Rend. Istit. Mat. Univ. Trieste
Vol. 54 (2022), Art. No. 13, 17 pages
DOI: 10.13137/2464-8728/34099

New examples of free projective curves

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Abstract. *It is known that a plane projective curve D consisting of a union of degree n curves in the same pencil with a smooth base locus is free if and only if D contains all the singular members of the pencil and its Jacobian ideal is locally a complete intersection. Here we generalizes this result to pencils having a singular base locus.*

Keywords: Free curves, derivation module, quasihomogeneous singularity.
MS Classification 2020: 14C21, 14N20, 32S22, 14H50.

1. Introduction

Let $R = \oplus_{k\geq 0} R_k = \boldsymbol{k}[x, y, z]$ be the graded ring in three indeterminates. The partial derivatives in these three variables are denoted ∂_x, ∂_y and ∂_z. The R graded-module of derivations is a rank 3 module $\mathrm{Der}_R = \oplus_{k\geq 0}[R_k\partial_x + R_k\partial_y + R_k\partial_z]$. The so-called Euler derivation is $\delta_E = x\partial_x + y\partial_y + z\partial_z$.

To a reduced homogeneous polynomial of degree $n \geq 1$, $f \in R_n$, one associates its module of tangent derivations:

$$\mathrm{Der}(f) = \{\delta \in \mathrm{Der}_R \mid \delta(f) \in (f)\}.$$

The Euler derivation belongs to $\mathrm{Der}(f)$ and there is a factorization

$$\mathrm{Der}(f) = R\delta_E \oplus \mathrm{Der}_0(f),$$

where

$$\mathrm{Der}_0(f) = \{\delta \in \mathrm{Der}_R \mid \delta(f) = 0\}.$$

Let $\nabla(f) = (\partial_x f, \partial_y f, \partial_z f)$ be the vector of partial derivatives. Then $\mathrm{Der}_0(f)$ is the kernel of the Jacobian map

$$R^3 \xrightarrow{\nabla(f)} R[n-1].$$

The modules $\mathrm{Der}(f)$ and $\mathrm{Der}_0(f)$ could also be defined in higher dimensions where instead of curves, we would have hypersurfaces. One reason to focus on curves is that the module $\mathrm{Der}_0(f)$ is locally free (its associated sheaf in $\mathbb{P}^2$ is reflexive and then it is a vector bundle for dimensional reasons). In some very particular cases, these modules can also be free (see the definition below). This was first pointed out in [4] for reduced hypersurfaces and studied in [7] for line arrangements (finite sets of distinct lines in $\mathbb{P}^2$) presenting a very special combinatorics; for instance, a union of lines invariant under the action of some reflection group or the Hesse arrangement of 12 lines through the 9 inflection points of a smooth cubic curve (see [2] for detailed examples). Actually, in [2], Terao conjectures that freeness of hyperplane arrangements depends only on its combinatorics. This conjecture is still unsolved even for line arrangements; this is certainly because we do not know enough examples of free line arrangements and more generally of free curves to clearly understand what distinguishes a free curve from a non free curve. Although combinatorics is not as relevant for general curves as for line arrangements, understanding why a curve is free, in addition to the interest of this result for itself, could help solve Terao's conjecture. Before going further on this subject, let us recall the definition of freeness for a reduced plane curve.

Definition 1.1. *The reduced curve $V(f)$ is free if and only if $\mathrm{Der}_0(f)$ (or equivalently $\mathrm{Der}(f)$) is a free module. More precisely $\mathrm{Der}_0(f)$ is free with exponents (a,b) if $\mathrm{Der}_0(f) = R[-a] \oplus R[-b]$ where a and b are integers verifying $0 \le a \le b$ and $a+b+1 = \deg(f)$ (or $\mathrm{Der}(f) = R[-1] \oplus R[-a] \oplus R[-b]$).*

Remark 1.2. A smooth curve of degree ≥ 2 is not free, an irreducible curve of degree ≥ 3 with only nodes and cusps as singularities is not free (see [1, Example 4.5]). Few examples of free curves are known and of course very few families of free curves are known. One such family can be found in [6, Prop. 2.2].

One method to produce free curves given in [8] (suggested by E. Artal-Bartolo and J. Cogolludo-Agustin in a personal communication), consists in taking the union of all the singular curves in a generic pencil of curves of the same degree ; generic means here that the base locus is smooth. More precisely, it was proved that:

Theorem 1.3. *Let f, g two reduced polynomials in R_n such that $B = V(f) \cap V(g)$ consists in n^2 distinct points. Denote by D_k the union of $k \geq 2$ curves and by D^s the union of all the singular curves of the pencil $\langle f, g\rangle$ of degree n curves generated by f and g. Then D_k is free with exponents $(2n-2, n(k-2)+1)$ if and only if $D^s \subset D_k$ and the singularities of D^s are quasihomogeneous.*

Let us first give some classical examples.

Example 1.4. The Braid arrangement defined by $xyz(x-y)(x-z)(y-z) = 0$ is the union of the three singular curves of the pencil $\langle (x-y)z, y(x-z)\rangle$. It is free with exponents $(2,3)$.

Example 1.5. The Hesse arrangement defined by

$$\prod_{\epsilon=0,1,j,j^2} (x^3+y^3+z^3-\epsilon\, xyz) = 0$$

is the union of four triangles, that are all the singular curves of the pencil $\langle x^3 + y^3 + z^3, xyz\rangle$. It is free with exponents $(4, 7)$.

EXAMPLE 1.6. The Fermat arrangement defined by

$$(x^n - y^n)(y^n - z^n)(x^n - z^n) = 0$$

is the union of three sets of n concurent lines that are all the singular curves of the pencil $\langle x^n - y^n, y^n - z^n\rangle$. It is free with exponents $(n + 1, 2n - 2)$.

As a definition of quasihomogeneous singularity we follow the characterization given in [5]:

DEFINITION 1.7. *Let $f \in \mathbb{C}[x, y, z]$ a reduced polynomial. Let $C = V(f)$ its corresponding projective curve. A singular point $p \in V(f)$ is a quasi-homogeneous singularity if and only if $\tau_p(C) = \mu_p(C)$, where $\tau_p(C)$ and $\mu_p(C)$ are the Tjurina and Milnor numbers of C at p.*

REMARK 1.8. The definition being local one can assume that $p = (0, 0)$ and $\mathbb{C}\{x, y\}$ is the ring of convergent power series ; then $\tau_p(C) = \frac{\mathbb{C}\{x,y\}}{(\partial_x f, \partial_y f, f)}$ and $\mu_p(C) = \frac{\mathbb{C}\{x,y\}}{(\partial_x f, \partial_y f)}$. This implies in particular that $\tau_p(C) \leq \mu_p(C)$.

REMARK 1.9. When p is a smooth point of C, these numbers vanish.

REMARK 1.10. These numbers play a crucial role here. Indeed, denoting by $\mathcal{T}_f$ the *logarithmic tangent sheaf* associated to $V(f)$ which is the sheafification of $\mathrm{Der}_0(f)$, and by $\mathcal{J}_f$ the sheaf of ideals, called Jacobian ideal, image of the Jacobian map, one has

$$0 \longrightarrow \mathcal{T}_f \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{\nabla(f)} \mathcal{J}_f(n-1) \longrightarrow 0.$$

Since the curve $C = V(f)$ is reduced, its singular locus is a finite scheme and the Jacobian ideal defines a finite scheme of length

$$c_2(\mathcal{J}_f) = \sum_{p \in C} \tau_p(C).$$

The sum $\tau(C) := \sum_{p \in C} \tau_p(C)$ is called the total Tjurina number of C. This gives also the following relation:

$$c_2(\mathcal{T}_f) = (n-1)^2 - \tau(C).$$

The proof of Theorem 1.3 was based on the following observations:

1. there exists a canonical derivation $\delta = \det[\nabla f, \nabla g, \nabla] = \langle \nabla f \wedge \nabla g \mid \nabla\rangle$ (where $\langle \mid \rangle$ is the usual scalar product of vectors in $\mathbb{C}^3$) associated to a pencil $\langle f, g\rangle$ of degree n curves; this canonical derivation induces for any $k \geq 2$ a non zero section $s_k \in \mathrm{H}^0(\mathcal{T}_{D_k}(2n - 2))$;

2. the zero locus of this section s_k is empty if and only if $D_k \subset D^s$ and at each singular point p of D^s one has $\tau_p(D^s) = \mu_p(D^s)$.

The smoothness of the base locus B is necessary to certify that its contribution to the length of the Jacobian scheme is

$$\sum_{i=1}^{n^2}(k-1)^2 = n^2(k-1)^2.$$

1.1. Objectives

We would like to extend this construction of free curves to more general pencils, i.e. pencils with a singular base locus. Here we focus on two cases.

1. The fat case: pencils generated by two powers $\langle f^b, g^a \rangle$ where $V(f)$ and $V(g)$ are two curves of degree a and b such that $(a,b) = 1$ and $V(f) \cap V(g)$ is a smooth set of ab distinct points. In such pencils any curve is singular along the base locus B when $a > 1$ and $b > 1$. The interest for this case comes from the celebrated example of the two types of 6-cusped sextics with non-isomorphic fundamental groups given by Zariski [9]; indeed the six cusps belong to a smooth conic for the first type and do not belong to a conic for the second type. The sextic of the first type is a general curve in a pencil $\langle f^3, g^2 \rangle$ where $f = 0$ is a smooth conic and $g = 0$ is a smooth cubic.

2. The tangential case: pencils of degree n curves such that the general one is smooth but with a singular base locus B, i.e. $\mathrm{card}(B) < n^2$. The complete description of these pencils remains difficult and we will concentrate in this text on the case of pencils generated by conics.

2. The fat case

In this section we do not study all the singular pencils but only those defined by two multiple structures on reduced curves with primary degrees meeting along a smooth set. More precisely, we prove:

THEOREM 2.1. *Let a, b be two positive integers such that $\gcd(a,b) = 1$, $f \in R_a$, $g \in R_b$ be two reduced polynomials such that the corresponding curves $V(f)$ and $V(g)$ meet along ab distinct points. We consider the pencil $\mathcal{C}_{ab} = \langle f^b, g^a \rangle$ of degree ab curves. Then,*

1. *if $a > 1$ and $b > 1$ then all curves of $\mathcal{C}_{ab}$ are singular at B;*

2. *there is a finite number of curves in $\mathcal{C}_{ab}$, disjoint from $V(f^b)$ and $V(g^a)$, that are singular outside B. We call these curves the +singular curves and their union is denoted by D^{+s} ; the length of the scheme of all the singular points of these +singular curves, including the singularities of $V(f)$ and $V(g)$ when these generators are not smooth, is*
$$(a-1)^2 + (b-1)^2 + (a-1)(b-1);$$
3. *if $V(f)$ and $V(g)$ are smooth, a union D_k of k curves of the pencil $\mathcal{C}_{ab}$ is free with exponents $(a+b-2, kab-(a+b)+1)$ if and only if $D^{+s} \subset D_k$ and any singularity of D^{+s} outside B is quasihomogeneous;*
4. *if $V(f)$ is not smooth (resp. or/and $V(g)$), the curve $D_k \cup V(f)$ (resp. $D_k \cup V(g)$ and $D_k \cup V(f) \cup V(g)$) where D_k is a union of k curves of the pencil $\mathcal{C}_{ab}$ is free with exponents $(a+b-2, kab-b+1)$ (resp. $(a+b-2, kab-a+1)$, $(a+b-2, kab+1)$) if and only if $D^{+s} \subset D_k$ and any singularity of D^{+s} outside B is quasihomogeneous.*

Proof. Let us prove each assertion.

(1) If (x, y) is a local system of coordinates at any base point $p \in B$, then any curve of the pencil is contained in the ideal (x^a, y^b) then singular at p.

(2) Let us consider a curve $H = \lambda f^b + \mu g^a$ with $\lambda\mu \neq 0$ with a singular point $p \notin B$. Since p is singular we obtain $\nabla H(p) = 0$. We have by Liebniz's rule:
$$\nabla H(p) = b\lambda f^{b-1}(p)\nabla f(p) + a\mu g^{a-1}(p)\nabla g(p) = 0.$$
Since $p \notin B$, $f^{b-1}(p) \neq 0$ and $g^{a-1}(p) \neq 0$. This is equivalent to say that $\nabla f(p)$ and $\nabla g(p)$ are proportional, in other words that the two by two minors of the matrix $[\nabla f, \nabla g]$ vanish simultaneously at p. Moreover since f and g meet transversally at B, these minors do not vanish at any point in B. The scheme Γ of singular points outside B is then defined by the following exact sequence
$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(1-b) \oplus \mathscr{O}_{\mathbb{P}^2}(1-a) \xrightarrow{[\nabla f, \nabla g]} \mathscr{O}_{\mathbb{P}^2}^3 \xrightarrow{\nabla f \wedge \nabla g} \mathscr{J}_\Gamma(a+b-2) \longrightarrow 0.$$
Reciprocally, if $p \in \Gamma$ then one can find easily two non zero constants λ and μ such that $\nabla(\lambda f^b + \mu g^a)(p) = 0$. The length of Γ is the number by
$$c_2(\mathscr{J}_\Gamma) = (a-1)^2 + (b-1)^2 + (a-1)(b-1).$$

(3) Let D_k, defined by $H_k = 0$, be a union of k curves in the pencil that contains D^{+s}. We consider the canonical derivation
$$\delta = \det[\nabla f, \nabla g, \nabla] = \langle \nabla f \wedge \nabla g \mid \nabla \rangle.$$

Since by Liebniz's rule, we have $\delta(H_k)=0$ for any $k\geq 2$, this derivation induces a non zero section of $\mathrm{H}^0(\mathcal{T}_{D_k}(a+b-2))$ and gives a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{O}_{\mathbb{P}^2}(2-a-b) & = & \mathscr{O}_{\mathbb{P}^2}(2-a-b) & & & & \\
 & & \downarrow{\scriptstyle s_\delta} & & \downarrow{\scriptstyle [\nabla f\wedge\nabla g]} & & & & \\
0 & \longrightarrow & \mathcal{T}_{D_k} & \longrightarrow & \mathscr{O}^3_{\mathbb{P}^2} & \longrightarrow & \mathcal{J}_{D_k}(kab-1) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & \mathcal{J}_{Z(s_k)}(a+b-kab-1) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{J}_{D_k}(kab-1) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the sheaf $\mathcal{F}$ is a rank two sheaf singular along Γ, the scheme of +singular points defined above. Dualizing the last exact sequence we obtain:

$$\begin{array}{l}
0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(1-kab) \xrightarrow{[U,V]^t} \mathscr{O}_{\mathbb{P}^2}(1-a)\oplus\mathscr{O}_{\mathbb{P}^2}(1-b) \xrightarrow{[-V,U]} \mathscr{O}_{\mathbb{P}^2}(1-a-b+kab)\\
\longrightarrow \omega_{D_k} \longrightarrow \mathscr{O}_\Gamma \longrightarrow \mathscr{O}_{Z(s_k)} \longrightarrow 0,
\end{array}$$

where ω_{D_k} is the dualizing sheaf of the Jacobian scheme associated to D_k, U and V are the polynomials of degree $kab-a$ and $kab-b$ such that

$$\nabla H_k = U\nabla f + V\nabla g.$$

Denoting by T the complete intersection defined by $\{U=0\}\cap\{V=0\}$, we find finally a shorter exact sequence:

$$0 \longrightarrow \mathscr{O}_T \longrightarrow \omega_{D_k} \longrightarrow \mathscr{O}_\Gamma \longrightarrow \mathscr{O}_{Z(s_k)} \longrightarrow 0.$$

Cutting this exact sequence in two short exact sequences we get

$$0 \longrightarrow \mathscr{O}_T \longrightarrow \omega_{D_k} \longrightarrow \mathfrak{R} \longrightarrow 0 \quad (s1)$$

and

$$0 \longrightarrow \mathfrak{R} \longrightarrow \mathscr{O}_\Gamma \longrightarrow \mathscr{O}_{Z(s_k)} \longrightarrow 0. \quad (s2)$$

The complete intersection T is supported by B. Since $\Gamma\cap B=\emptyset$ the exact sequence $(s2)$ proves that the scheme $\mathfrak{R}$ is supported on a subset of Γ and does not meet B. The exact sequence $(s1)$ then shows that $\mathfrak{R}$ is supported by all the +singular points appearing in D_k.

If $D^{+s}\subset D_k$ both schemes $\mathfrak{R}$ and Γ have the same support ; if the singularities of D^{+s} are quasihomogeneous then these schemes coincide. The curves $V(f)$ and $V(g)$ meeting transversally, the scheme Γ is lci (see [8, proof of Theorem 2.7]) ; this proves that $\mathfrak{R}=\mathscr{O}_\Gamma$ and finally, this implies $Z(s_k)=\emptyset$. □

2.1. Example

Consider the pencil $\langle f^3, g^2\rangle$ of sextic curves where

$$C_f = V(f) = \{y^2 - xz = 0\} \text{ and } C_g = V(g) = \{x^3 + y^3 + z^3 = 0\}.$$

The smooth conic C_f and the smooth cubic C_g meet in six different points $p_i = (a_i^2, a_i, 1)$ where $a_i^6 + a_i^3 + 1 = 0$. All curves of this pencil are singular in the six points p_i. Let us describe now the +singular curves of this pencil with more details.

Proposition 2.2. *In the pencil $\langle f^3, g^2\rangle$ there are exactly five curves that are singular in a point not belonging to the p_i's. Two of these five curves $C_{1,0}$ and $C_{0,1}$ are defined respectively by the equation $f^3 = 0$ $g^2 = 0$, the three other are $C_{1,-1}$, $C_{4,1}$ and $C_{4,-3}$ defined respectively by the equations $f^3 - g^2 = 0$, $4f^3 + g^2$ and $4f^3 - 3g^2 = 0$.*
The additional singular point of $C_{1,-1}$ is $(0, 1, 0)$.
The additional singular points of $C_{4,1}$ are $(1, 0, 1)$, $(1, 0, j)$ and $(1, 0, j^2)$.
The additional singular points of $C_{4,-3}$ are $(\frac{-1}{2}, 1, \frac{-1}{2})$, $(\frac{-j^2}{2}, 1, \frac{-j}{2})$ and $(\frac{-j}{2}, 1, \frac{-j^2}{2})$.
The curve $C_{1,-1} \cup C_{4,1} \cup C_{4,-3}$ is free with exponents $(3, 14)$.

Proof. The singular points $p = (a, b, c) \neq p_i$ of $C_{\lambda,\mu} := \lambda f^3 + \mu g^2 = 0$ are those verifying:

$$\nabla(\lambda f^3 + \mu g^2)(p) = 3\lambda f^2(p)\nabla(f)(p) + 2\mu g(p)\nabla(g)(p) = \vec{0}.$$

- If $f(p) = 0$ then $(\lambda, \mu) = (1, 0)$ and the corresponding curve is $f^3 = 0$.
- If $g(p) = 0$ then $(\lambda, \mu) = (0, 1)$ and the corresponding curve is $g^2 = 0$.
- If $f(p)g(p) \neq 0$ then $\nabla(f)(p) = (-c, 2b, -a)$ and $\nabla(g)(p) = (3a^2, 3b^2, 3c^2)$ are proportional. More precisely, (a, b, c) verifies the equations:

$$\begin{cases} 3b(bc + 2a^2) &= 0 \\ c^3 - a^3 &= 0 \\ 3b(ab + 2c^2) &= 0. \end{cases}$$

  Solving this system by elementary computations, we find the additional singular points and the singular curves associated. According to Theorem 2.1 the curve $C_{1,-1} \cup C_{4,1} \cup C_{4,-3}$ is free with exponents $(3, 14)$. □

2.2. Example

This second example corresponds to the case (4) of the main theorem.

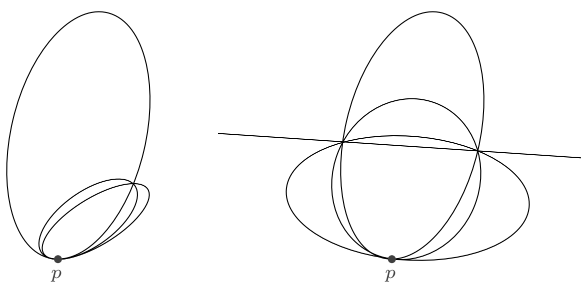
We consider the pencil $\langle f^3, g^2\rangle$ of sextic curves where

$$f(x,y,z) = x^2 + y^2 + z^2 \text{ and } g(x,y,z) = xyz.$$

The smooth conic $V(f)$ and the singular cubic $V(g)$ meet in six different points $(1,i,0),(1,-i,0),(1,0,i),(1,0,-i)$, $(0,1,i)$ and $(0,1,-i)$. Using the same method than in the previous example, we find that the locus $V(\nabla f \wedge \nabla g)$ consists in 7 points that are the three vertices of the triangle, $(1,0,0),(0,1,0)$, $(0,0,1)$ and the four singular points of $f^3 - 27g^2 = 0$. Then the curve $xyz(f^3 - 27g^2) = 0$ is free with exponents $(3,5)$.

3. The tangential case

The pencil is generated by two curves of degree n that do not meet transversally (i.e. the cardinality of the set B is $< n^2$). At the point $p \in B$ where $V(f)$ and $V(g)$ share the same tangent line, the canonical derivation $\delta = \det(\nabla f, \nabla g, \nabla)$ verifies $\delta(p) = 0$. This is the main difficulty here. Indeed the computation of the length of the Jacobian scheme becomes harder and we could have $\mu_p(H_k) \neq \tau_p(H_k)$ at such a point $p \in B$ for a union of k curves in the pencil. If $V(f)$ and $V(g)$ are two smooth conics such that B consists in a subscheme of length 3 and a distinct simple point. Then $V(fg(af+bg))$, where $V(af+bg)$ is also smooth, is free with exponents $(2,3)$. So it is possible for a union of smooth curves of the same pencil to be free. It is also possible to be free when instead of containing all the singular curves the union contains only some irreducible components of some singular curves. For instance, if $V(f)$ and $V(g)$ are two smooth conics tangent in a point p. Then $V(fg(af+bg)h)$ where $V(af+bg)$ is also smooth and $V(h)$ is the line passing through the two smooth points in B, is free with exponents $(2,4)$.



We will focus on pencil of conics. Our aim is to

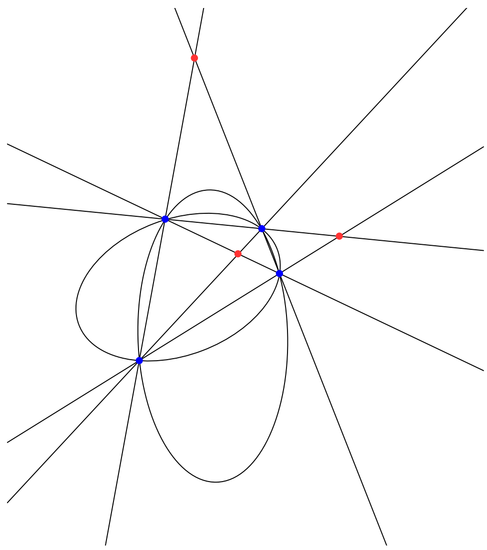
1. determine the “smaller” free union of conics for each kind of pencil;

2. compute the Tjurina numbers at the base points for any kind of pencil.

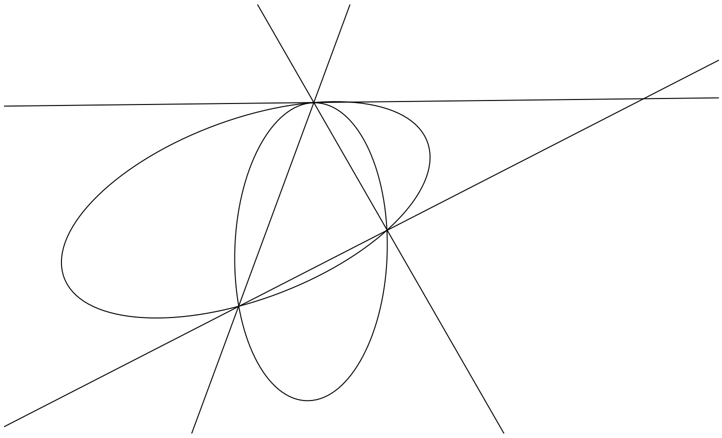
3.1. Pencil of conics

There are different regular pencils (the general conic of the pencil is smooth) generated by two conics C and D with no component in common. Let us precise now for any of this different pencils what generators $\langle f, g\rangle$ can be chosen. Recall that the canonical derivation is $\delta = \det[\nabla f, \nabla g, \nabla]$. The pencil is

1. **generic** when $C \cap D$ consists of 4 distinct points. Then, up to a linear transformation, C and D can be defined by $x^2 - z^2 = 0$ and $y^2 - z^2 = 0$. The canonical derivation δ has degree 2 ; among the intersection points appearing in the picture, the base points are blue and the singular points are red;

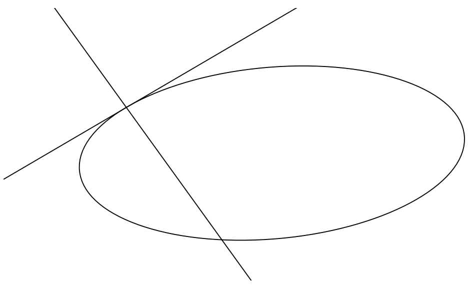


2. **tangent** when $C \cap D$ consists of 3 points, one double and two simple points. Then, up to a linear transformation, C and D can be defined by $x^2 - z^2 = 0$ and $yz = 0$. The canonical derivation δ has degree 2; now base points and singular points are not disjoint;

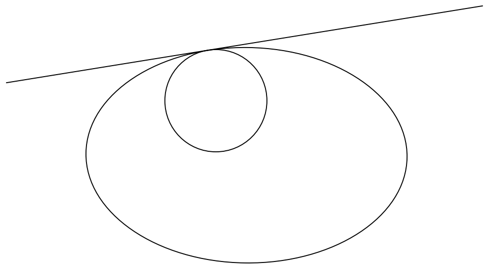


3. **bitangent** when $C \cap D$ consists of 2 double points. Then, up to a linear transformation, C and D can be defined by $x^2 - z^2 = 0$ and $y^2 = 0$. The canonical derivation δ can be factorized by y, i.e. $\delta = y\nu$ where the derivation ν has degree 1;

4. **osculating** when $C \cap D$ consists of 2 points, one simple and one triple point. Then, up to a linear transformation, C and D can be defined by $xy = 0$ and $y^2 - xz = 0$. The canonical derivation δ has degree 2;



5. **+osculating** when $C \cap D$ consists of one quadruple point. Then, up to a linear transformation, C and D can be defined by $y^2 - xz = 0$ and $x^2 = 0$. The canonical derivation δ can be factorized by x, i.e. $\delta = x\nu$ where the derivation ν has degree 1.

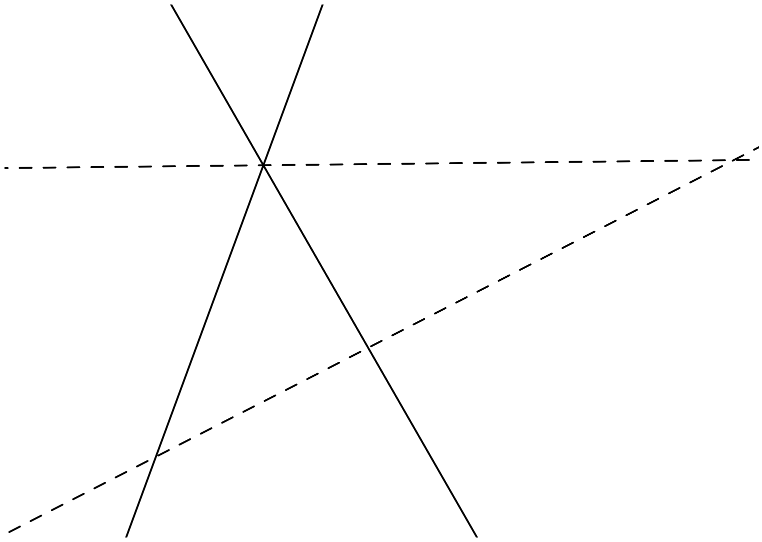


3.2. A free union of curves remains free by deleting a smooth curve

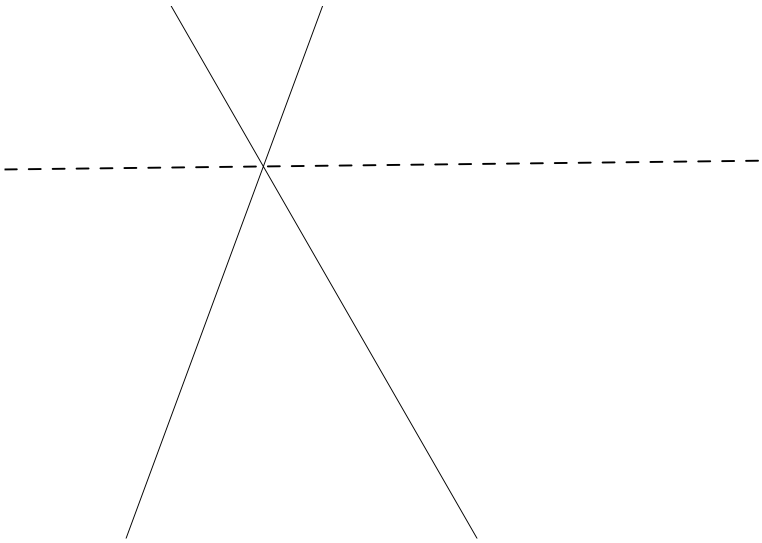
Proposition 3.1. *Assume that $\mathcal{A}$ is a union of curves $V(\lambda f+\mu g)$ of a regular pencil of degree n curves $\langle f,g\rangle$ in $\mathbb{P}^2$. Assume also that $\mathcal{A}$ contains a singular member $V(h_1h_2)$ ($h_1h_2\in\langle f,g\rangle$) which is a normal crossing divisor at the points $V(h_1)\cap V(h_2)$ and that $V(h_1)$ is smooth. Then if $\mathcal{A}$ is free the arrangement $\mathcal{A}\setminus V(h_1)$ is also free.*

Proof. Let δ be the canonical derivation associated to the pencil $\langle f,g\rangle$. If the pencil does not contain any multiple curve the degree of δ is $\alpha_n=2n-2$. If it contains a multiple curve then one can factorize it to define a new "canonical" derivation (vanishing along any curve of the pencil) with degree $\alpha_n<2n-2$. Since $V(h_1h_2)$ belongs to the pencil $\langle f,g\rangle$ one gets $\delta(h_1h_2)=\det(\nabla(f),\nabla(g),\nabla(h_1h_2))=0$. Then $h_1\delta(h_2)=-h_2\delta(h_1)$. Hence there exists a polynomial k such that $\delta(h_2)=-kh_2$ and $\delta(h_1)=kh_1$. The derivation $\delta'=\delta-\frac{k}{\deg(h_1)}\delta_E$ verifies $\delta'(h_1)=0$ and it has the same degree than δ. Since $V(h_1h_2)$ is a normal crossing divisor at $p\in V(h_1)\cap V(h_2)$ then $k(p)\neq 0$; indeed $h_1(p)=k(p)=0$ implies that $\delta(h_1)$ vanishes at p at the order two contradicting the normal crossing at p. Then $\delta'(p)\neq 0$ and the section induced by δ' does not vanish at p. Hence when the component $V(h_1)$ is deleted from $\mathcal{A}$, p is removed from the scheme defined by the Jacobian ideal $\mathcal{J}_{\nabla\mathcal{A}}$ and also removed from $Z(s_{\delta'})$ the zero scheme of the section induced by δ'. Then $Z(s_{\delta'})=\emptyset$ and $\mathcal{A}\setminus V(h_1)$ is also free. □

Example 3.2. The following arrangement of four lines can be seen as a union of two singular conics, the dashed one and the black one. It is free with exponents $(2,1)$.

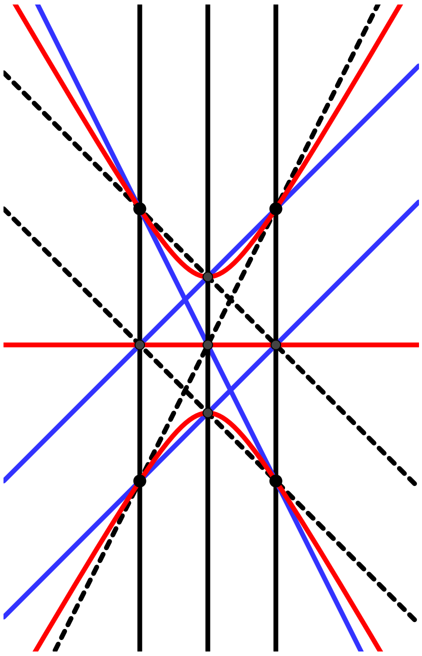


The following arrangement of lines is still free by Proposition 3.1 with exponents $(2, 0)$.



Example 3.3. Pappus arrangement consists in 9 lines given by the well known configuration 9_3. The 9 lines are the sides of the 3 triangles passing through 9 points. In the pencil generated by two triangles, singular curves are missing. In general three nodal cubics are missing but in the following example there is only one singular cubic missing: it consists in the union of a line union and a

smooth conic; indeed let us consider the pencil generated by one set of three concurrent lines and one triangle $[x(x^2 - z^2), (x + y)(x - 2y + z)(x - 2y - z)]$. It still contains another triangle $(x - y)(x + 2y - z)(x + 2y + z) = 0$ and a conic+line $y(3x^2 - 4y^2 + z^2) = 0$.



The union of all the singular members of the pencil is free with exponents $(4, 7)$ (by [8], Theorem 1.3). By Proposition 3.1 we obtain a new arrangement which is free with exponents $(4, 6)$ by removing the line from the conic+line member:

$$x(3x^2 - 4y^2 + z^2)(x^2 - y^2)(x^2 - z^2)((x + 2y)^2 - z^2)((x - 2y)^2 - z^2) = 0\,.$$

By Proposition 3.1 again, we obtain a new arrangement which is free with exponents $(4, 5)$ by removing the conic from the conic+line member:

$$xy(x^2 - y^2)(x^2 - z^2)((x + 2y)^2 - z^2)((x - 2y)^2 - z^2) = 0.$$

3.3. A free union of curves remains free by adding a smooth curve

Proposition 3.4. *Let C be a smooth curve in a pencil $\langle f, g\rangle$ of degree n curves, $\mathcal{A}$ be an arrangement of curves, or components of curves, of this pencil. Assume* *that the section of $\mathcal{T}_{\mathcal{A}}(\alpha_n)$ induced by the canonical derivation δ (of degree α_n) does not vanish. Then $\mathcal{A}$ is free with exponents $(\alpha_n, -\alpha_n - c_1(\mathcal{T}_{\mathcal{A}}))$ and $\mathcal{A}\cup C$ is free with exponents $(\alpha_n, -\alpha_n - c_1(\mathcal{T}_{\mathcal{A}\cup C}))$.*

Proof. There is a short exact sequence:

$$0 \longrightarrow \mathcal{T}_{\mathcal{A}\cup C} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{L} \longrightarrow 0,$$

where $\mathcal{L}$ is a line bundle over C. Indeed on an open affine neighborhood $U \subset \mathbb{P}^2$ the first arrow is given by a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a,b,c,d \in \mathcal{O}_U$ and $C_{|U} = \{ad-bc=0\}$. Assuming that the rank of $\mathcal{L}_p$ is >1 at some $p \in C$ means that $a(p)=c(p)=b(p)=d(p)=0$. But this would imply that $\nabla(ad-bc)(p)=0$ which contradicts the smoothness of C.

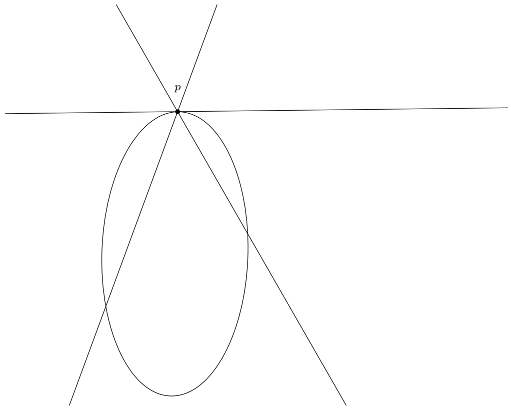
Since C belongs to the pencil the canonical derivation δ induces a non zero section of $\mathcal{T}_{\mathcal{A}}(\alpha_n)$ but also a non zero section of $\mathcal{T}_{\mathcal{A}\cup C}(\alpha_n)$. This gives the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^2}(-\alpha_n) & = & \mathcal{O}_{\mathbb{P}^2}(-\alpha_n) & & & & \\
 & & \downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s} & & & & \\
0 & \longrightarrow & \mathcal{T}_{\mathcal{A}\cup C} & \longrightarrow & \mathcal{T}_{\mathcal{A}} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{J}_{Z(s_1)}(c_1(\mathcal{T}_{\mathcal{A}\cup C})+\alpha_n) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(c_1(\mathcal{T}_{\mathcal{A}})+\alpha_n) & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $\mathcal{A}$ is free with exponents $(\alpha_n, -\alpha_n - c_1(\mathcal{T}_{\mathcal{A}})))$ and $\mathcal{L} = \mathcal{O}_C(c_1(\mathcal{T}_{\mathcal{A}})+\alpha_n))$. This proves

$$\mathcal{J}_{Z(s_{k+1})}(c_1(\mathcal{T}_{\mathcal{A}\cup C})+\alpha_n)) = \mathcal{O}_{\mathbb{P}^2}(c_1(\mathcal{T}_{\mathcal{A}\cup C})+\alpha_n). \qquad \square$$

EXAMPLE 3.5. By Proposition 3.4 the following arrangement (three concurrent lines with one of them tangent to a smooth conic) is free with exponents $(2,2)$. Computing the Chern classes of the logarithmic vector bundle associated, this implies that $\tau_p(\mathcal{A}\cup C) = 10$. Computing the Milnor number at p we find $\mu_p(\mathcal{A}\cup C) = 11$ showing that the tangent point p is not a quasihomogeneous singularity.



3.4. Tjurina number for pencils of conics

PROPOSITION 3.6. *Let p be the double point of a tangent pencil $\langle f, g\rangle$. Let $C_1, \dots, C_k$ be $k \geq 3$ smooth conics in the pencil $\langle f, g\rangle$. Then*

$$\tau_p\left(\bigcup_{i=1}^{k} C_i\right) = 2((k-1)^2+1).$$

Proof. By a direct computation, using for instance Macaulay 2, one can prove that the union of three smooth conics and a line through the two simple points of the base locus B is free with exponents $(2,4)$. Adding smooth conics of the same pencil does not change the freeness and the arrangement $\mathcal{A}$ consisting in $k \geq 3$ smooth conics plus one line through the two simple points is free with exponents $(2, 2k-2)$. Then

$$c_2(\mathcal{T}_{\mathcal{A}}) = 4k - 4 = (2k)^2 - \tau(\mathcal{A}).$$

The total Tjurina number is the sum of the two normal crossing singular points in B counting each of them as k^2 and the Tjurina number at the double point which is $\tau_p(\bigcup_{i=1}^{k} C_i)$. This means

$$\tau(\mathcal{A}) = 4k - 4 = 4k^2 - 2k^2 - \tau_p\left(\bigcup_{i=1}^{k} C_i\right),$$

proving the result. □

PROPOSITION 3.7. *Let p be one of the two double points of a bitangent pencil $\langle f, g\rangle$. Let $C_1, \dots, C_k$ be $k \geq 2$ smooth conics in the pencil $\langle f, g\rangle$. Then*

$$\tau_p\left(\bigcup_{i=1}^{k} C_i\right) = 2k^2 - 3k + 1.$$

Proof. By a direct computation, using for instance Macaulay 2, one can prove that the union of two smooth conics and the tangent lines along p and q, the two base points, is free with exponents $(1,4)$ (the degree of the canonical derivation is 1 instead of 2 because of the double line in the pencil). By Proposition 3.4 adding smooth conics of the same pencil does not change the freeness and the arrangement $\mathcal{A}$ consisting in $k \geq 2$ smooth conics plus the two tangent lines is still free with exponents $(1, 2k)$. Then

$$\begin{aligned} c_2(\mathcal{T}_{\mathcal{A}}) = 2k = (2k+1)^2 - \tau(\mathcal{A}) = (2k+1)^2 - 1 - \tau_p(\mathcal{A}) - \tau_q(\mathcal{A}) \\ = (2k+1)^2 - 1 - 2\tau_q(\mathcal{A}). \end{aligned}$$

Then we find $\tau_q(\mathcal{A}) = k(2k+1)$. By Proposition 3.1, removing one of these two lines we get a new free arrangement $\mathcal{A}'$ with exponents $(1, 2k-1)$. Then

$$c_2(\mathcal{T}_{\mathcal{A}'}) = 2k - 1 = (2k)^2 - \tau(\mathcal{A}') = (2k)^2 - \tau_p(\mathcal{A}') - \tau_q(\mathcal{A}').$$

Since $\tau_q(\mathcal{A}') = \tau_q(\mathcal{A}) = k(2k+1)$, we find $\tau_p(\mathcal{A}) = 2k^2 - 3k + 1$. At p the Tjurina number of $\mathcal{A}$ coincide with the one of k smooth conics in a bitangent pencil. This proves the assertion. □

PROPOSITION 3.8. *Let p be the triple point of an osculating pencil $\langle f, g \rangle$. Let $C_1, \ldots, C_k$ be $k \geq 3$ smooth conics in the pencil $\langle f, g \rangle$. Then*

$$\tau_p\left(\bigcup_{i=1}^{k} C_i\right) = 3((k-1)^2 + 1).$$

Proof. The union of three osculating smooth conics is a free divisor with exponents $(2,3)$. This is verified for instance with Macaulay2. Then adding smooth conics remains free, more precisely for $k \geq 3$ smooth osculating conics, this union is free with exponents $(2, n(k-2)+1)$. The second Chern class of the logarithmic bundle associated is $2 \times (n(k-2)+1)$. This number is also computed with the total Tjurina number. There are two points of intersection, p the osculating point and q where the k conics meet transversally. At q the Tjurina number is the Milnor number $(k-1)^2$. This gives $\tau_p(\bigcup_{i=1}^{k} C_i)$. □

PROPOSITION 3.9. *Let p be 4-uple point of a +osculating pencil $\langle f, g \rangle$. Let $C_1, \ldots, C_k$ be $k \geq 2$ smooth conics in the pencil $\langle f, g \rangle$. Then*

$$\tau_p\left(\bigcup_{i=1}^{k} C_i\right) = 4k^2 - 6k + 3.$$

Proof. The union of two +osculating smooth conics is a free divisor with exponents $(1,2)$. This is verified with Macaulay2. Then adding smooth conics remains free, more precisely for $k \geq 2$ smooth overosculating conics, this union is free with exponents $(1, 2(k-1))$. The second Chern class of the logarithmic bundle associated is $2(k-1)$. This number is also computed with the total Tjurina number. There is only one point of intersection, p. This gives $\tau_p(\bigcup_{i=1}^{k} C_i)$. □

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Received May 28, 2022
Revised August 25, 2022
Accepted September 16, 2022