Ortega, *The Prym map of degree-7 cyclic coverings*, Algebra Number Theory **10** (2016), 771–801.

[21] H. Lange and A. Ortega, *On the Prym map of cyclic coverings*, Arch. Math. **111** (2018), 261–631.

[22] V. Marcucci and J.C. Naranjo, *Prym varieties of double coverings of elliptic curves*, Int. Math. Res. Notices **6** (2014), 1689–1698.

[23] V. Marcucci and G.P. Pirola, *Generic Torelli for Prym varieties of ramified coverings*, Compos. Math. **148** (2012), 1147–1170.

[24] L. Masiewicki, *Universal properties of Prym varieties with an application to algebraic curves of genus five*, Trans. Amer. Math. Soc. **222** (1976), 221–240.

[25] D. Mumford, *Prym varieties I*, Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 325–350.

[26] D. Mumford, *On the Kodaira dimension of the Siegel modular variety*, Algebraic geometry – open problems (Ravello, 1982), Lecture Notes in Math., vol. 997, Springer, 1983, pp. 348–375.
[27] J.P. Murre, *Algebraic equivalence modulo rational equivalence on a cubic threefold*, Compositio Math. **25** (1972), 161–206.
[28] J.P. Murre, *Some results on cubic threefolds*, Classification of algebraic varieties and compact complex manifolds, Lecture Notes in Math., vol. 412, Springer, 1974, pp. 140–160.
[29] D.S. Nagaraj and S. Ramanan, *Polarisations of type* $(1, 2, \dots, 2)$ *on abelian varieties*, Duke Math. J. **80** (1995), 157–194.
[30] J.C. Naranjo and A. Ortega, *Global Prym–Torelli for double coverings ramified in at least six points*, J. Algebraic Geom. **31** (2022), 387–396.
[31] J.C. Naranjo, A. Ortega, and A. Verra, *Generic injectivity of the Prym map for double ramified coverings*, Trans. Amer. Math. Soc. **371** (2019), 3627–3646.
[32] S. Recillas, *Jacobians of curves with* g^1_4*'s are the Prym's of trigonal curves*, Bol. Soc. Mat. Mexicana (2) **19** (1974), 9–13.
[33] G. Welters, *Recovering the curve data from a general Prym variety*, Amer. J. Math. **109** (1987), 165–182.
[34] W. Wirtinger, *Untersuchen über Thetafunktionen*, Teubner, Leipzig, 1895.

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