On the study of decompositions
of forms in four variables

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Al mio amico Giorgio,
insieme al quale ho brindato ai recenti successi della Lupa

Abstract. *In the space of sextic forms in* 4 *variables with a decomposition of length* 18 *we determine and describe a closed subvariety which contains all non-identifiable sextics. The description of the subvariety is geometric, but one can derive from that an algorithm which can guarantee that a given form is identifiable.*

1. Introduction

The paper describes an application of geometric tools, mainly from the theory of finite sets in projective spaces, to the study of Waring decompositions of forms.

The tools have been introduced and employed, in a series of papers, mainly for forms of degree 4 or for forms in three variables (see [2, 3, 4, 5, 11]). Since quaternary forms of degree 5 are considered in a forthcoming paper [9], we turn now our attention to forms of degree 6 in four variables.

Our starting point is the same starting point of the celebrated Kruskal's criterion for the minimality and uniqueness of a decomposition (to be precise, in its version for symmetric tensors). We assume that we know a (Waring) expression of a form F in terms of powers of linear forms, as the one given in formula (1) below. The problem consists of determining if the expression is minimal, in which case it computes the Waring rank of F. In addition, one would like to know if the expression is unique (up to trivialities).

We attack the problem by considering the linear forms appearing in the expression as a set of points A in a projective space $\mathbb{P}^3$, and analyzing the existence of another set B, of length smaller or equal than the length of A, whose 6$-$Veronese image spans F.

It turns out that the union $Z = A \cup B$ must satisfy several geometric and algebraic restrictions. This makes it possible to analyze the situation up to rank 18. Indeed, we prove that when the length (= cardinality) of A is strictly smaller than 18 and A is sufficiently general in a very precise sense (see the statement of Proposition 4.4), then the expression is necessarily unique. The geometric situation in this case is similar to the one treated in Kruskal's criterion (which, by the way, even in its reshaped version described in [12], cannot work for $r > 14$ in the case of quaternary sextics).

The case $r = 18$ turns out to be different. For $r = 18$, even if A is completely general, there are forms in the span of the $6-$Veronese image of A for which a second decomposition B exists. We can be more specific: when A is general, so that cubic surfaces through A define a complete intersection irreducible curve C of degree 9, then B is forced to be residual to A in a complete intersection of C and a quartic surface. This allows us to parameterize the possible sets B, and thus parameterize a (locally closed) subvariety Γ of the span of $v_6(A)$, which contains the forms F of degree 6 in 4 variables, rank 18, which are not identifiable. The closure of Γ is the image of a map from a subspace of the projective space $\mathbb{P}((I_A)_4)$ to $\langle v_6(A) \rangle$. We refer to Theorem 5.5 for a more precise description.

In particular, we get that if F is a non-identifiable form, then the second decomposition B is bounded to an invariant curve C, defined by A. This is a case of *confinement* for decompositions of forms, as described in general in [1].

Since the generic rank of a form of degree 6 in four variables is 21, one may wonder what happens for the missing cases $r = 19, 20, 21$. For $r = 19$, the same procedure proves that a hypothetical second decomposition B must be bounded to the unique cubic surface defined by A, but we are not able to characterize it any more. For $r = 20, 21$ we have no precise characterization. This is probably due to the fact that the theory of finite sets in $\mathbb{P}^3$ is far from being completely understood, and also opens a series of questions on the structure of finite sets in higher dimensional spaces, which could suggest directions to investigators in the field.

2. Preliminaries

All polynomials in the paper are defined over the complex field.

We will often, by abuse, use the same letter to indicate a form in a polynomial ring, the projective hypersurface defined by the form, and the point defined by the form in the corresponding projective space.

Given a finite set A in a projective space, we denote by $\ell(A)$ its length (i.e. its cardinality).

Consider a form F of degree 6 in 4 variables, over the complex field.

Assume we know a Waring expression of F (of length r) as a linear combination of powers of linear forms

$$F = \sum_{i=1}^{r} a_i L_i^6 \tag{1}$$

but we do not know a priori if the expression is minimal or unique (up to trivialities). Thus we do not know if r is the (Waring) rank of F, and we do not know whether F is identifiable or not.

On the other hand, we can certainly assume that the expression is *non-redundant*, in the sense that the powers L_i^6's are linearly independent and no coefficient a_i is 0.

Call $A = \{L_1, \ldots, L_r\}$ the set of linear forms involved in the expression, considered as points in a projective space $\mathbb{P}^3$. If we denote with $v_d : \mathbb{P}^3 \to \mathbb{P}^N$ the d-Veronese map, the expression tells us that F (as we said above, identified by abuse with one point of the space $\mathbb{P}^{83}$ of sextic forms in $\mathbb{P}^3$) belongs to the span of the Veronese image $v_6(A)$. The non-redundancy of A is equivalent to saying that, for all *proper* subsets $A' \subset A$, F is not contained in the span of $v_6(A')$.

We have full control on the set A, so we may assume that we know all its invariants. Thus we can assume that

$$(*) \quad A \text{ is in } General\ Position\ (GP)$$

which, in this setting, means that all subsets of A have maximal Hilbert function.

Notice that if A has this property, then all subsets of A also have it.

REMARK 2.1. When $r \leq 14$, then the celebrated Kruskal's criterion, in its reshaped version (see [12]) guarantees that r is the rank of F, and the expression is unique (up to trivialities: product by a scalar or reordering).

Namely, if $u = \min\{r, 10\}$ then necessarily

$$r \leq \frac{u + u + u - 2}{2},$$

thus we can take a partition $6 = 2 + 2 + 2$ and consider F as a tensor of $Sym^2(\mathbb{C}^4) \otimes Sym^2(\mathbb{C}^4) \otimes Sym^2(\mathbb{C}^4)$. Since the second Kruskal's rank of A is u by the genericity assumption, then a direct application of Kruskal's criterion guarantees that (1) is the unique expression of F of length r.

When $r > 14$, we assume the existence of another expression

$$F = \sum_{i=1}^{s} a_i M_i^6, \quad s \leq r \tag{2}$$

and call $B = \{M_1, \ldots, M_s\}$ the consequent finite set in $\mathbb{P}^3$.

Again we may directly assume that also B is non-redundant.

When $r \geq 15$ the Kruskal's criterion cannot provide a proof of the minimality and uniqueness of the expression (1). Indeed in this case new expressions are possible. A finer geometrical analysis is required to understand the situation.

Call h_A, h_B, h_Z the Hilbert functions of A, B and $Z = A \cup B$ respectively.

By assumptions we know that the difference $Dh_A(i) = h_A(i) - h_A(i-1)$ is defined by the following table

i	0	1	2	3	4	5	$\ldots$
$Dh_A(j)$	1	3	6	$r - 10$	$\max\{0, r - 20\}$	0	$\cdots$

From [2, Proposition 2.19], we know that

$$\dim(\langle v_6(A) \rangle \cap \langle v_6(B) \rangle) = \ell(A \cap B) - 1 + h_Z^1(6).$$

where $h_Z^1(i)$ is defined by $h_Z^1(i) = \ell(Z) - h_Z(i)$.

In particular $h_Z(6) < \ell(Z)$ when A, B are disjoint.

We recall the Cayley-Bacharach property of Z from [5] and [2, Section 2.4].

REMARK 2.2. Since A, B are both non-redundant, if $A \cap B = \emptyset$ then the set Z satisfies the *Cayley-Bacharach property*. In particular for $j = 0, 1, 2, 3$,

$$\sum_{i=0}^{j} Dh_Z(i) \leq \sum_{i=0}^{j} Dh_Z(7 - j).$$

PROPOSITION 2.3. *Assume $r \leq 20$. Then $s = \ell(B) \geq r$. If $r = 15$ then A, B are disjoint. Moreover, for all r the ideals of A and Z agree up to degree 3.*

Proof. If $A \cap B = \emptyset$, then by Remark 2.2 we must have:

$$\ell(Z) = \ell(A) + \ell(B) \geq \sum_{i=0}^{7} Dh_Z(i)$$

$$\geq 2\sum_{i=0}^{3} Dh_Z(i) \geq 2\sum_{i=0}^{3} Dh_A(i) = 2\ell(A)$$

which proves $s \geq r$. If $r = s$, the inequalities become equalities, and this implies the result on the ideals of A and Z.

Assume $A \cap B \neq \emptyset$, i.e. assume $L_i = M_i$ for $i = 1, \ldots, j$, for some $j > 0$.

Then

$$F = a_1 L_1^6 + \cdots + a_j L_j^6 + a_{j+1} L_{j+1}^6 + \cdots + a_r L_r^6$$
$$= b_1 L_1^6 + \cdots + b_j L_j^6 + b_{j+1} M_{j+1}^6 + \cdots + b_r M_r^6.$$

Define F' by

$$F' = (a_1 - b_1)L_1^6 + \cdots + (a_j - b_j)L_j^6 + a_{j+1}L_{j+1}^6 + \cdots + a_r L_r^6$$
$$= b_{j+1}M_{j+1}^6 + \cdots + b_s M_s^6 .$$

F' has two disjoint decomposition. The former can have some vanishing coefficients, but its length in any case is at least $r - j$, while the latter has length $\leq s - j$.

If $r = 15$ we obtain a contradiction by the reshaped Kruskal's criterion (Remark 2.1) or by what we concluded above in the disjoint case. Then, arguing by induction on r, we get that $s \geq r$.

If A_0, B_0 are the two decompositions of F' defined above, then by induction the ideals of A_0 and $Z_0 = A_0 \cup B_0$ agree up to degree 3. Since A, B are obtained from A_0, B_0 by adding the same subset S, then also the ideals of A and Z agree up to degree 3. □

The minimality of the expression (1) proved in the previous result indeed also follows from [6, Theorem 1.2], or by [14, Theorem 3.1] .

3. The case $r = 15$

We know from Proposition 2.3 and its proof that if F has two decompositions A, B, then $A \cap B = \emptyset$.

We show an example in which the second decomposition B exists.

EXAMPLE 3.1. Assume that A is a general set of 15 points in a general elliptic quintic curve C. The 6-Veronese map maps C to a normal elliptic curve of degree 30 which spans a $\mathbb{P}^{29}$. In $\mathbb{P}^{29}$ a general point has two different decompositions with respect to the elliptic curve C (see [10, Proposition 5.2]). Thus one gets that a general F in the span of $v_6(A)$ has exactly two different decompositions.

It is easy indeed to construct examples of forms F with two decompositions of this type. A general set A of 15 points in an elliptic quintic and a general F in the span of $v_6(A)$ will do.

On the other hand, it is also simple to realize that a general set A of 15 points in $\mathbb{P}^3$ does not lie in an elliptic quintic. This is just a count of parameters: the Hilbert scheme of elliptic quintics has dimension $5 \cdot 4 = 20$, so the sets

of 15 points in such curves cannot depend on more than $20 + 15 = 35$ parameters; on the other hand, the family of sets of 15 points in $\mathbb{P}^3$ has dimension 45.

One can easily exclude that a given set A of 15 points in $\mathbb{P}^3$ lies in an elliptic quintic by considering the base locus of the system of cubics through A which, by assumption, has dimension 5.

PROPOSITION 3.2. *Assume $r = 15$ and assume that the base locus of the system of cubics through A contains no curves. Then A is the unique minimal decomposition of F.*

Proof. Assume there exists a second decomposition B of length ≤ 15. Arguing as in the final part of the proof of Proposition 2.3, since we can apply the reshaped Kruskal's criterion for decompositions of length ≤ 14, we see that A, B must be disjoint. We know that the ideal of $Z = A \cup B$ coincides with the ideal of A in degree 3. Since the base locus of the system of cubics through A contains no curves, then by Bézout Z has length at most 27. Thus $\ell(B) \leq 12$, which is excluded by Proposition 2.3. $\square$

One checks easily the dimension of the base locus of the system of cubics through A, by standard computer algebra packages.

4. The cases $r = 16, 17$

The situation for $r = 16, 17$ is quite similar to the case $r = 15$, except that now an intersection between the two decompositions is allowed.

EXAMPLE 4.1. Let A_0 be a general set of 15 points lying in a general elliptic quintic curve C. We saw in Example 3.1 that a general form F_0 in the span of $v_6(A_0)$ has a second decomposition B_0 of length 15, disjoint from A_0. If L_0 is a general linear form, then $\{L_0\} \cup A_0$ and $\{L_0\} \cup B_0$ are two different, non-disjoint, decompositions of length 16 of $L_0^6 + F_0$.

Arguing as in Proposition 2.3, one sees that these two decompositions are minimal, when A_0, B_0, L_0 are general.

Also examples with different disjoint decompositions are possible.

EXAMPLE 4.2. Let A be a general set of 16 points lying in a general rational quintic curve C. By Bézout, since C is irreducible, the ideal of C and the ideal of A agree in degree 3. The Veronese map v_6 maps C to $\mathbb{P}^{30}$. Since no curves are defective, a general point F of $\mathbb{P}^{30}$ has infinitely many (mostly disjoint) decompositions of length 16 with respect to $v_6(C)$.

Sets A of this type lie in the *Terracini locus*, as defined in [7]: the differential of the map from the abstract 16-secant variety to the space $\mathbb{P}^{83}$ of $v_6(\mathbb{P}^3)$ drops rank over a general $F \in \langle v_6(A) \rangle$.

EXAMPLE 4.3. Starting with forms with two decompositions of length 16, as e.g. in Example 4.2, and adding one point as in Example 4.1, one finds easily examples of non-disjoint different decompositions of length 17 for some sextics F.

As in the case $r = 15$, if the system of cubics through A has no curves in the base locus, then the decomposition A of F is unique.

PROPOSITION 4.4. *Assume $r = 16$ or $r = 17$ and assume that the base locus of the system of cubics through A contains no curves. Then A is the unique minimal decomposition of F.*

Proof. The proof is given only for $r = 16$, since the other case is completely analogous.

Assume there exists a second decomposition B of length 16. If $A \cap B = \emptyset$, since the ideal of $Z = A \cup B$ coincides with the ideal of A in degree 3, by Bézout Z has length at most 27. Thus $\ell(B) \leq 11$, which is excluded by Proposition 2.3.

If the intersection $A \cap B$ contains $j > 0$ points, then as above write

$$F = a_1 L_1^6 + \cdots + a_j L_j^6 + a_{j+1} L_{j+1}^6 + \cdots + a_{16} L_{16}^6$$
$$= b_1 L_1^6 + \cdots + b_j L_j^6 + b_{j+1} M_{j+1}^6 + \cdots + b_{16} M_{16}^6.$$

Define F' by

$$F' = (a_1 - b_1)L_1^6 + \cdots + (a_j - b_j)L_j^6 + a_{j+1} L_{j+1}^6 + \cdots + a_{16} L_{16}^6$$
$$= b_{j+1} M_{j+1}^6 + \cdots + b_{16} M_{16}^6.$$

F' has two disjoint decompositions, one for which A' is contained in A. Thus the system of cubics through A' has no curves in the base locus. Even if the length of A' is 15, we have a contradiction with Proposition 3.2. □

Since for $r \leq 17$ and A very general the system of cubics through A has no curves in the base locus, the previous proposition excludes the existence of a second decomposition, except for sets A contained in a Zariski closed subset of $(\mathbb{P}^3)^r$.

5. The case $r = 18$

For $r = 18$ and A general, the base locus of the system of cubics through A is a complete intersection curve C of degree 9 and genus 10. There is no way to use a strategy similar to the statement of Proposition 4.4 in order to prove the identifiability of F.

REMARK 5.1. From Proposition 3.2 and Proposition 4.4 it turns out that, when $r = 15, 16, 17$ and the system of cubics through A has no curves in the base

locus, then all forms F in the span of $v_6(A)$ are identifiable of (Waring) rank r, unless the decomposition A is redundant for F, i.e. unless F sits in the span of some strict subset of $v_6(A)$.

We can see immediately that the situation changes completely for $r = 18$.

EXAMPLE 5.2. Let A be a *general* set of 18 points in $\mathbb{P}^3$. Then A is contained in the complete intersection of two cubics G_1, G_2. Consider the complete intersection curve $C = G_1 \cap G_2$ and let G be a general quartic not containing C. The intersection of C with the surface G consists of 36 points $Z = A \cup B$. B is thus a set of 18 points in the curve C, disjoint from A. By the Cayley-Bacharach property of complete intersections, one knows that $h_Z^1(6) > 0$. Thus by [2, Proposition 2.19], we know that $\langle v_6(A) \rangle$ and $\langle v_6(B) \rangle$ meet in some point F. Such $F \in \langle v_6(A) \rangle$ has a second decomposition B of length 18.

REMARK 5.3. By [4, Proposition 3.9], when A, B are disjoint decompositions of F, then the sum of the homogeneous ideals $I_A + I_B$ does not coincide with the polynomial ring R in degree 6, and F is dual to $I_A + I_B$

Consider again the sets A, B described in Example 5.2.

The ideal of B can be found from G and the ideal of A as a result of the mapping cone process (see [13]). By the Minimal Resolution Conjecture, which holds in $\mathbb{P}^3$ (see [8]), a resolution of the ideal I_A is given by $0 \to R^8(-6) \to R^{18}(-5) \to R^2(-3) \oplus R^9(-4) \to I_A \to 0$. Combining with the Koszul complex of G_1, G_2, G one obtains a diagram

$$
\begin{array}{ccccccccc}
0 & \to & R(-10) & \xrightarrow{\alpha'} & R(-6) \oplus R^2(-7) & \xrightarrow{\beta'} & R^2(-3) \oplus R(-4) & \to & I_Z \to 0 \\
& & \gamma \downarrow & & \gamma' \downarrow & & \gamma'' \downarrow & & \\
0 & \to & R^8(-6) & \xrightarrow{\alpha} & R^{18}(-5) & \xrightarrow{\beta} & R^2(-3) \oplus R^9(-4) & \to & I_A \to 0
\end{array}
\tag{3}
$$

where the map γ'' is defined by G_1, G_2, G. From the diagram one obtains a resolution of I_B by the dual of the mapping cone:

$$
0 \to R^8(-6) \to R^{18}(-5) \xrightarrow{(\alpha \oplus \gamma')^\vee} R^2(-3) \oplus R^9(-4) \xrightarrow{(\alpha' \oplus \gamma)^\vee} I_B \to 0
$$

Thus there is a standard way to compute I_B, hence $I_A + I_B$, from I_A and G.

We have then all the ingredients to study the existence of a second decomposition for F.

PROPOSITION 5.4. *Assume that the decomposition A of length 18 of F, satisfying condition (∗), also satisfies the following condition: for all subsets $A' \subset A$ of length 17, the linear system of cubics through A' has base locus of dimension 0. Then any other decompositions of length 18 of F is disjoint from A.*

Proof. Assume there exists another decomposition B of length 18 with $\ell(A \cap B) = j > 0$. Then arguing as in the proof of Proposition 2.3 one finds another

sextic form F' with decompositions

$$F' = (a_1 - b_1)L_1^6 + \cdots + (a_j - b_j)L_j^6 + a_{j+1}L_{j+1}^6 + \cdots + a_{18}L_{18}^6$$
$$= b_{j+1}M_{j+1}^6 + \cdots + b_{18}M_{18}^6,$$

where $A = \{L_1, \ldots, L_{18}\}$ and $a_i, b_i \neq 0$ for all i. If some coefficient $a_i - b_i$, $i = 1, \ldots, j$, is non-zero, then the second decomposition of F' has length smaller than the first one, which is contained in A. We get a contradiction with Proposition 2.3. Thus $a_i = b_i$ for all $i = 1, \ldots, j$. But then F' has two disjoint decompositions of length $18 - j$, and one of them $A' = \{L_{j+1}, \ldots, L_{18}\}$ is contained in A. By assumption the system of cubics through A' has no curves in the base locus. Then we get a contradiction with either the Reshaped Kruskal's Criterion, or Proposition 3.2, or Proposition 4.4. $\square$

THEOREM 5.5. *Let F be a sextic in 4 variables, with a non-redundant decomposition A of length 18. Assume that A satisfies the following properties.*

(*) *A is in General Position;*

(**) *for all subsets $A' \subset A$ of length 17, the linear system of cubics through A' has base locus of dimension 0;*

(***) *the base locus of the pencil of cubics through A is an irreducible curve C.*

Then A is minimal, and any other decomposition B of length 18 of F (if any) is disjoint from A, and $Z = A \cup B$ is a complete intersection of surfaces of degrees $3, 3, 4$.

Proof. The unique thing that remains to prove is the last assertion, i.e. that $A \cup B$ is the intersection of C with a quartic surface.

If B exists, $Z = A \cup B$ lies in the pencil of cubics containing A, by Proposition 2.3. If all the quartics containing Z are composed with the pencil, then $h_Z(4) = 35 - 8 = 27$, so that $Dh_Z(4) = 9$. But then $Dh_Z(5) + Dh_Z(6) + Dh_Z(7) \leq 9 < Dh_Z(2) + Dh_Z(1) + Dh_Z(0)$, which contradicts the Cayley-Bacharach property. hence there is a quartic containing Z and not C. The claim follows. $\square$

REMARK 5.6. For a given form F and a decomposition A of length 18, one can produce a procedure which tests if A is unique i.e. if F is identifiable of rank 18, as follows.

1. Control if A is in GP.

2. Control that the system of cubics through any subset of length 17 of A has 0-dimensional base locus.

3. Control that the system of cubics through A has an irreducible nonic curve as base locus.

4. Consider a linear space W in $(I_A)_4$ orthogonal to the 8-dimensional subspace spanned by the cubics through A.

5. For all $G \in W$ compute the generators of the residue B of A in $G \cap C$, in terms of coordinates of $G \in W$.

6. Prove that for no choice of the coordinates of G the form F is dual to $I_A + I_B$.

Notice that the generators of I_B, mod the cubics containing C, are 9 quartics, by the resolution following diagram 3.

One of the most expensive points in the procedure is step (1), which requires to control that none of the $\binom{18}{8} = 43,758$ subsets of length 10 of A is contained in quadrics.

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