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Analysis of existence and non-existence of limit cycles for a family of Kolmogorov systems

Sarbast Hussein, Salah Benyoucef

Abstract. *The main objective of this paper is to study existence and non existence of limit cycles by using the idea of Green's theorem and inverse integrating factor method respectively, for some a significant family of Kolmogorov differential systems.*

Keywords: Sixteenth problem of Hilbert, planar differential system, Kolmogorov System, invariant curve, hyperbolic limit cycle, first integral.
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1. Introduction

We consider the following Kolmogorov system

$$\begin{cases} \dot{x} = xP(x,y), \\ \dot{y} = yQ(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials, the dot denotes derivative with respect to the time t, and the coefficients are real numbers.

Generally, Kolmogorov system is introduced as the structure of many natural phenomena models. Their applications can be appear in several fields such as, physics, biology, chemical reactions, hydrodynamics, fluid dynamics, economics, etc. for more detail see [1, 7, 16, 17].

One of the most important topics in qualitative theory of planar dynamical systems is related to the second part of the unsolved Hilbert 16th problem which consisted to study the maximum number of limit cycles and their relative distributions of the real planar polynomial system of degree n, see [12].

Many different methods have been used for proving the existence and nonexistence of limit cycles in simply connected region, for instance see [3, 11, 18]. In recent years, existence and nonexistence of limit cycle for some class of Kolmogorov system has been studied, see for instance [2, 4, 5, 6, 8, 13, 14]. In this paper we will give a unifying characterization on the invariant algebraic curves and first integrals to investigate existence and non existence of limit cycle for system (1).

Firstly, we need to give some necessary definitions. We define a vector field associated to the system (1) as follows

$$\mathcal{X} = xP(x,y)\frac{\partial}{\partial x} + y\,Q(x,y)\frac{\partial}{\partial y}.$$

Let $\mathcal{W} \subset \mathbb{R}^2$ be an open subset such that $\mathbb{R}^2 \backslash \mathcal{W}$ has zero Lebesgue measure. We say that a non-constant real function $\mathcal{H} = \mathcal{H}(x,y) : \mathbb{R}^2 \to \mathbb{R}$, is a *first integral* if $\mathcal{H}(x(t), y(t))$ is constant on all solutions $(x(t), y(t))$ of $\mathcal{X}$ contained in $\mathcal{W}$, i.e. $\mathcal{X}\mathcal{H}|_{\mathcal{W}} = 0$.

A polynomial $\mathcal{V}(x,y) \in \mathbb{R}[x,y]$, the ring of the real coefficient polynomials in x, y is called a *Invariant algebraic curve* for the system (1) if

$$\mathcal{X}\,\mathcal{V} = \mathcal{K}\,\mathcal{V}, \tag{2}$$

for some real polynomial $\mathcal{K}(x,y)$, which is called cofactor of $\mathcal{V}$.

The curve $\Gamma = \left\{(x,y) \in \mathbb{R}^2; \mathcal{V}(x,y) = 0\right\}$, is non-singular of system (1) if the equilibrium points of the system that satisfy the following system

$$\begin{cases} xP(x,y) = 0, \\ yQ(x,y) = 0, \end{cases} \tag{3}$$

are not contained on the curve Γ.

A solution $(x(t), y(t))$ for a differential system (1) is said to be T-periodic solution, if its satisfies

$$(x(t), y(t)) = (x(t+T), y(t+T)),$$

for all t, and for some $T > 0$.

A limit cycle is an isolated periodic solution of a differential equation, or is a T-periodic solution of system (1), isolated with respect to all other possible periodic solutions of the system and defined as

$$\gamma = \{(x(t), y(t)), t \in [0,T]\}.$$

Let γ be periodic orbit of system (1) of period T, then γ is an hyperbolic limit cycle if

$$\int_0^T \operatorname{div}(\mathcal{X})(\gamma(t))\,dt \neq 0,$$

for more detail see [18].

Let $\mathcal{W} \subset \mathbb{R}$ and $\Psi : \mathcal{W} \to \mathbb{R}$ be a function, Ψ is said to be an inverse integrating factor of (1) if it is not locally null and satisfies the partial differential equation

$$\mathcal{X}\Psi = \operatorname{div}(\mathcal{X})\,\Psi, \tag{4}$$

where $\operatorname{div}(\mathcal{X}) = \dfrac{\partial\,(xP(x,y))}{\partial x} + \dfrac{\partial\,(yQ(x,y))}{\partial y}$.

2. Main Results

As a main result, we have the following theorem,

THEOREM 2.1. *We consider Kolmogorov system of degree m $(m \geq 5)$*

$$\begin{cases} \dot{x} = x\left(\mathcal{V}\left(ax^{2n-1}y^{2k-1} + b\right) + \alpha x^{2n-1}y^{2n}\mathcal{V}_y\right), \\ \dot{y} = y\left(\mathcal{V}\left(cy^{2n-1}x^{2k-1} + d\right) - \alpha x^{2n}y^{2n-1}\mathcal{V}_x\right), \end{cases} \tag{5}$$

where $\mathcal{V} = \mathcal{V}(x, y)$, is a polynomial function, and $\mathcal{V}_x$ and $\mathcal{V}_y$ denotes the partial derivative of variables x and y respectively. The coefficients a, b, c, d, α are non zero reals, and the degree n and k are positive integers. Then the following statements are holds.

(1) Let $\Gamma = \left\{(x, y) \in \mathbb{R}^2,\ \mathcal{V}(x, y) = 0\right\}$, be a degree $l \geq 2$ invariant and non-singular curve of the differential system (5). If $b + d \neq 0$ and the bounded components of Γ do not intersect the axes $(x = 0, y = 0)$, then the system (5) admits all bounded components of Γ as hyperbolic limit cycles.

(2) If $b + d = 0$ the system is integrable with first integral

$$\mathcal{H} = \begin{cases} \exp\left(\frac{(-2cn+c)x^{-2n+2k} + (2an-a)y^{-2n+2k} - 2y^{-2n+1}b\,x^{-2n+1}(k-n)}{2\alpha(k-n)(2n-1)}\right)\mathcal{V} & \text{if } k \neq n \\ \frac{y^{\frac{a}{\alpha}}}{x^{\frac{c}{\alpha}}}\exp\left(\frac{-b}{(2n-1)\alpha x^{2n-1}y^{2n-1}}\right)\mathcal{V} & \text{if } k = n, \end{cases} \tag{6}$$

moreover the system has no limit cycle.

Proof of statement 1. Let $\Gamma = \left\{(x, y) \in \mathbb{R}^2,\ \mathcal{V}(x, y) = 0\right\}$ with degree l $(l \geq 2)$, be a non-singular of system (5) and the bounded components of Γ do not intersect the lines $(x = 0, y = 0)$. To show that all the bounded components of Γ are hyperbolic limit cycles of system (5), we will prove that Γ is an invariant curve of the system (5), and

$$\int_0^T \operatorname{div}(\mathcal{X})(\gamma(t))\,dt \neq 0,$$

see for instance Perko[15, Pages 216-217].

Its clearly $\mathcal{V}$ is an invariant curve of system (5), because

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} &= \mathcal{V}_x x\left(\mathcal{V}\left(ax^{2n-1}y^{2k-1} + b\right) + \alpha x^{2n-1}y^{2n}\mathcal{V}_y\right) \\ &\quad + \mathcal{V}_y y\left(U\left(cy^{2n-1}x^{2k-1} + d\right) - \alpha x^{2n}y^{2n-1}\mathcal{V}_x\right) \\ &= \mathcal{V}\left(b\,\mathcal{V}_x x + d\,y\,\mathcal{V}_y + a\,x^{2\,n}y^{2\,k-1}\,\mathcal{V}_x + c\,y^{2\,n}x^{2\,k-1}\mathcal{V}_y\right), \end{aligned}$$

where the cofactor is

$$\mathcal{K} = \left(a\,x^{2\,n-1}y^{2\,k-1} + b\right) x\mathcal{V}_x + \left(c\,y^{2\,n-1}x^{2\,k-1} - d\right) y\mathcal{V}_y.$$

To see $\int_0^T \operatorname{div}(\mathcal{X})\,(\gamma(t))\,dt$ is nonzero, we have show that

$$\int_0^T \operatorname{div}(\mathcal{X})\,(\gamma(t))\,dt = \int_0^T \mathcal{K}(x(t), y(t))dt, \tag{7}$$

is non zero (see for instance Giacomini & Grau [10, theo 2]).

$$\begin{aligned}
&\int_0^T \mathcal{K}(x(t), y(t))dt \\
&\quad = \oint_\Gamma \frac{\left(a\,x^{2\,n-1}y^{2\,k-1} + b\right)\,x\,\mathcal{V}_x}{-\alpha x^{2n}y^{2n}\mathcal{V}_x}dy + \oint_\Gamma \frac{\left(c\,y^{2\,n-1}x^{2\,k-1} + d\right)\,y\,\mathcal{V}_y}{\alpha x^{2n}y^{2n}\mathcal{V}_y}dx \\
&\quad = -\oint_\Gamma \frac{\left(a\,x^{2\,n-1}y^{2\,k-1} + b\right)}{\alpha x^{2n-1}y^{2n}}dy + \oint_\Gamma \frac{\left(c\,y^{2\,n-1}x^{2\,k-1} + d\right)}{\alpha x^{2n}y^{2n-1}}dx,
\end{aligned}$$

by applying the Green formula,

$$\begin{aligned}
&\oint_\Gamma \frac{\left(c\,y^{2\,n-1}x^{2\,k-1} + d\right)}{\alpha x^{2n}y^{2n-1}}dx - \oint_\Gamma \frac{\left(a\,x^{2\,n-1}y^{2\,k-1} + b\right)}{\alpha x^{2n-1}y^{2n}}dy \\
&\quad = \frac{1}{\alpha}\iint_{Int(\Gamma)} \left(\frac{\partial\left(\frac{\left(a\,x^{2\,n-1}y^{2\,k-1}+b\right)}{x^{2n-1}y^{2n}}\right)}{\partial x} + \frac{\partial\left(\frac{\left(c\,y^{2\,n-1}x^{2\,k-1}+d\right)}{x^{2n}y^{2n-1}}\right)}{\partial y} \right) dxdy \\
&\quad = -\frac{2n-1}{\alpha}\,(b+d)\iint_{Int(\Gamma)} \frac{1}{y^{2\,n}x^{2\,n}}dxdy,
\end{aligned}$$

where $Int\,(\Gamma)$ denotes the interior of Γ. As $\alpha \neq 0$, $b + d \neq 0$ and the bounded components of Γ do not intersect the lines $(x=0, y=0)$ then $\int_0^T \mathcal{K}(x(t), y(t))dt$ is non zero. □

To prove the second statement of Theorem 2.1, we will use the following Theorem.

THEOREM 2.2 ([11, Theorem 9]). *Let* $\Psi : \Omega \to \mathbb{R}$ *be an inverse integrating factor of system(1), if* $\Gamma \subset \Omega$ *is a limit cycle of (1) then* Γ *is contained in the set* $\Psi^{-1}(0) = \{(x, y) \in \Omega, \Psi(x, y) = 0\}$.

Proof of statement 2. For $d = -b$, we have a first integral in the form of equation (6). We separate the proof in two different cases . Firstly, if $k \neq n$

$$\mathcal{H}(x, y) = \exp\left(\frac{(-2cn+c)\,x^{-2n+2k} + (2an-a)\,y^{-2n+2k} - 2y^{-2n+1}b\,x^{-2n+1}\,(k-n)}{2\alpha\,(k-n)\,(2n-1)}\right)\mathcal{V}$$

$$
\begin{aligned}
&\frac{\partial \mathcal{H}}{\partial x}\dot{x} + \frac{\partial \mathcal{H}}{\partial y}\dot{y} \\
&\quad = \frac{1}{\alpha}\left(\left(by^{-2\,n+1}x^{-2\,n} - cx^{-2\,n+2\,k-1}\right)\,\mathcal{V} + \alpha\mathcal{V}_x\right) \\
&\qquad \exp\left(\frac{(-2cn+c)\,x^{-2n+2k} + (2an-a)\,y^{-2n+2k} - 2y^{-2n+1}b\,x^{-2n+1}\,(k-n)}{2\alpha\,(k-n)\,(2n-1)}\right) \\
&\qquad \left(x\left(\mathcal{V}\left(ax^{2n-1}y^{2k-1}+b\right) + \alpha x^{2n-1}y^{2n}\mathcal{V}_y\right)\right) \\
&\quad + \frac{1}{\alpha}\left(\left(ay^{-2\,n+2k-1} + b\,y^{-2n}x^{-2n+1}\right)\,\mathcal{V} + \alpha\mathcal{V}_y\right) \\
&\qquad \exp\left(\frac{(-2cn+c)\,x^{-2n+2k} + (2an-a)\,y^{-2n+2k} - 2y^{-2n+1}b\,x^{-2n+1}\,(k-n)}{2\alpha\,(k-n)\,(2n-1)}\right) \\
&\qquad \left(y\left(\mathcal{V}\left(cy^{2n-1}x^{2k-1}-b\right) - \alpha x^{2n}y^{2n-1}\mathcal{V}_x\right)\right) = 0.
\end{aligned}
$$

Therefore

$$
\dot{x}\frac{\partial \mathcal{H}}{\partial x} + \dot{y}\frac{\partial \mathcal{H}}{\partial y} = 0, \quad \text{then} \quad \frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}} = \frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}} = \Psi,
$$

where Ψ is an inverse integrating factor. Thus

$$
\frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}} = \frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}} = -\alpha x^{2n}y^{2n}\exp\left(\mathcal{U}(x,y)\right),
$$

where

$$
\mathcal{U}(x,y) = \frac{(2n-1)\left(-ay^{2k}x^{2n} + cy^{2n}x^{2k}\right) + 2bxy\,(k-n)}{2\alpha x^{2n}y^{2n}\,(2n-1)\,(k-n)}.
$$

According to Theorem 2.2, the system has no limit cycle because the set

$$
\begin{aligned}
\Psi^{-1}(0) = \Bigg\{ &(x,y)\in\mathbb{R}^2 \;\Big| \\
&-\alpha x^{2n}y^{2n}\exp\left(\frac{(2n-1)\left(-ay^{2k}x^{2n}+cy^{2n}x^{2k}\right)+2bxy\,(k-n)}{2x^{2n}y^{2n}\alpha\,(2n-1)\,(k-n)}\right) = 0\Bigg\}
\end{aligned}
$$

contains no closed curve.

Secondly, if $k = n$, then

$$
\mathcal{H}(x,y) = \frac{y^{\frac{a}{\alpha}}}{x^{\frac{c}{\alpha}}}\exp\left(\frac{-b}{(2n-1)\,\alpha x^{2n-1}y^{2n-1}}\right)\mathcal{V}(x,y).
$$

is first integral and satisfies the following equation

$$\frac{\partial \mathcal{H}}{\partial x}\dot{x}+\frac{\partial \mathcal{H}}{\partial y}\dot{y}=\left(\left(-\frac{1}{x^{\frac{1}{\alpha}(c+\alpha+2n\alpha)}}\frac{y^{\frac{1}{\alpha}(a-2n\alpha)}}{\alpha}\left(cx^{2n}y^{2n}-bxy\right)\right)\mathcal{V}+\frac{y^{\frac{a}{\alpha}}}{x^{\frac{c}{\alpha}}}\mathcal{V}_x\right)$$
$$\exp\left(\frac{-b}{(2n-1)\,\alpha x^{2n-1}y^{2n-1}}\right)\left(x\left(\mathcal{V}\left(ax^{2n-1}y^{2n-1}+b\right)+\alpha x^{2n-1}y^{2n}\mathcal{V}_y\right)\right)$$
$$+\left(\left(\frac{1}{x^{\frac{1}{\alpha}(c+2n\alpha)}y^{\frac{1}{\alpha}(\alpha-a+2n\alpha)}\alpha}\left(ax^{2n}y^{2n}+bxy\right)\right)\mathcal{V}+\frac{y^{\frac{a}{\alpha}}}{x^{\frac{c}{\alpha}}}\mathcal{V}_y\right)$$
$$\exp\left(\frac{-b}{(2n-1)\,\alpha x^{2n-1}y^{2n-1}}\right)\left(y\left(\mathcal{V}\left(cy^{2n-1}x^{2n-1}-b\right)-\alpha x^{2n}y^{2n-1}\mathcal{V}_x\right)\right)=0.$$

Thus

$$\frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}}=\frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}}=-\frac{x^{\frac{1}{\alpha}(c+2n\alpha)}}{y^{\frac{1}{\alpha}(a-2n\alpha)}}\alpha\exp\left(bx^{1-2n}\frac{y^{1-2n}}{\alpha\,(2n-1)}\right).$$

By using Theorem 2.2, the system has no limit cycle because the set

$$\Psi^{-1}(0)=\left\{(x,y)\in\mathcal{R}^2 \mid -\frac{x^{\frac{1}{\alpha}(c+2n\alpha)}}{y^{\frac{1}{\alpha}(a-2n\alpha)}}\alpha\exp\left(bx^{1-2n}\frac{y^{1-2n}}{\alpha\,(2n-1)}\right)=0\right\}$$

contains no closed curve. □

Now, we present two examples for illustrating the result.

Example 2.3. Let $a=b=c=d=\alpha=n=1$, $\mathcal{V}(x,y)=2\left(x^2+y^2-2\right)^2-4x^2y^2+2xy+1$. The system (5) reduced to

$$\begin{cases}\dot{x}=x\Big((2\,(x^2+y^2-2)^2-4\,x^2\,y^2+2\,xy+1)\,(xy+1)\\ \qquad +y^2x\left(8\,(x^2+y^2-2)y-8\,x^2y+2\,x\right)\Big),\\ \dot{y}=y\Big((2\,(x^2+y^2-2)^2-4\,x^2\,y^2+2\,xy+1)\,(yx+1)\\ \qquad -x^2y\left(8\,(x^2+y^2-2)x-8\,xy^2+2\,y\right)\Big),\end{cases}\tag{8}$$

$\Gamma=\left\{(x,y)\in\mathbb{R}^2,\ 2\left(x^2+y^2-2\right)^2-4x^2y^2+2xy+1=0\right\}$, does not intersect the axes $(x=0,y=0)$, and $b+d\neq 0$, then the system (8) admits all bounded components of Γ as hyperbolic limit cycles. So the system (8) admits four limit cycles represented by the curve $2\left(x^2+y^2-2\right)^2-4x^2y^2+2xy+1=0$ and nine singular points where $(0,0)$ is an unstable node, $(0.16868,-1.4441))$ is saddle point, $(-0.16868,1.4441)$ is a saddle point, $(1.1170,1.4373)$ is a strong unstable focus, $(-1.1170, -1.4373)$ is a strong unstable focus, $(1.3788, -1.5843)$ is a strong unstable focus, $(-1.3788, 1.5843)$ is a strong unstable focus, $(1.4235, 0.46345)$ is a saddle point, $(-1.4235, -0.46345)$ is a saddle point. See Figure 1.

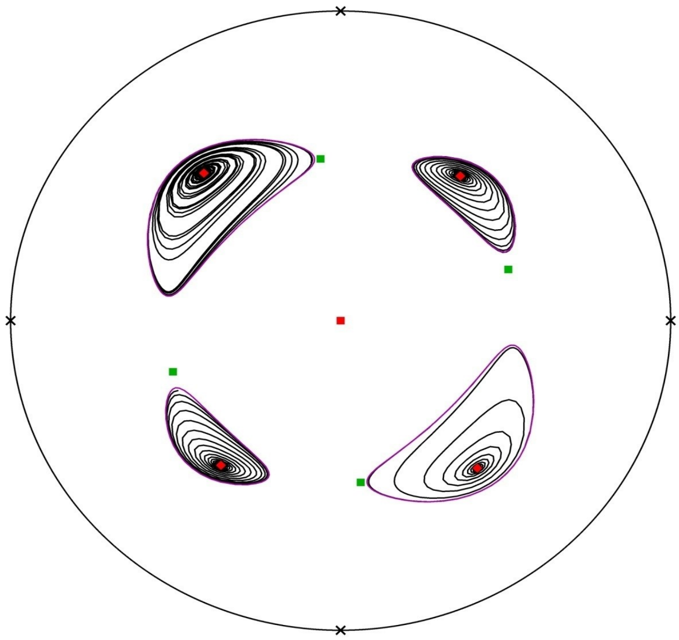

Figure 1: Limit cycles and singular points of system(8).

EXAMPLE 2.4. Let $a = b = c = k = 1, d = -1, n = 2, \alpha = \frac{1}{2}$, and $\mathcal{V}(x, y) = (x-2)^2 + (y-2)^2 - 1$. Then system (5) becomes as follows

$$\begin{cases} \dot{x} = x\left(\left((x-2)^2 + (y-2)^2 - 1\right)\left(x^3 y + 1\right) + x^3 y^4 (y-2)\right), \\ \dot{y} = y\left(\left((x-2)^2 + (y-2)^2 - 1\right)\left(y^3 x - 1\right) - x^4 y^3 (x-2)\right), \end{cases} \tag{9}$$

it has a first integral as follows

$$\mathcal{H}(x, y) = \exp\left(-\frac{1}{3x^3y^3}\left(3x^3y - 3xy^3 + 2\right)\right)\left((x-2)^2 + (y-2)^2 - 1\right),$$

It's clearly aforementioned $\mathcal{H}(x, y)$ satisfies the definition of *first integral.* Then

$$\frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}} = \frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}} = -\frac{1}{2}x^4y^4 \exp\left(\frac{1}{3x^3y^3}\left(3x^3y - 3xy^3 + 2\right)\right),$$

and the set

$$\Psi^{-1}(0) = \left\{ (x,y) \in \mathbb{R}^2 \mid -\frac{1}{2}x^4y^4 \exp\left(\frac{1}{3x^3y^3}\left(3x^3y - 3xy^3 + 2\right)\right) = 0 \right\}$$

contains no closed curve. The system (9) admits three singular points, where $(0,0)$ is a saddle point, $(1.85773, 2.110525)$ is a strong stable focus and $(1.99589, 0.777487)$ is a saddle point. The circle $(x-2)^2 + (y-2)^2 - 1 = 0$ is an invariant curve for system, but the system has not a limit cycle. See Figure 2.

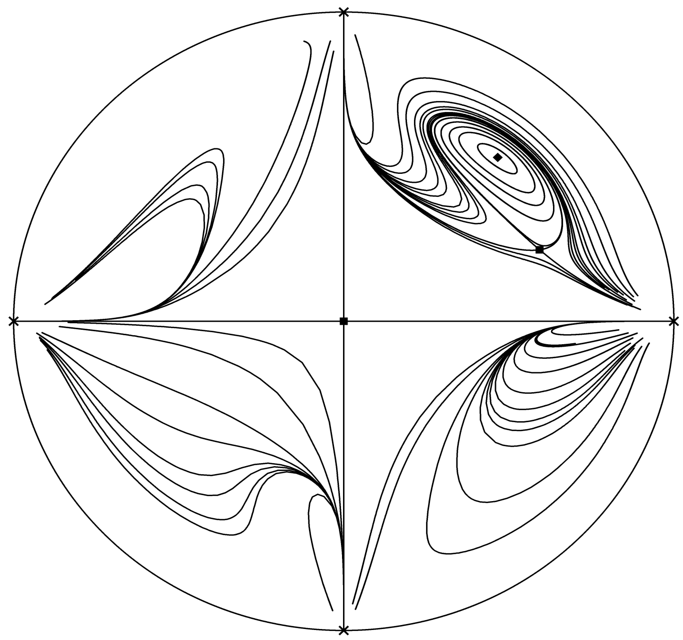

Figure 2: Phase portraits of system(9) in Poincaré disk.

3. Conclusion

In this paper, we investigate existence and non nonexistence of limit cycle for a class of Kolmogorov system (1). We characterized all conditions for the suggested system in order to find hyperbolic limit cycle. In addition, for investigating non existence limit cycle the general form of the first integral for system (1) has been found under suitable conditions of the coefficients.

Remark 3.1. All figures are plotted on the Poincaré disc by using polynomial planar phase portraits program, see for instance [9, pages 233-257].

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Authors' addresses:

Sarbast Hussein
Department of Mathematics
Faculty of Science, Soran University
Soran, 44008. Kurdistan Region-Iraq
E-mail: sarbast.mikael@soran.edu.iq

Salah Benyoucef
Department of Mathematics
Laboratory of Applied Mathematics
University of Setif 1, 19000, Algeria
E-mail: saben21@yahoo.fr

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