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# Ramification and discriminants of vector bundles and a quick proof of Bogomolov's theorem

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*Dedicated to Giorgio Ottaviani on the occasion of his sixtieth birthday.*

Abstract. *By analyzing degeneracy loci over projectivized vector bundles, we recompute the degree of the discriminant locus of a vector bundle and provide a new proof of the Bogomolov instability theorem.*

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## 0. Introduction

Let $X$ be an $n$-dimensional smooth complex projective variety and let $E$ be a globally generated vector bundle on $X$ of rank $e \leqslant n$. The projective space $\mathbb{P}^r = \mathbb{P}\big(H^0(X,E)^*\big)$ parameterizes sections of $E$ up to scalars. The *discriminant* of $E$ is the locus in $\mathbb{P}^r$, typically a hypersurface, defined by

$$\Delta(E) \coloneqq \big\{ s \in \mathbb{P}^r \;\big|\; \text{the zero scheme Zeroes}(s) \text{ of } s \text{ is singular} \big\}.$$

The closed algebraic set Zeroes$(s)$ is understood to have its natural scheme structure: when $e = n$, $\Delta(E)$ consists of those sections that vanish at something other than $\int c_n(E)$ distinct points. There are various situations where it is of interest to calculate the degree of $\Delta(E)$. This comes up, for instance, in connection with eigenvalues of tensors [2]. In [1], the first author derives a formula for the degree when $e = n$ and $X = \mathbb{P}^n$.

The first purpose of this note is to give a very quick derivation of a formula for the (virtual) degree of $\Delta(E)$ reproving some results from [10]. For example, when $e = n$, we show that the expected degree of $\Delta(E)$ is given by

$$\delta(E) = \int_X \big(K_X + c_1(E)\big)\; c_{n-1}(E) + n\, c_n(E)\,.$$

If each section $s$ in $\Delta(E)$ is singular at several points, then the actual degree of the discriminant hypersurface is smaller than its postulated one. However, when $E$ is very ample and 1-jet spanned, we also show that $\Delta(E)$ is irreducible of the expected degree.

As one might expect, the basic idea is to compute the class of the singular locus of the universal zero-locus over $\mathbb{P}^r$. It turns out that a somewhat related computation leads to an extremely quick proof of the Bogomolov instability theorem for vector bundles of rank 2 on an algebraic surface, reducing the statement in effect to the Riemann–Hurwitz formula. The existence of a proof along these lines seems to have been known to the experts, but as far as we can tell it is not generally familiar. We therefore take this occasion to present the argument. Some time ago, Langer [9, Appendix] gave an even quicker, but related proof, using the fact that stability is preserved under pulling back by generically finite morphisms.

The formula for the ramification locus is derived in Section 1. In Section 2, we show that, when $E$ is very ample and 1-jet spanned, the discriminant locus is irreducible of the expected degree. The proof of the Bogomolov instability theorem occupies Section 3.

### Conventions

We work throughout over the complex numbers $\mathbb{C}$. For any vector space $V$ or vector bundle $E$, $\mathbb{P}(V)$ or $\mathbb{P}(E)$ denotes the projective space of one-dimensional quotients. Given a smooth variety $X$, the Chow ring of $X$ is $A^\bullet(X)$ (or, if the reader prefers, this is the even cohomology ring $H^{2\bullet}(X,\mathbb{Z})$). We write $c_i(E)$ and $s_i(E)$ for the $i$-th Chern and Segre classes of a vector bundle $E$ whereas $c(E)$ and $s(E)$ are the corresponding total Chern and Segre classes. Following [6, Example 3.2.7], we use the notation $c(E-F) \coloneqq c(E)/c(F) = c(E)\ s(F)$ for the "difference" of the total Chern classes of two bundles. Finally, given a class $\alpha$ in $A^\bullet(X)$, the component of $\alpha$ in codimension $k$ is $\alpha_k \in A^k(X)$.

## 1. Ramification Locus

In this section, we derive a formula for ramification class of certain morphisms from projectivized vector bundles. To be more explicit, fix an $n$-dimensional smooth complex projective variety $X$ and consider a globally-generated vector bundle $E$ on $X$ of rank $e$ such that $e \leqslant n$.

Let $V_E \coloneqq H^0(X,E)$ be the $\mathbb{C}$-vector space of global sections of $E$ and set $r \coloneqq \dim_{\mathbb{C}} V_E - 1$. The trivial vector bundle on $X$ with fibre $V_E$ is $V_E \otimes_{\mathbb{C}} \mathscr{O}_X$ and the kernel of the evaluation map $\mathrm{ev}_E \colon V_E \otimes_{\mathbb{C}} \mathscr{O}_X \to E$ is $M_E \coloneqq \mathrm{Ker}(\mathrm{ev}_E)$. It follows that $M_E$ is a vector bundle of rank $r-e+1$ sitting in the short exact sequence

$$0 \longrightarrow M_E \longrightarrow V_E \otimes_{\mathbb{C}} \mathscr{O}_X \xrightarrow{\mathrm{ev}_E} E \longrightarrow 0\,.$$

Applying the duality functor $(-)^* := \mathscr{H}om(-, \mathscr{O}_X)$, we obtain the short exact sequence

$$0 \longrightarrow E^* \longrightarrow (V_E \otimes_{\mathbb{C}} \mathscr{O}_X)^* \longrightarrow M_E^* \longrightarrow 0\,.$$

The surjective map onto $M_E^*$ identifies the projectivization $\mathbb{P}(M_E^*)$ with a closed subscheme in the product $\mathbb{P}\big((V_E \otimes_{\mathbb{C}} \mathscr{O}_X)^*\big) = X \times \mathbb{P}(V_E^*)$ where $V_E^*$ is the dual vector space of $V_E$. Thus, we have $\mathbb{P}(M_E^*) = \{(x,[s]) \in X \times \mathbb{P}(V_E^*) \mid s(x) = 0\}$. Let $p_E \colon X \times \mathbb{P}(V_E^*) \to X$ be the projection onto the first factor. We also use $p_E$ for the restriction to $\mathbb{P}(M_E^*)$. Let $q_E \colon \mathbb{P}(M_E^*) \to \mathbb{P}(V_E^*)$ be the restriction of the projection from $X \times \mathbb{P}(V_E^*)$ onto the second factor $\mathbb{P}(V_E^*)$. When the vector bundle $E$ is unnecessary, we omit the subscripts on $V$, $M$, $p$, and $q$.

Guided by Example 14.4.8 in [6], the *ramification locus* $R(q)$ of the map $q \colon \mathbb{P}(M^*) \to \mathbb{P}(V^*)$ is the $(r-1)$-st degeneracy locus of the induced differential $dq \colon q^*\Omega_{\mathbb{P}(V^*)} \to \Omega_{\mathbb{P}(M^*)}$;

$$\begin{aligned} R(q) &:= \big\{ x \in \mathbb{P}(M^*) \;\big|\; \text{rank of map } dq \text{ at the point } x \text{ is at most } r-1 \big\} \\ &= \operatorname{Zeroes}\big(\textstyle\bigwedge^r dq\big)\,. \end{aligned}$$

Since $\mathbb{P}(V^*)$ and $\mathbb{P}(M^*)$ have dimension $r$ and $n+r-e$, the subscheme $R(q)$ has codimension at most $\big(r-(r-1)\big)\big(n+r-e-(r-1)\big) = n-e+1$; see [6, p. 242]. The next proposition provides a formula for the *ramification class* $[R(q)]$ in the Chow ring $A^\bullet\big(\mathbb{P}(M^*)\big)$.

Proposition 1.1. *When the ramification locus $R(q)$ has codimension $n-e+1$, its class in $A^\bullet\big(\mathbb{P}(M^*)\big)$ is $[R(q)] = \big\{c(p^*\Omega_X)\; s\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)\big\}_{n-e+1}$ and the degree of its pushforward is*

$$\deg q_*[R(q)] = \int_X p_*\Big([R(q)]\; c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^{r-1}\Big)\,.$$

*Proof.* Since $R(q)$ has codimension $n-e+1$, the Thom–Porteous formula [6, Theorem 14.4] establishes that $[R(q)] = c_{n-e+1}\big(\Omega_{\mathbb{P}(M^*)} - q^*\Omega_{\mathbb{P}(V^*)}\big)$. Hence, it suffices to prove that

$$c_{n-e+1}\big(\Omega_{\mathbb{P}(M^*)} - q^*\Omega_{\mathbb{P}(V^*)}\big) = c_{n-e+1}\big(p^*\Omega_X - p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)\,.$$

By combining the two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}_{\mathbb{P}(M^*)}/\mathscr{I}^2_{\mathbb{P}(M^*)} & \xrightarrow{\;\delta\;} & \Omega_{X \times \mathbb{P}(V^*)}\big|_{\mathbb{P}(M^*)} & \longrightarrow & \Omega_{\mathbb{P}(M^*)} & \longrightarrow & 0 \\ 0 & \longrightarrow & q^*\Omega_{\mathbb{P}(V^*)} & \longrightarrow & \Omega_{X \times \mathbb{P}(V^*)}\big|_{\mathbb{P}(M^*)} & \xrightarrow{\;\theta\;} & p^*\Omega_X & \longrightarrow & 0\,, \end{array}$$

we obtain the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & q^*\Omega_{\mathbb{P}(V^*)} & \overset{dq}{\searrow} & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{I}_{\mathbb{P}(M^*)}/\mathscr{I}^2_{\mathbb{P}(M^*)} & \xrightarrow{\ \delta\ } & \Omega_{X\times\mathbb{P}(V^*)}\big|_{\mathbb{P}(M^*)} & \longrightarrow & \Omega_{\mathbb{P}(M^*)} & \longrightarrow & 0\,. \\
 & & & \searrow & \downarrow{\scriptstyle\theta} & & & & \\
 & & & & p^*\Omega_X & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

The snake lemma shows that $\operatorname{Coker}(dq) \cong \operatorname{Coker}(\theta \circ \delta)$, so we deduce that

$$c_{n-e+1}\big(\Omega_{\mathbb{P}(M^*)} - q^*\Omega_{\mathbb{P}(V^*)}\big) = c_{n-e+1}\big(p^*\Omega_X - \mathscr{I}_{\mathbb{P}(M^*)}/\mathscr{I}^2_{\mathbb{P}(M^*)}\big)\,.$$

It remains to show that the conormal bundle $\mathscr{I}_{\mathbb{P}(M^*)}/\mathscr{I}^2_{\mathbb{P}(M^*)}$ on $\mathbb{P}(M^*)$ is isomorphic to the vector bundle $p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)$. As a closed subscheme of $X \times \mathbb{P}(V^*)$, the projectivization $\mathbb{P}(M^*)$ is the zero scheme of a regular section of $p^*E \otimes \mathscr{O}_{X\times\mathbb{P}(V^*)}(1)$; see [6, Appendix B.5.6]. Tensoring the Koszul complex associated to this regular section with $\mathscr{O}_{\mathbb{P}(M^*)}$ produces the desired isomorphism $p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1) \cong \mathscr{I}_{\mathbb{P}(M^*)} \otimes \mathscr{O}_{\mathbb{P}(M^*)} \cong \mathscr{I}_{\mathbb{P}(M^*)}/\mathscr{I}^2_{\mathbb{P}(M^*)}$.

To prove the second part, observe that $\mathscr{O}_{\mathbb{P}(M^*)}(1) = q^*\mathscr{O}_{\mathbb{P}(V^*)}(1)$; see [11, Example 6.1.5]. It follows from the projection formula that the degree of push-forward is

$$
\begin{aligned}
\deg q_*[R(q)] &= \int_{\mathbb{P}(V^*)} q_*[R(q)]\; c_1\big(\mathscr{O}_{\mathbb{P}(V^*)}(1)\big)^{r-1} \\
&= \int_{\mathbb{P}(M^*)} q^*\Big(q_*[R(q)]\; c_1\big(\mathscr{O}_{\mathbb{P}(V^*)}(1)\big)^{r-1}\Big) \\
&= \int_{\mathbb{P}(M^*)} [R(q)]\; c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^{r-1} \\
&= \int_X p_*\Big([R(q)]\; c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^{r-1}\Big)\,.
\end{aligned}
$$

□

In the following examples, we examine three special cases that express ramification class as a polynomial in the Chern classes for $E$ and $\Omega_X$. For all nonnegative integers $i$, the defining short exact sequence of the kernel bundle $M_E$ shows that $p_*\, c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^{r-e+i} = s_i(M) = c_i(E)$.

EXAMPLE 1.2 ($e = 1$). Suppose that the vector bundle $E$ has rank 1. When ramification locus $R(q)$ has codimension $n$, Proposition 1.1 implies that

$$\begin{aligned}
[R(q)] &= \big\{c(p^*\Omega_X)\; s\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)\big\}_n \\
&= \sum_{i=0}^{n} c_{n-i}(p^*\Omega_X)\; (-1)^i c_1\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)^i \\
&= \sum_{i=0}^{n} c_{n-i}(p^*\Omega_X)\; \sum_{j=0}^{i} \binom{i}{j} c_1(p^*E)^j\, c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^{i-j},
\end{aligned}$$

and $\deg q_*[R(q)] = \displaystyle\sum_{i=0}^{n} (i+1) \int_X c_{n-i}(\Omega_X)\; c_1(E)^i.$

EXAMPLE 1.3 ($n = e$). Suppose that the rank of the vector bundle $E$ equals the dimension of its underlying variety $X$. When $R(q)$ has codimension 1, Proposition 1.1 implies that

$$\begin{aligned}
[R(q)] &= \big\{c(p^*\Omega_X)\; s\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)\big\}_1 \\
&= c_1(p^*\Omega_X) - c_1\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big) \\
&= c_1(p^*\Omega_X) + c_1(p^*E) + n\, c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)
\end{aligned}$$

and $\deg q_*[R(q)] = \displaystyle\int_X \big(c_1(\Omega_X) + c_1(E)\big)\, c_{n-1}(E) + n\, c_n(E).$

EXAMPLE 1.4 ($e = n-1$). Suppose that the rank of $E$ is the dimension of $X$ minus 1. Observe that $s_2(p^*E) = s_1(p^*E)^2 - c_2(p^*E^*) = c_1(p^*E)^2 - c_2(p^*E)$ and

$$\begin{aligned}
& s_2\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big) \\
&= \binom{n}{n-2} c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^2 - n\, s_1(p^*E^*)\; c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big) + s_2(p^*E^*) \\
&= \binom{n}{2} c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^2 - n\, c_1(p^*E)\; c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big) + c_1(p^*E)^2 - c_2(p^*E);
\end{aligned}$$

see [6, p. 50 and Example 3.1.1]. When $R(q)$ codimension 2, Proposition 1.1 implies that

$$\begin{aligned}
& [R(q)] \\
&= \big\{c(p^*\Omega_X)\; s\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)\big\}_2 \\
&= c_2(p^*\Omega_X) + c_1(p^*\Omega_X)\; s_1\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big) + s_2\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big) \\
&= c_2(p^*\Omega_X) + c_1(p^*\Omega_X)\Big(c_1(p^*E) + (n-1)\, c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)\Big) \\
&\quad + \binom{n}{2} c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big)^2 - n\, c_1(p^*E)\; c_1\big(\mathscr{O}_{\mathbb{P}(M^*)}(1)\big) + c_1(p^*E)^2 - c_2(p^*E)
\end{aligned}$$

and

$$\deg q_*[R(q)] = \int_X \big(c_2(\Omega_X) + c_1(\Omega_X)\, c_1(E) + c_1(E)^2 - c_2(E)\big)\, c_{n-2}(E) \\ + \big((n-1)\, c_1(\Omega_X) + n\, c_1(E)\big)\, c_{n-1}(E)\,.$$

## 2. Discriminant Locus of a Vector Bundle

This section determines the degree of the discriminant of a vector bundle. As in the first section, $X$ is an $n$-dimensional smooth complex projective variety $X$ and $E$ is a globally-generated vector bundle on $X$ of rank $e \leqslant n$. Set $V_E := H^0(X, E)$, let $M_E$ be the kernel of $\mathrm{ev}_E \colon V_E \otimes_{\mathbb{C}} \mathscr{O}_X \to E$, and write $q_E \colon \mathbb{P}(M_E^*) \to \mathbb{P}(V_E^*)$ for composition of the inclusion $\mathbb{P}(M_E^*) \to X \times \mathbb{P}(V_E^*)$ and the projection $X \times \mathbb{P}(V_E^*) \to \mathbb{P}(V_E^*)$ onto the second factor.

The *discriminant locus* $\Delta(E)$ of the vector bundle $E$ is the reduced scheme structure on the image of the ramification locus $R(q_E)$ under the map $q_E$. A section $s$ in $V_E^*$ is *nonsingular* if its zero scheme $\mathrm{Zeroes}(s)$ is nonsingular and has codimension $e$ in $\mathbb{P}(V_E^*)$; otherwise it is *singular*. With this terminology, one verifies that

$$\Delta(E) := \big\{ [s] \in \mathbb{P}(V_E^*) \;\big|\; \text{the section } s \text{ is singular} \big\}\,.$$

The *defect* of the vector bundle $E$ is the integer $\mathrm{def}(E) := \operatorname{codim} \Delta(E) - 1$, the *expected degree* of the discriminant locus $\Delta(E)$ is $\delta(E) := \deg(q_E)_*[R(q_E)]$, and the *coefficient* of $R(q_E)$ in $[R(q_E)_{\mathrm{red}}]$ is the unique positive integer $m_E$ such that $[R(q_E)] = m_E\, [R(q_E)_{\mathrm{red}}]$ in the Chow ring $A^\bullet\big(\mathbb{P}(M_E^*)\big)$.

The significance of these numerical invariants becomes clear with an additional hypothesis.

Remark 2.1. Assume that the ramification locus $R(q_E)$ is irreducible and has dimension $r - 1$ (or equivalently codimension $n - e + 1$). It follows that the discriminant locus $\Delta(E)$ is also irreducible. For the function fields $\mathbb{C}\big(R(q_E)\big)$ and $\mathbb{C}\big(\Delta(E)\big)$ of the reduced schemes $R(q_E)_{\mathrm{red}}$ and $\Delta(E)$, the degree of the field extension is $\big[\mathbb{C}\big(R(q_E)\big) : \mathbb{C}\big(\Delta(E)\big)\big]$ and the degree of $R(q_E)$ over $\Delta(E)$ is

$$\deg R(q_E)/\Delta(E) := \begin{cases} \big[\mathbb{C}\big(R(q_E)\big) : \mathbb{C}\big(\Delta(E)\big)\big] & \text{if } \dim \Delta(E) = r - 1 \\ 0 & \text{if } \dim \Delta(E) < r - 1. \end{cases}$$

The definition of the pushforward of a cycle gives

$$(q_E)_*[R(q_E)] = m_E\, \big(\deg R(q_E)/\Delta(E)\big)\, [\Delta(E)]\,;$$

see [6, Section 1.4]. Hence, we have $\mathrm{def}(X) > 0$ if and only if $\delta(E) = 0$. When $R(q_E)$ is integral and birational to $\Delta(E)$, we also have $\deg \Delta(E) = \delta(E)$.

Although the next result is likely known to experts, we could not find an adequate reference.

THEOREM 2.2. *Assume that $X$ an $n$-dimensional smooth projective variety $X$ and let $E$ be a very ample vector bundle on $X$ of rank $e \leqslant n$. Let $\pi\colon \mathbb{P}(E) \to X$ be the projective bundle associated to $E$ and let $L \coloneqq \mathscr{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle on the projectivization $\mathbb{P}(E)$.*

- *The discriminant locus $\Delta(E)$ of the vector bundle $E$ is isomorphic to the discriminant locus $\Delta(L)$ of the line bundle $L$. In particular, the discriminant locus $\Delta(E)$ is irreducible.*
- *When the discriminant locus $\Delta(E)$ is a hypersurface, the reduced scheme $R(q_E)_{\mathrm{red}}$ is birational to $\Delta(E)$ and*

$$\deg \Delta(E) = m_E \left\{c(p^*\Omega_X)\, s\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M_E^*)}(-1)\big)\right\}_{n-e+1} .$$

*Proof.* The canonical isomorphism $V_E = H^0(X,E) \xrightarrow{\cong} H^0\big(\mathbb{P}(E), L\big) = V_L$ induces an isomorphism $\varphi\colon \mathbb{P}(V_L^*) \to \mathbb{P}(V_E^*)$. It is enough to show that the restriction of $\varphi$ to the discriminant locus $\Delta(L)$ yields an isomorphism from $\Delta(L)$ to $\Delta(E)$. To accomplish this, it suffices to prove that a section $s$ in $V_E^*$ is singular if and only if the corresponding section $\tilde{s}$ in $V_L^*$ is singular. As this assertion is local, we may assume that $X$ is affine and $E \cong \bigoplus_{i=1}^{e} \mathscr{O}_X$. Hence, there exist $f_1, f_2, \dots, f_e \in H^0(X, \mathscr{O}_X)$ such that $s = (f_1, f_2, \dots, f_e)$ and $\tilde{s} = f_1\, x_1 + f_2\, x_2 + \cdots + f_e\, x_e$ where $x_1, x_2, \dots, x_e$ are homogeneous coordinates of $\mathbb{P}^{e-1} = \mathbb{P}(V_E^*)$. The assertion now follows from a local calculation of derivatives as appears in [1, Subsection 3.2].

The same calculation shows that restriction of the map

$$\pi \times \varphi\colon \mathbb{P}(E) \times \mathbb{P}(V_L^*) \to X \times \mathbb{P}(V_E^*)$$

to $R(q_L)_{\mathrm{red}}$ is a birational map from $R(q_L)_{\mathrm{red}}$ to $R(q_E)_{\mathrm{red}}$. When $\Delta(L)$ is a hypersurface, Proposition 3.2 in [7] demonstrates that reduced scheme $R(q_L)_{\mathrm{red}}$ is birational to discriminant locus $\Delta(L)$. It follows that the reduced scheme $R(q_E)_{\mathrm{red}}$ is birational to discriminant locus $\Delta(E)$. Finally, the degree formula is an immediate consequence of Remark 2.1. □

To prove that the ramification locus is reduced, we first record a general observation about degeneracy loci. Consider three vector bundles $A$, $B$, and $C$ on a smooth projective variety $X$ together with an injective vector bundle morphism $\mu\colon A \otimes B^* \to C$. Let $\varpi\colon \mathbb{P}(C) \to X$ be the projective bundle associated to $C$, let $\eta\colon \varpi^*C \to \mathscr{O}_{\mathbb{P}(C)}(1)$ be the natural surjective morphism, and let $\tilde{\mu}\colon \varpi^*(A \otimes B^*) \to \mathscr{O}_{\mathbb{P}(C)}(1)$ be the composition of $\mu$ with $\eta$. The map $\tilde{\mu}$ corresponds to the morphism $\mu'\colon \varpi^*A \to \varpi^*B \otimes \mathscr{O}_{\mathbb{P}(C)}(1)$ via tensor-hom adjunction.

Lemma 2.3. *For any nonnegative integer $k$, the $k$-th degeneracy locus*

$$D_k(\mu') \coloneqq \operatorname{Zeroes}\bigl(\textstyle\bigwedge^{k+1} \mu'\bigr)$$

*is reduced and Cohen–Macaulay of codimension* $\bigl(\operatorname{rank}(A) - k\bigr)\bigl(\operatorname{rank}(B) - k\bigr)$.

*Proof.* As the assertion is local, we may assume that $X$ is affine and the three vector bundles are trivial. Let $U$, $V$, and $W$ be complex vector spaces such that $A = U \otimes_{\mathbb{C}} \mathscr{O}_X$, $B = V \otimes_{\mathbb{C}} \mathscr{O}_X$, and $C = W \otimes_{\mathbb{C}} \mathscr{O}_X$. For each nonnegative integer $k$, let $D_k(U, V)$ be the locus of points in $\mathbb{P}(U \otimes_{\mathbb{C}} V^*) = \mathbb{P}\bigl(\operatorname{Hom}_{\mathbb{C}}(U, V)\bigr)$ whose corresponding linear transformations from $U$ to $V$ have rank at most $k$.

Consider the projective bundle $\rho \colon \mathbb{P}(A \otimes B^*) \to X$ associated to $A \otimes B^*$. On $Y \coloneqq \mathbb{P}(A \otimes B^*)$, the surjective morphism $\theta \colon \rho^*(A \otimes B^*) \to \mathscr{O}_Y(1)$ corresponds to the morphism $\theta' \colon \rho^* A \to \rho^* B \otimes \mathscr{O}_Y(1)$ whose $k$-th degeneracy locus $D_k(\theta')$ is $X \times D_k(U, V)$. In particular, $D_k(\theta')$ is reduced and Cohen–Macaulay of codimension $\bigl(\operatorname{rank}(A) - k\bigr)\bigl(\operatorname{rank}(B) - k\bigr)$.

Let $Q$ be the cokernel of the map $\mu \colon A \otimes B^* \to C$. It follows that $\mathbb{P}(Q)$ is a subbundle of $\mathbb{P}(C)$. Let $\psi \colon \mathbb{P}(C) - \mathbb{P}(Q) \to Y$ be the associated trivial affine bundle over $X$. Since the map $\mu' \colon \varpi^* A \to \varpi^* B \otimes \mathscr{O}_{\mathbb{P}(C)}(1)$ is nonzero away from $\mathbb{P}(Q)$, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(C) - \mathbb{P}(Q) & \xrightarrow{\ \psi\ } & Y \\ & \searrow^{\varpi} \quad \swarrow_{\rho} & \\ & X & \end{array}$$

with the property that $\mu' = \psi^*(\theta')$. Hence, the $k$-th degeneracy locus $D_k(\mu')$ is the "cone" over $D_k(\theta')$ in $\mathbb{P}(C)$ with vertex $\mathbb{P}(Q)$; it is the product of $X$ and the cone over $D_k(U, V)$ in $\mathbb{P}(W)$ with vertex $\mathbb{P}\bigl(W/(U \otimes V^*)\bigr)$. We conclude that the $k$-th degeneracy locus $D_k(\mu')$ is also reduced and Cohen–Macaulay of codimension $\bigl(\operatorname{rank}(A) - k\bigr)\bigl(\operatorname{rank}(B) - k\bigr)$. $\square$

To ensure that the ramification locus $R(q_E)$ is reduced, we rely on a stronger hypothesis than $E$ being very ample. To define this condition, we use the first jet bundle $J_1(E)$ that parametrizes the first-order Taylor expansions of the sections of $E$. More precisely, let $\mathscr{J}$ be the ideal sheaf defining the diagonal embedding $X \hookrightarrow X \times X$ and let $\operatorname{pr}_1, \operatorname{pr}_2 \colon \operatorname{Zeroes}(\mathscr{J}^2) \to X$ be the restrictions of the projections $X \times X \to X$ to the closed subscheme $\operatorname{Zeroes}(\mathscr{J}^2) \subset X \times X$. The *first jet bundle* is $J_1(E) \coloneqq (\operatorname{pr}_1)_* \operatorname{pr}_2^* E$; this is also called the bundle of principal parts in [6, Example 2.5.6]. The vector bundle $J_1(E)$ has rank $n + 1$ and sits in the short exact sequence

$$0 \longrightarrow \Omega_X \otimes E \longrightarrow J_1(E) \longrightarrow E \longrightarrow 0\,.$$

The vector bundle $E$ is 1-*jet spanned* if the evaluation map $V_E \otimes_{\mathbb{C}} \mathscr{O}_X \to J_1(E)$ is surjective; see [3, Subsection 1.3]. With this concept, we have the following corollary.

Corollary 2.4. *Let $X$ be an $n$-dimensional smooth projective variety and let $E$ be a very ample vector bundle of rank $e \leqslant n$. Assuming that the vector bundle $E$ is 1-jet spanned, the ramification locus $R(q_E)$ is reduced and Cohen-Macaulay of codimension $n - e + 1$, so $\Delta(E) = (q_E)_*[R(q_E)]$. Furthermore, the discriminant locus $\Delta(E)$ is a hypersurface if and only if we have $\delta(E) > 0$. When $\Delta(E)$ is a hypersurface, the degree of discriminant locus is*

$$\deg \Delta(E) = \big\{c(p^*\Omega_X)\ s\big(p^*E^* \otimes \mathscr{O}_{\mathbb{P}(M^*)}(-1)\big)\big\}_{n-e+1}\,.$$

*Proof.* By Theorem 2.2 and Lemma 2.3, it suffices to show the existence of an injective vector bundle morphism from $E^* \otimes (\Omega_X)^*$ to $M_E^*$ or equivalently a surjective map from $M_E$ to $E \otimes \Omega_X$. To establish this, we combine the defining short exact sequence for $M_E$ with the canonical short exact sequence for $J_1(E)$ to obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_E & \longrightarrow & V_C \otimes_{\mathbb{C}} \mathscr{O}_X & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_X \otimes E & \longrightarrow & J_1(E) & \longrightarrow & E & \longrightarrow & 0\,. \end{array}$$

Since $E$ is 1-jet spanned, the second vertical map is surjective. Hence, the snake lemma implies that the first vertical map is also surjective. □

Remark 2.5. Remark 0.3.2 in [4] establishes that, for any very ample line bundle $L$ on an $m$-dimensional smooth projective variety $Y$, we have $\operatorname{def}(L) > 0$ if and only if $c_m\big(J_1(L)\big) = 0$. When the discriminant locus $\Delta(L)$ is a hypersurface, this remark also shows that $\deg \Delta(L) = \int_Y c_m\big(J_1(L)\big)$.

Given an $n$-dimensional smooth projective variety $X$ and a very ample vector bundle $E$ on $X$ of rank $e \leqslant n$, Lanteri and Muñoz compute the top Chern class of the first jet bundle of the line bundle $L \coloneqq \mathscr{O}_{\mathbb{P}(E)}(1)$. More precisely, when $Y = \mathbb{P}(E)$, Proposition 1.1 in [10] expresses $c_{n+e-1}\big(J_1(L)\big)$ as a polynomial in the Chern classes of $E$ and the tangent bundle $T_X$. Under the assumption that the vector bundle $E$ is 1-jet spanned, Corollary 2.4 provides a different formula for the degree of $\Delta(E)$.

Example 2.6. Let $L$ be a very ample line bundle on a smooth projective variety $X$. The line bundle $L$ is 1-jet spanned; see [3, Subsection 1.3]. When the discriminant locus $\Delta(L)$ is a hypersurface, Example 1.2 and Corollary 2.4 show that

$$\deg \Delta(L) = \sum_{i=0}^{n} (i+1) \int_X c_{n-i}(\Omega_X)\ c_1(L)^i\,.$$

Thus, we recover the degree of the classical discriminant; see [7, Example 3.12].

Our second corollary focuses on vector bundles whose rank equals the dimension of their underlying variety. Part of this result provides an alternative proof for Proposition 2.2 in [10].

COROLLARY 2.7. *Let $X$ be a $n$-dimensional smooth complex projective variety. For any very ample vector bundle $E$ of rank $n$ on $X$, the discriminant locus $\Delta(E)$ is irreducible and* $\operatorname{def}(E) > 0$ *if and only if $X = \mathbb{P}^n$ and $E = \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^n}(1)$. Assuming that the vector bundle $E$ is 1-jet spanned and $(X, E) \neq \big(\mathbb{P}^n, \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^n}(1)\big)$, the discriminant locus $\Delta(E)$ is an irreducible hypersurface of degree*

$$\int_X \big(c_1(\Omega_X) + c_1(E)\big)\, c_{n-1}(E) + n\, c_n(E)\,.$$

*Proof.* Theorem 2.2 and Example 1.3 show that the discriminant locus $\Delta(E)$ is irreducible and $\operatorname{def}(E) > 0$ if and only if

$$\delta(E) = \int_X \big(c_1(\Omega_X) + c_1(E)\big)\, c_{n-1}(E) + n\, c_n(E) = 0\,.$$

When $(X, E) = \big(\mathbb{P}^n, \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^n}(1)\big)$, we have $\delta(E) = \big((-n-1) + n\big)n + n = 0$ and $\operatorname{def}(E) > 0$. Hence, it suffices to show that, for any very ample $E$ excluding $\bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^n}(1)$, we have $\delta(E) > 0$. If $E$ is 1-jet spanned as well as very ample, then Corollary 2.4 shows that $\deg \Delta(E) = \delta(E)$.

Since $E$ is very ample, we have $\int_X c_n(E) > 0$; see [5, Proposition 2.2]. Thus, it is enough to prove that $\int_X \big(c_1(\Omega_X) + c_1(E)\big)\, c_{n-1}(E) \geqslant 0$. Let $K_X$ be the canonical divisor on $X$ and let $D$ be the Cartier divisor associated to $\det(E)$. Since $E$ is very ample, $D$ is also. Moreover, Theorem 2 in [13] establishes that the adjoint divisor $K_X + D$ is nef unless $(X, E) = \big(\mathbb{P}^n, \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^n}(1)\big)$. The very ampleness of the vector bundle $E$ implies that $c_{n-1}(E) \neq 0$; again see [5, Proposition 2.2]. We deduce that $c_{n-1}(E)$ is the class of a curve $C$ by a Bertini-type argument; see [8, Theorem B]. It follows that

$$\int_X \big(c_1(\Omega_X) + c_1(E)\big) c_{n-1}(E) = (K_X + D) \cdot C \geqslant 0\,. \qquad \square$$

To illustrate this corollary, we recompute the degree of the discriminant locus for nonnegative twists of the tangent bundle on $\mathbb{P}^n$; see [2, Corollary 4.2] and [1, Example 4.9].

EXAMPLE 2.8. Let $d$ be a nonnegative integer and let $T_{\mathbb{P}^n}$ be the tangent bundle on $\mathbb{P}^n$. We have $c_1(\Omega_{\mathbb{P}^n}) = c_1\big(\mathscr{O}_{\mathbb{P}^n}(-n-1)\big)$. From the Euler sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \bigoplus_{i=1}^n \mathscr{O}_{\mathbb{P}^n}(1) \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0\,,$$

we deduce that

$$\int_{\mathbb{P}^n} c_i\big(T_{\mathbb{P}^n}(d)\big) = \sum_{j=0}^{i} \binom{n-j}{i-j} d^{i-j} \binom{n+1}{j}$$

for all nonnegative integers $i$. Combining Propositions 2.1–2.3 in [3], the Euler sequence also shows that vector bundle $T_{\mathbb{P}^n}(d)$ is very ample and 1-jet spanned. Thus, Corollary 2.7 establishes that the discriminant locus $\Delta\big(T_{\mathbb{P}^n}(d)\big)$ is an irreducible hypersurface and

$$\begin{aligned} \deg \Delta\big(T_{\mathbb{P}^n}(d)\big) &= nd \sum_{j=0}^{n-1} (n-j)\, d^{n-1-j} \binom{n+1}{j} + n \sum_{j=0}^{n} d^{n-j} \binom{n+1}{j} \\ &= n \sum_{j=0}^{n} d^{n-j} (n+1-j) \binom{n+1}{n+1-j} \\ &= n(n+1) \sum_{j=0}^{n} d^{n-j} \binom{n}{j} = n(n+1)(d+1)^n \,. \end{aligned}$$

## 3. Bogomolov Instability Theorem

In this section, we use calculations involving the discriminant divisor of a multi-section to give a simple proof of the Bogomolov instability theorem for vector bundles having rank 2 on an algebraic surface. At the very least, it was known to experts that one could give an argument along these lines. However, since it fits well with the themes of this note and is not widely known, we felt it worthwhile to include it here. We refer the reader to [9] for another approach having several points of contact with the present proof.

Let $X$ be a smooth complex projective surface. We consider a vector bundle $E$ of rank 2 on $X$, and denote by $D$ a Cartier divisor associated to $\det(E)$. The vector bundle $E$ is *Bogomolov unstable* if there exist a divisor $A$ and a finite scheme $W \subset X$ (possibly empty) such that the sequence

$$0 \longrightarrow \mathscr{O}_X(A) \longrightarrow E \longrightarrow \mathscr{O}_X(D-A) \otimes \mathscr{I}_W \longrightarrow 0 \,,$$

is exact, $(2A-D)^2 > 4 \operatorname{length}(W)$, and $(2A-D)\cdot H > 0$ for some (or any) ample divisor $H$ on $X$. Roughly speaking, being Bogomolov unstable means that the vector bundle $E$ contains an unexpectedly positive subsheaf.

Bogomolov's theorem asserts that instability is detected numerically.

THEOREM 3.1. *The vector bundle $E$ is Bogomolov unstable if and only if*

$$\int_X c_1(E)^2 - 4\, c_2(E) > 0 \,.$$

The defining exact sequence for a Bogomolov unstable vector bundle gives

$$\int_X c_2(E) = \mathrm{length}(W) + A \cdot (D - A)\,,$$

so the inequality holds. Thus, the essential content of the Theorem 3.1 is the converse statement: the inequality implies the existence of a destabilizing subsheaf $\mathscr{O}_X(A)$.

For our proof of this implication, suppose that $\int_X \big(c_1(E)^2 - 4\,c_2(E)\big) > 0$. Let $\pi\colon \mathbb{P}(E) \to X$ the projectivization of $E$, so $\dim \mathbb{P}(E) = 3$. The starting point, as in other arguments, is the next lemma.

Lemma 3.2. *When the vector bundle $E$ satisfies the inequality in Theorem 3.1, the line bundle $\mathscr{O}_{\mathbb{P}(E)}(2) \otimes \pi^*\mathscr{O}_X(-D)$ on $\mathbb{P}(E)$ is big. In other words, there is a positive number $C > 0$ such that, for all sufficiently large integers $m$, we have*

$$h^0\big(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(2m) \otimes \pi^*\mathscr{O}_X(-mD)\big) = h^0\big(X, \mathrm{Sym}^{2m}(E) \otimes \mathscr{O}_X(-mD)\big) \geqslant C\, m^3\,.$$

*Idea of proof.* The asymptotic Riemann–Roch theorem [11, Theorem 1.1.24] shows that

$$\chi\big(X, \mathrm{Sym}^{2m}(E) \otimes \mathscr{O}_X(-mD)\big) = \tfrac{1}{3}\big(c_1^2(E) - 4\,c_2(E)\big)\, m^3 + O(m^2)\,.$$

The assertion follows via Serre duality and the fact that the vector bundle $\mathrm{Sym}^{2m}(E) \otimes \mathscr{O}_X(-mD)$ has trivial determinant; see [12, Proposition 2]. □

Now let $H$ be an ample divisor on $X$. By an argument of Kodaira [11, Proposition 2.2.6], it follows from the lemma that, for all sufficiently large integers $m$, we have $H^0\big(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(2m) \otimes \pi^*\mathscr{O}_X(-mD - H)\big) \neq 0$. Fix one such integer $m$ and choose nonzero section

$$s \in H^0\big(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(2m) \otimes \pi^*\mathscr{O}_X(-mD - H)\big)\,.$$

Let $Z \coloneqq \mathrm{Zeroes}(s)$ be the zero locus of the global section $s$. The subscheme $Z$ is a divisor on $\mathbb{P}(E)$ of relative degree $2m$ over $X$.

We study the irreducible components of $Z$ with the aim of singling out a particularly interesting one. To begin, let $Z_0 \subset \mathbb{P}(E)$ denote the union of any "vertical" components of $Z$: $Z_0$ is the preimage under $\pi$ of the zeroes of a section of $\mathscr{O}_X(-A_0)$ for some anti-effective divisor $A_0$ on $X$. Write $Z_1, Z_2, \ldots, Z_t$ in $\mathbb{P}(E)$ for the remaining irreducible components of $Z$ allowing repetitions to account for multiplicities. In other words, each $Z_i \subset \mathbb{P}(E)$ is a reduced and irreducible divisor that is defined by a section of $\mathscr{O}_{\mathbb{P}(E)}(d_i) \otimes \pi^*\mathscr{O}_X(-A_i)$ for some divisor $A_i$ on $X$ and positive integer $d_i$. By construction, the divisor $A_0 + A_1 + \cdots + A_t$ is linearly equivalent to $mD + H$ and $d_1 + d_2 + \cdots + d_t = 2m$, so the divisor $\sum_{i \geqslant 1}\big(A_i - \frac{d_i}{2} D\big)$ is numerically equivalent to $H - A_0$. Since $-A_0$

is an effective divisor, it follows that $\left(\sum_{i\geqslant 1} 2\,A_i - d_i\,D\right)\cdot H > 0$. By reindexing the components if necessary, we may assume that $(2\,A_1 - d_1\,D)\cdot H > 0$.

The idea is to consider the discriminant divisor $\Delta \subseteq X$ over which the fibre of the map $Z_1 \to X$ is not $d_1$ distinct points. Specifically, Proposition 3.3 shows that the class of $\Delta$ is given by $\delta = d_1(d_1-1)\,D - 2(d_1-1)\,A_1$ and $\delta$ is either effective or zero, so $\delta\cdot H \geqslant 0$. However, if $d_1 > 1$, then this would contradict the assumption that $(2\,A_1 - d_1\,D)\cdot H > 0$. Thus, we have $d_1 = 1$ and $Z_1$ is defined by a (necessarily saturated) section in $H^0\big(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(1)\otimes \pi^*\mathscr{O}_X(-A_1)\big)$. The corresponding section in $H^0\big(X, E\otimes\mathscr{O}_X(-A)\big)$ defines a closed subscheme $W$ of $X$ and gives rise to a short exact sequence

$$0 \longrightarrow \mathscr{O}_X(A_1) \longrightarrow E \longrightarrow \mathscr{O}_X(D-A_1)\otimes\mathscr{I}_W \longrightarrow 0\,.$$

The inequality $\int_X c_1(E)^2 - 4\,c_2(E) > 0$ implies that $(2A-D)^2 > 4\,\operatorname{length}(W)$ and $(2A-D)\cdot H > 0$. Therefore, we have established that the vector bundle $E$ is unstable.

It remains to prove the following proposition.

Proposition 3.3. *Let $E$ be a vector bundle on $X$ having rank 2 and satisfying $\det(E) = \mathscr{O}_X(D)$, let $\pi\colon \mathbb{P}(E)\to X$ be the projectivization of $E$, and consider a reduced and irreducible divisor*

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}(E) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

*defined by a section of $\mathscr{O}_{\mathbb{P}(E)}(d)\otimes\pi^*\mathscr{O}_X(-A)$ for some positive integer $d$. The locus $\Delta(f)\subseteq X$ of points $x\in X$ over which the fibre $f^{-1}(x)$ fails to consist of $d$ distinct points supports an effective divisor in the class*

$$\delta = d(d-1)\,D + 2(d-1)\,A\,.$$

*In particular, this class is effective or zero.*

*Proof.* Consider the set $\Gamma := \big\{y \in Y \;\big|\; f \text{ is not étale at } y\big\}$. The map $f$ is generically étale because $Y$ is reduced. It follows that $\Gamma$ has dimension 1 (or is empty) and $\Delta(f) = f(\Gamma)$. We claim that, viewed as a cycle of codimension 2 on $\mathbb{P}(E)$, $\Gamma$ supports the effective class

$$\gamma := \Big((d-2)\,c_1\big(\mathscr{O}_{\mathbb{P}(E)}(1)\big) + \pi^*(D-A)\Big)\cdot\Big(d\,c_1\big(\mathscr{O}_{\mathbb{P}(E)}(1)\big) - \pi^*A\Big)\,. \qquad (*)$$

There are at least two ways to confirm this claim.

- As a cycle on $Y$, $\gamma$ is the class of the first degeneracy locus of the induced differential $df\colon f^*\Omega_X \to \Omega_Y$, so

$$\gamma = c_1(\Omega_Y - f^*\Omega_X) = c_1\big(\mathcal{O}_Y(K_Y - f^*K_X)\big)$$

  which is the class of the relative canonical divisor $K_{Y/X} \coloneqq K_Y - f^*K_X$. The adjunction formula shows that $K_{Y/X} = (K_{\mathbb{P}(E)/X} + Y)|_Y$. Thus, as a cycle on $\mathbb{P}(E)$, we have $\gamma = [(K_{\mathbb{P}(E)/X} + Y)|_Y] \cap [Y]$. Since we also have $[Y] = c_1\big(\mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*\mathcal{O}_X(-A)\big)$, the equation $(*)$ follows from the equality $[K_{\mathbb{P}(E)/X}] = c_1\big(\mathcal{O}_{\mathbb{P}(E)}(-2) \otimes \pi^*\mathcal{O}_X(D)\big)$; see [11, Section 7.3.A].
- The section $s$ in $H^0\big(X, \mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*\mathcal{O}_X(-A)\big)$ defining $Y$ lifts to a section of the first relative jet bundle of $ds \in H^0\big(\mathbb{P}(E), J_1^\pi\big(\mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*\mathcal{O}_X(-A)\big)\big)$, and $\Gamma = \text{Zeroes}(ds)$. From the canonical short exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}(E)/X}(d) \otimes \pi^*\mathcal{O}_X(-A) \longrightarrow J_1^\pi\big(\mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*\mathcal{O}_X(-A)\big) \longrightarrow \mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*\mathcal{O}_X(-A) \longrightarrow 0$$

  we see that $\gamma = c_2\big(J_1^\pi\big(\mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*\mathcal{O}_X(-A)\big)\big)$, which again establishes the equation $(*)$.

It remains to check that $\pi_*(\gamma) = \delta$. This follows from the Grothendieck relation

$$c_1\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big)^2 - \pi^*\big(c_1(E)\big) \cdot c_1\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big) + \pi^*\big(c_2(E)\big) = 0\,,$$

$\pi_*\big(\pi^*(\alpha) \cdot c_1\big(\mathcal{O}_{\mathbb{P}(E)}(1)\big)\big) = \alpha$, and $\pi_*\big(\pi^*(\beta)\big) = 0$ for any classes $\alpha \in A^1(X)$ and $\beta \in A^2(X)$. □

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