Examples of non-effective rays at the boundary of the Mori cone of blow-ups of the plane

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We dedicate this paper to Giorgio Ottaviani on his 60th birthday

Abstract. *In this paper we prove that no multiple of the linear system of plane curves of degree $d \geqslant 4$ with a point of multiplicity $d - m$ (with $2 \leqslant m \leqslant d$) and $m(2d - m)$ simple general points is effective.*

1. Introduction

Alex Massarenti and Massimiliano Mella asked us the following question. Consider 13 general points $p_0, \ldots, p_{12}$ in the projective plane and consider the class of a quartic curve with a singular point at p_0 and passing through $p_1, \ldots, p_{12}$. Is it the case that no multiple of this class is effective?

In trying to answer this question we got aware of the fact that we are able to prove the following more general result.

Theorem 1.1. *Let d be any integer and $p_0, \ldots, p_{m(2d-m)}$ general points in the plane with $m \leqslant d$. Consider the class (or system) $\xi_{d,m}$ of plane curves of degree d with a point of multiplicity at least $d - m$ at p_0 and passing through $p_1, \ldots, p_{m(2d-m)}$. Fix $k \geqslant 1$.*

(a) *For any $d \geqslant 4$ and any m with $2 \leqslant m \leqslant d$, the class $k\xi_{d,m}$ is not effective.*

(b) *For any d, the system $\xi_{d,1}$ is a pencil of rational curves and $\xi_{d,0}$ is composed with the pencil of lines through the point p_0 and has dimension d. The multiple linear systems $k\xi_{d,1}$ are composed with the corresponding pencil and have dimension k. There is no member of these systems that contains an irreducible curve which is not a component of a member of this pencil. The same is true for the system $\xi_{2,2}$.*

(c) *For $d = 3$, the systems $\xi_{3,3}$ and $\xi_{3,2}$ coincide with the system of cubics through 9 general points, which consists of a unique cubic C. The systems $k\xi_{3,3}$ and $k\xi_{3,2}$ consist of the unique curve kC.*

A few remarks are in order. First statements (b) and (c) in the theorem are trivial (we stated them for completeness and we will inductively use them in the proof of (a) which is the core of the theorem. Secondly, if $m = d$ or $m = d - 1$ the statement is Nagata's theorem for d^2 general points in the plane (see [4]), hence the theorem can be viewed as a generalization of Nagata's theorem. So for the proof of (a) we may and will assume $2 \leqslant m \leqslant d - 2$.

Let X_n be the blow–up of the projective plane at n general points. Let $\mathcal{L}_d(m_1, \ldots, m_n)$ be the linear system on X_n corresponding to plane curves of degree d with general points of multiplicities at least $m_1, \ldots, m_n$.

If we blow–up $p_0, \ldots, p_{m(2d-m)}$ we get the surface $X_{m(2d-m)+1}$ and $\xi_{d,m}$ can be interpreted as an element in $\mathrm{Pic}(X_{m(2d-m)+1})$; note that $\xi_{d,m}^2 = 0$. Moreover $\xi_{d,m}$ is nef. Indeed we consider a general plane curve C of degree d with a point p_0 of multiplicity $d - m$ and we can fix $m(2d - m)$ general points on C. If we blow up p_0 and the $m(2d-m)$ chosen points, the proper transform of C is an irreducible curve with 0 self–intersection, and therefore it is nef on the blow–up. Since nefness is an open condition, this is true for the general class $\xi_{d,m}$.

Our result says that there is no positive number k such that $\mathcal{L}_{kd}(k(d - m), k^{m(2d-m)})$ is non–empty (the exponential notation for repeated multiplicities is clear). If we set $N_1(X_{m(2d-m)+1}) = \mathrm{Pic}(X_{m(2d-m)+1}) \otimes_\mathbb{Z} \mathbb{R}$, then $\xi_{d,m}$ generates a rational ray in $N_1(X_{m(2d-m)+1})$ that is not effective (see [1, §3.1]) and therefore it sits in the boundary of the Mori cone of $X_{m(2d-m)+1}$. Such a ray, if rational in $N_1(X_{m(2d-m)+1})$, is called a *good ray* in [1, §3.2] whereas, if irrational, it is called a *wonderful ray*. So far no wonderful ray has been discovered[1]. However, proving that a given ray is good is in general not easy, and in [1] the authors were able to exhibit some examples. Therefore it is interesting to find good rays, and in this paper we make a new contribution in this direction.

Our proof uses the degeneration technique we introduced for analyzing the dimension of such linear system (see, e.g., [2]). We briefly recall this in Section 2. The proof is by induction on m, the case $m = 2$ being the critical one. We prove the $m = 2$ case in Section 5. This particular example relies on a subtlety that requires us to analyze, more deeply than what we did in [3], the case in which there are multiple (-1)–curves splitting off a linear system in the limit. This we describe in Section 3. Finally in Section 5 we finish the proof of Theorem 1.1.

We notice that the surprising phenomenon that allows us to make the final analysis of the limit linear systems in the case $m = 2$ is that we eventually end up with curves of a certain degree tn in the plane with n^2 points of multiplicity t,

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[1]After this paper was finished wonderful rays have been exhibited in the preprint "Irrational nef rays at the boundary of the Mori cone for very general blowups of the plane" (arXiv:2201.08634), by J. Roé and the two authors of the present paper.

which is currently the only case in which Nagata's conjecture is proven (see [4]). Indeed we use here an argument inspired by the original one of Nagata (see l.c.) to deal with these cases.

The ideas in this note can be generalized to prove more general similar results about general linear systems with zero self–intersection and we will do this in a forthcoming paper.

2. The degeneration method

In this section we briefly recall the degeneration technique that we use to analyse planar linear systems (see [2]). We want to study a linear system $\mathcal{L}_d(m_1, \ldots, m_n)$. To do this we consider a trivial family $\mathbb{P}^2 \times \mathbb{D} \to \mathbb{D}$ over a disc $\mathbb{D}$. In the central fibre over $0 \in \mathbb{D}$ we blow–up a line R producing a new family $\mathcal{X} \to \mathbb{D}$ with an exceptional divisor $F \cong \mathbb{F}_1$ and the proper transform $P \cong \mathbb{P}^2$ of the original central fibre. The new central fibre consists now of $F \cup P$, with F, P transversely intersecting along the line R, which is the (-1)–curve in F.

Next we fix a general points on P and b general points of F, so that $a+b = n$. Consider sections of the family $\mathcal{X} \to \mathbb{D}$ extending these n points to general points on the general fibre. Blowing up these sections, we have a degeneration of X_n to the union of an X_a (the blow–up of P at the a general points) and of an X_{b+1} (the blow–up of F at the b general points).

Since there is an obvious map $\pi : \mathcal{X} \to \mathbb{P}^2$, we have the bundle $\mathcal{O}_{\mathcal{X}}(d) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(d))$. This bundle restricts to the general fibre to $\mathcal{O}_{\mathbb{P}^2}(d)$. On the central fibre it restricts to the bundle $\mathcal{O}_{\mathbb{P}^2}(d)$ on P and to $\mathcal{O}_F(df)$, where f is the class of a fibre of the ruling of F over $\mathbb{P}^1$. This is a limit of the line bundle on the general fibre; there are other limits obtained by *twisting* by $\mathcal{O}_{\mathcal{X}}(-lP)$, i.e., by tensoring the above limit bundle by $\mathcal{O}_{\mathcal{X}}(-lP)$, with l an integer. This restricts to $\mathcal{O}_{\mathbb{P}^2}(d+l)$ on P and to $\mathcal{O}_F(df - lR)$ on F. So we have a discrete set of limits of $\mathcal{L}_d(m_1, \ldots, m_n)$, depending on all choices for a, b, l and distribution of the multiplicities among the $a + b$ points on the central fibre.

A section of a limit line bundle is given by a pair of sections on P and F, that restrict equally to the double curve R. We will call this the *naive matching condition.* Such a section could be identically zero on one of the components of the central fibre, and in this case a matching section on the other component corresponds to a section of the linear system (called a *kernel linear system*) of curves on the other component containing the double curve R. One way to prove emptiness of the system on the general fibre is to find a, b, a distribution of the multiplicities and a twisting parameter l such that there is no section of the limit line bundle on the central fibre that verifies the naive matching condition and that is non–zero on at least one of the two components.

An alternative approach to proving the emptiness of the linear system on the general fibre is the following. Suppose that the system is non–empty on the general fibre. Then for every choice of a, b and a distribution of the multiplicities, there will be a limit curve which must be the zero of a section of a limit line bundle given by some particular twist parameter l, this section being not identically zero on both P and F. As we said, naively the matching condition means that the two curves restrict equally to R. However we will see in the next section that when the curves are non–reduced the matching conditions are more subtle. We will call these conditions *refined matching conditions*. Hence, to prove that the system on the general fibre is empty, it suffices to find a, b and a distribution of the multiplicities so that for no l there is a limit curve as above, i.e., a pair of curves on P and F satisfying the refined matching. This will be the approach we will use in the proof of the case $m = 2$.

Clearly the former approach is easier than the latter, which however could be necessary if the naive approach fails for every twist, which could be the case.

3. Refined matching conditions

In this section we will perform an analysis, needed later, which is a generalization of the concepts of 1–*throws* and 2–*throws* considered in [3].

Suppose that a (-1)-curve C lives on a component P in a degeneration with two components P and F in the central fibre of a family $\mathcal{X} \to \mathbb{D}$, intersecting transversely along a curve R, and suppose we are given a line bundle $\mathcal{L}$ on $\mathcal{X}$. Suppose that the intersection number of C with the restriction of $\mathcal{L}$ to P is $-s$. Suppose in addition that C meets transversely at m points the double curve R.

For $m = 1$ we have the 1–throw considered in [3], which reveals that the appropriate matching conditions for a curve on F to be a limit is that it must have a point of multiplicity s at the intersection point of C with R, not simply an intersection multiplicity s.

Now suppose that $m > 1$. We blow up C in the threefold $\mathcal{X}$ m times, thus obtaining a new threefold $\mathcal{X}'$ and a new family $\mathcal{X}' \to \mathbb{D}$. This blows up F m times at each of the m intersection points of C with R, for a total of m^2 blow–ups. We denote by $\bar{F}$ the resulting surface.

These blow–ups create m ruled surfaces $Q_{m-1}, Q_{m-2}, \ldots, Q_1, Q_0$ which are stacked one on the other. In the central fiber of $\mathcal{X}'$, Q_i appears with multiplicity $m - i$, for $i = 0, \ldots, m - 1$. One checks that $Q_i \cong \mathbb{F}_i$, with non–positive section B_i and disjoint non–negative section S_i; on Q_i we have $B_i^2 = -i$, $S_i^2 = i$, and $S_i \sim B_i + if$, where f denotes as usual the fibre class and $\sim$ is the linear equivalence. Q_0 meets the surface P in a section B_0 (equal to C on P), with $B_0^2 = 0$. Each Q_i meets Q_{i+1} so that S_i (on Q_i) is identified with B_{i+1} (on Q_{i+1}). Each Q_i also meets the other component $\bar{F}$ in m fibers of the ruling, corresponding to the m points where C meets R.

The normal bundle of Q_0 in $\mathcal{X}'$ is $(-1/m)(B_0+(m-1)S_0+mf) = -B_0-f$. For $1 < i < m-1$, the normal bundle of Q_i in $\mathcal{X}'$ is $(-1/(m-i))((m-i+1)B_i + (m-i-1)S_i + mf) = -2B_i - (i+1)f$. For $i = m-1$, the normal bundle of Q_{m-1} in $\mathcal{X}'$ is $(-1)(2B_{m-1}+mf) = -2B_{m-1} - mf$.

When we pull back the bundle $\mathcal{L}$ to $\mathcal{X}'$, this pull back $\mathcal{L}'$ restricts to $-sf$ on each Q_i. At this point we make the additional assumption that s is divisible by m: write $s = hm$. Twist $\mathcal{L}'$ by $\mathcal{O}_{\mathcal{X}'}(-h(\sum_{i=0}^{m-1}(m-i)Q_i))$. Let us analyze the restriction of this new bundle on each component of the central fibre.

First we consider the surface P, on which the original curve C sits. Since the only exceptional surface that meets P is Q_0, we are twisting the restriction of the bundle on P by $-hmQ_0 = -sQ_0$, and since Q_0 restricts to C on P this removes sC from the restriction of the bundle on P, and then this restriction is trivial on C.

The restriction to Q_0 is

$$-sf - hmQ_0|_{Q_0} - h(m-1)Q_1|_{Q_0} = -sf - s(-B_0-f) - h(m-1)S_0 = hB_0.$$

For $1 < i < m-1$, the restriction to Q_i is

$$-sf - h(m-i+1)Q_{i-1}|_{Q_i} - h(m-i)Q_i|_{Q_i} - h(m-i-1)Q_{i+1}|_{Q_i} =$$
$$-sf - h(m-i+1)B_i - h(m-i)(-2B_i - (i+1)f) - h(m-i-1)S_i = 0.$$

Finally for $i = m-1$ the restriction to Q_{m-1} is

$$-sf - 2hQ_{m-2}|_{Q_{m-1}} - hQ_{m-1}|_{Q_{m-1}} = 0.$$

The above analysis shows that the bundle is now trivial on $Q_{m-1}, Q_{m-2}, \ldots, Q_1$, and non–trivial only on Q_0, where it consists of hB_0, i.e., h horizontal sections. Therefore the matching divisor on $\bar{F}$ does not meet any of the exceptional divisors of the first $m-1$ blow–ups, and meets only the last ones h times at each of the m points. Moreover, there is a *correspondence* on the divisors on the final exceptional curves, namely they must all agree with h horizontal sections. In other words, any one of these intersections determines all the other $m-1$ ones. This behaviour of the curves on $\bar{F}$ means that the curve on F must have at each of the m points of the intersection of C and R, m infinitely near points of multiplicity h along R. We denote this phenomenon by $[h^m]_R$. Hence the matching conditions for the curves on F can be written as $([h^m]_R)^m$, plus the correspondence.

We can summarize what we proved in this section in the following statement:

PROPOSITION 3.1. *Suppose we have a semistable degeneration of surfaces $\pi : \mathcal{X} \to \mathbb{D}$ over a disc $\mathbb{D}$ (i.e., $\mathcal{X}$ is smooth, all fibres of π are smooth except perhaps for the one over 0, that has normal crossings) and a line bundle $\mathcal{L}$ on $\mathcal{X}$ which restricts to line bundles on every component of the central fibre. Let P be a*

component of the central fibre containing a (-1)–curve C which is not a double curve and intersects the double curve R transversely at m points $p_1, \dots, p_m$ that are not triple points of the central fibre. Suppose that $C \cdot \mathcal{L} = -hm$, with $h > 0$. Then any curve on the central fibre that is a limit of a curve in the general fibre in the linear system determined by the restriction of $\mathcal{L}$, must satisfy the following conditions: for every point p_i the curve on the component different from P has the singularity of type $[h^m]_R$ at p_i and the final h infinitely near points to the p_i's of order m correspond in the sense described above.

4. The proof of the case $m = 2$

We focus in this section on the case $m = 2$. We will consider the degeneration described in Section 2 and first we want to describe the distribution of the multiplicities and the limit linear systems on P and F. For convenience we set $n = d - 1$ and note that there are $4n$ simple points in the case $m = 2$, which we will distribute evenly among P and F. So the limit linear systems will be

$$\mathcal{L}_P := \mathcal{L}_{kn+t}(k(n-1), k^{2n}), \quad \mathcal{L}_F := \mathcal{L}_{k(n+1)}(kn + t, k^{2n})$$

with t the twisting parameter.

In order to prove Theorem 1.1(a), for $m = 2$, we will use the refined matching approach. This requires that we prove that for any twisting parameter t there is no limit curve satisfying the refined matching conditions stated in Proposition 3.1.

Consider the curve class (useful on both P and F) equal to $\mathcal{L}_n(n-1, 1^{2n})$. We note that this linear system is of dimension 0 and consists of a unique (-1)–curve C.

The linear system $\mathcal{L}_{kn}(k(n-1), k^{2n})$ is equal to $|kC|$, and has dimension 0. Therefore if $t < 0$, then $\mathcal{L}_P$ is empty. Hence we may assume $t \geqslant 0$.

Let us analyze $\mathcal{L}_F$. The lines through the first point and through any one of the other $2n$ points split off with multiplicity t. The residual system has the form $\mathcal{L}_{k(n+1)-2nt}(kn + t - 2tn, (k - t)^{2n})$. Now we intersect this system with C and get $-t(n-1)$. So C splits $t(n-1)$ times. The further residual system is $\mathcal{L}'_F = \mathcal{L}_{(n+1)(k-tn)}(n(k-tn), (k-tn)^{2n})$. For $\mathcal{L}'_F$ to be effective, one needs $t \leqslant k/n$, which we will assume from now on. A sequence of n quadratic transformations (each based at the first point and at two of the $2n$ points) brings $\mathcal{L}'_F$ to the complete linear system $\mathcal{L}_{k-tn}$.

As for $\mathcal{L}_P$, one sees that C splits off with multiplicity $k-tn$ and the residual system is $\mathcal{L}'_P = \mathcal{L}_{t(n^2+1)}(tn(n-1), (tn)^{2n})$. A sequence of n quadratic transformations (each based at the first point and at two of the $2n$ points) brings this system to $\mathcal{L}_{t(n+1)}(t^{2n})$.

Let us see what the refined matching implies on $\mathcal{L}'_P$ or rather on its Cremona transform $\mathcal{L}_{t(n+1)}(t^{2n})$.

Each of the $2n$ lines splitting t times from $\mathcal{L}_F$ are (-1)–curves meeting the double curve R once. Hence in the notation of the previous section $m = 1$ and $s = h = t$ and therefore we are imposing $2n$ points of multiplicity t to the linear system $\mathcal{L}_{t(n+1)}(t^{2n})$. These points are located along the curve T, the Cremona image of R on P, which is easy to see to be equal to a curve of degree $n + 1$, with a point of multiplicity n.

Also the curve C splits $t(n-1)$ times from $\mathcal{L}_F$ and meets the double curve R transversely at $n - 1$ points. In the notation of the previous section $m = n - 1$, $s = t(n - 1) = tm$ hence $h = t$. Therefore we are imposing to $\mathcal{L}_{t(n+1)}(t^{2n})$ also the multiple points $([t^{n-1}]_T)^{n-1}$, plus the correspondence.

Eventually the resulting system on P is Cremona equivalent to the system $\mathcal{L}$ of plane curves of degree $t(n+1)$, with $(n+1)^2$ points of multiplicity t, plus the correspondence. These $(n+1)^2$ points are distributed in $2n$ general points, $2n$ general points on T, and $n - 1$ general points of type $[t^{n-1}]_T$.

Assume now $t > 0$. We want to prove that $\mathcal{L}$ is empty and therefore $\mathcal{L}_P$ is empty. To prove this, we need the following:

LEMMA 4.1. *For any $t > 0$ the linear system of plane curves of degree $t(n+1)$ with $n - 1$ general points of type $[t^{n-1}]_T$, with $2n$ general points of multiplicity t on T and $2n$ additional general points of multiplicity t consists of at most one element.*

Proof. We specialize the configuration of the imposed multiple points to $n - 1$ general points of type $[t^{n-1}]_T$, and with $4n$ more general points of multiplicity t on T. This is a total $(n+1)^2$ points (some of them are infinitely near) forming a divisor D on T supported on the smooth locus of T. By generality, for no positive integer t, tD belongs to $|\mathcal{O}_T(t(n+1))|$. So any curve of degree $t(n+1)$ with the above multiple points on T must contain T. To the residual curve we may apply the same argument, so T recursively splits; by induction we conclude that the only possible member of the system is tT. This implies the assertion. $\square$

To prove that $\mathcal{L}$ is empty, we notice that the possible unique curve satisfying the multiplicity conditions imposed on $\mathcal{L}$ (see Lemma 4.1) will not satisfy the required correspondence as soon as $m = n - 1 \geqslant 2$, i.e., $n \geqslant 3$, hence $\mathcal{L}_P$ is empty.

Finally we have to deal with the case $t = 0$. In this case we take $k = hn$ and $\mathcal{L}_P$ consists of the unique curve hnC. Now C is a (-1)–curve that intersects R transversely at n points. Therefore, in the notation of Section 3, we have $m = n$, $s = k$. So the refined matching implies that we eventually have to impose to $\mathcal{L}_F$, or rather to its Cremona transform $\mathcal{L}_{hn}$, n points of type $[h^n]_{T'}$, where T' is the Cremona image of the double curve R, plus the correspondence. Note that T' is a curve of degree n with a point of multiplicity $n - 1$.

By the same argument as in Lemma 4.1, we see that the only curve verifying all the above conditions is hT', so in the original analysis it is hR plus some exceptional curves which appear in the Cremona transformation. However, as we saw in the refined matching analysis, on P the bundle is now trivial, so the corresponding section on P has to vanish identically on P. This shows that there is no limit curve in the case $t = 0$ either.

Eventually we have seen that for any twisting parameter t there is no limiting curve verifying the refined matching conditions, finishing the proof of Theorem 1.1(a) for $m = 2$ and $d \geqslant 4$.

5. The proof for $m > 2$

In this section we will complete the proof of Theorem 1.1(a) in the case $m \geqslant 3$, arguing by induction on m (the case $m = 2$ for all $d \geqslant 4$ is the starting case of the induction). For this we will again use the degeneration as in Section 2 and the naive matching approach will be sufficient.

Let us describe the limit linear systems we will use, i.e.,

$$L_P = \mathcal{L}_{k(d-2)}(k(d-m), k^{(m-2)(2d-m-2)}) = k\xi_{d-2,m-2},$$

$$L_F = \mathcal{L}_{kd}(k(d-2), k^{4d-4}) = k\xi_{d,2}.$$

By the $m = 2$ case, L_F is empty, and therefore also the kernel system is empty. Hence it suffices to show that the kernel system on P is also empty.

First consider the case $m = 3$. Then, by Theorem 1.1(b), L_P is composed with a pencil of rational curves, and the kernel system is empty because it consists of the members of L_P that vanish along the double curve R, which is a general line on P. This proves the $m = 3$ case for all $d \geqslant 5$ (remember that $m \leqslant d - 2$).

Next assume $m \geqslant 4$, and therefore, since $m \leqslant d - 2$, we have $d \geqslant 6$. Then, by induction, L_P is empty and hence also the kernel system is empty, finishing the proof of Theorem 1.1.

Acknowledgements

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